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Efficient bond price approximations in non-linear equilibrium-based term structure models

Abstract: This paper develops an efficient method to compute higher-order perturbation approximations of bond prices. At third order, our approach can significantly shorten the approximation process and its precision exceeds the log-normal method and a procedure using consol bonds. The efficiency gains greatly facilitate any estimation which is illustrated by considering a long-run risk model for the US. Allowing for an unconstrained intertemporal elasticity of substitution enhances the model's fit, and we see further improvements when incorporating stochastic volatility and external habits.

Keywords: DSGE model; habit model; higher order perturbation method; long-run risk; stochastic volatility.

JEL classifiactions: C68; E0.

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1 Introduction

Understanding the economic mechanisms behind the term structure and its embedded term premium has long been an active area of research. It is of profound interest to explain variation in the yield curve, and how it relates to consumption growth, inflation, and the economy in general.

These and other questions have been widely studied in equilibrium-based term structure models where consumption and inflation follow exogenously given processes, e.g., in the endowment models by Backus, Gregory, and Zin (1989), Wachter (2006), and Piazzesi and Schneider (2007). To solve for bond prices within these models, researchers are typically restricted in their choice of preferences and the driving processes for consumption and inflation.[1]

Analyzing bond prices in production-based equilibrium models, i.e., in dynamic stochastic general equilibrium (DSGE) models, is even more difficult and often constrained by the fact that closed-form solutions, in general, are unavailable. This leaves researchers with a challenging numerical problem that standard approximation methods struggle to deal with. For example, the well-known log-linear approximation is inadequate as it restricts term premia to zero contrary to existing evidence [see Campbell and Shiller (1991) and Cochrane and Piazzesi (2005)].[2]

Higher order perturbations methods have been shown to be an attractive approximation approach for DSGE models (Arouba, Fernández-Villaverde, and Rubio-Ramírez 2005; Caldara et al. 2012, among others), and the method may also be useful in relation to endowment economies. However, the standard perturbation method is computationally demanding for term structure models because it solves for all bond prices simultaneously. The present paper addresses this problem by proposing a more efficient perturbation method to

1 See for instance Tsionas (2003), Martin (2008), Anh, Dai, and Singleton (2010), among others.
2 Other solution methods include value function iteration, finite elements, and Chebyshev polynomials, but these are typically considered infeasible for medium-scale DSGE models.

approximate bond prices to arbitrary order within the broad class of DSGE models considered in Schmitt-Grohé and Uribe (2004), including both endowment and production-based economies. We focus on the standard case where bond prices with maturities beyond one period do not affect the rest of the economy. This assumption is always fulfilled in endowment models and is typically also satisfied in production-based economies.[3] The solution we advocate splits the approximation problem into two steps. In the first step, standard perturbation packages can be used to solve a reduced version of the model *without* bond prices having maturities of more than one period. The second step defines a perturbation problem exclusively for bond prices. Relying on information from the first step, we then recursively solve for the remaining bond prices and as a result significantly increase the speed of the approximation process. We emphasize that our method gives *exactly* the same numerical values for bond prices as when all bonds are solved for simultaneously in the standard "one-step" perturbation routine.

A simulation study shows that our method implies substantial computational gains. With five countries in the real business cycle (RBC) model presented in Juillard and Villemot (2011), our method is between 10 and 38 times faster than the one-step perturbation routine with a 10-year and 20-year yield curve, respectively. In general, the efficiency gains of our method depend positively on the maturity of the approximated yield curve and positively on the number of state variables in the model. Due to the memory efficient nature of our method, it is also shown that it enables us to solve larger yield curve models than possible using the one-step perturbation routine.

We also assess the accuracy of our method using closed-form solutions for bond prices in a consumption endowment model with habits (Zabczyk 2014). In line with Arouba, Fernández-Villaverde, and Rubio-Ramírez (2005) and Caldara et al. (2012), we find that a third-order approximation delivers a high level of accuracy and clearly outperforms alternative methods like the log-normal approach (Jermann 1998; Doh 2011) and the consol method proposed in Rudebusch and Swanson (2008).

To illustrate the efficiency gains and flexibility of our approach, we finally apply it to an extended version of the long-run risk model by Piazzesi and Schneider (2007) estimated on US data by Simulated Method of Moments. We first relax the constraint that the intertemporal elasticity of substitution (IES) is equal to one, and it is shown that this model is able to match the level of the yield curve and generate sufficient variability and persistence in all yields. Allowing for stochastic volatility in consumption growth and inflation substantially improves the model's performance. In addition to the aforementioned moments, the model is now able to match all contemporaneous cross-correlations between the yield curve and the two macro variables and most cross auto-correlations (i.e., moments of the form $E[x_t, y_{t-1}]$). Extending the model with external habit formation further enhances its performance so it almost perfectly matches the auto-correlation in consumption growth and the ability of inflation to forecast future consumption growth. We therefore conclude that it may be beneficial to include habit formation in a long-run risk model.

The remainder of this paper is organized as follows. Section 2 presents our approximation method and provides simple expressions for bond prices up to third order. Section 3 documents the gains in speed at third order implied by our method, while Section 4 assesses its accuracy. Section 5 contains our empirical application and Section 6 concludes.

2 Approximating bond prices

We discuss the principle behind our method in Section 2.1. The perturbation problem for bond prices is formally defined in Section 2.2 and we introduce our notation in Section 2.3. The following sections derive recursive formulas for bond price derivatives up to third order at the steady state. We finally discuss various extensions of our method in Section 2.7. To facilitate the use of our work, Matlab codes are pro-

3 A non-exhaustive list includes the work by Wu (2006), Uhlig (2007), De Paoli, Scott, and Weeken (2010), Hordahl, Tristani, and Vestin (2008), Amisano and Tristani (2009), Bekaert, Cho, and Moreno (2010), Binsbergen, et al. (2012), and Rudebusch and Swanson (2012).

vided which implement the suggested procedure to third order. A brief introduction to these codes is available in Appendix B.

2.1 An overview of the proposed method

To describe the principle behind our method, recall that equilibrium conditions for most models can be written as

$$E_t[\tilde{\mathbf{f}}(\mathbf{y}_{t+1}^{all},\mathbf{y}_t^{all},\mathbf{x}_{t+1},\mathbf{x}_t)]=\mathbf{0}. \tag{1}$$

Here, $\mathbf{y}_t^{all}$ contains the full set of non-predetermined variables, including all bond prices, and $\mathbf{x}_t$ denotes an $n_x\times 1$ vector of predetermined state variables. Moreover, let $P^{t,k}$ denote the price in period t of a zero-coupon bond maturing in k periods with a face value of 1. The price of this bond satisfies the fundamental pricing equation [see Cochrane (2001)]

$$P^{t,k}=E_t[\mathcal{M}\times P^{t+1,k-1}] \tag{2}$$

for $k=1,2,\ldots,K$ where $\mathcal{M}$ is the stochastic discount factor. Given the recursive structure in (2) and the assumption that bond prices beyond maturity 1 do not affect the economy, we then re-express (1) as

$$E_t[\mathbf{f}(\mathbf{y}_{t+1},\mathbf{y}_t,\mathbf{x}_{t+1},\mathbf{x}_t)]=\mathbf{0} \tag{3}$$

$$E_t[f(P^{t+1,k-1},P^{t,k},\mathcal{M})]=0 \text{ for } k=1,2,\ldots,K \tag{4}$$

Here, $\mathbf{y}_t$ denotes an $n_y\times 1$ vector of all non-predetermined variables *except* bond prices with maturities exceeding one period. The idea behind our approximation method is to solve the problem in (3) independently of (4) *and* to recursively solve for bond prices using (4). That is, we first solve the problem in (3) by standard perturbation. Given this solution, we then solve the second perturbation problem for all bond prices. On account of this structure, we refer to our method as "perturbation-on-perturbation" (POP).[4]

Our method is related to the one applied in Hordahl, Tristani, and Vestin (2008) which uses an "approximation order-matching" argument to derive a second-order accurate solution to bond prices. We consider a slightly more general setup than in Hordahl, Tristani, and Vestin (2008), as we allow for general transformations of variables (nesting their log-specification) and introduce no restrictions on the functional form of the stochastic discount factor. In addition, we provide a third-order approximation to bond prices which is of great economic interest as it allows for time-varying term premia as emphasized in Andreasen (2012b), Binsbergen et al. (2012), and Rudebusch and Swanson (2012).

2.2 The perturbation problem

The exact solution to the part of the model without bond prices with maturities exceeding one period in (3) is given by (Schmitt-Grohé and Uribe 2004)

$$\mathbf{y}_t=\mathbf{g}(\mathbf{x}_t,\sigma) \tag{5}$$

$$\mathbf{x}_{t+1}=\mathbf{h}(\mathbf{x}_t,\sigma)+\sigma\eta\epsilon_{t+1} \tag{6}$$

4 Binsbergen et al. (2012) independently apply a related method to compute interest rates in a version of the neoclassical growth model. The method and formulas we provide are not model specific and our approach nests their procedure.

Here, $\boldsymbol{\epsilon}_{t+1}$ is a vector of n_ϵ innovations with the properties $\boldsymbol{\epsilon}_{t+1} \sim \mathcal{IID}(\mathbf{0}, \mathbf{I})$ and σ is an auxiliary perturbation parameter scaling the square root of the covariance matrix $\boldsymbol{\eta}$. The assumption that innovations only enter linearly in (6) is without loss of generality as shown in Andreasen (2012b). We illustrate this point in our empirical application in Section 5 by considering state variables featuring stochastic volatility.

In the first step of the POP method, the solution to (5)–(6) is approximated to N-th order using standard perturbation packages. In endowment economies, this means solving for the first bond price, whereas the law of motion for the state variables are known. In production-based economies, we additionally solve for the law of motion for all endogenous state variables and variables such as consumption and inflation, which are functions of the structural shocks to the economy.

In the second step of the POP method all remaining bond prices are computed. Since continuously compounded interest rates are linear functions of the logarithm of bond prices, we focus on the popular log-transformation and re-write the fundamental equation in (2) as

$$\exp(p^{t,k}) = E_t[\mathcal{M} \times \exp(p^{t+1,k-1})] \tag{7}$$

where $p^{t,k} \equiv \ln(P^{t,k})$.[5] We emphasize that this expression for bond prices is identical to the one in (2) but often convenient when computing numerical approximations. Appendix A considers the case of a general transformation, which includes the identity mapping and corresponds to using (2) for the approximation.[6]

In deriving the perturbation approximation to $p^{t,k}$ we exploit two facts. Firstly, the functional form of the stochastic discount factor $\mathcal{M}(\mathbf{y}_{t+1}, \mathbf{y}_t, \mathbf{x}_{t+1}, \mathbf{x}_t)$ is known.[7] Secondly, since any bond price is non-predetermined, it is a function of $\mathbf{x}_t$ and σ. That is, $p^{t,k} := p^k(\mathbf{x}_t, \sigma)$ where k denotes the maturity of the bond. Using these insights and substituting (5)–(6) into (7), we define the function

$$F^k(\mathbf{x},\sigma) := E[\exp(p^{t,k}(\mathbf{x},\sigma)) - \mathcal{M}(\mathbf{g}(\mathbf{h}(\mathbf{x},\sigma)+\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1},\sigma), \mathbf{g}(\mathbf{x},\sigma), \mathbf{h}(\mathbf{x},\sigma)+\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}, \mathbf{x}) \\ \times \exp(p^{t+1,k-1}(\mathbf{h}(\mathbf{x},\sigma)+\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1},\sigma))]$$

for $k=1, 2, \ldots, K$ where the time index for the state vector is suppressed. Hence, $F^k(\mathbf{x}, \sigma) \equiv 0$ for all values of $\mathbf{x}$ and σ, implying that all derivatives of $F^k(\mathbf{x}, \sigma)$ must also equal zero. That is,

$$F^k_{\mathbf{x}^i \sigma^j}(\mathbf{x},\sigma) = 0 \quad \forall \mathbf{x}, \sigma, i, j \tag{8}$$

where $F^k_{\mathbf{x}^i \sigma^j}(\mathbf{x},\sigma)$ denotes the derivative of F^k with respect to $\mathbf{x}$ taken i times and with respect to σ taken j times.

2.3 Notation

A few remarks in relation to our notation are appropriate before deriving the bond price recursions. We adopt the convention that indices α and γ relate to elements of $\mathbf{x}$, while β and ϕ correspond to elements of $\mathbf{y}$ and $\boldsymbol{\epsilon}$, respectively. Subscripts on these indices will capture the sequence in which derivatives are taken. For example, α_1 corresponds to the first time a function is differentiated with respect to $\mathbf{x}$, while α_2 is used when differentiating with respect to $\mathbf{x}$ the second time.

5 If the one-period interest rate r_t is included in the solution to the first perturbation step, then we immediately have $p^1 = -r$ with a log-transformation and we do not explicitly need to report p^1 when solving (5)–(6).

6 For the accuracy study in Section 4, unreported results show that the log-transformation is more accurate than using a "level" approximation based on (2).

7 We assume that the variables in the first block of the model, i.e., $\mathbf{x}$ and $\mathbf{y}$, have also been transformed. Accordingly, $\mathcal{M}$ and all its derivatives are known functions of the *transformed* variables. For example, for CRRA utility we would have $\mathcal{M}(c_{t+1}, c_t) = \beta \exp(-\gamma c_{t+1}) / \exp(-\gamma c_t)$.

In most of the subsequent derivations we use the tensor notation. For instance, $[p_{\mathbf{x}}^{k}]_{\gamma_1}$ denotes the γ_1-th element of the $1\times n_x$ vector of derivatives of p^k with respect to $\mathbf{x}$. Similarly, the derivative of $\mathbf{h}$ with respect to $\mathbf{x}$ is an $n_x\times n_x$ matrix and $\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_1}^{\gamma_1}$ is the element of this matrix located at the intersection of the γ_1-th row and the α_1-th column. Also, $\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_1}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_1}^{\gamma_1}=\sum_{\gamma_1=1}^{n_x}(\partial p^{k-1}/\partial \mathbf{x}_{\gamma_1})(\partial \mathbf{h}^{\gamma_1}/\partial \mathbf{x}_{\alpha_1})$ while $\left[p_{\mathbf{xx}}^{k-1}\right]_{\gamma_1\gamma_2}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_2}^{\gamma_2}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_1}^{\gamma_1}=\sum_{\gamma_1=1}^{n_x}\sum_{\gamma_2=1}^{n_x}(\partial^2 p^{k-1}/\partial \mathbf{x}_{\gamma_1}\partial \mathbf{x}_{\gamma_2})(\partial \mathbf{h}^{\gamma_2}/\partial \mathbf{x}_{\alpha_2})(\partial \mathbf{h}^{\gamma_1}/\partial \mathbf{x}_{\alpha_1})$ where, for instance, $\mathbf{h}^{\gamma_1}$ denotes the γ_1-th function of mapping $\mathbf{h}$ and $\mathbf{x}_{\alpha_1}$ is the α_1-th element of vector $\mathbf{x}$.

