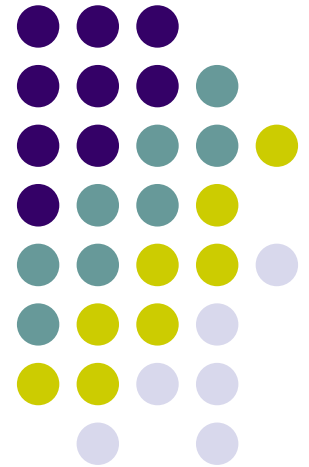


# Efficient Algorithms for the Longest Path Problem

Ryuhei UEHARA (JAIST)

Yushi UNO (Osaka Prefecture University)





# The Longest Path Problem

- Finding a longest (vertex disjoint) path in a given graph
- Motivation (comparing to Hamiltonian path):
  - ... Approx. Algorithm, Parameterized Complexity
  - ... More practical/natural
  - ... More difficult(?)



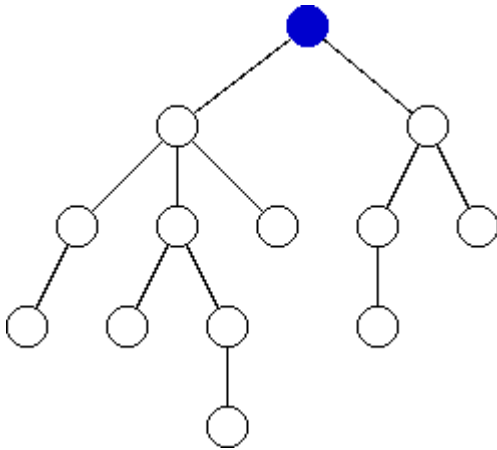
# The Longest Path Problem

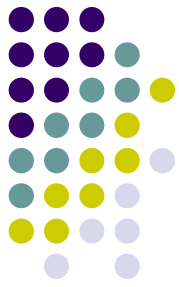
- Known (hardness) results;
  - We cannot find a path of length  $n - n^\epsilon$  in a given Hamiltonian graph in poly-time unless  $P=NP$  [Karger, Motwani, Ramkumar; 1997]
  - We can find  $O(\log n)$  length path [Alon, Yuster, Zwick; 1995]  
( $\Rightarrow O((\log n / \log \log n)^2)$  [Björklund, Husfeldt; 2003])
  - Approx. Alg. achieves  $O(n / \log n)$  [AYZ95]  
( $\Rightarrow O(n(\log \log n / \log n)^2)$  [BH03])
- Exponential algorithm [Monien 1985]



# The Longest Path Problem

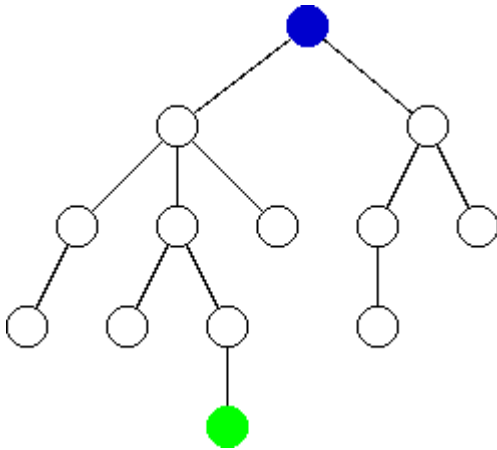
- Known polynomial time algorithm;
  - Dijkstra's Alg.(196?): Linear alg. for finding a longest path in a tree;





# The Longest Path Problem

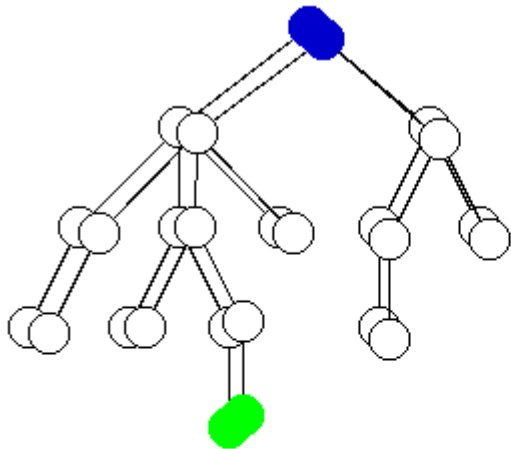
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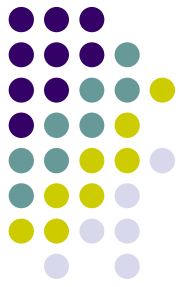




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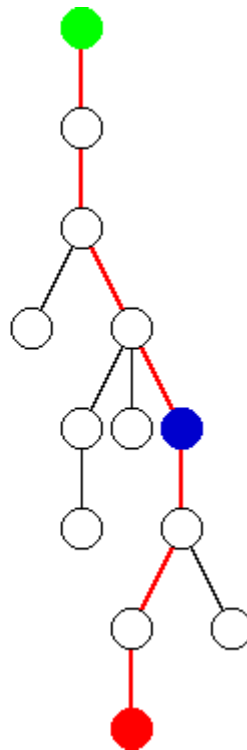
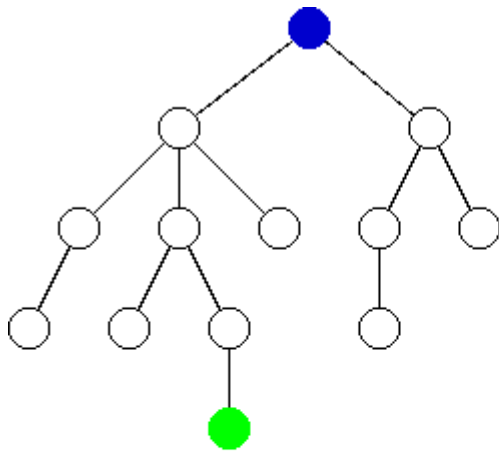
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# The Longest Path Problem

- Known polynomial time algorithm;
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# Approaches to the Efficient Algs to Longest Path Problem



1. Extension of the Dijkstra's algorithm
  - Weighted trees (linear), block graphs (linear), cacti ( $O(n^2)$ ).  
(ISAAC 2004)
2. Graph classes s.t. Hamiltonian Path can be found in poly time
  - Some graph classes having interval representations (bipartite permutation, interval biconvex graphs)  
(ISAAC 2004)
3. Dynamic programming to the graph classes that have tree representations (on going)
  - Cacti(linear), ...



