

# Efficiency Statistics and Bounds for Systems with Broken Time-Reversal Symmetry

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Universal properties of the statistics of stochastic efficiency for mesoscopic time-reversal symmetry broken energy transducers are revealed in the Gaussian approximation. We also discuss how the second law of thermodynamics restricts the statistics of stochastic efficiency. The tight-coupling (reversible) limit becomes unfavorable, characterized by an infinitely broad distribution of efficiency at *all times*, when time-reversal symmetry breaking leads to an asymmetric Onsager response matrix. The underlying physics is demonstrated through the integer quantum Hall effect and further elaborated in a triple-quantum-dot three-terminal thermoelectric engine.

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*Introduction.*— Nonequilibrium phenomena of macroscopic systems are described by their responses to external perturbations and follow the laws of thermodynamics [1]. Statistical fluctuations of a measurable quantity are negligible in macroscopic systems but in very small (“mesoscopic”) systems they play an essential role [2]. Importantly, in such a mesoscopic regime a single measurement of a physical quantity (e.g., electrical current) does *not* follow the laws of thermodynamics (although its average over many different measurements does). Well known examples are the Jarzynski equality [3–6] and the fluctuation theorem, which states that a stochastic negative entropy production may show up, though exponentially unlikely compared to its corresponding positive entropy production process [7–9]. An important consequence of the fluctuation theorem is that the “stochastic energy efficiency” of a very small engine can be larger than the Carnot efficiency, although the average efficiency over many measurements is smaller or equal to the Carnot efficiency, in accord with the second law of thermodynamics [1, 2, 10, 12, 14–17]. The study of stochastic efficiency of very small energy transducers is pertinent to understanding of mesoscopic thermoelectric energy conversion in normal electron [18, 19] and Cooper pair islands [20], biological photosynthesis in an individual reaction unit [21], and photo-mechanical energy conversion in quantum optomechanical systems [22].

Analyzing stochastic efficiency, it was recently shown that the Carnot efficiency is the least likely stochastic efficiency [10], later found to be solely the consequence of the fluctuation theorem [23] for time-reversal symmetric (TRS) energy transducers [2]. Breaking time-reversal symmetry can shift the least likely efficiency away from the Carnot efficiency [2, 12, 14]. However, little is known about efficiency statistics of time-reversal symmetry broken (TRB) mesoscopic energy transducers, which is the gap we want to fill in this Letter. Another fundamental question concerns how the second law of thermodynamics

restricts the statistics of efficiency for both TRS and TRB energy transducers. In this Letter we address these problems for systems operating in the linear-response regime where fluctuations can be well described within the Gaussian approximation. One of our key results is that in the reversible (or “tight-coupling”) limit the distribution of stochastic efficiencies becomes infinitely broad at *all times*, thus, the average efficiency loses its meaning even in the infinite long time limit. This anomaly occurs only for reversible TRB energy transducers of which the Onsager response matrix is asymmetric. We discuss this anomaly using the example of the integer quantum Hall effect. We further demonstrate our result with a triple-quantum-dot (QD) thermoelectric engine where a full-counting statistics method confirms our Gaussian theory.

*Efficiency statistics in the Gaussian approximation.*— We consider a generic situation in which there are two energy output channels (“1” and “2”). Each of the channels has a thermodynamic “current” and a conjugated driving force (i.e., affinity). The time-integrated currents are denoted by  $J_i$  ( $i = 1, 2$ ) while the time-intensive current is defined as  $I_i = J_i/t$  with  $t$  the total time of operation. The affinities are associated with the properties of the reservoirs (e.g., temperatures and electrochemical potentials) and hence fluctuate negligibly. In contrast, the currents may fluctuate considerably. A small TRB machine can be characterized in the linear-response regime by  $\bar{I}_i = M_{ij}A_j$  ( $i, j = 1, 2$ ), or  $\vec{\bar{I}} = \hat{M}\vec{A}$  with  $\vec{\bar{I}} = (\bar{I}_1, \bar{I}_2)$  and  $\vec{A} = (A_1, A_2)$ . As well, in this regime the statistics of the currents at long time  $t$  can be described within the Gaussian approximation by the distribution  $P_t(\vec{I}) = \frac{t\sqrt{\det((\hat{M}^{-1})_{sym})}}{4\pi} \exp(-\frac{t}{4}\delta\vec{I}^T \cdot \hat{M}^{-1} \cdot \delta\vec{I})$  [24]. Here  $\det((\hat{M}^{-1})_{sym})$  is the determinant of the symmetric part of the inverse of the Onsager response matrix  $\hat{M}$  and the superscript “ $T$ ” denotes transpose. While averaged quantities are represented with a bar over the symbols throughout this paper,  $\delta\vec{I} = \vec{I} - \vec{\bar{I}}$  repre-

sents fluctuations of the currents. From the probability distribution of stochastic currents we calculate the distribution of efficiency  $P_t(\eta)$  [1]. We then obtain the large deviation function (LDF) of the stochastic efficiency  $\mathcal{G}(\eta) \equiv -\lim_{t \rightarrow \infty} t^{-1} \ln[P_t(\eta)]$ . The scaled LDF is derived in the Supplementary Material. It is given by

$$J(\eta) \equiv \frac{\mathcal{G}(\eta)}{\bar{S}_{tot}} = \frac{J(\eta_C) (\eta + \alpha^2 + \alpha q r + \alpha q \eta)^2}{(1 + \alpha^2 + \alpha q r + \alpha q) (\eta^2 + \alpha^2 + \alpha q \eta + \alpha q r \eta)}, \quad (1)$$

where  $\bar{S}_{tot} = \sum_i \bar{I}_i A_i$  is the average total entropy production rate and

$$J(\eta_C) \equiv \frac{4 - q^2(1+r)^2}{16(1-q^2r)} \quad (2)$$

is the *scaled LDF at Carnot efficiency*. Here,

$$q \equiv \frac{M_{21}}{\sqrt{M_{22}M_{11}}}, \quad r \equiv \frac{M_{12}}{M_{21}}, \quad \alpha \equiv \frac{A_1 \sqrt{M_{11}}}{A_2 \sqrt{M_{22}}}, \quad (3)$$

are dimensionless parameters that characterize the responses of the system and the applied affinities. We term  $q$  the degree of coupling [25],  $r$  the TRB parameter [26], and  $\alpha$  the affinity parameter [5]. In addition, the efficiency is defined as  $\eta = -I_1 A_1 / (I_2 A_2)$  [2, 5, 24, 25]. For thermal engines,  $\eta = \tilde{\eta} / \tilde{\eta}_C$ , with the standard definition of energy efficiency  $\tilde{\eta} = W/Q$  and  $\tilde{\eta}_C$  the original Carnot efficiency. In our scheme, efficiency is scaled so that the Carnot (reversible) efficiency corresponds to  $\eta_C \equiv 1$ .

The second law of thermodynamics requires that [5, 26]  $\bar{S}_{tot} \geq 0$  and hence  $M_{11}, M_{22} \geq 0$  and  $M_{11}M_{22} \geq (M_{21} + M_{12})^2/4$ , i.e.,

$$-2 \leq q(1+r) \leq 2. \quad (4)$$

Equality is attained only in the “tight-coupling” limit [25] where the average efficiency reaches its upper bound. It has been proposed [26] that breaking time-reversal symmetry can open the possibility of achieving Carnot efficiency at finite output power and improving the efficiency at maximum output power to overcome the Curzon-Ahlborn limit [28]. In contrast to the average efficiency discussed in previous works [3, 5, 6, 8, 26, 32, 33], the LDF  $J(\eta)$  allows us to examine the statistics of efficiency fluctuations. Particularly, we will show that in TRB systems the tight-coupling limit becomes unfavourable as the efficiency distribution becomes infinitely broad at *all times*.