For simplicity, we also use superscripts t and t+1 on functions p^k, $\mathbf{h}$, $\mathbf{g}$, and their derivatives to indicate the arguments at which they are evaluated. When these superscripts are omitted, functions are evaluated at the steady state, i.e., at $(\mathbf{x}, \sigma)=(\mathbf{x}_{ss}, 0)$. For example, for $f\in\{p^k, \mathbf{g}, \mathbf{h}\}$

$$\begin{array}{lll} f^t:=f(\mathbf{x}_t,\sigma) & f^{t+1}:=f(\mathbf{x}_{t+1},\sigma) & f:=f(\mathbf{x}_{ss},0) \\ f_{\mathbf{x}}^t:=(\partial f/\partial \mathbf{x})|_{(\mathbf{x}_t,\sigma)} & f_{\mathbf{x}}^{t+1}:=(\partial f/\partial \mathbf{x})|_{(\mathbf{x}_{t+1},\sigma)} & f_{\mathbf{x}}:=(\partial f/\partial \mathbf{x})|_{(\mathbf{x}_{ss},0)}. \end{array}$$

2.4 First order terms

To find the first-order derivatives of $p^k(\mathbf{x}, \sigma)$ with respect to $\mathbf{x}$, we start by differentiating $F^k(\mathbf{x}, \sigma)$ with respect to $\mathbf{x}$. Exploiting (8) we rewrite $[F_{\mathbf{x}}^k(\mathbf{x}_t,\sigma)]_{\alpha_1}=0$ as

$$\exp\left(p^k\right)\left[p_{\mathbf{x}}^{t,k}\right]_{\alpha_1}-\left[\mathcal{M}_{\mathbf{x}}\right]_{\alpha_1}\exp\left(p^{t+1,k-1}\right)-\mathcal{M}\exp\left(p^{t+1,k-1}\right)\left[p_{\mathbf{x}}^{t+1,k-1}\right]_{\gamma_1}\left[\mathbf{h}_{\mathbf{x}}^t\right]_{\alpha_1}^{\gamma_1}=0 \quad (9)$$

for $\alpha_1, \gamma_1=1, 2, \ldots, n_x$. Evaluating (9) in the deterministic steady state gives a set of equations which determine $[p_{\mathbf{x}}^k]_{\alpha_1}$ for $\alpha_1=1, 2, \ldots, n_x$ and $k=2, \ldots, K$. To see this, consider a bond with one period to maturity. The price of a maturing bond is 1 for all values of $(\mathbf{x}, \sigma)$, and all of its derivatives are therefore equal to zero, i.e., $p_{\mathbf{x}}^{t+1,0}=0$. Accordingly, evaluating (9) at the steady state and for $k=1$, this equation simplifies to

$$\exp(p^1)[p_{\mathbf{x}}^1]_{\alpha_1}=[\mathcal{M}_{\mathbf{x}}]_{\alpha_1}, \quad (10)$$

as $\exp(p^0)=P^0=1$. We know the value of $\mathcal{M}=\exp(p^1)$, and $[p_{\mathbf{x}}^1]_{\alpha_1}$ is given by the first perturbation step. Using the expression for $[\mathcal{M}_{\mathbf{x}}]_{\alpha_1}$ in (10), simple algebra implies

$$\mathbf{p}_{\mathbf{x}}^k=\mathbf{p}_{\mathbf{x}}^1+\mathbf{p}_{\mathbf{x}}^{k-1}\mathbf{h}_{\mathbf{x}}, \quad (11)$$

where $\mathbf{p}_{\mathbf{x}}^k$ denotes a $1\times n_x$ vector of derivatives of p^k with respect to $\mathbf{x}$. Given that $\mathbf{p}_{\mathbf{x}}^1$ and $\mathbf{h}_x$ are known from the first perturbation step, all remaining first-order derivatives of $p^k(\mathbf{x}, \sigma)$ with respect to $\mathbf{x}$ are then easily computed by iterating the linear and recursive system in (11) where $\left\{\mathbf{p}_{\mathbf{x}}^k\right\}_{k=2}^K$ are found by simple summation. In contrast, for the one-step perturbation method, Schmitt-Grohé and Uribe (2004) show that $\left\{\mathbf{p}_{\mathbf{x}}^k\right\}_{k=1}^K$ are determined from a quadratic and simultaneous system of equations which is substantially more complicated to solve [see for instance Klein (2000)].

The first-order derivatives of bond prices with respect to σ are found in a similar way.[8] That is, we exploit the fact that the derivative of $F^k(\mathbf{x}, \sigma)$ with respect to σ evaluated at the steady state equals zero, i.e.,

$$F_\sigma^k(\mathbf{x}_{ss},0)=E_t[\exp(p^k)[p_\sigma^k]-[\mathcal{M}_\sigma]\exp(p^{k-1})-\mathcal{M}\exp(p^{k-1})([p_{\mathbf{x}}^{k-1}]_{\gamma_1}([\mathbf{h}_\sigma]^{\gamma_1}+\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1})+[p_\sigma^{k-1}])]=0.$$

For the one-period bond, this reduces to

$$E_t[\exp(p^1)[p_\sigma^1]]=E_t[\mathcal{M}_\sigma] \quad (12)$$

8 We know from Schmitt-Grohé and Uribe (2004) that these derivatives are zero. Nevertheless, we solve for these terms to make subsequent derivations of higher-order derivatives more transparent.

as $\exp(p^0)=1$, $E_t[\boldsymbol{\epsilon}_{t+1}]=0$, and all the derivatives of p^0 are zero. The fact that $E_t[\mathcal{M}_\sigma]=0$ implies $[p_\sigma^1]=0$. Moreover, $\mathbf{h}_\sigma=0$ and this suffices to show that $p_\sigma^k=0$ for $k=2, 3, \ldots, K$.

2.5 Second order terms

All second-order terms are derived in a similar manner. Starting with second-order derivatives with respect to the state vector, we obtain

$$\begin{aligned}\left[F_{\mathbf{xx}}^k(\mathbf{x}_{ss},0)\right]_{\alpha_1,\alpha_2} &= \exp(p^k)\left[p_{\mathbf{x}}^k\right]_{\alpha_2}\left[p_{\mathbf{x}}^k\right]_{\alpha_1} + \exp(p^k)\left[p_{\mathbf{xx}}^k\right]_{\alpha_1\alpha_2} \\ &-\left[\mathcal{M}_{\mathbf{xx}}\right]_{\alpha_1\alpha_2}\exp(p^{k-1}) - \left[\mathcal{M}_{\mathbf{x}}\right]_{\alpha_1}\exp(p^{k-1})\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_2}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_2}^{\gamma_2} \\ &-\left[\mathcal{M}_{\mathbf{x}}\right]_{\alpha_2}\exp(p^{k-1})\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_1}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_1}^{\gamma_1} \\ &-\mathcal{M}\exp(p^{k-1})\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_2}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_2}^{\gamma_2}\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_1}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_1}^{\gamma_1} \\ &-\mathcal{M}\exp(p^{k-1})\left(p_{\mathbf{xx}}^{k-1}\right)_{\gamma_1\gamma_2}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_2}^{\gamma_2}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_1}^{\gamma_1} \\ &-\mathcal{M}\exp(p^{k-1})\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_1}\left[\mathbf{h}_{\mathbf{xx}}\right]_{\alpha_1\alpha_2}^{\gamma_1} = 0\end{aligned} \tag{13}$$

for $\alpha_1, \alpha_2, \gamma_1, \gamma_2 = 1, 2, \ldots, n_x$. To evaluate the right hand side of (13) we only need an expression for $\mathcal{M}_{\mathbf{xx}}$ as all other terms are known by now. Considering (13) when $k=1$, we have

$$[\mathcal{M}_{\mathbf{xx}}]_{\alpha_1\alpha_2} = \exp(p^1)[p_{\mathbf{xx}}^1]_{\alpha_1,\alpha_2} + \exp(p^1)[p_{\mathbf{x}}^1]_{\alpha_2}[p_{\mathbf{x}}^1]_{\alpha_1}, \tag{14}$$

as all derivatives of $p^0(\mathbf{x}, \sigma)$ are zero. Simple algebra then implies

$$\mathbf{p}_{\mathbf{xx}}^k = \mathbf{p}_{\mathbf{xx}}^1 + \mathbf{h}_{\mathbf{x}}'\mathbf{p}_{\mathbf{xx}}^{k-1}\mathbf{h}_{\mathbf{x}} + \sum\nolimits_{\gamma_1=1}^{n_x} p_{\mathbf{x}}^{k-1}(\gamma_1)\mathbf{h}_{\mathbf{xx}}(\gamma_1,:,:), \tag{15}$$

where we use the notation that $A(\gamma_1, \gamma_2, \ldots, \gamma_N)$ denotes an element on the intersection of dimensions γ_1, γ_2, and γ_N in matrix $\mathbf{A}$ and colons refer to entire dimensions. For example, $\mathbf{h}_{\mathbf{xx}}(\gamma_1, :, :)$ is an $n_x \times n_x$ matrix of second-order derivatives of the γ_1-th mapping of $\mathbf{h}$ evaluated at the steady state, and $\mathbf{p}_{\mathbf{xx}}^k$ is the $n_x \times n_x$ matrix of second-order derivatives of p^k with respect to $\mathbf{x}$. Given that $\{\mathbf{p}_{\mathbf{x}}^k\}_{k=1}^K$, $\mathbf{h}_{\mathbf{x}}$, $\mathbf{p}_{\mathbf{xx}}^1$, and $\mathbf{h}_{\mathbf{xx}}$ are known by now, all remaining second-order derivatives of $p^k(\mathbf{x}, \sigma)$ with respect to $\mathbf{x}$ are then easily computed by iterating the linear and recursive system in (15) using simple summation. On the other hand, for the one-step perturbation method considered in Schmitt-Grohé and Uribe (2004), $\{\mathbf{p}_{\mathbf{xx}}^k\}_{k=1}^K$, are determined from a linear system of simultaneous equations and therefore much more time consuming to compute than the solution we suggest in (15).

To find $p_{\sigma\sigma}^k$, we differentiate $F^k(\mathbf{x}, \sigma)$ twice with respect to σ and evaluate the expression in the steady state, giving

$$\begin{aligned}\left[F_{\sigma\sigma}(\mathbf{x}_{ss},0)\right] = E_t\Big[&-\exp(p^k)\left[p_{\sigma\sigma}^k\right] + \left[\mathcal{M}_{\sigma\sigma}\right]\exp(p^{k-1}) \\ &+\left[\mathcal{M}_\sigma\right]\exp(p^{k-1})\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2} \\ &+\left[\mathcal{M}_\sigma\right]\exp(p^{k-1})\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_1}\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1} \\ &+\mathcal{M}\exp(p^{k-1})\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_1}\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1} \\ &+\mathcal{M}\exp(p^{k-1})\left[p_{\mathbf{xx}}^{k-1}\right]_{\gamma_1\gamma_2}\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}[\boldsymbol{\eta}]_{\phi_1}^{\gamma_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1} \\ &+\mathcal{M}\exp(p^{k-1})\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_1}\left[\mathbf{h}_{\sigma\sigma}\right]^{\gamma_1} + \mathcal{M}\exp(p^{k-1})\left[p_{\sigma\sigma}^{k-1}\right]\Big] = 0\end{aligned} \tag{16}$$

where $\gamma_1, \gamma_2=1, 2, \dots, n_x$ and $\phi_1, \phi_2=1, 2, \dots, n_\epsilon$. To simplify (16) we have relied on the fact that the terms $\mathbf{h}_\sigma, p^k_{\boldsymbol{\sigma}}$, and $p^k_{\mathbf{x}\boldsymbol{\sigma}}$ are known to be zero (Schmitt-Grohé and Uribe 2004). Again, the important thing to observe is that (16) allows us to solve for $p^k_{\sigma\sigma}$ for $k=2, 3, \dots, K$. To show this, we first differentiate $\mathcal{M}$ with respect to σ to obtain

$$\left[\mathcal{M}_\sigma\right]=\left[\mathcal{M}_{\mathbf{y}_{t+1}}\right]_{\beta_2}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_2}_{\gamma_2}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}+\left[\mathcal{M}_{\mathbf{x}_{t+1}}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}. \tag{17}$$

Evaluating (16) at $k=1$ and exploiting the fact that all derivatives of $p^0(\mathbf{x}, \sigma)$ are zero gives

$$E_t[\mathcal{M}_{\sigma\sigma}]=[p^1_{\sigma\sigma}]\exp(p^1). \tag{18}$$

Combining the results in (17) and (18) to evaluate (16) we obtain

$$\begin{aligned}\left[p^k_{\sigma\sigma}\right]=&\exp\left(-p^k\right)\Big\{\left[p^1_{\sigma\sigma}\right]\exp\left(p^1\right)\exp\left(p^{k-1}\right)\\ &+2\left[\mathcal{M}_{\mathbf{y}_{t+1}}\right]_{\beta_1}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\exp\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\mathbf{I}\right]^{\phi_2}_{\phi_1}\\ &+2\left[\mathcal{M}_{\mathbf{x}_{t+1}}\right]_{\gamma_1}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\mathbf{I}\right]^{\phi_2}_{\phi_1}\\ &+\exp\left(p^1\right)\exp\left(p^{k-1}\right)\left(\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\mathbf{I}\right]^{\phi_1}_{\phi_2}+\left[p^{k-1}_{\mathbf{xx}}\right]_{\gamma_1\gamma_2}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi}\left[\mathbf{I}\right]^{\phi_1}_{\phi_2}\right)\\ &+\exp\left(p^1\right)\exp\left(p^{k-1}\right)\left(\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}\left[\mathbf{h}_{\sigma\sigma}\right]^{\gamma_1}+\left[p^{k-1}_{\sigma\sigma}\right]\right)\Big\}\end{aligned} \tag{19}$$

As discussed previously, the derivatives of the stochastic discount factor $\mathcal{M}_{\mathbf{y}_{t+1}}$ and $\mathcal{M}_{\mathbf{x}_{t+1}}$ are straightforward to compute from the known functional form of $\mathcal{M}$. In the standard case of $\mathcal{M}:=\beta\lambda_{t+1}/(\lambda_t\pi_{t+1})$ where β is the discount factor, λ_t denotes marginal utility of consumption, and π_t is inflation, simple manipulations imply

$$p^k_{\sigma\sigma}=p^1_{\sigma\sigma}+p^{k-1}_{\sigma\sigma}+\mathbf{p}^{k-1}_{\mathbf{x}}\mathbf{h}_{\sigma\sigma}+\mathrm{trace}(\boldsymbol{\eta}'\mathbf{p}^{k-1}_{\mathbf{xx}}\boldsymbol{\eta})+\mathbf{p}^{k-1}_{\mathbf{x}}\boldsymbol{\eta}\boldsymbol{\eta}'(\mathbf{p}^{k-1}_{\mathbf{x}})'+2(\boldsymbol{\lambda}_{\mathbf{x}}-\boldsymbol{\pi}_{\mathbf{x}})\boldsymbol{\eta}\boldsymbol{\eta}'(\mathbf{p}^{k-1}_{\mathbf{x}})'. \tag{20}$$

Here, $\boldsymbol{\lambda}_{\mathbf{x}}$ and $\boldsymbol{\pi}_{\mathbf{x}}$ denote $1\times n_x$ matrices of first-order derivatives for λ_t and π_t with respect to $\mathbf{x}$ evaluated in the steady state, respectively. Hence, also $\left\{p^k_{\sigma\sigma}\right\}^K_{k=2}$ are obtained by iterating a linear and recursive system, whereas these bond price derivatives in the one-step perturbation method are determined from a linear system of simultaneous equations.

We finally emphasize that our perturbation method only uses the restrictions implied by $[F^k_{\mathbf{xx}}(\mathbf{x}_{ss},0)]_{\alpha_1,\alpha_2}=0$ and $[F_{\sigma\sigma}(\mathbf{x}_{ss},0)]=0$ to solve for $\left\{\mathbf{p}^k_{\mathbf{xx}}\right\}^K_{k=2}$ and $\left\{\mathbf{p}^k_{\sigma\sigma}\right\}^K_{k=2}$, respectively. Hence, we do not rely on an "approximation order-matching" argument as in Hordahl, Tristani, and Vestin (2008) or the pruning scheme by Kim et al. (2008). Similar to the one-step perturbation routine, such additional assumptions are unnecessary for the POP method because the steady state ($\mathbf{x}_{ss}$, $\sigma=0$) is a fixed-point in (6), ensuring that terms beyond the considered approximation order are not present when bond prices are computed recursively.