# Approaches to the Efficient Algs to Longest Path Problem



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# 1. Ex of Dijkstra's Alg

Bulterman et.al. (*IPL*,2002) showed that the correctness of Dijkstra's alg stands for;

1. For each  $u, v$ ,  
length of the **shortest** path between  $u$  and  $v$   
= length of the **longest** path between  $u$  and  $v$
2. For each  $u, v, w$ ,  
 $d(u, v) \leq d(u, w) + d(w, v)$
3. For each  $u, v, w$ ,  
 $d(u, v) = d(u, w) + d(w, v)$  if and only if  
 $w$  is on the **unique** path between  $u$  and  $v$



# 1. Ex of Dijkstra's Alg

Construct  $G'=(V',E')$  from  $G=(V,E)$  s.t.:

- $V \subseteq V'$
- For each  $u, v \in V$ ,  
length of the **shortest path** between  $u, v$  on  $G'$   
= length of the **longest path** between  $u, v$  on  $G$
- For each  $u, v \in V$ ,  
the **shortest path** between  $u, v$  on  $G'$  is **unique**

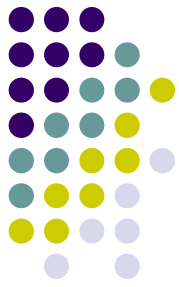


# 1. Ex of Dijkstra's Alg

Theorem: ExDijkstra finds a longest path if  $G$  and  $G'$  satisfy the conditions.

ExDijkstra:  $G=(V,E)$  and  $G'=(V',E')$

1. pick any vertex  $w$  in  $V$ ;
2. find  $x \in V$  with  $\max\{d(w,x)\}$  on  $G'$ ;
3. find  $y \in V$  with  $\max\{d(x,y)\}$  on  $G'$ ;
4.  $x$  and  $y$  are the endpoints of the longest path in  $G$ , and  $d(x,y)$  on  $G'$  is its length.



# 1. Ex of Dijkstra's Alg (Summary)

Theorem: Vertex/edge weighted tree (linear)

Theorem: Block graph ( $O(|V|+|E|)$ )

Theorem: **Cactus** ( $O(|V|^2)$ )

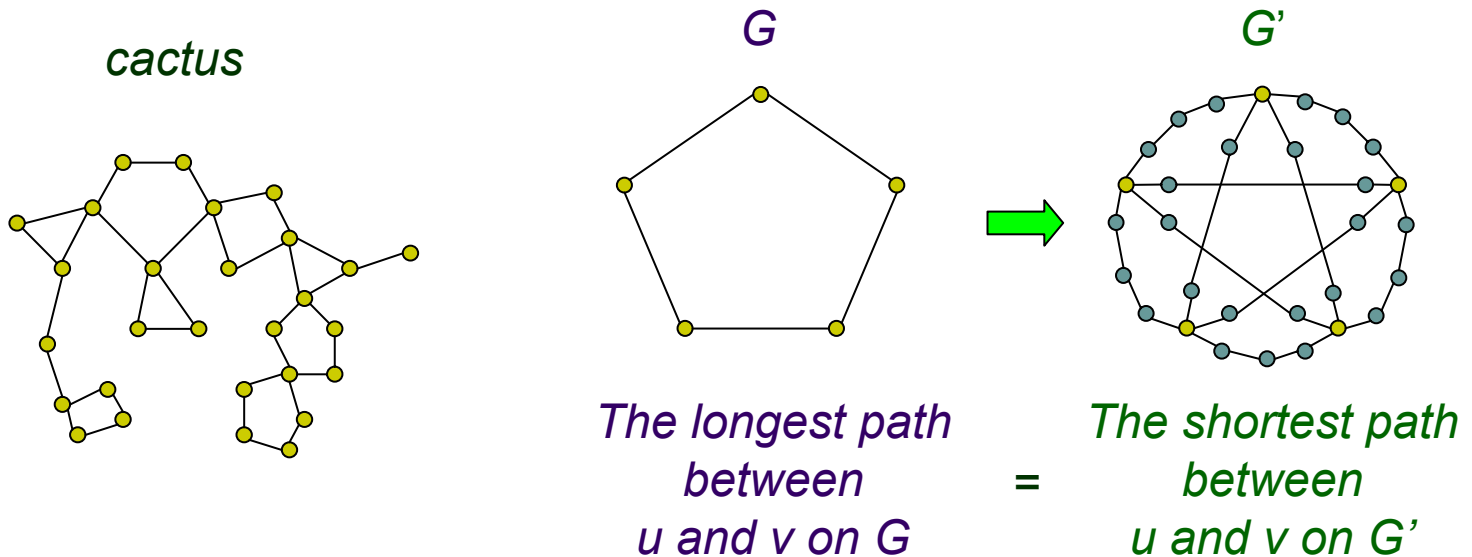


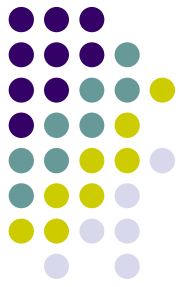
# 1. Ex of Dijkstra's Alg (Cacti)

Cactus:

Each block is a *cycle*

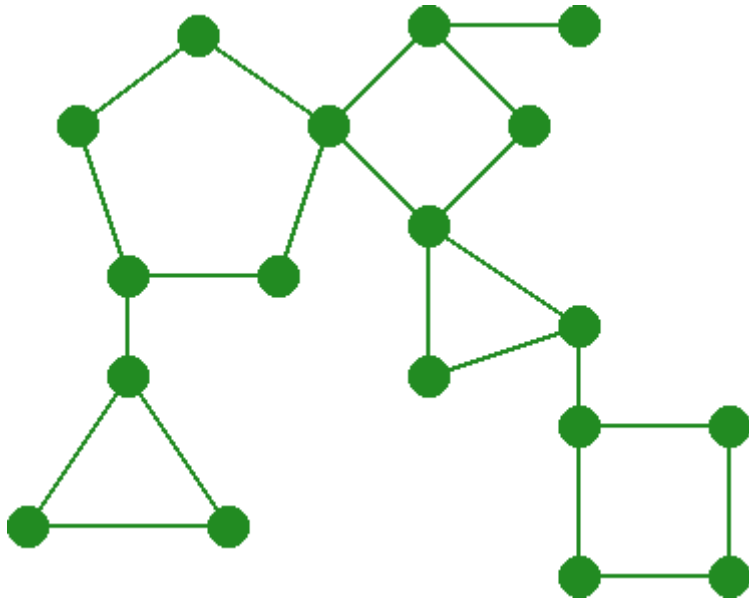
Two cycle share at most one vertex which is a separator