The LDF in TRB systems, Eq. (1), is a key expression in our work. As we discuss below, its shape can be characterized by three quantities: the average value of efficiency  $\bar{\eta}$ , the least probable efficiency  $\eta^*$ , and the width of the distribution around the average,  $\sigma_\eta$ . In what follows we investigate these properties, particularly, under two (separate) experimentally relevant conditions of

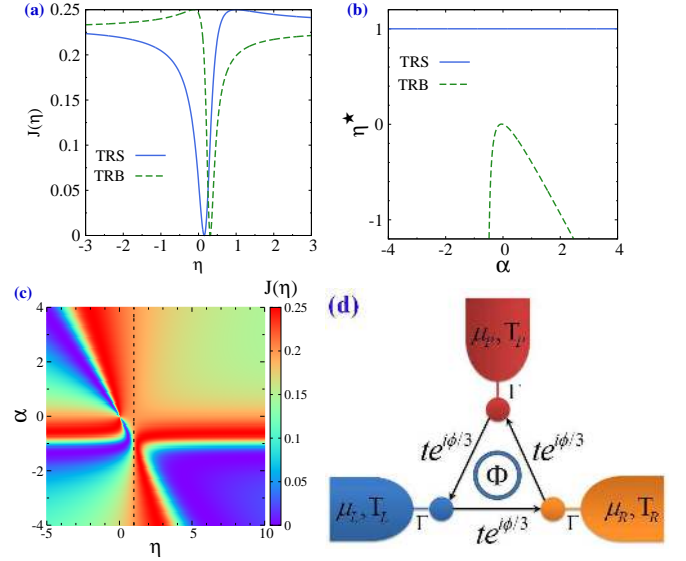


FIG. 1. Efficiency statistics for TRS and TRB systems. (a) An example of the LDF  $J(\eta)$  for TRS (blue,  $r = 1$ ) and TRB (green,  $r = 1.6$ ) cases with  $\alpha = -0.3$  and  $q = 0.7$ . (b) The least probable efficiency  $\eta^*$  for these two systems at different  $\alpha$ . (c) LDF  $J(\eta)$  as a function of  $\alpha$  and  $\eta$  for a TRB system with  $r = 1.6$  and  $q = 0.7$ . Note that the least probable efficiency (the maximum of the LDF)  $\eta^*$  shifts away from the Carnot efficiency [ $\eta_C \equiv 1$ , labeled as the dashed line in (c)] for TRB systems as demonstrated in the panels (a), (b), and (c). In panel (c) the minimum of  $J(\eta)$  for  $\eta \gg 1$  with large negative  $\alpha$  is associated with the reversed machine of which the actual efficiency is  $1/\eta$ , instead of  $\eta$ . (d) Our analysis is exemplified on a triple-QD thermoelectric device, see Fig. 4 and text.

maximum average efficiency and maximum average output power.

We begin with some general properties of the LDF. First, the LDF at the Carnot efficiency  $J(\eta_C)$  is *independent of affinities but it is solely determined by the response coefficients*. It is also invariant under time-reversal operation, which turns  $r \rightarrow 1/r$  and  $q \rightarrow qr$  [23, 26].  $J(\eta_C)$  can be suppressed by breaking time-reversal symmetry, particularly in approaching the tight-coupling limit. Second, in the TRS limit,  $r = 1$ , Eq. (1) goes back to results obtained in Refs. [1, 2, 10, 15]. In more general situations we find that  $0 \leq J(\eta) \leq 1/4$  is guaranteed by the thermodynamic bound (4) [see Supplementary Material]. Moreover,  $J(\eta)$  has only one minimum and one maximum. While the minimum  $J(\bar{\eta}) = 0$  is reached at the average efficiency  $\bar{\eta} = -\alpha(\alpha + qr)/(\alpha q + 1)$ , the maximum value  $J(\eta^*) = 1/4$  is realized at the least probable efficiency

$$\eta^* = 1 + \frac{q(r-1)(1 + \alpha q + \alpha q r + \alpha^2)}{q - qr - 2\alpha + q^2(1+r)\alpha}. \quad (5)$$

In the TRS limit, the least likely efficiency is *always* identical to the Carnot efficiency,  $\eta^* = \eta_C \equiv 1$  [1, 2, 10, 15].

For TRB systems, in contrast, we find here that  $\eta^*$  depends on the parameters  $q$ ,  $r$ , and  $\alpha$ , see Fig. 1(a), (b), and (c). We note that  $\eta^*$  diverges at  $\alpha_c = \frac{q(1-r)}{2-q^2-q^2r}$ . For  $|r| > 1$ ,  $\alpha_c$  produces a positive average efficiency and output power, relevant for device operation. Besides,  $\eta^* \xrightarrow{r \rightarrow \infty} \infty$  for all  $\alpha$  and  $q$ . It has been shown that this limit is achievable for a triple-QD thermoelectric device [see Fig. 1(d) for schematic of the device] when  $M_{21} \rightarrow 0$  and  $M_{12} \neq 0$  take place simultaneously [6], see also Supplementary Materials.

The width of the distribution around the average efficiency,  $\sigma_\eta$ , is another key characteristic of efficiency fluctuations. Expanding  $J(\eta)$  around its minimum  $\bar{\eta}$ , one writes  $J(\eta) \simeq \frac{1}{2\sigma_\eta^2}(\eta - \bar{\eta})^2 + \mathcal{O}((\eta - \bar{\eta})^3)$ , to provide here

$$\sigma_\eta = \frac{2\sqrt{2}|\alpha|(1-q^2r)(1+\alpha^2+\alpha q+\alpha qr)}{(1+\alpha q)^2\sqrt{4-q^2(1+r)^2}}. \quad (6)$$

We now proceed to describe the properties of  $\bar{\eta}$ ,  $\eta^*$  and  $\sigma_\eta$  under conditions for optimized (average) operations.

*Efficiency fluctuations at maximum average efficiency.*— We obtain the familiar form for the maximum average efficiency

$$\bar{\eta} = \bar{\eta}_{max} = r \frac{\sqrt{ZT+1}-1}{\sqrt{ZT+1}+1} = r \left( \frac{1-\sqrt{1-q^2r}}{1+\sqrt{1-q^2r}} \right) \quad (7)$$

when  $\alpha = -qr/(1+\sqrt{1-q^2r})$  [5], expressed in terms of the figure of merit for energy conversion  $ZT = q^2r/(1-q^2r)$  [5, 26]. The thermodynamic upper bound of the average efficiency, reached at the tight-coupling limit,  $|q(1+r)| \rightarrow 2$ , is [5, 26]

$$\bar{\eta}_{bound} = \min\{r^2, 1\}. \quad (8)$$

One of our key results here is that under the maximum average efficiency condition, the least probable efficiency (5) reduces to ( $\eta_C \equiv 1$ )

$$\eta^* = r. \quad (9)$$

In the TRS limit this recovers recent findings that Carnot efficiency is the least probable stochastic efficiency [2, 10]. The maximum average efficiency, the upper bound of the average efficiency, and the least probable efficiency are plotted in Fig. 2(a) as a function of  $r$  for  $q = 0.5$ , and we observe that

$$\eta^* \geq \bar{\eta}_{bound} \geq \bar{\eta}_{max}, \quad \forall r \geq 0, \quad (10a)$$

$$\eta^* < \bar{\eta}_{max} \leq \bar{\eta}_{bound}, \quad \forall r < 0. \quad (10b)$$

The least probable efficiency  $\eta^*$  coincides with  $\bar{\eta}_{bound}$  only in the TRS limit,  $r = 1$ , or for the TRB case with  $r = 0$ .

The width parameter  $\sigma_\eta$  is plotted in Fig. 2(b) where the white region in the figure is forbidden by the second

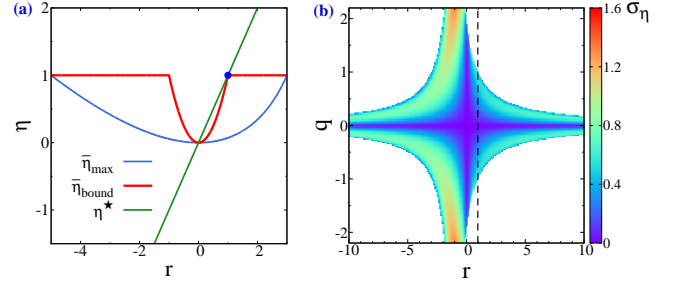


FIG. 2. Efficiency statistics under the maximum average efficiency condition. (a) Maximum average efficiency  $\bar{\eta}_{max}$  (blue curve), thermodynamic upper bound of the average efficiency  $\bar{\eta}_{bound}$  (red curve), and the least probable efficiency  $\eta^*$  (green curve) for  $q = 0.5$ . The blue dot represents the TRS point,  $r = 1$ . (b) Width of efficiency distribution  $\sigma_\eta$ . The dashed line represents the TRS limit,  $r = 1$ . The white region is forbidden by the thermodynamic bound (4).

law of thermodynamics according to (4). It is small when  $q$  or  $r$  are small, corresponding to the “weak-coupling limit” when the average efficiency is small. Physically, this can be understood as that two nearly uncoupled currents are very unlikely to have collective fluctuations which result in a considerably large efficiency.