2.6 Third order terms

The expressions for the third-order terms $\mathbf{p}^k_{\mathbf{xxx}}, \mathbf{p}^k_{\mathbf{x}\sigma\sigma}$, and $\mathbf{p}^k_{\sigma\sigma\sigma}$ are somewhat more involved and we therefore defer their derivation to Appendix A.[9] Here we show that

$$\begin{aligned}p^k_{\mathbf{xxx}}(\alpha_1,\alpha_2,\alpha_3)=&p^1_{\mathbf{xxx}}(\alpha_1,\alpha_2,\alpha_3)+\textstyle\sum_{\gamma_1=1}^{n_x}\mathbf{h}_{\mathbf{x}}(\gamma_1,\alpha_1)\mathbf{h}_{\mathbf{x}}(:,\alpha_2)'\mathbf{p}^{k-1}_{\mathbf{xxx}}(\gamma_1,:,:)\mathbf{h}_{\mathbf{x}}(:,\alpha_3)\\ &+\mathbf{h}_{\mathbf{x}}(:,\alpha_1)'\mathbf{p}^{k-1}_{\mathbf{xx}}\mathbf{h}_{\mathbf{xx}}(:,\alpha_2,\alpha_3)+\mathbf{h}_{\mathbf{xx}}(:,\alpha_1,\alpha_3)'\mathbf{p}^{k-1}_{\mathbf{xx}}\mathbf{h}_{\mathbf{x}}(:,\alpha_2)\\ &+\mathbf{h}_{\mathbf{xx}}(:,\alpha_1,\alpha_2)'\mathbf{p}^{k-1}_{\mathbf{xx}}\mathbf{h}_{\mathbf{x}}(:,\alpha_3)+\mathbf{p}^{k-1}_{\mathbf{x}}\mathbf{h}_{\mathbf{xxx}}(:,\alpha_1,\alpha_2,\alpha_3)\end{aligned} \tag{21}$$

9 As shown in Andreasen (2012b) $\mathbf{p}^k_{\mathbf{xx}\sigma}=\mathbf{0}$.

for $k=2, 3, \dots, K$ and $\alpha_1, \alpha_2, \alpha_3=1, 2, \dots, n_x$. When $\mathcal{M}:=\beta\lambda_{t+1}/(\lambda_t\pi_{t+1})$, the general formulas for $\mathbf{p}^k_{\sigma\sigma\mathbf{x}}$ and $\mathbf{p}^k_{\sigma\sigma\sigma}$ reported in Appendix A simplify to

$$\begin{aligned}\mathbf{p}^k_{\sigma\sigma\mathbf{x}} = & \mathbf{p}^1_{\sigma\sigma\mathbf{x}} - 2(\boldsymbol{\lambda}_\mathbf{x} - \boldsymbol{\pi}_\mathbf{x})\eta\eta'(\mathbf{p}^{k-1}_\mathbf{x})'\mathbf{p}^1_\mathbf{x} + 2\mathbf{p}^{k-1}_\mathbf{x}\eta\eta'(\boldsymbol{\lambda}'_\mathbf{x}\boldsymbol{\pi}_\mathbf{x} - \boldsymbol{\lambda}'_\mathbf{x}\boldsymbol{\lambda}_\mathbf{x}) \\ & + 2\mathbf{p}^{k-1}_\mathbf{x}\eta\eta'(\boldsymbol{\lambda}'_\mathbf{x}\boldsymbol{\lambda}_\mathbf{x} - \boldsymbol{\lambda}'_\mathbf{x}\boldsymbol{\pi}_\mathbf{x} - \boldsymbol{\pi}'_\mathbf{x}\boldsymbol{\lambda}_\mathbf{x} + \boldsymbol{\pi}'_\mathbf{x}\boldsymbol{\pi}_\mathbf{x} + \mathbf{g}^\lambda_\mathbf{xx} - \mathbf{g}^\pi_\mathbf{xx} + \mathbf{p}^{k-1}_\mathbf{xx})\mathbf{h}_\mathbf{x} \\ & + 2(\boldsymbol{\lambda}_\mathbf{x} - \boldsymbol{\pi}_\mathbf{x})\eta\eta'\mathbf{p}^{k-1}_\mathbf{xx}\mathbf{h}_\mathbf{x} + \sum_{\gamma_1=1}^{n_x}\eta(\gamma_1, :)\eta'\mathbf{p}^{k-1}_\mathbf{xxx}(\gamma_1, :, :)\mathbf{h}_\mathbf{x} \\ & + \mathbf{h}'_{\sigma\sigma}\mathbf{p}^{k-1}_\mathbf{xx}\mathbf{h}_\mathbf{x} + \mathbf{p}^{k-1}_\mathbf{x}\mathbf{h}_{\sigma\sigma\mathbf{x}} + \mathbf{p}^{k-1}_{\sigma\sigma\mathbf{x}}\mathbf{h}_\mathbf{x}\end{aligned} \tag{22}$$

and

$$\begin{aligned}p^k_{\sigma\sigma\sigma} = & p^1_{\sigma\sigma\sigma} + \mathbf{p}^{k-1}_\mathbf{x}\mathbf{h}_{\sigma\sigma\sigma} + p^{k-1}_{\sigma\sigma\sigma} \\ & + \sum_{\phi_1=1}^{n_e} 3(\boldsymbol{\eta}(:,\phi_1)m^3(\boldsymbol{\epsilon}_{t+1}(\phi_1)))'(\boldsymbol{\lambda}'_\mathbf{x}\boldsymbol{\lambda}_\mathbf{x} - \boldsymbol{\lambda}'_\mathbf{x}\boldsymbol{\pi}_\mathbf{x} - \boldsymbol{\pi}'_\mathbf{x}\boldsymbol{\lambda}_\mathbf{x} + \boldsymbol{\pi}'_\mathbf{x}\boldsymbol{\pi}_\mathbf{x})\boldsymbol{\eta}(:,\phi_1)\mathbf{p}^{k-1}_\mathbf{x}\boldsymbol{\eta}(:,\phi_1) \\ & + \sum_{\phi_1=1}^{n_e} 3(\boldsymbol{\eta}(:,\phi_1)m^3(\boldsymbol{\epsilon}_{t+1}(\phi_1)))'(\boldsymbol{\lambda}_\mathbf{xx} - \boldsymbol{\pi}_\mathbf{xx})\boldsymbol{\eta}(:.\phi_1)\mathbf{p}^{k-1}_\mathbf{x}\boldsymbol{\eta}(:,\phi_1) \\ & + \sum_{\phi_1=1}^{n_e} 3(\boldsymbol{\lambda}_\mathbf{x} - \boldsymbol{\pi}_\mathbf{x})\mathbf{g}_\mathbf{x}\boldsymbol{\eta}(:,\phi_1)m^3(\boldsymbol{\epsilon}_{t+1}(\phi_1))\mathbf{p}^{k-1}_\mathbf{x}\boldsymbol{\eta}(:,\phi_1)\mathbf{p}^{k-1}_\mathbf{x}\boldsymbol{\eta}(:,\phi_1) \\ & + \sum_{\phi_1=1}^{n_e} 3(\boldsymbol{\lambda}_\mathbf{x} - \boldsymbol{\pi}_\mathbf{x})\boldsymbol{\eta}(:,\phi_1)m^3(\boldsymbol{\epsilon}_{t+1}(\phi_1))\boldsymbol{\eta}(:,\phi_2)'\mathbf{p}^{k-1}_\mathbf{xx}\boldsymbol{\eta}(:,\phi_3) \\ & + \sum_{\phi_1=1}^{n_e} \mathbf{p}^{k-1}_\mathbf{x}\boldsymbol{\eta}(:,\phi_1)\mathbf{p}^{k-1}_\mathbf{x}\boldsymbol{\eta}(:,\phi_1)\mathbf{p}^{k-1}_\mathbf{x}\boldsymbol{\eta}(:,\phi_1)m^3(\boldsymbol{\epsilon}_{t+1}(\phi_1)) \\ & + \sum_{\phi_1=1}^{n_e} 3\mathbf{p}^{k-1}_\mathbf{x}\boldsymbol{\eta}(:,\phi_1)(\boldsymbol{\eta}(:,\phi_1)m^3(\boldsymbol{\epsilon}_{t+1}(\phi_1)))'\mathbf{p}^{k-1}_\mathbf{xx}\boldsymbol{\eta}(:,\phi_1) \\ & + \sum_{\phi_1=1}^{n_e} \boldsymbol{\eta}(:,\phi_1)'\mathbf{p}^{k-1}_\mathbf{xxx}(\gamma_1,:,:)\boldsymbol{\eta}(:,\phi_1)\boldsymbol{\eta}(\gamma_1,\phi_1)m^3(\boldsymbol{\epsilon}_{t+1}(\phi_1))\end{aligned} \tag{23}$$

for $k=2, 3, \dots, K$. Here, $m^3(\boldsymbol{\epsilon}_{t+1}(\phi_1))$ denotes the third moment of $\boldsymbol{\epsilon}_{t+1}(\phi_1)$ for $\phi_1=1, 2, \dots, n_{\boldsymbol{\epsilon}}$. Moreover, $\boldsymbol{\lambda}_\mathbf{xx}$ and $\boldsymbol{\pi}_\mathbf{xx}$ represent $n_x \times n_x$ matrices of second-order derivatives for λ_t and π_t with respect to $\mathbf{x}$ in the steady state, respectively.

2.7 Higher order terms and extensions

It is easy to see that our method can be applied to derive any Taylor series approximation to the considered class of models. Our method is only constrained by the property that the model can be split into two distinct parts: one containing all equations without bond prices beyond one period maturities and another consisting entirely of Euler-equations for the remaining bond prices. This assumption is always satisfied in endowment models but may not necessarily hold in production-based models, although this restriction is imposed in nearly all current DSGE models with a yield curve. However, the POP method may still be useful even if the condition does not hold. To see how, consider the case in which one is interested in the dynamics of the 10-year yield curve but it is only possible to separate out bond price Euler-equations of maturity >5 years. To apply our method, the model including bond prices of maturity up to 20 quarters (5 years) needs to be solved in the first step of the POP method. This gives all derivatives of bond prices for $k \le 20$. The remaining derivatives for bond prices with maturities between 5 and 10 years, i.e., $k=21, 22, \dots, 40$, can then be computed in the second step by starting the recursions derived in this paper at $k=20$. Hence, the POP method may also reduce the approximation time in this case, although its computational gain will be smaller than in the standard case.

Andreasen (2012a) presents another extension of the POP method in which the expected value of future non-predetermined variables are computed in the second perturbation step, making it possible to efficiently solve for expected future short interest rates, inflation rates, etc. This enables us to use the POP method to efficiently compute bond term premia when defined as the difference between the long interest rate and the average of expected future short rates.

3 Evaluating the computational gain

This section assesses the speed of the POP method and compares it to the standard one-step perturbation routine. To illustrate the numerical problem, consider a quarterly model with n state variables. Suppose we are interested in computing the 10-year interest rate from the price of a zero-coupon bond with the same maturity. This bond price is a function of n state variables and to approximate it to third order – i.e., by a third-order Taylor series expansion – would require computing

$$\underbrace{1}_{0^{\text{th}}\text{ Order Terms}} + \underbrace{n}_{1^{\text{st}}\text{ Order Terms}} + \underbrace{n\cdot\frac{n+1}{2}}_{2^{\text{nd}}\text{ Order Terms}} + \underbrace{n\cdot\frac{n+1}{2}\cdot\frac{n+2}{3}}_{3^{\text{rd}}\text{ Order Terms}}$$

distinct coefficients. Typically, the 10-year bond price is computed recursively along with all 40 intermediate bond prices, implying that we need to compute $40\left(1+n+\frac{n(n+1)}{2}+\frac{n(n+1)(n+2)}{6}\right)$ bond price derivatives.[10] For n equal to 5, 10, or 15, the 10-year interest rate then introduces respectively 2240, 11,440, or 32,640 additional coefficients to be computed. This may significantly increase the time required to derive the approximation and in some cases even make the problem too large to solve using standard solution packages because of memory constraints.[11]

Hence, the efficiency gains from the POP method depend mainly on the maximum maturity of bonds in the yield curve and the number of state variables in the model. To illustrate the effect of the maximum bond maturity, we report results corresponding to yield curves of maturities ranging from 5 to 20 years.[12] The relevance of the number of state variables is examined by studying the multi-country RBC model in Juillard and Villemot (2011) with an increasing number of countries.[13] All versions of this model are approximated to third order, and we use *Dynare++* as the standard perturbation package.

The efficiency gain from the POP method is measured by

$$\Psi=\frac{\text{Computing time using the one-step perturbation method}}{\text{Computing time using the POP method}}.$$

Table 1 reports estimates of execution time and computational improvement of the POP method based on 21 replications. For one country in the RBC model with five state variables, i.e., $n_x=5$, we find modest speed gains with $\Psi=1.42$ and $\Psi=2.52$ for a 10-year and 20-year yield curve, respectively. The computational gain is larger with two countries and eight state variables, where the POP method is more than eight times faster than the one-step perturbation method for a 20-year yield curve. In the case of five countries and 17 state variables, we find substantial efficiency gains as the POP method is 10 and 38 times faster than the one-step perturbation method with a 10-year and 20-year yield curve, respectively. Although nearly all versions of the model considered so far solve within a minute, the reported efficiency gains from the POP method are nevertheless essential when repeated approximations are required for estimation or extensive sensitivity analysis. For instance, in the five country model with a 20-year yield curve using the one-step perturbation method it takes roughly 38 h to compute 2000 function evaluations, whereas it only takes 1 h using the POP method.

10 Alternative non-recursive methods involve creating many auxiliary variables which similarly complicate the approximation problem.

11 These packages include *Dynare*, *Dynare++*, and *Perturbation AIM* [see Kamenik (2005) and Swanson, Anderson, and Levin (2005), respectively] and the set of routines accompanying Schmitt-Grohé and Uribe (2004).

12 The case of the 20-year yield curve is seldom considered in the literature, with the 10-year yield curve being the benchmark. However, from a computational perspective, approximating the 20-year yield curve is equivalent to: i) computing jointly the 10-year nominal and real yield curves, or ii) computing the 10-year yield curve and the corresponding term premia.

13 We adopt the calibration in Juillard and Villemot (2011) without labor, using power preferences with an intertemporal elasticity of substitution of two and a Cobb-Douglas production function.

Table 1 Gain in computing speed from the POP method.

	5-year	10-year	15-year	20-year
1-country model (n_x=5)				
One-step perturbation method (seconds)	0.18	0.25	0.36	0.53
POP method (seconds)	0.13	0.16	0.18	0.21
Speed gain	**1.42**	**1.56**	**1.93**	**2.52**
2-country model (n_x=8)				
One-step perturbation method (seconds)	0.51	1.01	1.96	3.16
POP method (seconds)	0.25	0.30	0.33	0.37
Speed gain	**2.05**	**3.42**	**6.04**	**8.50**
5-country model (n_x=17)				
One-step perturbation method (seconds)	5.32	14.38	34.53	67.99
POP method (seconds)	1.22	1.39	1.55	1.80
Speed gain	**4.35**	**10.33**	**22.35**	**37.75**
10-country model (n_x=32)				
One-step perturbation method (seconds)	67.07	189.93	NA	NA
POP method (seconds)	9.85	11.38	12.76	14.43
Speed gain	**6.81**	**16.68**	NA	NA
20-country model (n_x=62)				
One-step perturbation method (seconds)	NA	NA	NA	NA
POP method (seconds)	151.10	181.24	209.81	238.22
Speed gain	NA	NA	NA	NA

The reported numbers are averages from 21 Monte Carlo replications for third order approximations. All models are solved in Dynare++ and bond prices from the POP method are implemented in Matlab. NA indicates that we are unable to obtain a third order approximation in this case. All computations are done on an HP Blade Server with an Intel Xeon X5450 CPU, running Windows Server 2003 Standard x64 Edition (SP2) with 4GB of RAM and an HP Logical SCSI Disk Device.

We next consider ten countries in the RBC having 32 state variables. For 5-year and 10-year yield curves we see even larger efficiency improvements from the POP method when compared to previous versions of the model, but more importantly the one-step perturbation method is unable to solve for the 15-year and 20-year yield curves. This finding highlights another benefit of the POP method as it is more memory efficient than the one-step perturbation method, and therefore allows us to solve larger models than possible using the one-step perturbation method. We emphasize this point by finally considering 20 countries in the RBC model, where only the POP method is able to solve for a model with yield curves of the considered maturities.

4 Comparing solution accuracy

Our proposed POP method is faster to execute than traditional one-step perturbation, but there are other approximation methods which have become popular, in part due to their speed. This section compares the accuracy of the POP method to three alternatives.

The first alternative is the well-known first-order log-normal method proposed by Jermann (1998). The next alternative is the second-order log-normal method in Doh (2011), which extends Jermann's approach by combining a second-order perturbation approximation with bond prices derived from the log-normal assumption. The final alternative is the "consol" method proposed in Rudebusch and Swanson (2008) where consol bonds are used to approximate zero-coupon yields.

To assess the accuracy of the aforementioned methods, we use expressions for zero-coupon bond prices in a consumption endowment model with external habits as derived in Zabczyk (2014). We proceed by briefly introducing the habit model in Section 4.1, while Section 4.2 compares the approximation accuracy of the POP method to the three mentioned alternatives.