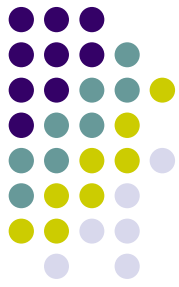




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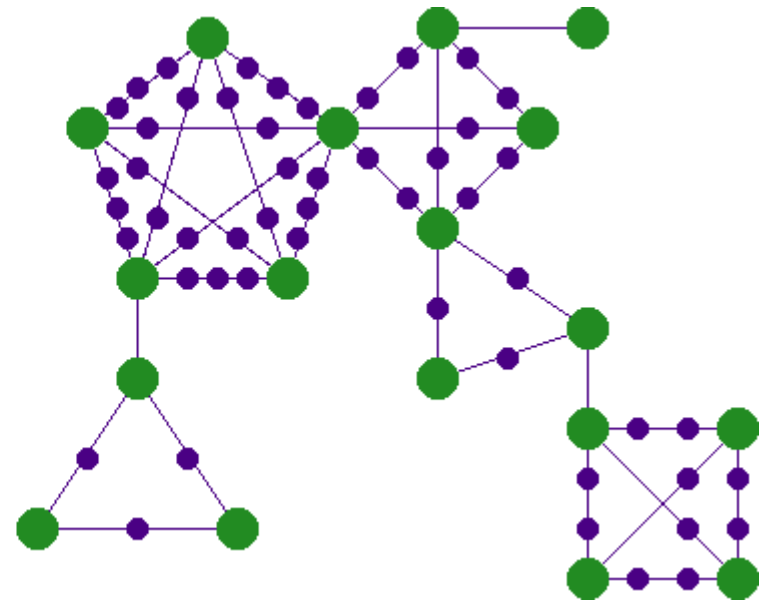
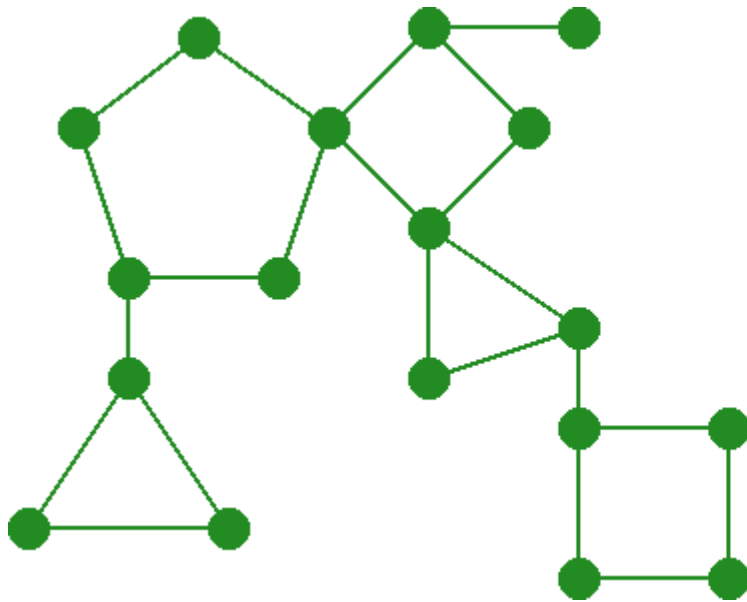
Sample





# 1. Ex of Dijkstra's Alg (Cacti)

Sample

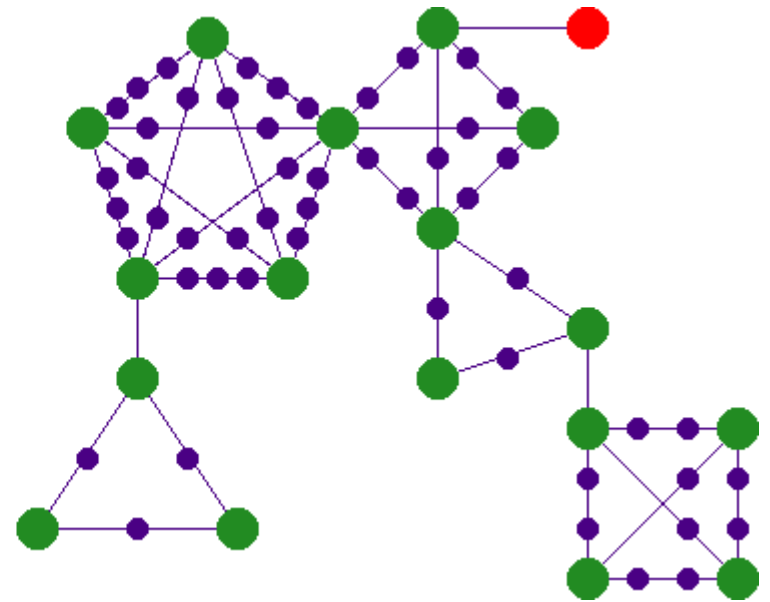
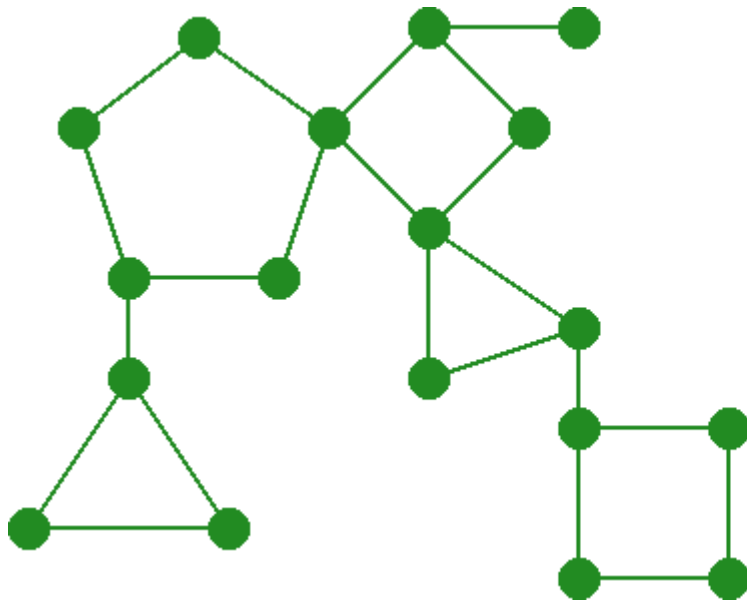






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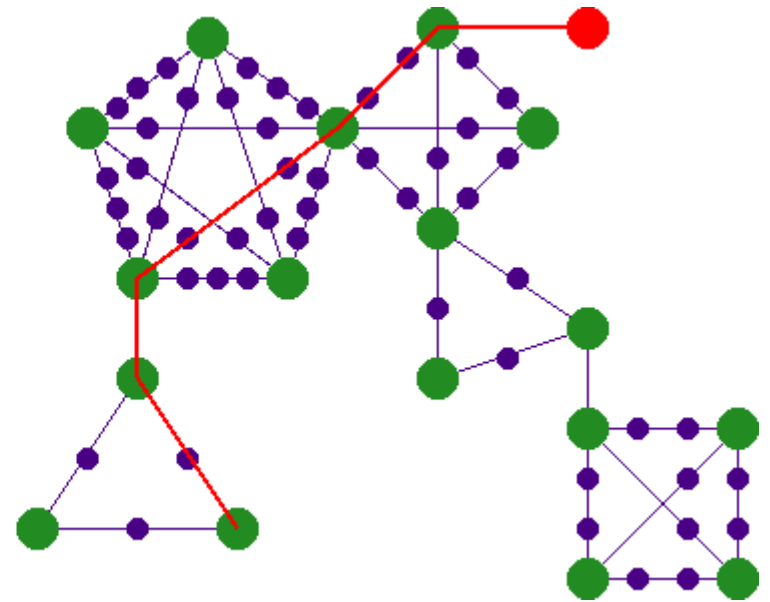
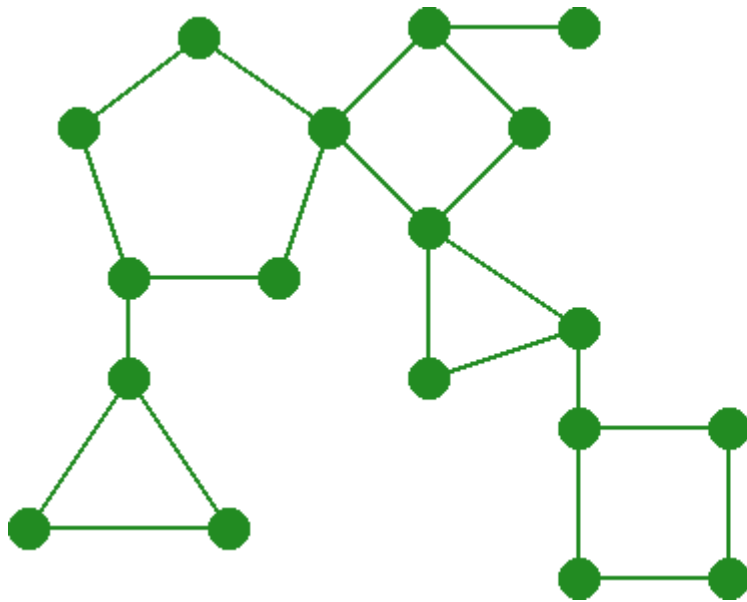
Sample