Approaching the tight-coupling limit  $|q(1+r)| \rightarrow 2$  [which cannot be depicted in Fig. 2(b)], we find that  $\sigma_\eta \rightarrow \infty$  for  $0 < |r| < 1$  and  $r = -1$ , while  $\sigma_\eta \rightarrow 0$  for other regimes. In the TRS limit,  $r = 1$ , our results agree with Ref. [1]. The singular behavior of  $\sigma_\eta$  in this limit can be understood by noticing that the denominator of Eq. (6) vanishes in the tight-coupling limit whereas the numerator is proportional to the (average) total entropy production rate. It has been shown [5, 26] that in the tight-coupling limit, for  $|r| < 1$ , the maximum average efficiency is attained at *finite* (average) total entropy production rate (thus the upper bound efficiency is *less than* 100%). In contrast, for  $|r| \geq 1$ , the maximum average efficiency is reached when the (average) total entropy production rate is zero (hence the upper bound efficiency is 100%). When  $\bar{S}_{tot} = 0$  the unscaled LDF  $\mathcal{G}(\eta) = \bar{S}_{tot}J(\eta)$  is always zero. Therefore, in the tight-coupling limit the distribution of stochastic efficiency is infinitely broad for  $|r| \geq 1$  as well. The only exception is the TRS limit,  $r = 1$ , where the width of efficiency distribution becomes zero[1].

Direct examination of Eq. (1) shows that in the tight-coupling limit  $J(\eta_C) = 0$  (except for  $r = 1$ , the TRS limit).  $J(\eta)$  thus vanishes whenever the average entropy production rate  $\bar{S}_{tot}$  is nonzero. However, when  $\bar{S}_{tot} = 0$ ,  $J(\eta)$  is ill-defined, nevertheless the unscaled LDF  $\mathcal{G}(\eta) = \bar{S}_{tot}J(\eta)$  is always zero. In addition, for  $r = 0$  and  $\alpha = 0$ ,  $J(\eta)$  is constant for all  $\eta$ . Therefore, the distribution of efficiency is infinitely broad for *any* TRB energy transducer in the tight-coupling limit.

An example that may help to understand the anomaly in the tight-coupling limit is the integer quantum Hall effect. For example, at filling factor  $\nu = 1$ , electrical transport in the quantum Hall insulator is described by

$$\begin{pmatrix} j_x \\ j_y \end{pmatrix} = \frac{e^2}{h} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \mathcal{E}_x \\ \mathcal{E}_y \end{pmatrix} \quad (11)$$

where  $j_{x/y}$  and  $\mathcal{E}_{x/y}$  are the electrical currents and fields along the  $x/y$  direction, respectively. Although the quantum Hall insulator does not conduct electron longitudinally, charge can be conducted via the Berry phase effect, or more physically through the chiral edge states. In this way, the system converts electrical energy in the  $x$  direction  $W_x = j_x \mathcal{E}_x = \frac{e^2}{h} \mathcal{E}_x \mathcal{E}_y$  to electrical energy in the  $y$  direction  $W_y = j_y \mathcal{E}_y = -\frac{e^2}{h} \mathcal{E}_x \mathcal{E}_y$  (and vice versa). The macroscopic efficiency  $\eta = -W_y/W_x$  is always 100%, and the output power  $-W_y$  is finite for nonzero (positive)  $\mathcal{E}_x \mathcal{E}_y$ .

However, in the Gaussian description, the distributions of the electrical currents  $j_x$  and  $j_y$  are singular because the transport is completely dissipationless. This also leads to singular distributions of the output power and efficiency. Casting into our parameters, the integer quantum Hall systems have  $q = \infty$  and  $r = -1$ . Nevertheless, in a finite system the longitudinal conductance is not vanishing (i.e.,  $q$  is finite). A further examination of current noises in a Hall measurement needs more careful treatment [34], which is beyond the scope of this work. We speculate that close to the tight-coupling limit, the Gaussian approximation is inadequate to describe fluctuations in the system. However, energy efficiency is much easier to measure in this system since the measurement of stochastic electric currents in mesoscopic systems is a rather mature technology[34]. It is much easier to measure stochastic efficiency distribution in this time-reversal symmetry broken system than in other known solid-state systems.

*Efficiency fluctuations at maximum average output power.*— We now turn our attention to another highly pursued situation, the maximum average output power condition [17] arrived at  $\alpha = -qr/2$ . Under this condition the average efficiency is [5, 26]

$$\bar{\eta}(W_{max}) = \frac{rZT}{2(ZT + 2)} = \frac{q^2 r^2}{4 - 2q^2 r}, \quad (12)$$

with the thermodynamic upper bound [5, 26]

$$\bar{\eta}_{bound}(W_{max}) = \frac{r^2}{1 + r^2}, \quad (13)$$

reached in the tight-coupling limit. Note that for  $r^2 > 1$  the efficiency at maximum power can be larger than 50% (the value of the Curzon-Ahlborn efficiency in the linear-response regime). The least likely efficiency is found to be

$$\eta^* = r \left( \frac{4 - 3q^2 r - q^2 r^2}{4 - 2q^2 r - 2q^2 r^2} \right). \quad (14)$$

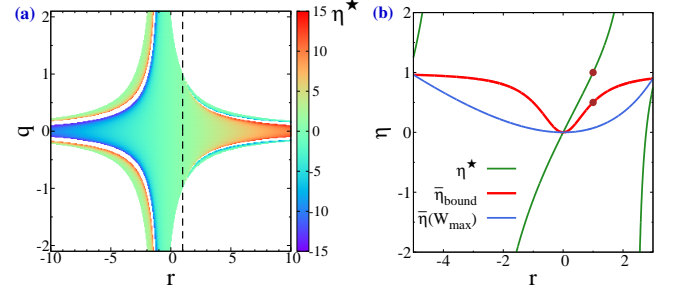


FIG. 3. Efficiency statistics under the maximum average power condition. (a) Least likely efficiency  $\eta^*$  as a function of  $r$  and  $q$ . The dashed line represents the TRS limit  $r = 1$ . The white region outside is forbidden by the thermodynamic bound (4). In contrast, the white region inside the color graphics region is due to divergences of  $\eta^*$  at special  $r$ - $q$  lines given by  $r = (-1 \pm \sqrt{1 + 8/q^2})/2$ . (b) Least probable efficiency  $\eta^*$ , average efficiency  $\bar{\eta}(W_{max})$  and its upper bound  $\bar{\eta}_{bound}(W_{max})$  at  $q = 0.5$  as a function of  $r$ . The dots identify values in the TRS limit,  $r = 1$ .

Calculations in Fig. 3 indicate that there are lines of singularities for  $\eta^*$ , varying with  $q$  and  $r$ . These singularity lines appear at  $r = (-1 \pm \sqrt{1 + 8/q^2})/2$ , and they reach to the thermodynamic bound in the  $r$ - $q$  plane at  $r = 1$  and  $q = \pm 1$ . The width of efficiency distribution under maximum average output power condition also shows the same singular behavior as that under the maximum average efficiency condition. In the Supplementary Material we explore  $J(\eta)$  under different situations and manifest its rich features.