4.1 The consumption endowment model with habits

We consider a representative agent with the utility function

$$U_0=\sum_{t=0}^{\infty}\beta^t E_0\left[\frac{(C_t-hZ_t)^{1-\gamma}-1}{1-\gamma}\right],$$

where C_t is consumption and $h\in[0, 1]$ controls the degree of external habit formation. The habit stock Z_t evolves as $\ln Z_t=(1-\phi)\ln Z_{t-1}+\phi\ln C_{t-1}$, where $\phi\in[0, 1]$ determines the habit persistence. Consumption growth is defined as $x_t:=\ln(C_t/C_{t-1})$, and x_t is assumed to follow the process $x_t=(1-\rho)\mu+\rho x_{t-1}+\xi_t$ where $\xi_t\sim\mathcal{NID}(0,\sigma_\xi^2)$. The exact solution to zero-coupon bond prices is provided in Appendix C.

The model is calibrated as follows. We consider the case with strong and highly persistent habits by letting $h=0.9$ and $\phi=0.10$. The value for the subjective discount factor is set to $\beta=0.9985$. The coefficients in the consumption process are determined from an OLS regression for US quarterly NIPA data on nondurables and services in the period 1961 Q2 to 2011 Q2. This implies $\mu=0.0073$, $\rho=0.5015$, and $\sigma_\xi=1.7152\times10^{-5}$. Two values are considered for the curvature parameter γ. We initially let $\gamma=3$ and subsequently study $\gamma=6$ to explore effects of stronger non-linearities in the model.

4.2 The accuracy of various approximation methods

Figure 1 plots the benchmark 10-year interest rate as a function of consumption growth when $\gamma=3$. The solid red line represents the exact solution and the other lines correspond to various approximation methods. The approximated solutions from one-step perturbation and the POP method are identical, by construction, and are referred to as the "perturbation method" throughout this section. We first note for the range of consumption growth considered that the third-order perturbation method delivers a highly accurate approximation, with small deviations from the exact solution only visible for consumption growth $x_t-\mu$ exceeding ±0.08. The second-order perturbation method is less accurate as it clearly deviates from the exact solution when the absolute value of $x_t-\mu$ exceeds 0.06. We also note that the first-order log-normal method captures the precautionary saving correction which is absent in the first-order perturbation solution. The consol method

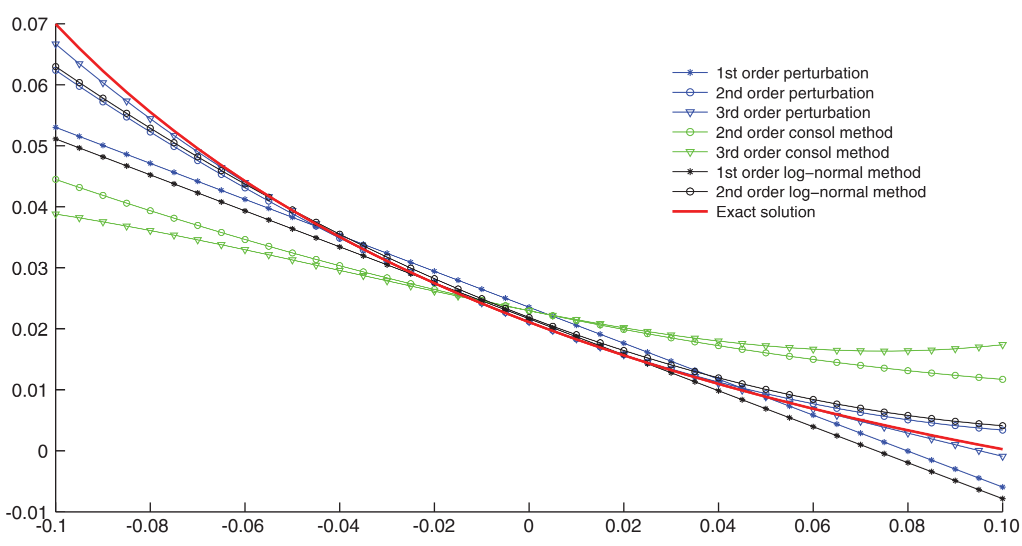

Figure 1 The function for the 10-year interest rate: $\gamma=3$.
The x-axis reports consumption growth in deviation from the deterministic steady state. The y-axis reports the value of the 10-year interest rate when expressed in quarterly terms.

at second and third order generates larger approximation errors than any of the other methods and displays substantial deviations from the exact solution.

In Figure 2, we turn to the case of stronger non-linearities with γ=6. The third-order perturbation approximation is also here close to the exact solution whereas the second-order perturbation method displays notable deviations when the absolute values of $x_t-\mu$ exceeds 0.06. We also observe that the second-order log-normal method lies marginally above the second-order perturbation method but the two methods perform otherwise very similarly. Finally, the consol method delivers very large errors which increase when moving from a second- to a third-order approximation.

These observations are also confirmed by Table 2 which reports the root mean squared errors (RMSEs) implied by the approximations in Figures 1 and 2. For the chosen calibration we see that the third-order approximation clearly outperforms all alternative methods. Note in particular the large reduction in RMSEs of more than 60% for both values of γ when moving from a second-order to a third-order approximation in the perturbation method.

Table 2 also reports the first two moments of the 10-year interest rate. For γ=3, the third-order perturbation method gives a mean value of 8.63% which is close to the exact mean of 8.61%.[14] The log-normal method performs surprisingly well at first-order with a mean of 8.66%, but its accuracy deteriorates at second order where the mean is overestimated by 29 basis points. The log-normal method includes terms beyond the considered approximation order, and our finding suggests that these additional terms may not necessarily improve the accuracy of the approximated solution. The standard deviation in the 10-year interest rate is reasonably matched by the considered approximations, except for the consol method which roughly underestimates this moment by 60 basis points.

Turning to the moments in the highly non-linear model with γ=6, the third-order perturbation method has a mean value of 15.09% which is slightly higher than the exact mean of 14.86%. For all other approximation methods we find larger deviations from the exact mean, in particular for the second-order log-normal method (15.52%) and the consol method (16.81%). The standard deviation of the 10-year interest rate is 5.32%

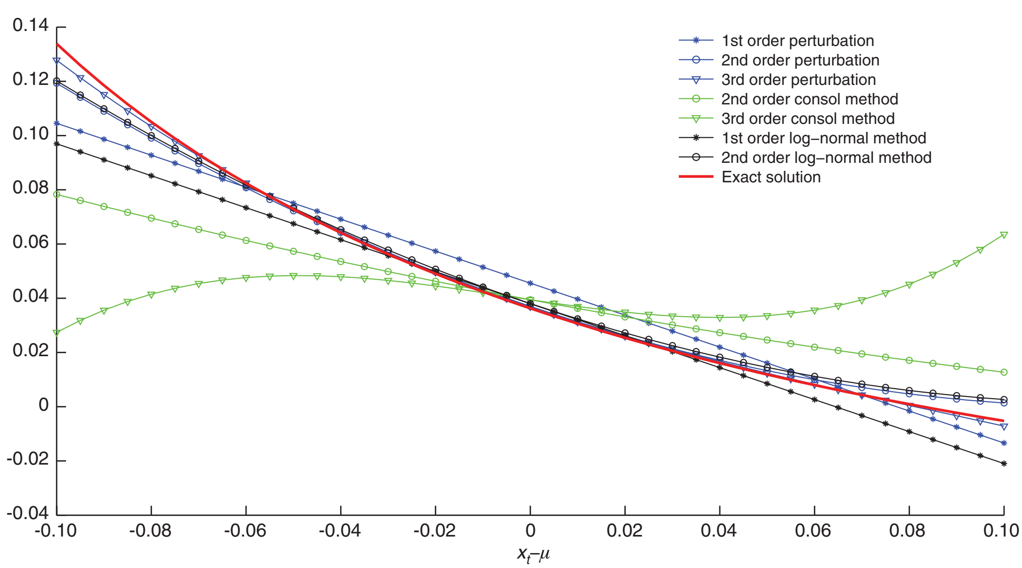

Figure 2 The function for the 10-year interest rate: γ=6.
The x-axis reports consumption growth in deviation from the deterministic steady state. The y-axis reports the value of the 10-year interest rate when expressed in quarterly terms.

14 The normality of consumption growth implies that all third moments in the approximations are zero. This explains why the mean values of the perturbation method and the consol method at third order are identical to the corresponding mean values at second order.

Table 2 Approximation accuracy for the 10-year interest rate.

	γ=3			γ=6		
	RMSE	Mean	Std	RMSE	Mean	Std
1st order perturbation	2.09	9.42	2.63	3.80	18.23	5.25
2nd order perturbation	0.89	8.63	2.64	1.72	15.09	5.27
3rd order perturbation	0.34	8.63	2.66	0.61	15.09	5.32
2nd order consol method	4.03	9.51	2.22	8.29	16.81	4.59
3rd order consol method	4.97	9.51	2.18	16.53	16.81	4.04
1st order log-normal method	2.50	8.66	2.63	4.82	15.20	5.25
2nd order log-normal method	0.89	8.90	2.63	1.79	15.52	5.26
Exact solution	–	8.61	2.80	–	14.86	5.59

The root mean squared errors (RMSE) for the approximations are computed for consumption growth at the following points: −0.1, −0.095, …, 0.095, 0.1. The RMSE in the table are in annualized percentage. All moments for the 10-year interest rate are expressed in annualized percentage and the moments are computed based on a simulated time series of 1,000,000 observations.

in the third-order perturbation method and slightly lower at 5.27% in second-order perturbation method. Both methods therefore differ somewhat from the exact solution with a standard deviation of 5.59%. However, all other methods deliver less accurate approximations for this moment than the third-order perturbation method.

5 Empirical application

To illustrate the usefulness of the POP model we finally apply it to a version of the endowment model by Piazzesi and Schneider (2007) with infinite horizon.[15] For tractability, Piazzesi and Schneider (2007) use Epstein-Zin-Weil preferences with the intertemporal elasticity of substitution restricted to one and they assume homoscedastic innovations to consumption growth and inflation. Our approximation method makes it straightforward to relax both constraints, and we additionally explore whether this long-run risk model can be further improved by introducing external habit formation. The following estimation exercise then studies to which degree this extended model is able to explain the dynamics of the US nominal yield curve jointly with consumption growth and inflation.

This section proceeds as follows. We describe our extended version of the model by Piazzesi and Schneider (2007) in Section 5.1, while Section 5.2 presents the data and our estimation approach. Section 5.3 discusses our findings and the features which allow us to match the data.

5.1 The model

Following Epstein and Zin (1989) and Weil (1990), we consider an infinitely lived representative agent with recursive preferences over an exogenously given consumption stream $\{c_t\}_{t=1}^{\infty}$. Allowing for external habit formation based on lagged consumption, the agent's value function V_t is given by

$$V_t=\begin{cases} \frac{1}{1-\rho}(C_t-hC_{t-1})^{1-\rho}+\beta(E_t[V_{t+1}^{1-\alpha}])^{\frac{1}{1-\alpha}} & 0<\rho<1 \\ (C_t-hC_{t-1})^{1-\beta}(E_t[V_{t+1}^{1-\alpha}])^{\frac{\beta}{1-\alpha}} & \rho=1 \\ \frac{1}{1-\rho}(C_t-hC_{t-1})^{1-\rho}-\beta(E_t[(-V_{t+1})^{1-\alpha}])^{\frac{1}{1-\alpha}} & \rho>1 \end{cases}, \quad (24)$$

15 Andreasen (2012a) shows that the POP method also makes it feasible to estimate a medium-scaled DSGE model with a whole yield curve approximated to third order.

where $\beta<1$ is the subjective discount factor. Without habits $h=0$, the case of $\rho=1$ corresponds to the specification in Piazzesi and Schneider (2007) where the intertemporal elasticity of substitution (IES) $1/\rho$ equals 1 and α determines the degree of relative risk aversion. In the two other cases, we follow the specification in Rudebusch and Swanson (2012) where higher values of α corresponds to greater risk aversion for $\rho<1$, and vice versa for $\rho>1$.[16] Using the results in Swanson (2012), and the related working paper, the relative risk aversion for $\rho\neq1$ is given by $(\rho+\alpha(1-\rho))/(1-h)$, and the IES is equal to $(1-h/\mu_c)/\rho$ at the steady state.

Given a complete market for state contingent claims, the nominal stochastic discount factor is easily shown to be

$$M_{t,t+1}=\begin{cases} \beta\left(\frac{C_{t+1}-hC_t}{C_t-hC_{t-1}}\right)^{-\rho}\left(\frac{(E_t[V_{t+1}^{1-\alpha}])^{\frac{1}{1-\alpha}}}{V_{t+1}}\right)^{\alpha}\frac{1}{\pi_{t+1}} & 0<\rho<1 \\ \beta\left(\frac{C_{t+1}-hC_t}{C_t-hC_{t-1}}\right)^{-1}\left(\frac{(E_t[V_{t+1}^{1-\alpha}])^{\frac{1}{1-\alpha}}}{V_{t+1}}\right)^{\alpha-1}\frac{1}{\pi_{t+1}} & \rho=1\;. \\ \beta\left(\frac{C_{t+1}-hC_t}{C_t-hC_{t-1}}\right)^{-\rho}\left(\frac{\left(E_t[(-V_{t+1})^{1-\alpha}]\right)^{\frac{1}{1-\alpha}}}{-V_{t+1}}\right)^{\alpha}\frac{1}{\pi_{t+1}} & \rho>1 \end{cases} \quad (25)$$

The model is closed by specifying the law of motion for consumption growth $\Delta c_t:=\ln(C_t/C_{t-1})$ and the inflation rate π_t. The system we consider is given by

$$\begin{bmatrix}\Delta c_t\\ \pi_t\end{bmatrix}=\begin{bmatrix}\mu_c\\ \mu_\pi\end{bmatrix}+\begin{bmatrix}x_{1,t-1}\\ x_{2,t-1}\end{bmatrix}+\begin{bmatrix}\sigma_{1,t} & 0\\ \omega & \sigma_{2,t}\end{bmatrix}\begin{bmatrix}e_{1,t}\\ e_{2,t}\end{bmatrix}, \quad (26)$$

where

$$\begin{bmatrix}x_{1,t}\\ x_{2,t}\end{bmatrix}=\begin{bmatrix}\phi_{11} & \phi_{12}\\ \phi_{21} & \phi_{22}\end{bmatrix}\begin{bmatrix}x_{1,t-1}\\ x_{2,t-1}\end{bmatrix}+\begin{bmatrix}k_{11} & k_{12}\\ k_{21} & k_{22}\end{bmatrix}\begin{bmatrix}\sigma_{1,t} & 0\\ \omega & \sigma_{2,t}\end{bmatrix}\begin{bmatrix}e_{1,t}\\ e_{2,t}\end{bmatrix} \quad (27)$$

and $\begin{bmatrix}e'_{1,t} & e'_{2,t}\end{bmatrix}'\sim\mathcal{NID}(\mathbf{0},\mathbf{I})$. This system is more general than the one assumed in Piazzesi and Schneider (2007) as we allow for time-variation in the standard deviations $\sigma_{1,t}$ and $\sigma_{2,t}$ to the innovations. To ensure strictly positive volatility processes for $\sigma_{1,t}$ and $\sigma_{2,t}$ with well-defined properties, we follow Chernov et al. (2003) and let

$$\sigma_{i,t}\equiv\begin{cases}\exp\{v_{i,t}\} & \text{for } v_{i,t}\leq\bar{v}_i\\ \exp\{\bar{v}_i\}\sqrt{1+\bar{v}_i+v_{i,t}} & \text{for } v_{i,t}>\bar{v}_i\end{cases} \quad (28)$$

$$v_{i,t+1}=(1-\rho_{\sigma_i})v_i+\rho_{\sigma_i}v_t+\epsilon_{\sigma_i,t+1} \quad (29)$$

for $i=\{1, 2\}$ where $\begin{bmatrix}\epsilon'_{\sigma_1,t} & \epsilon'_{\sigma_2,t}\end{bmatrix}'\sim\mathcal{NID}(\mathbf{0},diag(\sigma^2_{\sigma_1},\sigma^2_{\sigma_2}))$. This specification implies that the volatility is log-normally distributed when $v_{i,t}$ is lower than $\bar{v}_i$ but not when $v_{i,t}$ exceeds $\bar{v}_i$.[17] The lower part of the function in

16 The specification for $\rho<1$ is equivalent to the slightly more traditional way of introducing recursive preferences $U_t=\left[C_t^{1-\psi}+\beta(E_t[U_{t+1}^{1-\gamma}])^{\frac{1-\psi}{1-\gamma}}\right]^{\frac{1}{1-\psi}}$ if we let $U_t=V_t^{\frac{1}{1-\psi}}$ and $\alpha=\frac{\gamma-\psi}{1-\psi}$. A similar equivalence holds for $\rho>1$.