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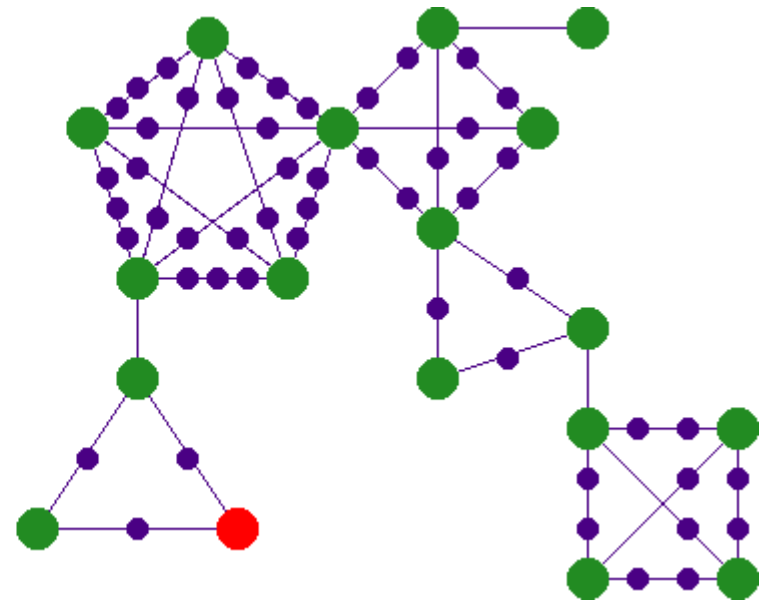
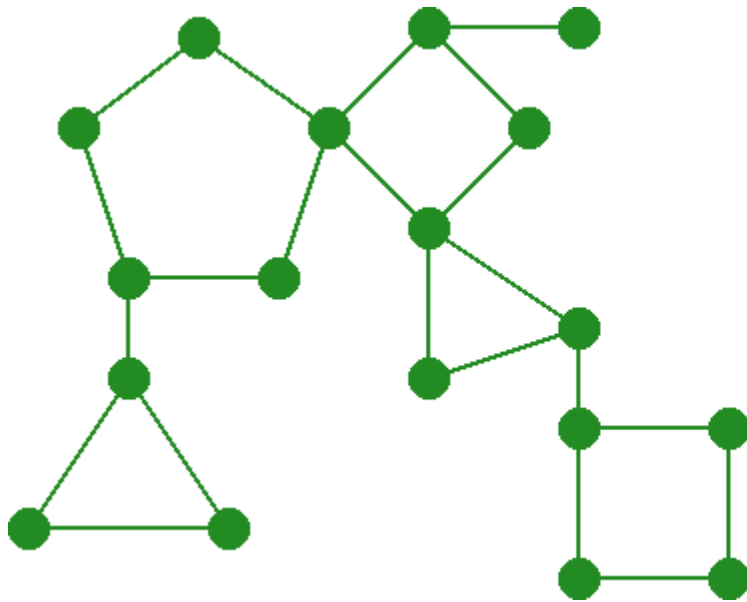
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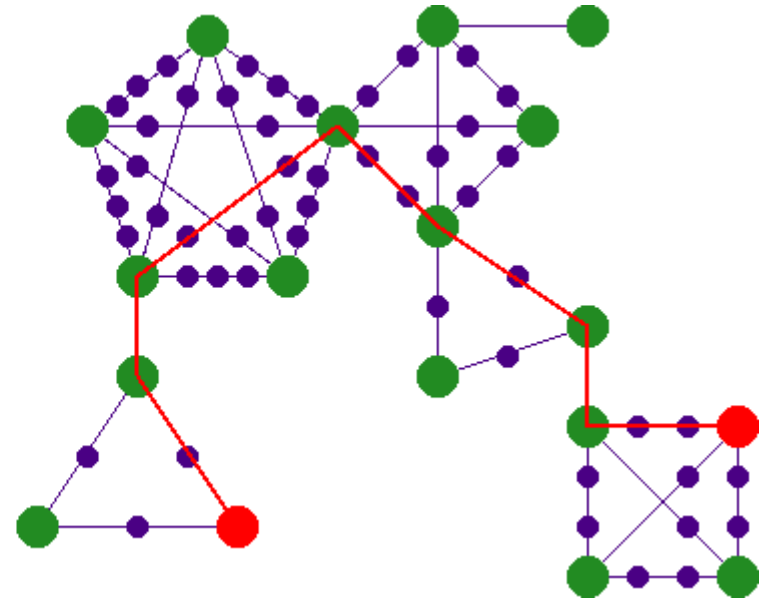
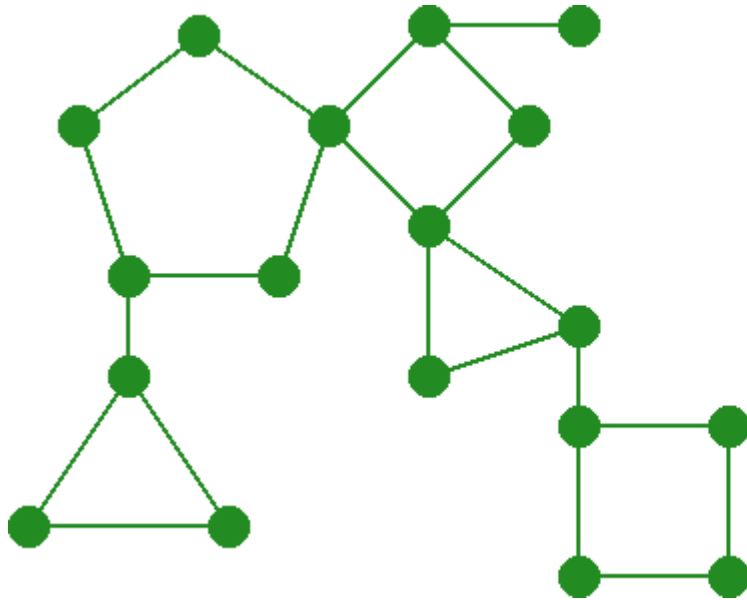
Sample