*TRB thermoelectric transport in three-terminal systems.*— We exemplify our analysis within a mesoscopic triple-QD thermoelectric device, see Fig. 1(d). The affinities for a two-terminal thermoelectric device are the electrochemical potential  $A_1 = \frac{\mu_L - \mu_R}{eT_R}$  and the temperature difference  $A_2 = \frac{1}{T_R} - \frac{1}{T_L}$ , where  $e$  is the electronic charge,  $T_{L,R}$  and  $\mu_{L,R}$  denote the temperatures and chemical potentials in the left ( $L$ ) and right ( $R$ ) electronic reservoirs. In order to receive  $M_{12} \neq M_{21}$ , we introduce a third (probe  $P$ ) terminal and employ the constraints that the average thermal and electrical currents flowing out of the probe terminal are zero [8, 35]. These conditions set the temperature  $T_P$  and chemical potential  $\mu_P$  in the probe. Each QD is coupled through elastic tunneling to the nearby reservoir thus we employ the indices 1/2/3 to identify the leads  $L/R/P$ , respectively. Hopping between QDs are affected by the magnetic flux  $\Phi$  piercing through at the center with  $\phi = 2\pi\Phi/\Phi_0$  ( $\Phi_0$  is flux quantum). The system is described by the Hamiltonian  $\hat{H} = \hat{H}_{qd} + \hat{H}_{lead} + \hat{H}_{tun}$  where  $\hat{H}_{qd} = \sum_{i=1,2,3} E_i d_i^\dagger d_i + (te^{i\phi/3} d_{i+1}^\dagger d_i + \text{H.c.})$ ,  $\hat{H}_{lead} = \sum_{i=1,2,3} \sum_k \varepsilon_k c_{ik}^\dagger c_{ik}$ , and  $\hat{H}_{tun} = \sum_{i,k} V_k d_i^\dagger c_{ik} + \text{H.c.}$

This noninteracting model has been analyzed thoroughly in Ref. [6] using the Landauer-Büttiker approach,



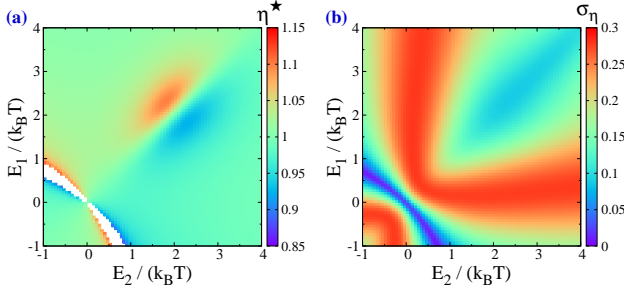


FIG. 4. Triple-QD thermoelectric systems at the maximum average output power condition: (a) Least probable efficiency  $\eta^*$  and (b) width of efficiency distribution  $\sigma_\eta$ . The QD connected to the probe terminal is set at  $E_3 = 2$ .  $\phi = \pi/2$ ,  $\Gamma = 0.5$ , and  $t = -0.2$ . The white region in (a) depicts very large or very small (negative)  $\eta^*$  values which are not properly incorporated into the figure.

to study transport properties and the average efficiency. We recall here that the transmission function is given by  $\hat{S}(\omega) = -\hat{1} + i\Gamma\hat{G}^r(\omega)$  where  $\hat{G}^r(\omega) = [(\omega + i\Gamma/2)\hat{1} - \hat{H}_{qd}]$  is the retarded Green's function of the quantum dots and the damping rate  $\Gamma = 2\pi \sum_k |V_{ik}|^2 \delta(\omega - \varepsilon_{ik})$  is assumed to be a constant (independent of energy) for all three leads. Using  $\phi = \pi/2$ ,  $\Gamma = 0.5$  and  $t = -0.2$  (the equilibrium chemical potential is set at zero, the energy unit is  $k_B T$ ), we calculate transport coefficients and substitute them into Eq. (1) to obtain the LDF of stochastic efficiency. We then calculate the least probable efficiency  $\eta^*$  and the width of distribution  $\sigma_\eta$  under the maximum average output power condition, see Fig. 4. We find that around  $E_1 + E_2 = 0$ ,  $|\eta^*|$  becomes very large. The underlying physics is that when  $E_1 + E_2$  is close to zero, the transport coefficient  $|M_{21}|$  can become very small while  $|M_{12}|$  is still finite, to yield a very large  $|r|$  [6]. Accordingly, the least probable efficiency  $|\eta^*|$  becomes very large as we showed in Fig. 3(b). In addition, at this special region the width  $\sigma_\eta$  becomes very small, in accordance with Eq. (6), for  $\alpha = -qr/2$  ( $|qr| \ll 1$ ,  $q \rightarrow 0$ ). These calculations, based on the Gaussian approximation, agree with a careful full-counting statistics analysis with vertex corrections [7], carried out in the linear-response regime [see Supplementary Material].

*Future perspectives.*— Our analysis lays the foundation for studies of efficiency statistics beyond the linear-response approximation, as well as to experimental works via, e.g., mesoscopic quantum Hall systems [33] or TRB thermoelectric engines.

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## SUPPLEMENTARY MATERIALS

### I. LDF IN TRB SYSTEMS AND ITS PROPERTIES

#### IA. Derivation of the LDF for TRB systems

The LDF of efficiency fluctuations was derived in Ref. [1], specific to TRS systems. Here we extend this work and obtain the LDF for TRB systems. We begin by introducing the probability distribution function (PDF) of the stochastic currents  $I_i$  ( $i = 1, 2$ )

$$P_t(\vec{I}) = \frac{t\sqrt{\det((\hat{M}^{-1})_{sym})}}{4\pi} \exp\left(-\frac{t}{4}\delta\vec{I}^T \cdot \hat{M}^{-1} \cdot \delta\vec{I}\right). \quad (15)$$

Replacing the stochastic currents with stochastic entropy production rates  $S_i = I_i A_i$  ( $i = 1, 2$ ) at given affinities  $A_1$  and  $A_2$ , one finds the PDF of the entropy production [1, 2]

$$P_t(S_1, S_2) = \frac{t\sqrt{\det((\hat{C}^{-1})_{sym})}}{4\pi\bar{S}_{tot}} \exp\left[-\frac{t}{4\bar{S}_{tot}}\delta\vec{S}^T \cdot \hat{C}^{-1} \cdot \delta\vec{S}\right], \quad (16)$$

where  $\delta\vec{S} = \vec{S} - \bar{\vec{S}}$ ,  $\vec{S} = (S_1, S_2)^T$  is the stochastic entropy production,  $\bar{\vec{S}} = (\bar{S}_1, \bar{S}_2)$  is the averaged (macroscopic) entropy production rate, and  $\det((\hat{C}^{-1})_{sym})$  is the determinant of the symmetric part of the inverse of matrix  $\hat{C}$ . The macroscopic total entropy production rate is  $\bar{S}_{tot} = \bar{S}_1 + \bar{S}_2$ . Here,  $C_{ij} = A_i M_{ij} A_j / \bar{S}_{tot}$  ( $i, j = 1, 2$ ). The PDF of stochastic efficiency is

$$P_t(\eta) = \int dS_1 dS_2 \delta\left(\eta + \frac{S_1}{S_2}\right) P_t(S_1, S_2) = \int dS_2 |S_2| P_t(-\eta S_2, S_2). \quad (17)$$

A direct calculation yields an expression

$$P_t(-\eta S_2, S_2) = \frac{t\sqrt{\det((\hat{C}^{-1})_{sym})}}{4\pi\bar{S}_{tot}} \exp\left[-\frac{t}{4\bar{S}_{tot}}[a(\eta)S_2^2 + 2b(\eta)S_2 + c]\right], \quad (18)$$

with the coefficients

$$a(\eta) = (C_{11} + \eta(C_{12} + C_{21}) + \eta^2 C_{22}) / \det(\hat{C}), \quad c = \bar{S}_{tot}^2, \quad (19a)$$

$$b(\eta) = \{\bar{S}_1(C_{12} + C_{21}) - 2\bar{S}_2 C_{11} + \eta[2C_{22}\bar{S}_1 - (C_{12} + C_{21})\bar{S}_2]\} / [2\det(\hat{C})]. \quad (19b)$$

The coefficients  $C_{ij}$  sum up to unity  $\sum_{ij} C_{ij} = 1$ .  $\det(\hat{C}) = C_{11}C_{22} - C_{12}C_{21}$  is the determinant of  $\hat{C}$ . Note that  $C_{12} \neq C_{21}$  because of time-reversal symmetry breaking. The full probability distribution of the stochastic efficiency is now found to be

$$P_t(\eta) = \frac{\sqrt{\det((\hat{C}^{-1})_{sym})} \exp[-t\bar{S}_{tot}/4]}{\pi a(\eta)} \left[1 + \sqrt{\pi t \bar{S}_{tot} h(\eta)} \exp[t\bar{S}_{tot} h^2(\eta)] \operatorname{erf}(\sqrt{t \bar{S}_{tot} h(\eta)})\right], \quad (20)$$

with  $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$  being the error function and

$$h(\eta) = \frac{-b(\eta)}{2\bar{S}_{tot}\sqrt{a(\eta)}}. \quad (21)$$

The large deviation function of stochastic efficiency is obtained from

$$J(\eta) \equiv -\frac{\lim_{t \rightarrow \infty} \ln[P_t(\eta)]}{t\bar{S}_{tot}} = \frac{1}{4} - h^2(\eta). \quad (22)$$