17 See Andreasen (2010) for a detailed discussion of this issue.

(28) is chosen to ensure a smooth splicing at $\bar{v}_i$ where the level and the first derivative of the upper and lower part of the function in (28) are identical. As suggested by Andreasen (2010), we let $\bar{v}_i > v_i$ where v_i is the steady state level of $v_{i,t}$, implying that the mean of the volatility process is $\sigma_i = \ln v_i$ for $i=\{1, 2\}$. It is further assumed that $\begin{bmatrix} \epsilon'_{\sigma_1,t} & \epsilon'_{\sigma_2,t} \end{bmatrix}'$ is uncorrelated with $\begin{bmatrix} e'_{1,t} & e'_{2,t} \end{bmatrix}'$ at all leads and lags. Thus, by restricting $\sigma_{\sigma_1} = \sigma_{\sigma_2} = 0$, we recover the system in Piazzesi and Schneider (2007) without stochastic volatility.[18]

5.2 Data and estimation approach

The model is estimated on quarterly US data from 1961 Q2 to 2011 Q2. The nominal yield curve is represented by the 1-year, 5-year, and 10-year interest rates from Gürkaynak, Sack, and Wright (2007).[19] As in Piazzesi and Schneider (2007), we leave out the yield on a quarterly bond because it is likely to be influenced by liquidity effects which are not included in the model. Our measures of consumption and inflation follow Piazzesi and Schneider (2007); that is we use NIPA data on nondurables and services to construct a consumption series, and its price deflator defines our inflation rate. All time series are expressed in annual growth rates and stored in $\mathbf{y}_t$ with dimension 5×1.

Our model has 21 parameters which are jointly denoted by $\boldsymbol{\theta}$. We want to explore whether the considered model can match the following four stylized features in the data: i) the average level of the yield curve and the two macro variables, ii) the variability in these variables, iii) the contemporaneous correlations among yields and the macro variables, and iv) the cross auto-correlations in these variables. This leads us to initially consider the following 45 moments

$$\mathbf{m} = \frac{1}{T} \sum_{t=1}^{T} \begin{bmatrix} \{y_{i,t}\}_{i=1}^{5} \\ \{y_{i,t}^2\}_{i=1}^{5} \\ vech(\mathbf{y}_t \mathbf{y}'_t) \\ vec(\mathbf{y}_t \mathbf{y}'_{t-1}) \end{bmatrix}. \tag{30}$$

For the versions of the model featuring stochastic volatility, the above moments are extended with fourth moments $\{y_{i,t}^4\}_{i=1}^{5}$. This is done partly to ensure that the presence of stochastic volatility is consistent with the tail behavior of the data, and partly to provide identification of the parameters describing the volatility processes.

Model moments are computed from a simulated time series of 500,000 observations, and $\boldsymbol{\theta}$ is then determined by minimizing the quadratic distance between sample and model moments [see Duffie and Singleton (1993)]. We adopt the standard procedure where an initial estimate of $\boldsymbol{\theta}$ is obtained with a diagonal weighting matrix to find the optimal weighting matrix which we use for the reported estimates of $\boldsymbol{\theta}$.

5.3 Empirical results

As a natural benchmark, we first restrict our model to resemble the one in Piazzesi and Schneider (2007) and apply their estimation procedure. That is, we let μ_c and μ_π equal the sample averages for consumption growth and inflation, respectively, and we estimate the system in (26) and (27) without stochastic volatility

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18 Rewriting the model in terms of consumption growth and a de-trended value function, it is straightforward to see that this model fits into our framework with the state vector consisting of $x_{1,t-1}$, $x_{2,t-1}$, $v_{1,t}$, $v_{2,t}$, $e_{1,t}$, and $e_{2,t}$.

19 As in Rudebusch and Swanson (2012), observations for the 10-year interest rate from 1961 Q3 to 1971 Q3 are computed by extrapolation of the estimated yield curves in Gürkaynak, Sack, and Wright (2007).

by maximum likelihood on demeaned data.[20] The IES is restricted to 1 (ρ=1), while β and α are calibrated to match the moments in (30). We find β=0.9997 and α=66, which are close to the values of β=1.005 and α=59 in Piazzesi and Schneider (2007).

To evaluate the empirical fit of the model, Table 3 shows various statistics related to our moment conditions in (30). We first note that the model-implied level of the 10-year yield curve exceeds the empirical level and that the model only generates an average slope of 14 basis points versus 102 basis points in the data. This is contrary to Piazzesi and Schneider (2007) as their calibrated model perfectly matches the level of the 5-year yield curve with an average slope of nearly 100 basis points in their sample. Two features explain this difference. Firstly, we impose the standard requirement in models with infinite horizon of β<1, whereas Piazzesi and Schneider (2007) in their model with finite horizon let β>1 to obtain a sufficiently low level for all yields. Secondly, consumption growth and inflation are not as negatively correlated in our sample as in the one considered by Piazzesi and Schneider (2007), ranging from 1952 Q2 to 2005 Q4.[21] This explains why our model generates a lower slope for the average yield curve compared to Piazzesi and Schneider (2007).[22] Turning to the second moments, we find similarly to Piazzesi and Schneider (2007) that the model generates too low variability and persistence in all yields. Table 3 also suggests that the model implies too high contemporaneous and lagged correlations between the two macro variables and the yield curve, as indicated by the bold figures in Table 3.

We next consider the benefit of relaxing the constraint on IES while simultaneously estimating the system for Δc_t and π_t without stochastic volatility using the moment conditions in (30). The estimates are reported in Table 4 in the column marked by $\mathcal{M}_1^{\text{IES}}$ where we find ρ=7.93. This implies a very low IES of 0.13 which appears to be inconsistent with micro-economic evidence [see for instance Beaudry and Wincoop (1996) and Vissing-Jorgensen (2002)]. The degree of relative risk aversion is estimated to 873 which is extremely high. Table 3 shows that the model matches the level of the entire yield curve and generates sufficient variability and persistence in all yields. We also note that consumption growth and its lagged value are uncorrelated with the yield curve as in the data. However, this improved behavior comes at the expense of too low auto-correlation in inflation (0.50 vs. 0.79) and consumption growth (0.01 vs. 0.50). It is also worth noticing that the model overstates the negative co-movement between consumption growth and inflation (−0.60 vs. −0.21), and lagged inflation is not negatively correlated with consumption growth, meaning that high inflation does not predict low future consumption growth in this version of the model.

To further improve the model, we next introduce stochastic volatility in consumption growth and inflation. As shown in the column named $\mathcal{M}_2^{\text{IES\&SV}}$ in Table 4, we find clear evidence of stochastic volatility with $\sigma_{\sigma_1}=0.196$ and $\sigma_{\sigma_2}=0.011$, and both volatility processes display high persistence with ρ_1=0.90 and ρ_2=0.99. We now find ρ=1.08 and hence a fairly realistic IES of 0.93, which is close to the calibrated value of 1 in Piazzesi and Schneider (2007). The degree of relative risk aversion is 157 and still high, but nevertheless substantially lower than in the previous specification. In addition to these slightly more realistic features, the model matches all mean values, standard deviations, and contemporaneous cross-correlations as seen in the fourth column in Table 3. Values for kurtosis also appear largely consistent with the data, although the model generates too low kurtosis in inflation (3.06 vs. 5.00). We also see that the model is successful in matching most cross auto-correlations. Its main shortcomings relate to: insufficient persistence in consumption growth (0.31 vs. 0.50), too high auto-correlation for inflation (0.86 vs. 0.79), too high correlation between inflation and the lagged 5-year yield (0.59 vs. 0.49), and a too negative correlation between inflation and lagged consumption growth (−0.20 vs. −0.07).

Our final extension introduces external habits into the model in an attempt to further improve its performance. We estimate the degree of habit formation to h=0.63 which jointly with the other estimates for

20 We are grateful to Monika Piazzesi and Martin Schneider for making their codes publicly available.

21 A similar observation is made in Benigno (2007), showing that the size of the negative correlation between consumption growth and inflation in the US varies over time.

22 To further illustrate this effect, we examine what value of α is required in our infinite horizon model to generate an average slope for the 10-year yield curve of 102 basis points when using the calibrated system in Piazzesi & Schneider (2007) and our updated calibration of this system using their procedure. With β=0.997, we find α=7 when relying on the calibration in Piazzesi and Schneider (2007) and α=320 when using our updated calibration.

Table 3 Empirical and model moments.

	Data	$\mathcal{M}_0^{PSModel}$	$\mathcal{M}_1^{IES}$	$\mathcal{M}_2^{IES\&SV}$	$\mathcal{M}_3^{IES\&SV\&Habit}$
Mean (in pct)					
$r_{4,t}$	5.73	**6.96**	5.76	5.77	5.88
$r_{20,t}$	6.37	**7.02**	6.39	6.39	6.51
$r_{40,t}$	6.75	**7.10**	6.78	6.76	6.88
π_t	3.95	3.95	3.92	3.92	4.01
Δc_t	2.97	2.97	2.97	2.99	3.01
Standard deviation (in pct)					
$r_{4,t}$	3.04	**1.77**	2.93	2.95	2.95
$r_{20,t}$	2.70	**0.85**	2.62	2.68	2.65
$r_{40,t}$	2.47	**0.47**	2.28	2.46	2.43
π_t	2.59	2.60	2.37	2.47	2.47
Δc_t	1.92	1.92	1.79	1.90	1.88
Kurtosis					
$r_{4,t}$	3.74	–	–	3.00	3.01
$r_{20,t}$	3.48	–	–	3.01	3.01
$r_{40,t}$	3.54	–	–	3.01	3.01
π_t	5.00	–	–	**3.06**	**3.08**
Δc_t	4.60	–	–	5.00	**3.39**
Contemp. cross correlations					
$corr(r_{4,t}, r_{20,t})$	0.965	0.998	0.980	0.983	0.968
$corr(r_{4,t}, r_{40,t})$	0.921	0.997	0.976	0.928	0.917
$corr(r_{4,t}, \pi_t)$	0.602	**0.815**	**0.704**	0.584	0.559
$corr(r_{4,t}, \Delta c_t)$	0.054	**0.351**	0.041	0.043	−0.0155
$corr(r_{20,t}, r_{40,t})$	0.987	0.999	0.999	0.981	0.987
$corr(r_{20,t}, \pi_t)$	0.542	**0.847**	**0.675**	0.544	0.530
$corr(r_{20,t}, \Delta c_t)$	0.057	**0.296**	0.040	0.050	0.0294
$corr(r_{40,t}, \pi_t)$	0.519	**0.854**	**0.670**	0.488	0.486
$corr(r_{40,t}, \Delta c_t)$	0.005	**0.284**	0.040	0.053	0.031
$corr(\pi_t, \Delta c_t)$	−0.210	−0.208	**−0.608**	−0.219	−0.238
Cross auto-correlation					
$corr(r_{4,t}, r_{4,t-1})$	0.941	**0.882**	0.965	0.939	0.954
$corr(r_{4,t}, r_{20,t-1})$	0.913	0.881	0.996	0.921	0.932
$corr(r_{4,t}, r_{40,t-1})$	0.874	0.881	**0.998**	0.868	0.884
$corr(r_{4,t}, \pi_{t-1})$	0.607	**0.728**	0.659	0.571	0.538
$corr(r_{4,t}, \Delta c_{t-1})$	0.079	**0.296**	0.040	0.044	0.061
$corr(r_{20,t}, r_{4,t-1})$	0.928	0.886	0.958	0.921	0.933
$corr(r_{20,t}, r_{20,t-1})$	0.959	**0.889**	0.988	0.937	0.965
$corr(r_{20,t}, r_{40,t-1})$	0.948	0.889	0.989	0.920	0.953
$corr(r_{20,t}, \pi_{t-1})$	0.556	**0.759**	**0.655**	0.532	0.517
$corr(r_{20,t}, \Delta c_{t-1})$	0.060	**0.252**	0.039	0.050	0.045
$corr(r_{40,t}, r_{4,corrt-1})$	0.893	0.886	0.955	0.868	0.885
$corr(r_{40,t}, r_{20,t-1})$	0.952	**0.890**	0.984	0.920	0.953
$corr(r_{40,t}, r_{40,t-1})$	0.965	**0.890**	0.985	0.940	0.965
$corr(r_{40,t}, \pi_{t-1})$	0.535	**0.765**	**0.652**	0.477	0.475
$corr(r_{40,t}, \Delta c_{t-1})$	0.008	**0.242**	0.039	0.053	0.040
$corr(\pi_t, r_{4,t-1})$	0.562	**0.717**	**0.706**	0.634	0.607
$corr(\pi_t, r_{20,t-1})$	0.493	**0.739**	**0.692**	**0.591**	**0.575**
$corr(\pi_t, r_{40,t-1})$	0.466	**0.743**	**0.688**	0.530	0.526
$corr(\pi_t, \pi_{t-1})$	0.785	0.782	**0.498**	**0.863**	**0.874**
$corr(\pi_t, \Delta c_{t-1})$	−0.066	−0.071	0.028	**−0.198**	−0.017
$corr(\Delta c_t, r_{4,t-1})$	−0.006	**0.132**	**0.108**	−0.006	−0.030
$corr(\Delta c_t, r_{20,t-1})$	0.030	0.098	0.075	0.000	−0.016
$corr(\Delta c_t, r_{40,t-1})$	−0.008	**0.091**	**0.071**	0.005	−0.013
$corr(\Delta c_t, \pi_{t-1})$	−0.174	−0.175	**0.088**	−0.168	−0.205
$corr(\Delta c_t, \Delta c_{t-1})$	0.499	0.508	**0.005**	**0.306**	0.519

All model moments are computed based on a simulated time series of 500,000 observations. Figures in bold indicate notable deviations between model moments and the corresponding empirical moments.

Table 4 Model estimates.

	$\mathcal{M}_1^{\text{IES}}$	$\mathcal{M}_2^{\text{IES\&SV}}$	$\mathcal{M}_3^{\text{IES\&SV\&Habit}}$
β	0.9977 (0.0201)	0.9965 (0.0021)	0.9958 (0.0022)
α	−124.85 (62.37)	−2047.82 (105.12)	−807.77 (244.14)
ρ	7.9320 (1.2068)	1.0762 (0.0259)	1.2714 (0.0705)
$\mu_c \times 10^{-2}$	0.7415 (0.0194)	0.7468 (0.0198)	0.7516 (0.0228)
$\mu_\pi \times 10^{-2}$	0.9806 (0.0367)	0.9802 (0.0384)	1.0025 (0.0352)
k_{11}	0.9210 (0.1896)	0.1436 (0.0483)	0.3017 (0.0492)
k_{12}	0.9820 (0.2030)	0.1254 (0.0971)	−0.4276 (0.1202)
k_{21}	0.2748 (0.1828)	−0.1023 (0.0742)	−0.1677 (0.0613)
k_{22}	0.3167 (0.1995)	−0.7799 (0.1619)	−0.7327 (0.1420)
ϕ_{11}	−0.1839 (0.0414)	0.9561 (0.0348)	0.6874 (0.0253)
ϕ_{12}	0.1241 (0.0225)	−0.0028(0.0062)	−0.0518 (0.0056)
ϕ_{21}	1.5134 (0.3873)	0.0333 (0.0395)	0.0607 (0.0259)
ϕ_{22}	1.1416 (0.0280)	0.9827 (0.0072)	0.9911 (.0048)
$\sigma_1 \times 10^{-2}$	0.4404 (0.0119)	0.3538 (0.0384)	0.3770 (0.0154)
$\sigma_2 \times 10^{-2}$	0.0750 (0.0098)	0.1561 (0.0147)	0.1120 (0.0150)
$\omega \times 10^{-2}$	−0.4138 (0.0115)	−0.0316 (0.0182)	−0.1286 (0.0143)
ρ_1	–	0.8972 (0.0130)	0.9579 (0.0107)
ρ_2	–	0.9923 (0.0022)	0.963 (0.0031)
σ_{σ_1}	–	0.1960 (0.0663)	0.0586 (0.0152)
σ_{σ_2}	–	0.0108 (0.0021)	0.0535 (0.0078)
h	–	–	0.6344 (0.0260)

All estimates are computed based on a simulated time series of 500,000 observations. Standard errors are for the optimal weighting matrix, estimated by the Newey-West estimator using 10 lags.

this model $\mathcal{M}_3^{\text{IES\&SV\&Habit}}$ leads to a low IES of 0.29 and a high degree of relative risk aversion of 603. It is also interesting to note that the long-run risk aspect as captured by the Epstein-Zin-Weil coefficient remains highly important (α=−807) even when we allow for external habit formation. This indicates that long-run risk is an essential component for the model's ability to match the data. The last column in Table 3 shows that adding habits to the model preserves the satisfying properties of the previous model, except for the excess kurtosis in consumption growth. The main effect of habits is to address two of the four short-comings of $\mathcal{M}_2^{\text{IES\&SV}}$. That is, habits allow the model to match the auto-correlation in consumption growth (0.49 vs. 0.52) and to generate basically no correlation between inflation and lagged consumption growth as found in the data (−0.02 vs. −0.07).