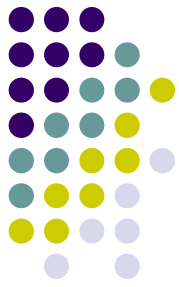




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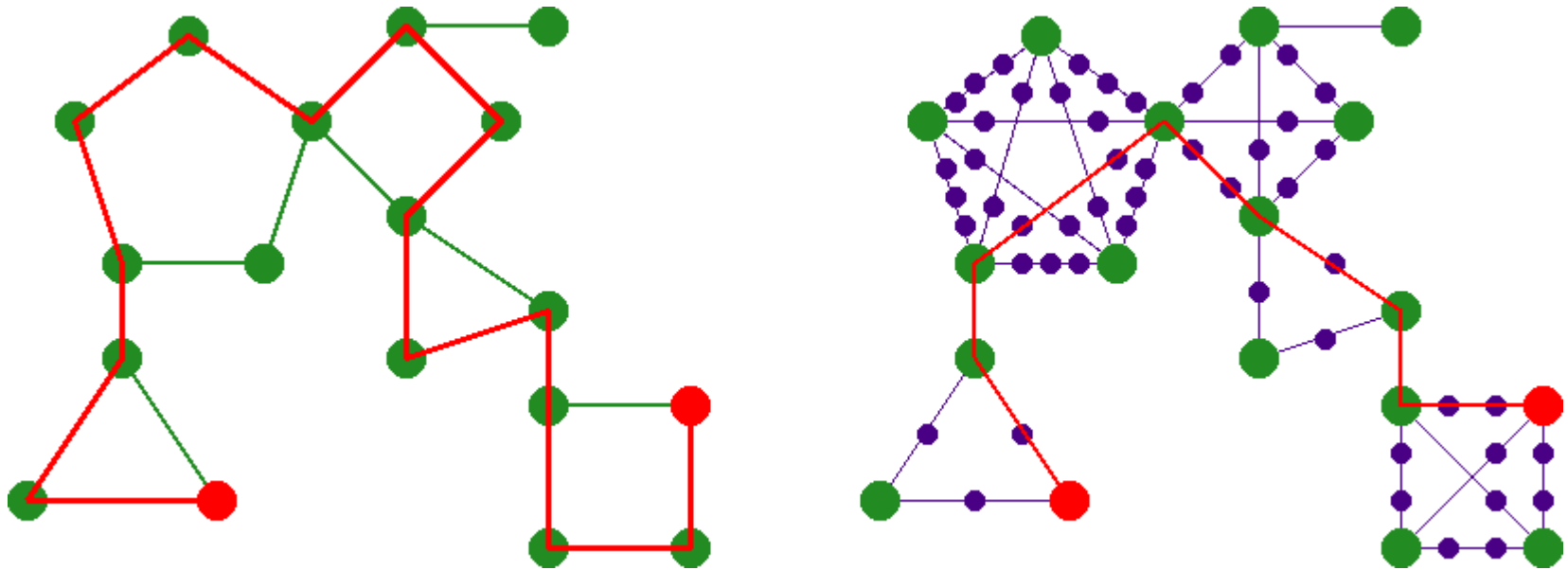
Sample





# 1. Ex of Dijkstra's Alg (Cacti)

Sample



# Graph classes s.t. Hamiltonian Path can be found in poly time



Fact 1:

Hamiltonian Path is NP-hard on a **chordal graph**.

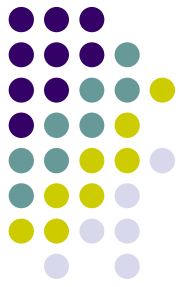
(In fact, strongly chordal split graph[Müller, 1997].)

Fact 2:

Hamiltonian Path is solvable on an **interval graph** in linear time. [Damaschke, 1993].

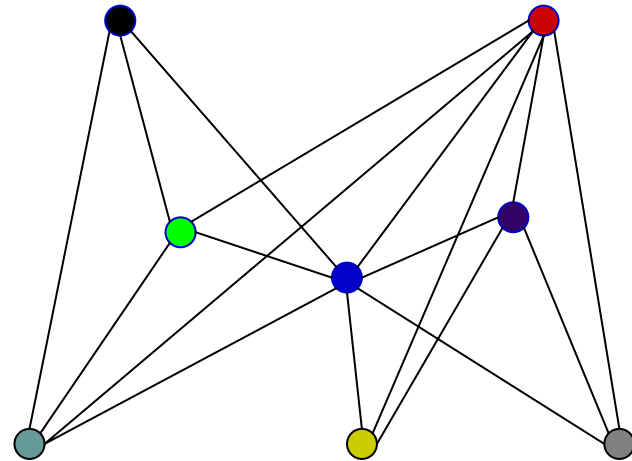
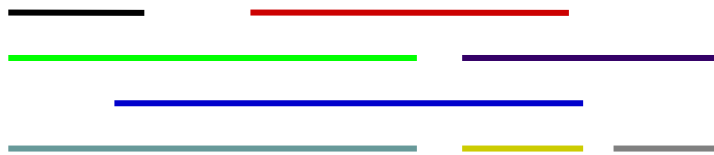
Our goal:

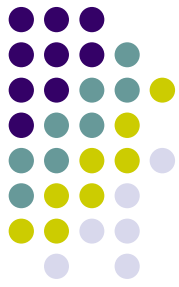
Poly-time algorithm for Longest Path on an **interval graph**.



# Interval Graphs

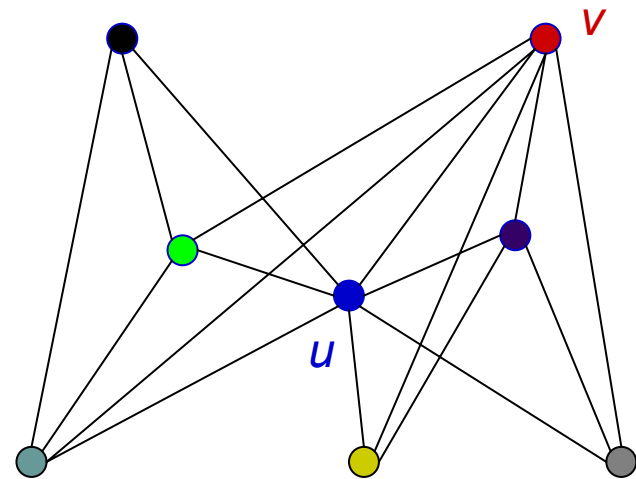
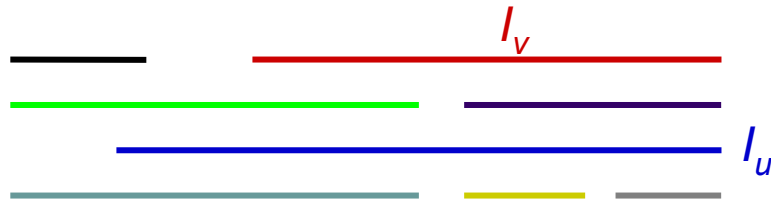
- An interval graph  $G=(V,E)$  has an interval representation s.t.  $\{u,v\} \in E$  iff  $I_u \cap I_v \neq \emptyset$





# Interval Graphs

- An interval graph  $G=(V,E)$  has an interval representation s.t.  $\{u,v\} \in E$  iff  $I_u \cap I_v \neq \emptyset$



Hamiltonian Path: linear time solvable.