By substituting the following parametrization,

$$q \equiv \frac{M_{21}}{\sqrt{M_{22}M_{11}}}, \quad r \equiv \frac{M_{12}}{M_{21}}, \quad \alpha \equiv \frac{A_1\sqrt{M_{11}}}{A_2\sqrt{M_{22}}}, \quad (23)$$

we find that

$$a(\eta) = \frac{(1 + \alpha q + \alpha r q + \alpha^2)(\alpha^2 + \alpha q(1 + r)\eta + \eta^2)}{\alpha^2(1 - q^2 r)}, \quad (24a)$$

$$b(\eta) = M_{22}A_2^2 \frac{(1 + \alpha q + \alpha r q + \alpha^2)(\alpha^2 q(r - 1) + \alpha(q^2(1 + r)(r - \eta) + 2(\eta - 1)) + q(r - 1)\eta)}{2\alpha(1 - q^2 r)}, \quad (24b)$$

$$\bar{S}_{tot} = M_{22}A_2^2(1 + \alpha q + \alpha r q + \alpha^2), \quad (24c)$$

$$J(\eta) = \frac{[4 - q^2(1 + r)^2](\alpha^2 + \eta + \alpha q r + \alpha q \eta)^2}{16(1 - q^2 r)(1 + \alpha^2 + \alpha q + \alpha q r)(\alpha^2 + \alpha q \eta + \alpha q r \eta + \eta^2)}. \quad (24d)$$

### IB. Thermodynamic bounds on the LDF

We prove here that the inequalities  $0 \leq J(\eta) \leq 1/4$  are guaranteed by the thermodynamic bound

$$|q(1 + r)| \leq 2. \quad (25)$$

First,  $4(1 - q^2 r) \geq 4 - q^2(1 + r)^2 \geq 0$ , which guarantees the positive semi-definiteness of the prefactors of the numerator and denominator in Eq. (24d). Second,  $1 + \alpha^2 + \alpha q + \alpha q r = \left(\alpha + \frac{q(1+r)}{2}\right)^2 + \left(1 - \frac{q^2(1+r)^2}{4}\right) \geq 0$ . This is consistent with the fact that this term originates from the average total entropy production rate. The last term in the denominator is also not less than zero, since  $\alpha^2 + \alpha q \eta + \alpha q r \eta + \eta^2 = \left(\alpha + \eta \frac{q(1+r)}{2}\right)^2 + \eta^2 \left(1 - \frac{q^2(1+r)^2}{4}\right) \geq 0$ . Therefore  $J(\eta)$  is guaranteed to be greater than zero. Using exactly the same arguments, one can show that

$$\frac{1}{4} - J(\eta) = \frac{[\alpha^2 q(r - 1) + \alpha(q^2(1 + r)(r - \eta) + 2(\eta - 1)) + q(r - 1)\eta]^2}{16(1 - q^2 r)(1 + \alpha^2 + \alpha q + \alpha q r)(\alpha^2 + \alpha q \eta + \alpha q r \eta + \eta^2)} \geq 0. \quad (26)$$

Therefore,  $0 \leq J(\eta) \leq 1/4$ . From the above we find that  $J(\eta) = 1/4$  is reached only at

$$\eta^* = 1 + \frac{q(r - 1)(1 + \alpha q + \alpha q r + \alpha^2)}{q - q r - 2\alpha + q^2(1 + r)\alpha}. \quad (27)$$

We also find that there is only one minimum of  $J(\eta)$  which is the macroscopic efficiency  $\bar{\eta}$ , and only one maximum of  $J(\eta)$  which is precisely the least probable efficiency  $\eta^*$  given above. This is confirmed by solving the extremum equation  $\partial_\eta J(\eta) = 0$  where we find only two solutions: one is  $\bar{\eta}$ , the other is  $\eta^*$ . This property determines the basic-generic shape of the LDF curve.

### IC. $J(\eta)$ under time-reversal operation

In the main text we defined the following parameters

$$r = \frac{M_{12}}{M_{21}}, \quad q = \frac{M_{21}}{\sqrt{M_{22}M_{11}}}. \quad (28)$$

Under time-reversal operation,  $\phi \rightarrow -\phi$ , the above parameters transform as follows

$$r(-\phi) \rightarrow \frac{1}{r}, \quad q(-\phi) \rightarrow q r, \quad (29)$$

where we have used Onsager's reciprocity relation  $M_{12}(\phi) = M_{21}(-\phi)$ . Denoting the LDF of efficiency for the reversed magnetic field ( $\phi \rightarrow -\phi$ ) by  $\tilde{J}(\eta)$ , we obtain that

$$J(\eta) - \tilde{J}(\eta) = \frac{J(\eta_C)}{1 + \alpha^2 + \alpha q r + \alpha q} \frac{\alpha^2 q^2 (r^2 - 1)(1 - \eta^2) + 2(\alpha^2 + \eta)\alpha q (r - 1)(1 - \eta)}{\alpha^2 + \alpha q \eta + \alpha q r \eta + \eta^2}. \quad (30)$$

We find that for TRB systems Carnot efficiency ( $\eta_C = 1$ ) appears as a special point where the distributions become invariant under time-reversal operation, i.e.,  $J(\eta_C) = \tilde{J}(\eta_C)$ . For TRS systems ( $r = 1$ ) this equality trivially holds for all values of efficiency.

#### II. $J(\eta)$ at various limits

In this section we illustrate the rich behavior of  $J(\eta)$  at various limits: (i) weak coupling limit  $r \rightarrow 0$  or  $q \rightarrow 0$ ; (ii) tight-coupling limit with maximum macroscopic efficiency for  $0 < |r| < 1$ ,  $|r| > 1$ , and  $r = \pm 1$ ; (iii) tight coupling limit with maximum macroscopic output power for  $|r| \neq 0, 1$ , and for  $r = \pm 1$ . Results are plotted in Fig. 5.

Fig. 5(a) shows that in the weak-coupling limit ( $r \rightarrow 0$ ), the LDF experiences a sharp transition from the minimum  $\bar{\eta} \rightarrow 0$  to the maximum  $\eta^* \rightarrow 0$ . Therefore,  $J(\eta)$  behaves like a derivative of the Dirac delta function. In contrast, when  $q \rightarrow 0$  but  $r$  is finite, Fig. 5 (b) shows that  $J(\eta)$  develops an infinitely narrow dip near  $\eta = 0$ , i.e., it behaves like the Dirac delta function itself. Note that in the weak coupling regime with  $q \rightarrow 0$  and/or  $r \rightarrow 0$ , the affinity parameter  $\alpha \rightarrow 0$ . As a result, the behavior of  $J(\eta)$  is very similar either under the maximum macroscopic efficiency condition or the maximum output power condition as both the efficiency and the output power go to 0.

In the tight-coupling limit, the behavior is quite different under those two conditions. We first examine the maximum macroscopic efficiency condition. Fig. 5(c) shows that for  $0 < |r| < 1$ , the width of efficiency distribution  $\sigma_\eta$  tends to infinity, while the maximum value of  $J(\eta)$  at  $\eta^* = r$  develops a very sharp peak. In contrast, for  $|r| > 1$ , as shown in Fig. 5(e), the width of efficiency distribution approaches zero, while the width at the least probable efficiency becomes infinite. The  $r = 1$  situation, see Fig. 5(d), demonstrates a sharp transition from the minimum value to the maximum point, resembling the behavior of the derivative of the Dirac delta function. For  $r = -1$ , the tight coupling limit,  $|q(1 + r)| \rightarrow 2$  is pushed to  $q \rightarrow \infty$ . Therefore, in this situation for any finite  $q$  the behavior of  $J(\eta)$  shows a regular behavior. Nevertheless, as shown in Fig. 5(f), the distribution of  $\eta$  is quite broad for  $r = -1$ . This case is relevant to recent studies on ‘‘chiral thermoelectrics’’ (e.g., Nernst engines) where, however, a much stronger bound on  $q$  was obtained [3, 4].