To further explore the effects of habits in a long-run risk model, Figure 3 plots additional moments not included in the estimation of $\mathcal{M}_2^{\text{IES\&SV}}$ and $\mathcal{M}_3^{\text{IES\&SV\&Habit}}$ along with the 95 percentage confidence intervals for sample moments. We see that both models match the persistence in yields up to ten lags, and generate slightly higher auto-correlation in inflation than seen in the data. However, the inflation persistence in both models are within the 95 percentage confidence intervals for the first six lags and are in this sense consistent with the data. The two models differ more in terms of the persistence in consumption growth where habit formation allows the model to better capture the overall shape of the empirical auto-correlation than the model without habits. Another difference appears in the correlation between inflation and lagged consumption growth where the model without habits generates somewhat lower correlation than found in the data and in the model with habits.

All versions of the model rely on a high level of risk aversion, whereas more plausible values of risk aversion are typically required to match moments for equities in long-run risk models [see Bansal and Yaron (2004)]. There are at least two possible explanations for our finding. Firstly, high risk aversion in Epstein-Zin-Weil preferences may proxy for model uncertainty as shown by Barillas, Hansen, and Sargent (2009). Secondly, Malloy, Moskowitz, and Vissing-Jørgensen (2009) document that variability in consumption for US stockholders is higher than the variation in aggregate consumption, and risk aversion is therefore estimated

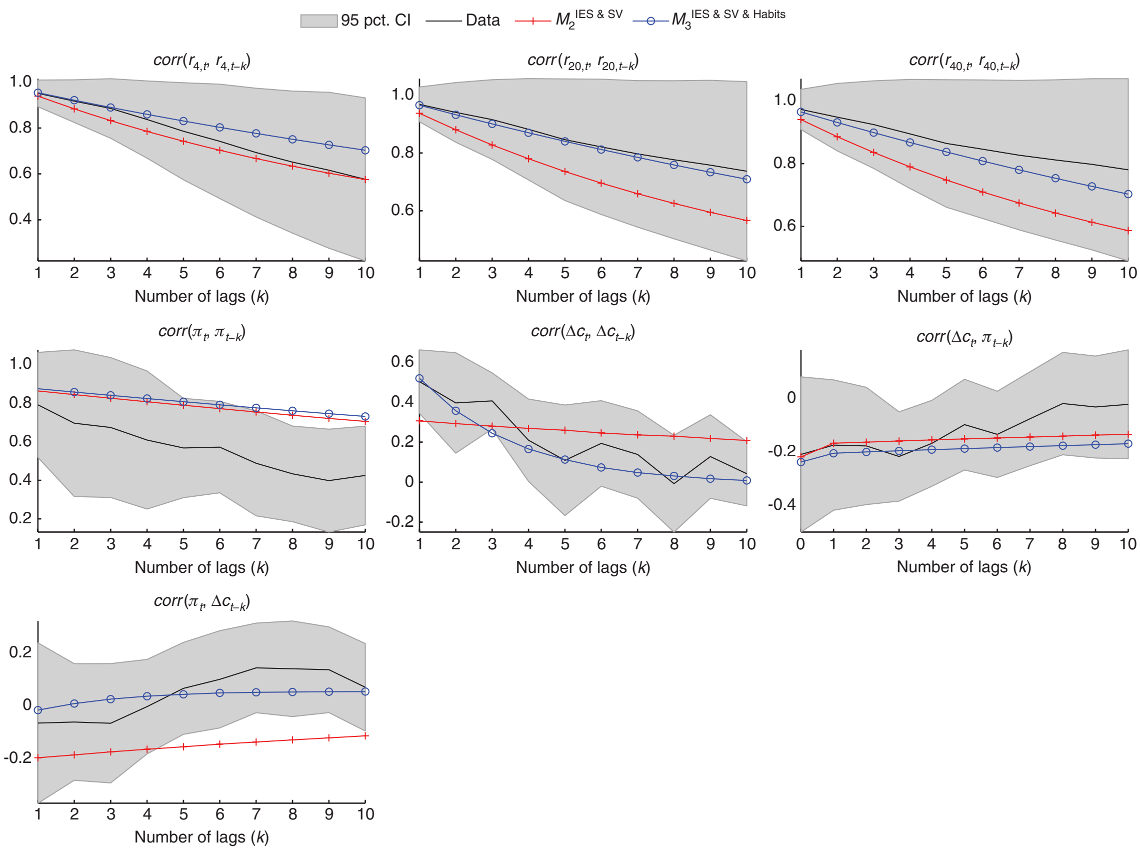

Figure 3 Studying the effects of habit formation via additional moments.
The confidence intervals are calculated by the circular block bootstrap with 10,000 blocks and a window of 50 observations.

to be significantly lower for stockholders when compared to a representative agent using aggregate consumption. In other words, the high risk aversion we find is likely to compensate for insufficient consumption variation within our model, partly because we ignore model uncertainty and partly because consumption variability for stockholders is higher than implied by aggregate consumption.

To summarize, we find that the long-run risk model by Piazzesi and Schneider (2007) benefits from an unconstrained IES in the case of homoscedastic innovations, and that this model matches the level of the yield curve and generates sufficient variability and persistence in all yields. The inclusion of stochastic volatility takes the IES close to unity and substantially improves the model's ability to match the considered moments. Extending the model with habit formation enhances its performance further, showing that it may be beneficial to introduce habits into a long-run risk model.

6 Conclusion

This paper proposes an efficient method to compute higher-order bond price approximations for a wide class of non-linear equilibrium-based term structure models. While the numerical values for bond prices using our formulas *exactly* match those derived using the standard one-step perturbation routine, a simulation study documents that execution times can be lowered substantially. In general, the improvement in speed depends positively on the maturity of the approximated yield curve and positively on the number of state variables in the model. Due to the memory efficient nature of our method, it is also shown that it enables us to solve larger models with a yield curve than possible using the one-step perturbation routine.

We also assess the accuracy of our perturbation method in a consumption endowment model with habits. Our results show that the third-order approximation to the 10-year interest rate is more accurate than a second-order solution and those of popular alternatives. It is also shown that interest rates approximated from prices of consol bonds can be less precise, even at third order, than those computed using the first-order log-normal approach.

The suggested method is finally applied to estimate the long-run risk model by Piazzesi and Schneider (2007) extended with an unconstrained IES, stochastic volatility, and external habit formation. We show that each of these extensions brings the model closer to the data, with the full model able to match nearly all of the considered moments. The application therefore illustrates some of the benefits associated with our method which allows for great flexibility in setting up equilibrium-based term structure models.

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Appendix A

A general transformation of bond prices

This appendix considers the general case of an invertible transformation function $R(\cdot)\in\mathcal{C}^N$, implying that $R(p^{t,k})\equiv P^{t,k}$. Here

$$F^k(\mathbf{x},\sigma)=E_t[R(p^{t,k}(\mathbf{x},\sigma))-\mathcal{M}(\mathbf{g}(\mathbf{h}(\mathbf{x},\sigma)+\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1},\sigma),\mathbf{g}(\mathbf{x},\sigma),\mathbf{h}(\mathbf{x},\sigma)+\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1},\mathbf{x}) \\ \times R(p^{t+1,k-1}(\mathbf{h}(\mathbf{x},\sigma)+\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1},\sigma))].$$

First order terms

Derivative of p^k with respect to x

Consider $[F_{\mathbf{x}}^k(\mathbf{x}_t,\sigma)]_{\alpha_1}=0$ which implies

$$R_p\left(p^k\right)\left[p_{\mathbf{x}}^{t,k}\right]_{\alpha_1}-\left[\mathcal{M}_{\mathbf{x}}\right]_{\alpha_1}R\left(p^{t+1,k-1}\right)-\mathcal{M}R_p\left(p^{t+1,k-1}\right)\left[p_{\mathbf{x}}^{t+1,k-1}\right]_{\gamma_1}\left[\mathbf{h}_{\mathbf{x}}^t\right]_{\alpha_1}^{\gamma_1}=0$$

For $k=1$ we have $R_p(p^1)[p_{\mathbf{x}}^1]_{\alpha_1}=[\mathcal{M}_{\mathbf{x}}]_{\alpha_1}$ and $M=R(p^1)$. Hence

$$R_p\left(p^k\right)\left[p_{\mathbf{x}}^k\right]_{\alpha_1}=\left[p_{\mathbf{x}}^1\right]_{\alpha_1}R_p\left(p^1\right)R\left(p^{k-1}\right)+R\left(p^1\right)R_p\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_1}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_1}^{\gamma_1}$$

Derivative of p^k with respect to σ

Computing $F_\sigma^k(\mathbf{x}_{ss},0)=0$ implies

$$E_t\left[R_p\left(p^k\right)\left[p_\sigma^k\right]-\left[\mathcal{M}_\sigma\right]R\left(p^{k-1}\right)-\mathcal{M}R_p\left(p^{k-1}\right)\left(\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_1}\left(\left[\mathbf{h}_\sigma\right]^{\gamma_1}+\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right)+\left[p_\sigma^{k-1}\right]\right)\right]=0$$

For $k=1$ we have $E_t[R_p(p^1)[p_\sigma^1]]=E_t[\mathcal{M}_\sigma]$. Similar arguments as in the text then implies $p_\sigma^k=0$ for all values of k.

Second order terms

Derivative of p^k with respect to (x, x)

We have that $[F_{\mathbf{xx}}^k(\mathbf{x}_{ss},0)]_{\alpha_1,\alpha_2}=0$ implies

$$\begin{aligned}
&R_{pp}\left(p^k\right)\left[p_{\mathbf{x}}^k\right]_{\alpha_2}\left[p_{\mathbf{x}}^k\right]_{\alpha_1}+R_p\left(p^k\right)\left[p_{\mathbf{xx}}^k\right]_{\alpha_1\alpha_2}\\
&\quad-\left[\mathcal{M}_{\mathbf{xx}}\right]_{\alpha_1\alpha_2}R\left(p^{k-1}\right)-\left[\mathcal{M}_{\mathbf{x}}\right]_{\alpha_1}R_p\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_2}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_2}^{\gamma_2}\\
&\quad-\left[\mathcal{M}_{\mathbf{x}}\right]_{\alpha_2}R_p\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_1}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_1}^{\gamma_1}\\
&\quad-\mathcal{M}R_{pp}\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_2}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_2}^{\gamma_2}\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_1}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_1}^{\gamma_1}\\
&\quad-\mathcal{M}R_p\left(p^{k-1}\right)\left[p_{\mathbf{xx}}^{k-1}\right]_{\gamma_1\gamma_2}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_2}^{\gamma_2}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_1}^{\gamma_1}\\
&\quad-\mathcal{M}R_p\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_1}\left[\mathbf{h}_{\mathbf{xx}}\right]_{\alpha_1\alpha_2}^{\gamma_1}\\
&\quad=0
\end{aligned}$$

The value of $\mathcal{M}$ equals $R(p^1)$ and $\mathcal{M}_{\mathbf{x}}$ is computed above. Moreover, for $k=1$ we have

$$\left[\mathcal{M}_{\mathbf{xx}}\right]_{\alpha_1\alpha_2}=R_p\left(p^1\right)\left[p_{\mathbf{xx}}^1\right]_{\alpha_1,\alpha_2}+R_{pp}\left(p^1\right)\left[p_{\mathbf{x}}^1\right]_{\alpha_2}\left[p_{\mathbf{x}}^1\right]_{\alpha_1}$$

Thus

$$\begin{aligned}
&R_p\left(p^k\right)\left[p_{\mathbf{xx}}^k\right]_{\alpha_1,\alpha_2}=-R_{pp}\left(p^k\right)\left[p_{\mathbf{x}}^k\right]_{\alpha_2}\left[p_{\mathbf{x}}^k\right]_{\alpha_1}\\
&\quad+\left(R_p\left(p^1\right)\left[p_{\mathbf{xx}}^1\right]_{\alpha_1,\alpha_2}+R_{pp}\left(p^1\right)\left[p_{\mathbf{x}}^1\right]_{\alpha_2}\left[p_{\mathbf{x}}^1\right]_{\alpha_1}\right)R\left(p^{k-1}\right)\\
&\quad+\left[p_{\mathbf{x}}^1\right]_{\alpha_1}R_p\left(p^1\right)R_p\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_2}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_2}^{\gamma_2}\\
&\quad+\left[p_{\mathbf{x}}^1\right]_{\alpha_2}R_p\left(p^1\right)R_p\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_1}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_1}^{\gamma_1}\\
&\quad+R\left(p^1\right)R_{pp}\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_2}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_2}^{\gamma_2}\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_1}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_1}^{\gamma_1}\\
&\quad+R\left(p^1\right)R_p\left(p^{k-1}\right)\left[p_{\mathbf{xx}}^{k-1}\right]_{\gamma_1\gamma_2}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_2}^{\gamma_2}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_1}^{\gamma_1}\\
&\quad+R\left(p^1\right)R_p\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_1}\left[\mathbf{h}_{\mathbf{xx}}\right]_{\alpha_1\alpha_2}^{\gamma_1}
\end{aligned}$$

Derivative of p^k with respect to (σ, σ)

Next, $[F_{\sigma\sigma}(\mathbf{x}_{ss},0)]=0$ implies

$$
\begin{aligned}
E_t\Big[&-R_p\left(p^k\right)\left[p^k_{\sigma\sigma}\right]+\left[\mathcal{M}_{\sigma\sigma}\right]R\left(p^{k-1}\right)\\
&+\left[\mathcal{M}_{\sigma}\right]R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\\
&+\left[\mathcal{M}_{\sigma}\right]R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\\
&+\mathcal{M}R_{pp}\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\\
&+\mathcal{M}R_{p}\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{xx}}\right]_{\gamma_1\gamma_2}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\\
&+\mathcal{M}R_{p}\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}\left[\mathbf{h}_{\sigma\sigma}\right]^{\gamma_1}+\mathcal{M}R_{p}\left(p^{k-1}\right)\left[p^{k-1}_{\sigma\sigma}\right]\Big]\\
&=0
\end{aligned}
$$

For $k=1$ we have $E_t[\mathcal{M}_{\sigma\sigma}]=[p^1_{\sigma\sigma}]R_p(p^1)$. Using this and previous results, we then obtain

$$
\begin{aligned}
R_p\left(p^k\right)\left[p^k_{\sigma\sigma}\right]=&\left[p^1_{\sigma\sigma}\right]R_p\left(p^1\right)R\left(p^{k-1}\right)\\
&+2\left[\mathcal{M}_{\mathbf{y}_{t+1}}\right]_{\beta_1}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\mathbf{I}\right]^{\phi_2}_{\phi_1}\\
&+2\left[\mathcal{M}_{\mathbf{x}_{t+1}}\right]_{\gamma_1}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\mathbf{I}\right]^{\phi_2}_{\phi_1}\\
&+R\left(p^1\right)R_{pp}\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\mathbf{I}\right]^{\phi_1}_{\phi_2}\\
&+R\left(p^1\right)R_{p}\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{xx}}\right]_{\gamma_1\gamma_2}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\mathbf{I}\right]^{\phi_1}_{\phi_2}\\
&+R\left(p^1\right)R_{p}\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}\left[\mathbf{h}_{\sigma\sigma}\right]^{\gamma_1}\\
&+R\left(p^1\right)R_{p}\left(p^{k-1}\right)\left[p^{k-1}_{\sigma\sigma}\right]
\end{aligned}
$$