Longest Path: ????

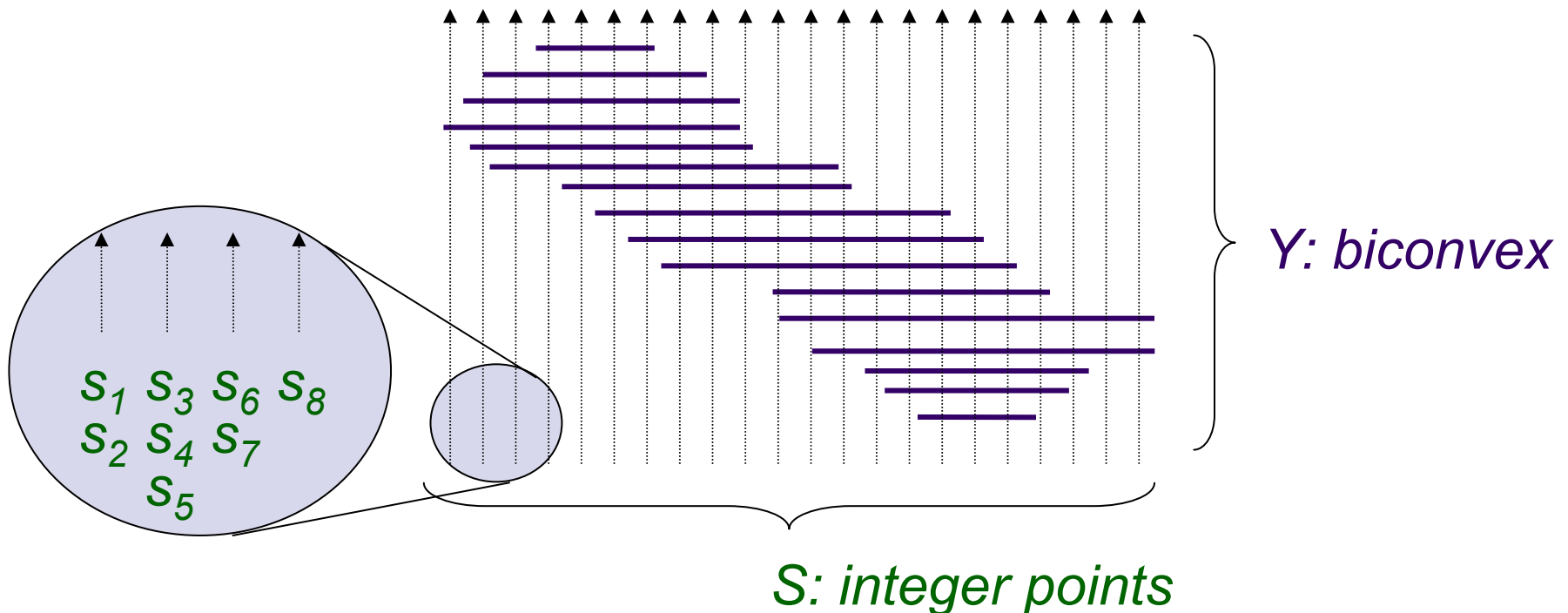
⇒ Restricted interval graphs...





# Restricted Interval Graphs

- An interval **biconvex** graph  $G=(S \cup Y, E)$  has an interval representation *s.t...*





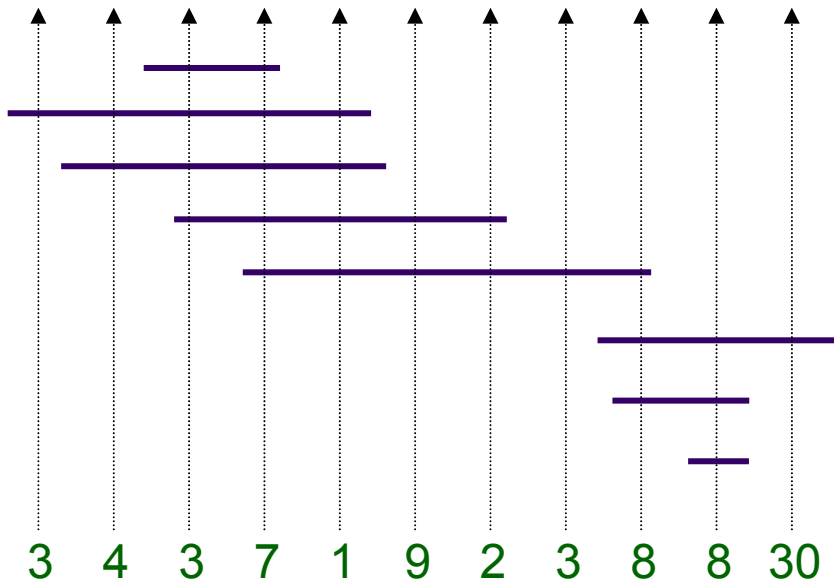
# Restricted Interval Graphs

- Interval **biconvex** graph  $G=(S \cup Y, E)$  is introduced [Uehara, Uno; 2004] from graph theoretical viewpoints;
  - ✓ *Natural analogy of biconvex graphs (bipartite graph class)*
  - ✓ *Generalization of proper interval graphs*
  - ✓ *Generalization of threshold graphs*
  - ✓ *Best possible class longest path can be found in poly time...*

# Poly-time alg for longest path on an interval biconvex graph (idea)



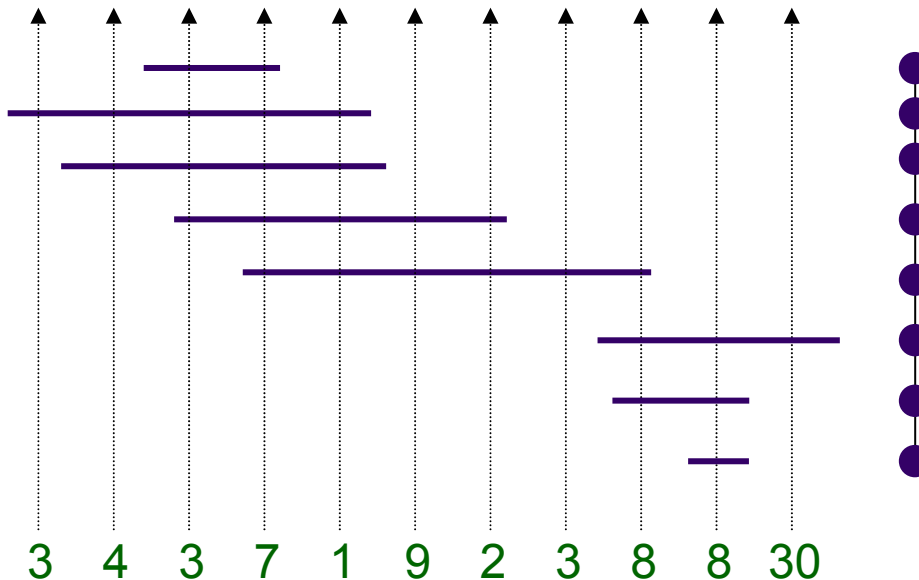
- Find the trivial longest path  $P$  on  $G[Y]$ ;
- Embed the vertices in  $S$  into  $P$  as possible;
- Adjust endpoints if necessary.