Under the maximum macroscopic output power condition, the behavior of  $J(\eta)$  for  $r \rightarrow 0$ ,  $q \rightarrow 0$  cases is identical to that observed in Fig. 5(a) and (b). The tight-coupling limit with  $0 < |r| < 1$  also shows features similar to Fig. 5(c). However, when  $r = 1$  in the tight-coupling limit, see Fig. 5(g), the width around the macroscopic efficiency ( $\bar{\eta} = 0.5$ ) approaches zero while the width around the maximum value of  $J(\eta)$ , at  $\eta^* = 1$ , becomes infinite. Fig. 5(h) focuses on the  $|r| > 1$  case for which the width of efficiency distribution tends to infinity, while the maximum of  $J(\eta)$  becomes a sharp peak. Fig 5(i) shows that at  $r = -1$ ,  $J(\eta)$  becomes an extremely broad distribution.

#### IE. Width of efficiency distribution under general conditions

The width of efficiency distribution at arbitrary  $\alpha$  is found to be

$$\sigma_\eta = \frac{2\sqrt{2}|\alpha|(1 - q^2 r)(1 + \alpha^2 + \alpha q + \alpha q r)}{(1 + \alpha q)^2 \sqrt{4 - q^2(1 + r)^2}}. \quad (31)$$

In Fig. 6 we display  $\sigma_\eta$  as a function of the affinity parameter  $\alpha$  and the TRB parameter  $r$  for  $q = 0.5$ . We find that the width of efficiency distribution grows with  $|\alpha|$ ; at  $\alpha \sim 0$  it is very small. Besides, it tends to very large values in approaching the tight-coupling limit.

#### IF. Width of efficiency distribution under the maximum macroscopic power condition

Fig. 2(b) in the main text illustrates the behavior of the width of efficiency distribution  $\sigma_\eta$  under the condition of maximum macroscopic efficiency. Here we complement this result and display in Fig. 7(a) the behavior of  $\sigma_\eta$  under the condition of maximum macroscopic power. We see in this figure that the features of  $\sigma_\eta$  are qualitatively similar in



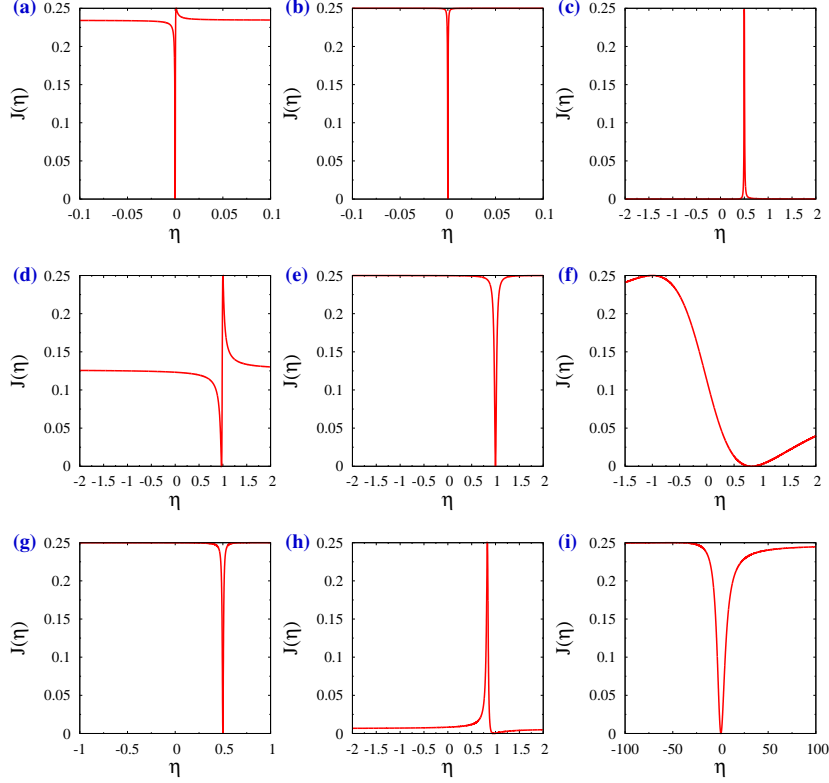


FIG. 5. Illustrations of the rich behavior of  $J(\eta)$  at various limits. (a)-(b) Weak-coupling limit. (a) The  $r \rightarrow 0$  limit with  $r = 0.001$  and  $q = 0.5$ . (b) The  $q \rightarrow 0$  limit with finite  $r$ . We employ  $q = 0.001$  and  $r = 0.5$ . (c)-(f) Several examples for achieving the tight-coupling limit  $|q(1+r)| \rightarrow 2$  with maximum macroscopic efficiency. (c)  $r = 0.5$ ,  $q = 1.3332$  (d)  $r = 1$ ,  $q = 0.9999$ , (e)  $r = 5$ ,  $q = 0.3332$ , (f)  $r = -1$ ,  $q = 10$ . (g)-(i) Examples for the tight-coupling limit with maximum macroscopic output power. (g)  $r = 1$ ,  $q = 0.9999$ , (h)  $r = 5$ ,  $q = 0.3332$ , (i)  $r = -1$ ,  $q = 10$ . Note that the scales of the horizontal axes in these figures are different.

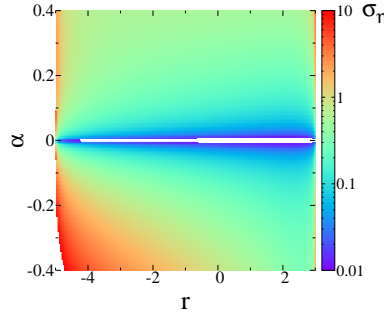


FIG. 6. Width of efficiency distribution  $\sigma_\eta$  as a function of the affinity parameter  $\alpha$  and the TRB parameter  $r$  for  $q = 0.5$ . The boundaries,  $r = -5$  and  $r = 3$ , satisfy the tight-coupling limit. White regions appear either because  $\sigma_\eta$  is too small (around  $\alpha = 0$ ), or because it is too large (around  $r = -5$  or  $3$ ).

both cases. In particular, at the small  $q^2 r$  limit,  $\sigma_\eta$  for both cases become quantitatively similar, which is consistent with the understanding that the maximum macroscopic efficiency and maximum macroscopic power conditions are close at this limit[5]. Nonetheless, we find that  $\sigma_\eta$  diverges in the tight-coupling limit for all  $r$  except at  $r = 0, \pm 1$ . In Fig. 7(b) we examine a particular case with  $q = 0.5$ . The tight-coupling limit is then reached for  $r = -5$  or  $r = 3$ , where  $\sigma_\eta$  is shown to diverge.

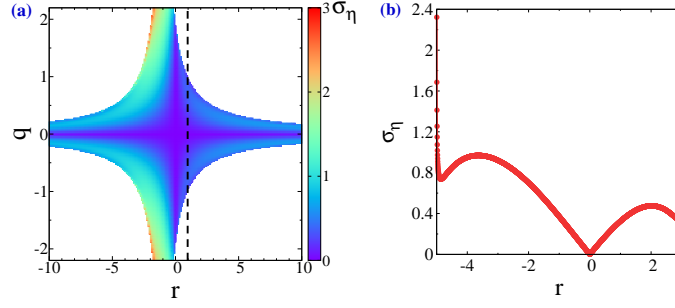


FIG. 7. width of efficiency distribution at the maximum macroscopic output power condition. (a) Width of efficiency distribution  $\sigma_\eta$  as a function of  $r$  and  $q$ . (b) Particular example of  $\sigma_\eta$  with  $r$  at  $q = 0.5$ .

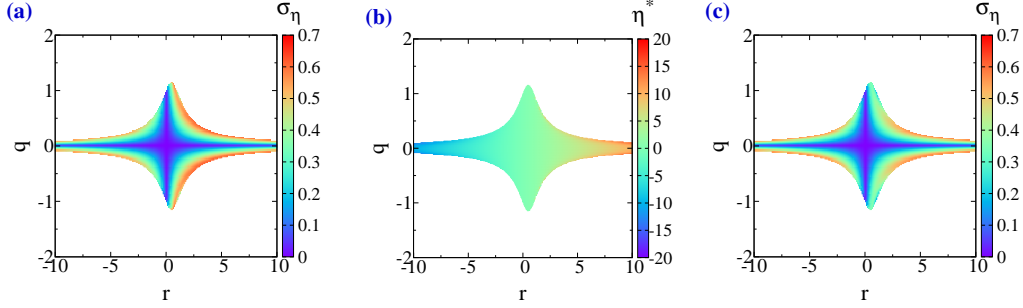


FIG. 8. Efficiency statistics while considering the quantum bound on transport coefficients. (a) Width of efficiency distribution  $\sigma_\eta$  under the maximum macroscopic efficiency condition. (b) Least probable efficiency  $\eta^*$  and (c) width of distribution  $\sigma_\eta$  under the maximum macroscopic output power condition.