Third order terms

Derivative of p^k with respect to (x, x, x)

Applying the chain rule to the definition of F^k one can show that $[F_{\mathbf{xxx}}(\mathbf{x}_{ss},0)]_{\alpha_1\alpha_2\alpha_3}=0$ equals

$$
\begin{aligned}
R_p\left(p^k\right)\left[p^k_{\mathbf{xxx}}\right]_{\alpha_1\alpha_2\alpha_3}=&-R_{ppp}\left(p^k\right)\left[p^k_{\mathbf{x}}\right]_{\alpha_3}\left[p^k_{\mathbf{x}}\right]_{\alpha_2}\left[p^k_{\mathbf{x}}\right]_{\alpha_1}\\
&-R_{pp}\left(p^k\right)\left[p^k_{\mathbf{xx}}\right]_{\alpha_2\alpha_3}\left[p^k_{\mathbf{x}}\right]_{\alpha_1}\\
&-R_{pp}\left(p^k\right)\left[p^k_{\mathbf{x}}\right]_{\alpha_3}\left[p^k_{\mathbf{xx}}\right]_{\alpha_1\alpha_2}\\
&-R_{pp}\left(p^k\right)\left[p^k_{\mathbf{x}}\right]_{\alpha_3}\left[p^k_{\mathbf{xx}}\right]_{\alpha_1\alpha_2}\\
&+\left[\mathcal{M}_{\mathbf{xxx}}\right]_{\alpha_1\alpha_2\alpha_3}R\left(p^{k-1}\right)\\
&+\left(R_p\left(p^1\right)\left[p^1_{\mathbf{xx}}\right]_{\alpha_1,\alpha_2}+R_{pp}\left(p^1\right)\left[p^1_{\mathbf{x}}\right]_{\alpha_2}\left[p^1_{\mathbf{x}}\right]_{\alpha_1}\right)R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\\
&+\left(R_p\left(p^1\right)\left[p^1_{\mathbf{xx}}\right]_{\alpha_1\alpha_3}+R_{pp}\left(p^1\right)\left[p^1_{\mathbf{x}}\right]_{\alpha_3}\left[p^1_{\mathbf{x}}\right]_{\alpha_1}\right)R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\\
&+\left[p^1_{\mathbf{x}}\right]_{\alpha_1}R_p\left(p^1\right)/R\left(p^0\right)R_{pp}\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\\
&+\left[p^1_{\mathbf{x}}\right]_{\alpha_1}R_p\left(p^1\right)/R\left(p^0\right)R_{p}\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{xx}}\right]_{\gamma_2\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\\
&+\left[p^1_{\mathbf{x}}\right]_{\alpha_1}R_p\left(p^1\right)/R\left(p^0\right)R_{p}\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}\left[\mathbf{h}_{\mathbf{xx}}\right]^{\gamma_2}_{\alpha_2\alpha_3}
\end{aligned}
$$

$$
\begin{aligned}
&+\left(R_p(p^1)[p^1_{\mathbf{xx}}]_{\alpha_2,\alpha_3}+R_{pp}(p^1)[p^1_{\mathbf{x}}]_{\alpha_3}[p^1_{\mathbf{x}}]_{\alpha_2}\right)R_p(p^{k-1})[p^{k-1}_{\mathbf{x}}]_{\gamma_1}[\mathbf{h}_{\mathbf{x}}]^{\gamma_1}_{\alpha_1}\\
&\quad+[p^1_{\mathbf{x}}]_{\alpha_2}R_p(p^1)/R(p^0)R_{pp}(p^{k-1})[p^{k-1}_{\mathbf{x}}]_{\gamma_3}[\mathbf{h}_{\mathbf{x}}]^{\gamma_3}_{\alpha_3}[p^{k-1}_{\mathbf{x}}]_{\gamma_1}[\mathbf{h}_{\mathbf{x}}]^{\gamma_1}_{\alpha_1}\\
&\quad+[p^1_{\mathbf{x}}]_{\alpha_2}R_p(p^1)/R(p^0)R_p(p^{k-1})[p^{k-1}_{\mathbf{xx}}]_{\gamma_1\gamma_3}[\mathbf{h}_{\mathbf{x}}]^{\gamma_3}_{\alpha_3}[\mathbf{h}_{\mathbf{x}}]^{\gamma_1}_{\alpha_1}\\
&\quad+[p^1_{\mathbf{x}}]_{\alpha_2}R_p(p^1)/R(p^0)R_p(p^{k-1})[p^{k-1}_{\mathbf{x}}]_{\gamma_1}[\mathbf{h}_{\mathbf{xx}}]^{\gamma_1}_{\alpha_1\alpha_3}\\
&\quad+[p^1_{\mathbf{x}}]_{\alpha_3}R_p(p^1)/R(p^0)R_{pp}(p^{k-1})[p^{k-1}_{\mathbf{x}}]_{\gamma_2}[\mathbf{h}_{\mathbf{x}}]^{\gamma_2}_{\alpha_2}[p^{k-1}_{\mathbf{x}}]_{\gamma_1}[\mathbf{h}_{\mathbf{x}}]^{\gamma_1}_{\alpha_1}\\
&\quad+R(p^1)R_{ppp}(p^{k-1})[p^{k-1}_{\mathbf{x}}]_{\gamma_3}[\mathbf{h}_{\mathbf{x}}]^{\gamma_3}_{\alpha_3}[p^{k-1}_{\mathbf{x}}]_{\gamma_2}[\mathbf{h}_{\mathbf{x}}]^{\gamma_2}_{\alpha_2}[p^{k-1}_{\mathbf{x}}]_{\gamma_1}[\mathbf{h}_{\mathbf{x}}]^{\gamma_1}_{\alpha_1}\\
&\quad+R(p^1)R_{pp}(p^{k-1})[p^{k-1}_{\mathbf{xx}}]_{\gamma_2\gamma_3}[\mathbf{h}_{\mathbf{x}}]^{\gamma_3}_{\alpha_3}[\mathbf{h}_{\mathbf{x}}]^{\gamma_2}_{\alpha_2}[p^{k-1}_{\mathbf{x}}]_{\gamma_1}[\mathbf{h}_{\mathbf{x}}]^{\gamma_1}_{\alpha_1}\\
&\quad+R(p^1)R_{pp}(p^{k-1})[p^{k-1}_{\mathbf{x}}]_{\gamma_2}[\mathbf{h}_{\mathbf{xx}}]^{\gamma_2}_{\alpha_2\alpha_3}[p^{k-1}_{\mathbf{x}}]_{\gamma_1}[\mathbf{h}_{\mathbf{x}}]^{\gamma_1}_{\alpha_1}\\
&\quad+R(p^1)R_{pp}(p^{k-1})[p^{k-1}_{\mathbf{x}}]_{\gamma_2}[\mathbf{h}_{\mathbf{x}}]^{\gamma_2}_{\alpha_2}[p^{k-1}_{\mathbf{xx}}]_{\gamma_1\gamma_3}[\mathbf{h}_{\mathbf{x}}]^{\gamma_3}_{\alpha_3}[\mathbf{h}_{\mathbf{x}}]^{\gamma_1}_{\alpha_1}\\
&\quad+R(p^1)R_{pp}(p^{k-1})[p^{k-1}_{\mathbf{x}}]_{\gamma_2}[\mathbf{h}_{\mathbf{x}}]^{\gamma_2}_{\alpha_2}[p^{k-1}_{\mathbf{x}}]_{\gamma_1}[\mathbf{h}_{\mathbf{xx}}]^{\gamma_1}_{\alpha_1\alpha_3}\\
&\quad+[p^1_{\mathbf{x}}]_{\alpha_3}R_p(p^1)/R(p^0)R_p(p^{k-1})[p^{k-1}_{\mathbf{xx}}]_{\gamma_1\gamma_2}[\mathbf{h}_{\mathbf{x}}]^{\gamma_2}_{\alpha_2}[\mathbf{h}_{\mathbf{x}}]^{\gamma_1}_{\alpha_1}\\
&\quad+R(p^1)R_{pp}(p^{k-1})[p^{k-1}_{\mathbf{x}}]_{\gamma_3}[\mathbf{h}_{\mathbf{x}}]^{\gamma_3}_{\alpha_3}[p^{k-1}_{\mathbf{xx}}]_{\gamma_1\gamma_2}[\mathbf{h}_{\mathbf{x}}]^{\gamma_2}_{\alpha_2}[\mathbf{h}_{\mathbf{x}}]^{\gamma_1}_{\alpha_1}\\
&\quad+R(p^1)R_p(p^{k-1})[p^{k-1}_{\mathbf{xxx}}]_{\gamma_1\gamma_2\gamma_3}[\mathbf{h}_{\mathbf{x}}]^{\gamma_3}_{\alpha_3}[\mathbf{h}_{\mathbf{x}}]^{\gamma_2}_{\alpha_2}[\mathbf{h}_{\mathbf{x}}]^{\gamma_1}_{\alpha_1}\\
&\quad+R(p^1)R_p(p^{k-1})[p^{k-1}_{\mathbf{xx}}]_{\gamma_1\gamma_2}[\mathbf{h}_{\mathbf{xx}}]^{\gamma_2}_{\alpha_2\alpha_3}[\mathbf{h}_{\mathbf{x}}]^{\gamma_1}_{\alpha_1}\\
&\quad+R(p^1)R_p(p^{k-1})[p^{k-1}_{\mathbf{xx}}]_{\gamma_1\gamma_2}[\mathbf{h}_{\mathbf{x}}]^{\gamma_2}_{\alpha_2}[\mathbf{h}_{\mathbf{xx}}]^{\gamma_1}_{\alpha_1\alpha_3}\\
&\quad+[p^1_{\mathbf{x}}]_{\alpha_3}R_p(p^1)/R(p^0)R_p(p^{k-1})[p^{k-1}_{\mathbf{x}}]_{\gamma_1}[\mathbf{h}_{\mathbf{xx}}]^{\gamma_1}_{\alpha_1\alpha_2}\\
&\quad+R(p^1)R_{pp}(p^{k-1})[p^{k-1}_{\mathbf{x}}]_{\gamma_3}[\mathbf{h}_{\mathbf{x}}]^{\gamma_3}_{\alpha_3}[p^{k-1}_{\mathbf{x}}]_{\gamma_1}[\mathbf{h}_{\mathbf{xx}}]^{\gamma_1}_{\alpha_1\alpha_2}\\
&\quad+R(p^1)R_p(p^{k-1})[p^{k-1}_{\mathbf{xx}}]_{\gamma_1\gamma_3}[\mathbf{h}_{\mathbf{x}}]^{\gamma_3}_{\alpha_3}[\mathbf{h}_{\mathbf{xx}}]^{\gamma_1}_{\alpha_1\alpha_2}\\
&\quad+R(p^1)R_p(p^{k-1})[p^{k-1}_{\mathbf{x}}]_{\gamma_1}[\mathbf{h}_{\mathbf{xxx}}]^{\gamma_1}_{\alpha_1\alpha_2\alpha_3}
\end{aligned}
$$

Note that we can also eliminate $[\mathcal{M}_{\mathbf{xxx}}]_{\alpha_1\alpha_2\alpha_3}$ from this expression. Again, the trick is to observe that for $k=1$ we have $P^0=1$ for all values of $(\boldsymbol{x}_t, \sigma)$ and so all derivatives have to equal zero. Thus

$$
\begin{aligned}
R_p(p^1)[p^1_{\mathbf{xxx}}]_{\alpha_1\alpha_2\alpha_3}&=-R_{ppp}(p^1)[p^1_{\mathbf{x}}]_{\alpha_3}[p^1_{\mathbf{x}}]_{\alpha_2}[p^1_{\mathbf{x}}]_{\alpha_1}\\
&\quad-R_{pp}(p^1)[p^1_{\mathbf{xx}}]_{\alpha_2\alpha_3}[p^1_{\mathbf{x}}]_{\alpha_1}\\
&\quad-R_{pp}(p^1)[p^1_{\mathbf{x}}]_{\alpha_2}[p^1_{\mathbf{xx}}]_{\alpha_1\alpha_3}\\
&\quad-R_{pp}(p^1)[p^1_{\mathbf{x}}]_{\alpha_3}[p^1_{\mathbf{xx}}]_{\alpha_1\alpha_2}\\
&\quad+[\mathcal{M}_{\mathbf{xxx}}]_{\alpha_1\alpha_2\alpha_3}R(p^0)
\end{aligned}
$$

$$\Updownarrow$$

$$
\begin{aligned}
&\left(R_p(p^1)[p^1_{\mathbf{xxx}}]_{\alpha_1\alpha_2\alpha_3}+R_{ppp}(p^1)[p^1_{\mathbf{x}}]_{\alpha_3}[p^1_{\mathbf{x}}]_{\alpha_2}[p^1_{\mathbf{x}}]_{\alpha_1}+R_{pp}(p^1)[p^1_{\mathbf{xx}}]_{\alpha_2\alpha_3}[p^1_{\mathbf{x}}]_{\alpha_1}\right.\\
&\left.R_{pp}(p^1)[p^1_{\mathbf{x}}]_{\alpha_2}[p^1_{\mathbf{xx}}]_{\alpha_1\alpha_3}+R_{pp}(p^1)[p^1_{\mathbf{x}}]_{\alpha_3}[p^1_{\mathbf{xx}}]_{\alpha_1\alpha_2}\right)=[\mathcal{M}_{\mathbf{xxx}}]_{\alpha_1\alpha_2\alpha_3}
\end{aligned}
$$

because $R(p^0)=1$.