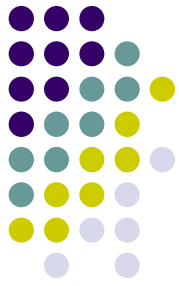
# Poly-time alg for longest path on an interval biconvex graph (idea)



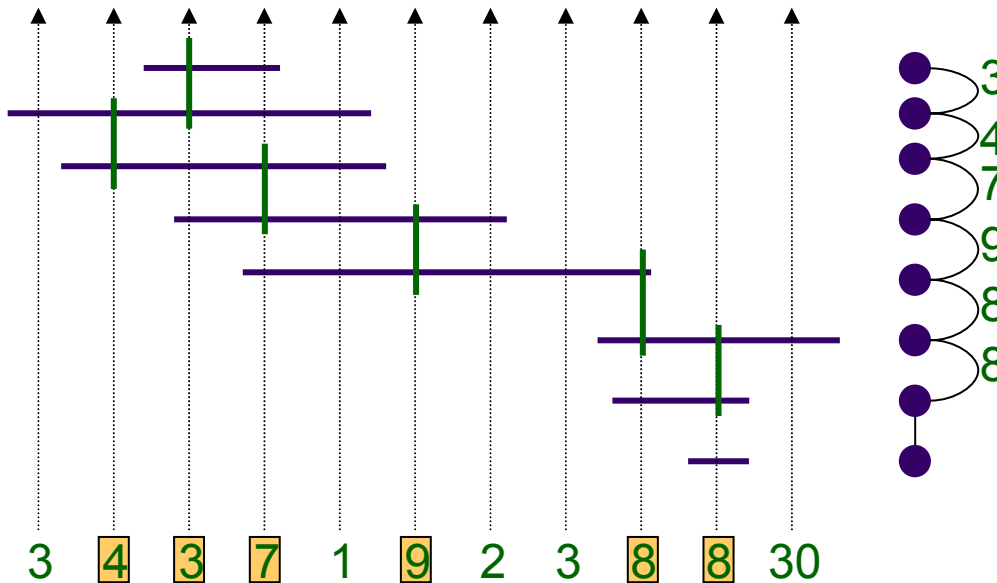
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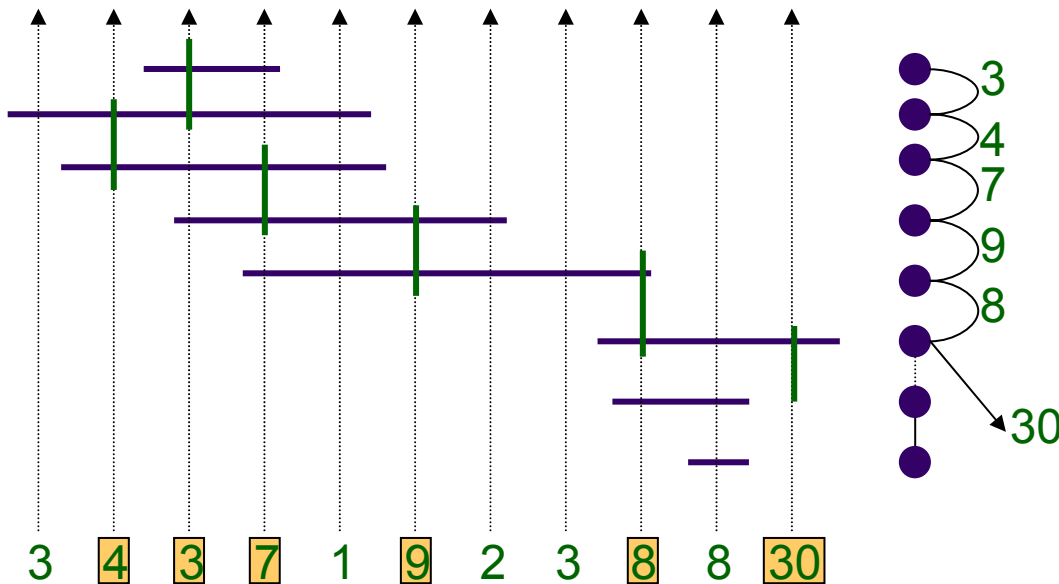
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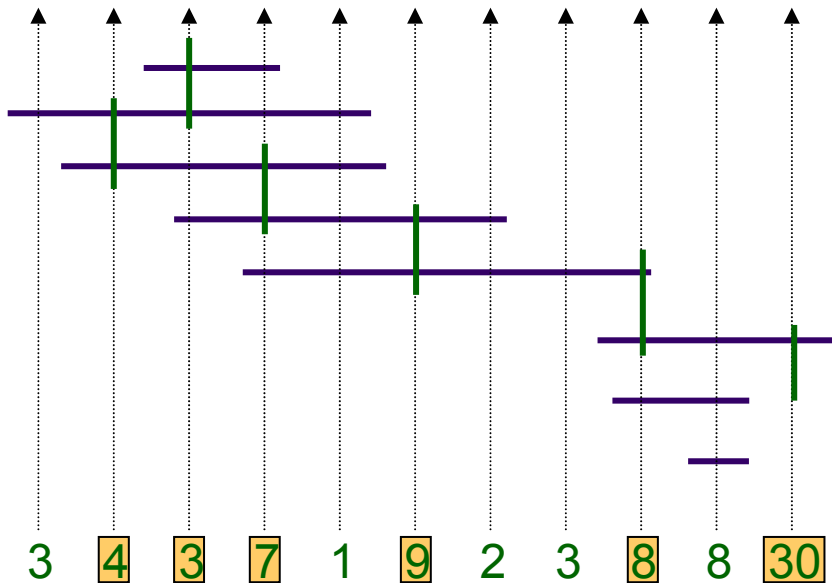
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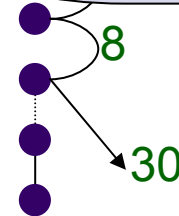


- Find the trivial longest path  $P$  on  $G[Y]$ ;
- Embed the vertices in  $S$  into  $P$  as possible;
- Adjust endpoints if necessary.



✓ How can we determine the vertices in  $S$ ?