## II. QUANTUM MECHANICAL BOUND ON EFFICIENCY FLUCTUATIONS

We examine here the effect of the quantum mechanical bound, which was introduced in Ref. [3], on efficiency fluctuations. Within our parameters the bound comes up as  $1 + q^2 r - q^2 - q^2 r^2 \geq 0$ , affecting the width of efficiency distribution fluctuation  $\sigma_\eta$ . The bound was obtained by considering a three-terminal Landauer-Büttiker model, and it is a direct result of the unitarity of the scattering matrix. In Fig. 8 we plot (a)  $\sigma_\eta$  under the maximum macroscopic efficiency condition, (b) the least probable efficiency  $\eta^*$ , and (c)  $\sigma_\eta$  under the maximum macroscopic power condition. The quantum bound regularizes the divergent behavior of  $\sigma_\eta$  under both conditions, as well as the divergency of  $\eta^*$  for the maximum power condition. This is because the above bound prohibits us from meeting the tight-coupling limit, unless  $r = 1$ , see  $J(\eta)$  in Fig. 5(d). However, note that the least probable efficiency can still diverge at the following affinity parameter, even within the quantum mechanical bound,

$$\alpha_c = \frac{q(1-r)}{2 - q^2 - q^2 r}. \quad (32)$$

At this value we receive positive average efficiencies and average output power when  $r > 1$  or  $r < -1$ . Inspecting Fig. 8 further, we find that the width of efficiency distribution under both maximum macroscopic efficiency and maximum macroscopic power conditions can become considerably large. In addition, the least probable efficiency  $\eta^*$  can deviate significantly from the Carnot efficiency. In fact, within the maximum macroscopic efficiency condition,  $\eta^*$  is allowed to diverge when  $r \rightarrow \infty$  [6]. We note that the above bound goes to the thermodynamic bound for  $N$ -terminal Landauer-Büttiker conductors, when  $N \rightarrow \infty$  [3]. However, the quantum bound may not necessarily hold when genuine many body interactions and inelastic processes (e.g., electron-electron scattering and electron-phonon scattering) are taken into account.

### III. COMPARISON BETWEEN THE GAUSSIAN APPROXIMATION AND A FULL COUNTING STATISTICS ANALYSIS FOR A TRIPLE-QDS THERMOELECTRIC MODEL

In this section we present a comparison between the LDFs as obtained under the Gaussian approximation and a full counting statistics analysis using a three-terminal thermoelectric model. A counting statistics method including a probe terminal has been recently developed by Utsumi *et al.* (though the discussion there is focused on heat currents fluctuations) [7]. In this approach the LDF of efficiency is calculated through the cumulant-generating function (CGF) defining counting fields for the particle number  $\xi_j$  and energy  $\lambda_j$  in each reservoir  $j = L, R, P$ . The CGF of the three-terminal system is given by [7]

$$\mathcal{F}_{3t} = \int \frac{d\omega}{2\pi} \ln \det[\hat{1} - \hat{f}(\omega)\hat{K}(\omega, \phi)]. \quad (33)$$

Here

$$\hat{f} = \text{diag}(f_L, f_R, f_P), \quad \hat{K} = \hat{1} - e^{i\hat{\theta}} \hat{S}^\dagger(\omega, \phi) e^{-i\hat{\theta}} \hat{S}(\omega, \phi), \quad \hat{\theta} = \text{diag}(\omega\lambda_L + \xi_L, \omega\lambda_R + \xi_R, \omega\lambda_P + \xi_P), \quad (34)$$

with “diag” denoting a diagonal matrix,  $f_j = [\exp(\frac{\omega - \mu_j}{T_j}) + 1]^{-1}$  ( $j = L, R, P$ ) is the Fermi distribution for  $j$ -th reservoir, and  $\hat{S}$  is the S-matrix of the triple-quantum-dot (QD) system.

By integrating out the short-time dynamics it was shown in Ref. [7] that the effective CGF for particle and energy transport between the L and R reservoirs (when the particle and energy currents flowing out of the probe terminal are zero) is given by

$$\mathcal{F}_{2t}(\lambda_L, \lambda_R, \xi_L, \xi_R, T_L, \mu_L, T_R, \mu_R) = \mathcal{F}_{3t}(\lambda_L, \lambda_R, \lambda_P^*, \xi_L, \xi_R, \xi_P^*, T_L, \mu_L, T_R, \mu_R, T_P^*, \mu_P^*). \quad (35)$$

Here  $\lambda_P^*$ ,  $\xi_P^*$ ,  $T_P^*$ , and  $\mu_P^*$  are determined by the saddle-point equations

$$\frac{\partial \mathcal{F}_{3t}}{\partial \lambda_P} = \frac{\partial \mathcal{F}_{3t}}{\partial \xi_P} = \frac{\partial \mathcal{F}_{3t}}{\partial T_P} = \frac{\partial \mathcal{F}_{3t}}{\partial \mu_P} = 0, \quad (36)$$

which maximize the probability of processes with zero energy and particle currents flowing out of the probe terminal.

In the linear-response regime, the effective two-terminal CGF can be approximated by a second-order expansion in  $\lambda_j$  and  $\xi_j$  as well as the affinities  $(\mu_j - \mu)/T$  and  $1/T - 1/T_j$  for  $j = L, R$ . Due to particle and energy conservation, the counting fields for the right reservoir can be regarded as redundant, hence we can set  $\lambda_R = \xi_R = 0$ . Furthermore we set  $T_R = T$  and  $\mu_R = \mu$ . The approximate two-terminal CGF is now given by

$$\mathcal{F}_{2t}(\vec{a}) = \frac{1}{2} \vec{a}^T \cdot \hat{\mathcal{R}} \vec{a}, \quad (37)$$

where

$$\vec{a}^T = (\lambda_L, \xi_L, A_1, A_2), \quad \hat{\mathcal{R}}_{\beta\beta'} = \left. \frac{\partial^2 \mathcal{F}_{2t}}{\partial a_\beta \partial a_{\beta'}} \right|_{\lambda_L = \xi_L = A_1 = A_2 = 0}, \quad (38)$$

with  $A_1 = (\mu_L - \mu_R)/T_R$  and  $A_2 = 1/T_R - 1/T_L$ .

The matrix  $\hat{\mathcal{R}}$  must be calculated from the second derivative tensor of the three-terminal CGF at equilibrium with *vertex corrections* as shown in Ref. [7],

$$\left. \frac{\partial^2 \mathcal{F}_{2t}}{\partial a_\beta \partial a_{\beta'}} \right|_{\lambda_L = \xi_L = A_1 = A_2 = 0} = \left( \frac{\partial^2 \mathcal{F}_{3t}}{\partial a_\beta \partial a_{\beta'}} - \frac{\partial^2 \mathcal{F}_{3t}}{\partial a_\beta \partial b_\gamma} U_{\gamma\gamma'} \frac{\partial^2 \mathcal{F}_{3t}}{\partial b_{\gamma'} \partial a_{\beta'}} \right) \Big|_{\lambda_L = \xi_L = \lambda_P = \xi_P = A_1 = A_2 = A_3 = A_4 = 0}, \quad (39)$$

where  $\vec{b} = (\lambda_P, \xi_P, A_3, A_4)$  with  $A_3 = (\mu_P - \mu_R)/T_R$ ,  $A_4 = 1/T_R - 1/T_P$ , and  $\hat{U} = \hat{B}^{-1}$  with

$$B_{\gamma\gamma'} = \left. \frac{\partial^2 \mathcal{F}_{3t}}{\partial b_\gamma \partial b_{\gamma'}} \right|_{\lambda_L = \xi_L = \lambda_P = \xi_P = A_1 = A_2 = A_3 = A_4 = 0}. \quad (40)$$

We now change variables

$$i\lambda_L \rightarrow \eta\zeta(T_L - T_R)/T^2, \quad i\xi_L \rightarrow \zeta(\mu_L - \mu_R)/T, \quad (41)$$

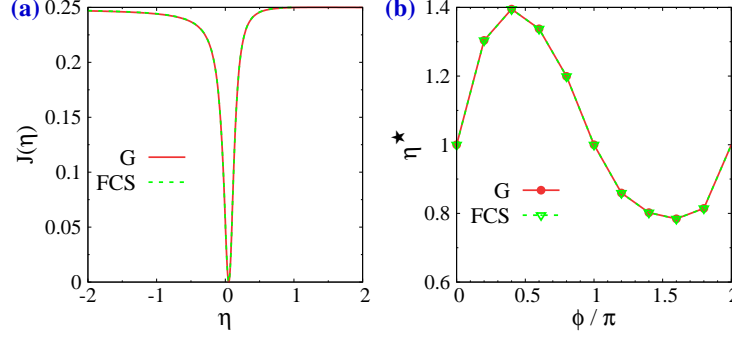


FIG. 9. Comparison between  $J(\eta)$  within the Gaussian approximation (labeled by “G”), and from a full counting statistics analysis (labeled by “FCS”) for the triple-QDs thermoelectric model. (a)  $J(\eta)$  for  $\phi = \pi/2$ . (b)  $\eta^*$  versus  $\phi$ . Parameters are  $E_1 = 1$ ,  $E_2 = 0$ ,  $E_3 = 1$ ,  $\Gamma = 1$ ,  $t = 0.4$ ,  $T_L = 1.125$ ,  $T_R = 1$ ,  $\mu_L = -0.01$ , and  $\mu_R = 0$ .

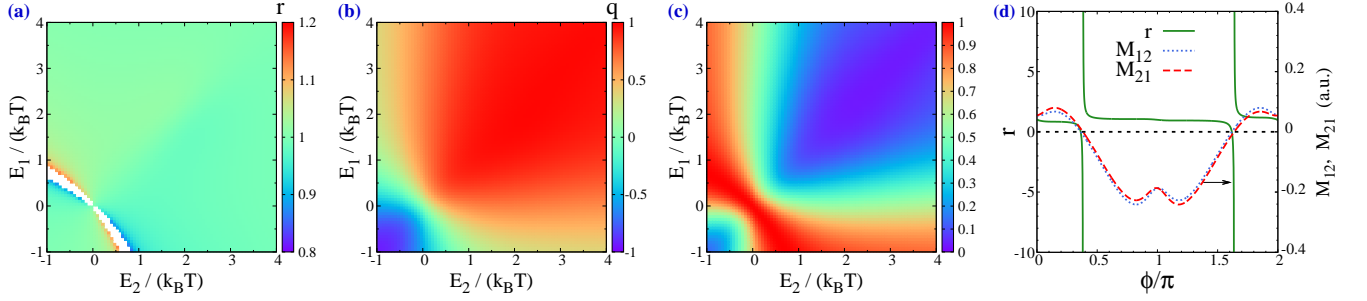


FIG. 10. Time-reversal symmetry breaking in a three-terminal triple QDs model for thermoelectrics. (a) TRB parameter  $r$  and (b) degree of coupling  $q$  as functions of the QDs energies  $E_1$  and  $E_2$ . (c) The function  $1 + q^2 r - q^2 - q^2 r^2$  for the same values of energies as in panel (a), demonstrating that the quantum bound is satisfied. In (a)-(c) we used the following parameters: the QD connected to the probe terminal has energy  $E_3 = 2$ ,  $\phi = \pi/2$ ,  $\Gamma = 0.5$ , and  $t = -0.2$  (energy unit is  $k_B T = 1$ ). (d) The example illustrates the appearance of  $r \rightarrow \infty$  at a certain flux ( $\phi$ ) value, when the parameters are chosen as  $E_1 = 0.2$ ,  $E_2 = -0.2$ ,  $E_3 = 1$ ,  $\Gamma = 0.5$ , and  $t = -1$ .

and obtain the LDF of efficiency fluctuations as[2]

$$\mathcal{G}(\eta) = -\min_{\zeta} \mathcal{F}_{2t}(-i\eta\zeta, -i\zeta(\mu_L - \mu_R)/T, A_1, A_2) \quad (42)$$

for any given  $A_1$  and  $A_2$ .

In Fig. 9(a) we plot the LDF  $J(\eta)$  as obtained from the Gaussian approximation [Eq. (24d)] and that from the full counting statistics method. The two functions perfectly match. In Fig. 9(b) we plot the least probable efficiency  $\eta^*$  calculated from the Gaussian approximation and the full counting statistics method. Again, the results from the two methods agree well with each other. These results confirm the validity of our analysis in the main text based on the Gaussian approximation, a consequence of the fluctuation theorem.

#### IV. CHARACTERIZATION OF THERMOELECTRIC TRANSPORT IN A TRIPLE-QDS SYSTEM

We calculate linear transport coefficients for a three-terminal thermoelectric model when the average thermal and electrical currents flowing out of the probe terminal  $P$  are zero. These conditions lead to  $\hat{M} = \hat{M}'_{LL} - \hat{M}'_{LP} \hat{M}'_{PP}{}^{-1} \hat{M}'_{PL}$ . Here  $\hat{M}'$  is the transport matrix for the total three-terminal system, i.e.,  $\vec{I}' = \hat{M}' \vec{A}'$  where  $\vec{I}' = (\vec{I}'_L, \vec{I}'_P)$  and  $\vec{A}' = (\vec{A}'_L, \vec{A}'_P)$  with  $\vec{I}'_\gamma = (I'_{\gamma e}, I'_{\gamma h})$  and  $\vec{A}'_\gamma = (A'_{\gamma e}, A'_{\gamma h})$  are the currents and affinities for terminals  $\gamma = L, P$ .  $I_e$  is the charge current,  $I_h$  is the heat current, and e.g.,  $A_{Le} = (\mu_L - \mu_R)/T_R$ ,  $A_{Lh} = 1/T_R - 1/T_L$ . Linear transport

coefficients are calculated from the expression

$$\hat{M}'_{\gamma\gamma'} = \int \frac{d\omega}{2\pi} [\delta_{\gamma\gamma'} - |S_{\gamma\gamma'}(\omega, \phi)|^2] \begin{pmatrix} 1 & \omega \\ \omega & \omega^2 \end{pmatrix} f_0(\omega)[1 - f_0(\omega)], \quad (43)$$

where  $S_{\gamma\gamma'}(\omega, \phi)$  ( $\gamma, \gamma' = L, P$ ) is the scattering matrix between terminals  $\gamma$  and  $\gamma'$ ,  $\phi = 2\pi\Phi/\Phi_0$ .  $\Phi$  and  $\Phi_0$  are the magnetic flux in our triple-quantum-dots (QDs) system and the flux quantum, respectively. The Fermi distribution  $f_0(\omega) = [\exp(\frac{\omega}{T}) + 1]^{-1}$  corresponds to an equilibrium state with the chemical potential set at  $\mu = 0$ . Onsager reciprocal symmetry originates from the symmetry  $S_{\gamma\gamma'}(\omega, \phi) = S_{\gamma'\gamma}(\omega, -\phi)$ .

The transmission function is obtained from the relation  $\hat{S}(\omega, \phi) = -\hat{1} + i\Gamma\hat{G}^r(\omega)$ . Here  $\hat{1}$  is a  $3 \times 3$  identity matrix and  $\hat{G}^r(\omega) = [(\omega + i\Gamma)\hat{1} - \hat{H}_{qd}]^{-1}$  is the retarded Green's function of the QDs. The hybridization energy  $\Gamma = 2\pi \sum_k |V_k|^2 \delta(\omega - \varepsilon_k)$  is assumed to be a constant (independent of energy) for all three QDs.

We calculate the transport coefficients and then determine the TRB parameter  $r$  and the degree of coupling  $q$ . In Fig. 10 we plot these parameters against  $E_1$  and  $E_2$ . We further confirm the “quantum bound” on linear transport coefficients discovered by Brandner *et al.* for three-terminal TRB conductors[8], which using our parametrization casts into the form as  $1 + q^2r - q^2 - q^2r^2 \geq 0$ . In Fig. 10(c) we plot it and show that it is always greater than zero in our parameter region. To illustrate the divergence of  $r$ , we plot it in Fig. 10(d) as a functions of the magnetic flux  $\phi$ . We find that  $r$  diverges when  $M_{21}$  goes to zero with a finite  $M_{12}$ . This happen when  $\phi \simeq 0.4\pi$  or  $\phi \simeq 1.6\pi$ .

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