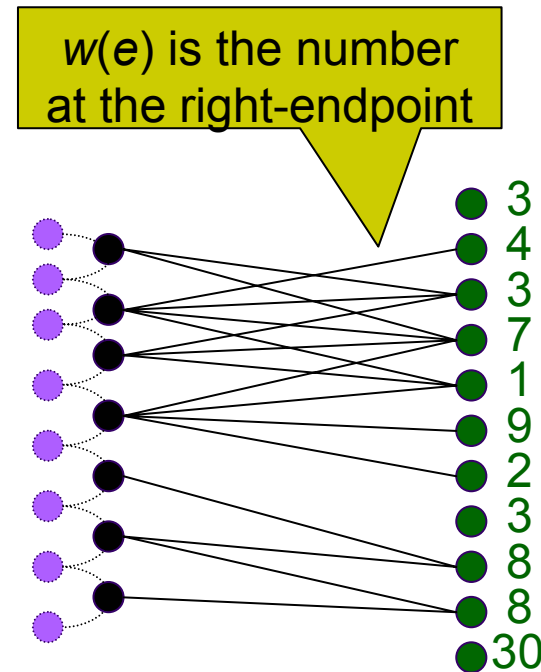
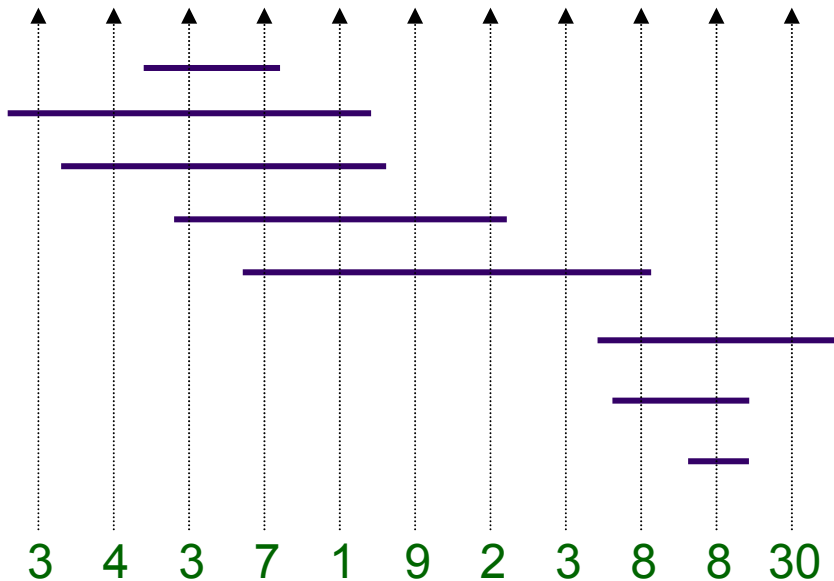
✓ Where do we embed them?



# Poly-time alg for longest path on an interval biconvex graph (idea)



- Embed the vertices in  $S$  into  $P$  as possible;

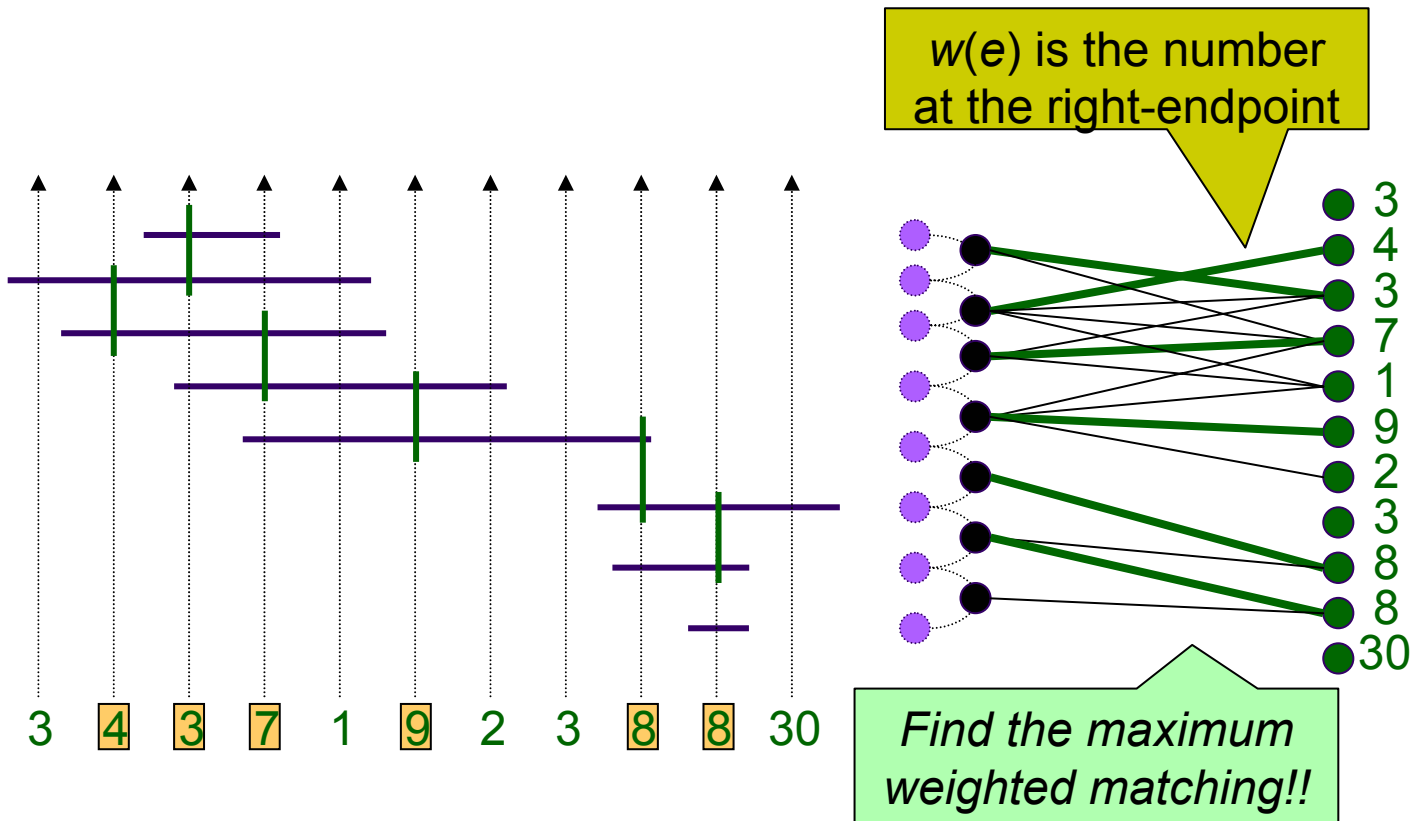




# Poly-time alg for longest path on an interval biconvex graph (idea)



- Embed the vertices in  $S$  into  $P$  as possible;





# Open Problems

- Longest Path on an interval graph??
  - Combination of *DP/Dijkstra* and *weighted maximum matching* on MPQ-tree representation?
  - Related to the following open problem?  
Hamiltonian Path *with a start point* on an interval graph?  
[Damaschke, 1993].
- Extension to
  - Longest *cycle* on some graph classes
  - *Hamiltonian cycle/path* on some graph classes