Thus we get for $k>1$

$$
\begin{aligned}
R_p\left(p^k\right)\left[p^k_{\mathbf{xxx}}\right]_{\alpha_1\alpha_2\alpha_3} = & -R_{ppp}\left(p^k\right)\left[p^k_{\mathbf{x}}\right]_{\alpha_3}\left[p^k_{\mathbf{x}}\right]_{\alpha_2}\left[p^k_{\mathbf{x}}\right]_{\alpha_1} \\
& -R_{pp}\left(p^k\right)\left[p^k_{\mathbf{x}}\right]_{\alpha_2\alpha_3}\left[p^k_{\mathbf{x}}\right]_{\alpha_1} \\
& -R_{pp}\left(p^k\right)\left[p^k_{\mathbf{x}}\right]_{\alpha_2}\left[p^k_{\mathbf{x}}\right]_{\alpha_1\alpha_3} \\
& -R_{pp}\left(p^k\right)\left[p^k\right]_{\alpha_3}\left[p^k_{\mathbf{xx}}\right]_{\alpha_1\alpha_2} \\
& +(R_p\left(p^1\right)\left[p^1_{\mathbf{xxx}}\right]_{\alpha_1\alpha_2\alpha_3}+R_{ppp}\left(p^1\right)\left[p^1_{\mathbf{x}}\right]_{\alpha_3}\left[p^1_{\mathbf{x}}\right]_{\alpha_2}\left[p^1_{\mathbf{x}}\right]_{\alpha_1}+R_{pp}\left(p^1\right)\left[p^1_{\mathbf{xx}}\right]_{\alpha_2\alpha_3}\left[p^1_{\mathbf{x}}\right]_{\alpha_1} \\
& +R_{pp}\left(p^1\right)\left[p^1_{\mathbf{x}}\right]_{\alpha_2}\left[p^1_{\mathbf{xx}}\right]_{\alpha_1\alpha_3}+R_{pp}\left(p^1\right)\left[p^1_{\mathbf{x}}\right]_{\alpha_3}\left[p^1_{\mathbf{xx}}\right]_{\alpha_1\alpha_2})R\left(p^{k-1}\right) \\
& +\left(R_p\left(p^1\right)\left[p^1_{\mathbf{xx}}\right]_{\alpha_1,\alpha_2}+R_{pp}\left(p^1\right)\left[p^1_{\mathbf{x}}\right]_{\alpha_2}\left[p^1_{\mathbf{x}}\right]_{\alpha_1}\right)R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} \\
+ & \left(R_p\left(p^1\right)\left[p^1_{\mathbf{xx}}\right]_{\alpha_1\alpha_3}+R_{pp}\left(p^1\right)\left[p^1_{\mathbf{x}}\right]_{\alpha_3}\left[p^1_{\mathbf{x}}\right]_{\alpha_1}\right)R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2} \\
& +\left[p^1_{\mathbf{x}}\right]_{\alpha_1}R_p\left(p^1\right)/R\left(p^0\right)R_{pp}\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2} \\
& +\left[p^1_{\mathbf{x}}\right]_{\alpha_1}R_p\left(p^1\right)/R\left(p^0\right)R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{xx}}\right]_{\gamma_2\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2} \\
& +\left[p^1_{\mathbf{x}}\right]_{\alpha_1}R_p\left(p^1\right)/R\left(p^0\right)R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}\left[\mathbf{h}_{\mathbf{xx}}\right]^{\gamma_2}_{\alpha_2\alpha_3} \\
+ & \left(R_p\left(p^1\right)\left[p^1_{\mathbf{xx}}\right]_{\alpha_2,\alpha_3}+R_{pp}\left(p^1\right)\left[p^1_{\mathbf{x}}\right]_{\alpha_3}\left[p^1_{\mathbf{x}}\right]_{\alpha_2}\right)R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1} \\
& +\left[p^1_{\mathbf{x}}\right]_{\alpha_2}R_p\left(p^1\right)/R\left(p^0\right)R_{pp}\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1} \\
& +\left[p^1_{\mathbf{x}}\right]_{\alpha_2}R_p\left(p^1\right)/R\left(p^0\right)R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{xx}}\right]_{\gamma_1\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1} \\
& +\left[p^1_{\mathbf{x}}\right]_{\alpha_2}R_p\left(p^1\right)/R\left(p^0\right)R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}\left[\mathbf{h}_{\mathbf{xx}}\right]^{\gamma_1}_{\alpha_1\alpha_3} \\
& +\left[p^1_{\mathbf{x}}\right]_{\alpha_3}R_p\left(p^1\right)/R\left(p^0\right)R_{pp}\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1} \\
& +R\left(p^1\right)R_{ppp}\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1} \\
& +R\left(p^1\right)R_{pp}\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{xx}}\right]_{\gamma_2\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1} \\
& +R\left(p^1\right)R_{pp}\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}\left[\mathbf{h}_{\mathbf{xx}}\right]^{\gamma_2}_{\alpha_2\alpha_3}\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1} \\
& +R\left(p^1\right)R_{pp}\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[p^{k-1}_{\mathbf{xx}}\right]_{\gamma_1\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1} \\
& +R\left(p^1\right)R_{pp}\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}\left[\mathbf{h}_{\mathbf{xx}}\right]^{\gamma_1}_{\alpha_1\alpha_3} \\
& +\left[p^1_{\mathbf{x}}\right]_{\alpha_3}R_p\left(p^1\right)/R\left(p^0\right)R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{xx}}\right]_{\gamma_1\gamma_2}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1} \\
& +R\left(p^1\right)R_{pp}\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[p^{k-1}_{\mathbf{xx}}\right]_{\gamma_1\gamma_2}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1} \\
& +R\left(p^1\right)R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{xxx}}\right]_{\gamma_1\gamma_2\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1} \\
& +R\left(p^1\right)R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{xx}}\right]_{\gamma_1\gamma_2}\left[\mathbf{h}_{\mathbf{xx}}\right]^{\gamma_2}_{\alpha_2\alpha_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1} \\
& +R\left(p^1\right)R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{xx}}\right]_{\gamma_1\gamma_2}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{h}_{\mathbf{xx}}\right]^{\gamma_1}_{\alpha_1\alpha_3} \\
& +\left[p^1_{\mathbf{x}}\right]_{\alpha_3}R_p\left(p^1\right)/R\left(p^0\right)R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}\left[\mathbf{h}_{\mathbf{xx}}\right]^{\gamma_1}_{\alpha_1\alpha_2} \\
& +R\left(p^1\right)R_{pp}\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}\left[\mathbf{h}_{\mathbf{xx}}\right]^{\gamma_1}_{\alpha_1\alpha_2} \\
& +R\left(p^1\right)R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{xx}}\right]_{\gamma_1\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[\mathbf{h}_{\mathbf{xx}}\right]^{\gamma_1}_{\alpha_1\alpha_2} \\
& +R\left(p^1\right)R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}\left[\mathbf{h}_{\mathbf{xxx}}\right]^{\gamma_1}_{\alpha_1\alpha_2\alpha_3}
\end{aligned}
$$

With a log-transformation $R(p^{t,k})=M^k$, $R_p(p^{t,k})=M^k$, $R_{pp}(p^{t,k})=M^k$, and $R_{ppp}(p^{t,k})=M^k$ in the deterministic steady state. Using the expressions for first and second order derivatives of bond prices derived above, we get, after simplifying, the expression stated in the body of the text.

Derivative of p^k with respect to $(\sigma, \sigma, \mathbf{x})$

It is possible to show that $[F_{\sigma\sigma\mathbf{x}}(\mathbf{x}_{ss},0)]_{\alpha_3}=0$ implies

$$
\begin{aligned}
E_t\Big\{&-R_{pp}(p^k)[p^k_{\mathbf{x}}]_{\alpha_3}[p^k_{\sigma\sigma}]-R_p(p^k)[p^k_{\sigma\sigma\mathbf{x}}]_{\alpha_3}\\
&+[\mathcal{M}_{\sigma\sigma\mathbf{x}}]R(p^{k-1})+[\mathcal{M}_{\sigma\sigma}]R_p(p^{k-1})[p^{k-1}_{\mathbf{x}}]_{\gamma_3}[\mathbf{h}_{\mathbf{x}}]^{\gamma_3}_{\alpha_3}\\
&+2[\mathcal{M}_{\sigma\mathbf{x}}]_{\alpha_3}R_p(p^{k-1})[p^{k-1}_{\mathbf{x}}]_{\gamma_2}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\epsilon}_{t+1}]^{\phi_2}\\
&+2[\mathcal{M}_{\sigma}]R_{pp}(p^{k-1})[p^{k-1}_{\mathbf{x}}]_{\gamma_3}[\mathbf{h}_{\mathbf{x}}]^{\gamma_3}_{\alpha_3}[p^{k-1}_{\mathbf{x}}]_{\gamma_2}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\epsilon}_{t+1}]^{\phi_2}\\
&+2[\mathcal{M}_{\sigma}]R_p(p^{k-1})[p^{k-1}_{\mathbf{xx}}]_{\gamma_2\gamma_3}[\mathbf{h}_{\mathbf{x}}]^{\gamma_3}_{\alpha_3}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\epsilon}_{t+1}]^{\phi_2}\\
&+[\mathcal{M}_{\mathbf{x}}]_{\alpha_3}R_{pp}(p^{k-1})[p^{k-1}_{\mathbf{x}}]_{\gamma_2}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\epsilon}_{t+1}]^{\phi_2}[p^{k-1}_{\mathbf{x}}]_{\gamma_1}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\\
&+\mathcal{M}R_{ppp}(p^{k-1})[p^{k-1}_{\mathbf{x}}]_{\gamma_3}[\mathbf{h}_{\mathbf{x}}]^{\gamma_3}_{\alpha_3}[p^{k-1}_{\mathbf{x}}]_{\gamma_2}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\epsilon}_{t+1}]^{\phi_2}[p^{k-1}_{\mathbf{x}}]_{\gamma_1}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\\
&+\mathcal{M}R_{pp}(p^{k-1})[p^{k-1}_{\mathbf{xx}}]_{\gamma_2\gamma_3}[\mathbf{h}_{\mathbf{x}}]^{\gamma_3}_{\alpha_3}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\epsilon}_{t+1}]^{\phi_2}[p^{k-1}_{\mathbf{x}}]_{\gamma_1}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\\
&+\mathcal{M}R_{pp}(p^{k-1})[p^{k-1}_{\mathbf{x}}]_{\gamma_2}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\epsilon}_{t+1}]^{\phi_2}[p^{k-1}_{\mathbf{xx}}]_{\gamma_1\gamma_3}[\mathbf{h}_{\mathbf{x}}]^{\gamma_3}_{\alpha_3}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\\
&+[\mathcal{M}_{\mathbf{x}}]_{\alpha_3}R_p(p^{k-1})[p^{k-1}_{\mathbf{xx}}]_{\gamma_1\gamma_2}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\epsilon}_{t+1}]^{\phi_2}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\\
&+\mathcal{M}R_{pp}(p^{k-1})[p^{k-1}_{\mathbf{x}}]_{\gamma_3}[\mathbf{h}_{\mathbf{x}}]^{\gamma_3}_{\alpha_3}[p^{k-1}_{\mathbf{xx}}]_{\gamma_1\gamma_2}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\epsilon}_{t+1}]^{\phi_2}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\\
&+\mathcal{M}R_p(p^{k-1})[p^{k-1}_{\mathbf{xxx}}]_{\gamma_1\gamma_2\gamma_3}[\mathbf{h}_{\mathbf{x}}]^{\gamma_3}_{\alpha_3}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\epsilon}_{t+1}]^{\phi_2}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1}\\
&+[\mathcal{M}_{\mathbf{x}}]_{\alpha_3}R_p(p^{k-1})[p^{k-1}_{\mathbf{x}}]_{\gamma_1}[\mathbf{h}_{\sigma\sigma}]^{\gamma_1}\\
&+\mathcal{M}R_{pp}(p^{k-1})[p^{k-1}_{\mathbf{x}}]_{\gamma_3}[\mathbf{h}_{\mathbf{x}}]^{\gamma_3}_{\alpha_3}[p^{k-1}_{\mathbf{x}}]_{\gamma_1}[\mathbf{h}_{\sigma\sigma}]^{\gamma_1}\\
&+\mathcal{M}R_p(p^{k-1})[p^{k-1}_{\mathbf{xx}}]_{\gamma_1\gamma_3}[\mathbf{h}_{\mathbf{x}}]^{\gamma_3}_{\alpha_3}[\mathbf{h}_{\sigma\sigma}]^{\gamma_1}\\
&+\mathcal{M}R_p(p^{k-1})[p^{k-1}_{\mathbf{x}}]_{\gamma_1}[\mathbf{h}_{\sigma\sigma\mathbf{x}}]^{\gamma_1}_{\alpha_3}\\
&+[\mathcal{M}_{\mathbf{x}}]_{\alpha_3}R_p(p^{k-1})[p^{k-1}_{\sigma\sigma}]\\
&+\mathcal{M}R_{pp}(p^{k-1})[p^{k-1}_{\mathbf{x}}]_{\gamma_3}[\mathbf{h}_{\mathbf{x}}]^{\gamma_3}_{\alpha_3}[p^{k-1}_{\sigma\sigma}]\\
&+\mathcal{M}R_p(p^{k-1})[p^{k-1}_{\sigma\sigma\mathbf{x}}]_{\gamma_3}[\mathbf{h}_{\mathbf{x}}]^{\gamma_3}_{\alpha_3}\Big\}=0
\end{aligned}
$$

$$\Updownarrow$$

$$
\begin{aligned}
R_p(p^k)[p^k_{\sigma\sigma\mathbf{x}}]_{\alpha_3}=&-R_{pp}(p^k)[p^k_{\mathbf{x}}]_{\alpha_3}[p^k_{\sigma\sigma}]\\
&+E_t[\mathcal{M}_{\sigma\sigma\mathbf{x}}]R(p^{k-1})\\
&+[p^1_{\sigma\sigma}]R_p(p^1)/R(p^0)R_p(p^{k-1})[p^{k-1}_{\mathbf{x}}]_{\gamma_3}[\mathbf{h}_{\mathbf{x}}]^{\gamma_3}_{\alpha_3}\\
&+2E_t\Big([\mathcal{M}_{\sigma\mathbf{x}}]_{\alpha_3}R_p(p^{k-1})[p^{k-1}_{\mathbf{x}}]_{\gamma_2}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\epsilon}_{t+1}]^{\phi_2}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+2E_t\left(\left[\mathcal{M}_\sigma\right]R_{pp}\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_3}^{\gamma_3}\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right)\\
&+2E_t\left(\left[\mathcal{M}_\sigma\right]R_{p}\left(p^{k-1}\right)\left[p_{\mathbf{xx}}^{k-1}\right]_{\gamma_2\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_3}^{\gamma_3}\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right)\\
&+\left[p_{\mathbf{x}}^{1}\right]_{\alpha_3}R_p\left(p^1\right)/R\left(p^0\right)R_{pp}\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_1}\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}\left[\mathbf{I}\right]_{\phi_2}^{\phi_1}\\
&+R\left(p^1\right)R_{ppp}\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_3}^{\gamma_3}\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_1}\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}\left[\mathbf{I}\right]_{\phi_2}^{\phi_1}\\
&+R\left(p^1\right)R_{pp}\left(p^{k-1}\right)\left[p_{\mathbf{xx}}^{k-1}\right]_{\gamma_2\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_3}^{\gamma_3}\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_1}\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}\left[\mathbf{I}\right]_{\phi_2}^{\phi_1}\\
&+R\left(p^1\right)R_{pp}\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\left[p_{\mathbf{xx}}^{k-1}\right]_{\gamma_1\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_3}^{\gamma_3}\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}\left[\mathbf{I}\right]_{\phi_2}^{\phi_1}\\
&+\left[p_{\mathbf{x}}^{1}\right]_{\alpha_3}R_p\left(p^1\right)/R\left(p^0\right)R_{p}\left(p^{k-1}\right)\left[p_{\mathbf{xx}}^{k-1}\right]_{\gamma_1\gamma_2}\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}\left[\mathbf{I}\right]_{\phi_2}^{\phi_1}\\
&+R\left(p^1\right)R_{pp}\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_3}^{\gamma_3}\left[p_{\mathbf{xx}}^{k-1}\right]_{\gamma_1\gamma_2}\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}\left[\mathbf{I}\right]_{\phi_2}^{\phi_1}\\
&+R\left(p^1\right)R_{p}\left(p^{k-1}\right)\left[p_{\mathbf{xxx}}^{k-1}\right]_{\gamma_1\gamma_2\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_3}^{\gamma_3}\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}\left[\mathbf{I}\right]_{\phi_2}^{\phi_1}\\
&+\left[p_{\mathbf{x}}^{1}\right]_{\alpha_3}R_p\left(p^1\right)/R\left(p^0\right)R_{p}\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_1}\left[\mathbf{h}_{\sigma\sigma}\right]^{\gamma_1}\\
&+R\left(p^1\right)R_{pp}\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_3}^{\gamma_3}\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_1}\left[\mathbf{h}_{\sigma\sigma}\right]^{\gamma_1}\\
&+R\left(p^1\right)R_{p}\left(p^{k-1}\right)\left[p_{\mathbf{xx}}^{k-1}\right]_{\gamma_1\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_3}^{\gamma_3}\left[\mathbf{h}_{\sigma\sigma}\right]^{\gamma_1}\\
&+R\left(p^1\right)R_{p}\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_1}\left[\mathbf{h}_{\sigma\sigma\mathbf{x}}\right]_{\alpha_3}^{\gamma_1}\\
&+\left[p_{\mathbf{x}}^{1}\right]_{\alpha_3}R_p\left(p^1\right)/R\left(p^0\right)R_{p}\left(p^{k-1}\right)\left[p_{\sigma\sigma}^{k-1}\right]\\
&+R\left(p^1\right)R_{pp}\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_3}^{\gamma_3}\left[p_{\sigma\sigma}^{k-1}\right]\\
&+R\left(p^1\right)R_{p}\left(p^{k-1}\right)\left[p_{\sigma\sigma\mathbf{x}}^{k-1}\right]_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_3}^{\gamma_3}
\end{aligned}
$$

where we have used

$$\mathcal{M}=R(p^1)$$

$$[\mathcal{M}_{\mathbf{x}}]_{\alpha_3}=[p_{\mathbf{x}}^1]_{\alpha_3}R_p(p^1)/R(p^0)$$

$$E_t[\mathcal{M}_{\sigma\sigma}]=[p_{\sigma\sigma}^1]R_p(p^1)/R(p^0)$$

We now compute the terms with derivatives of σ. Here we recall that

$$\left[\mathcal{M}_\sigma\right]\equiv\left(\left[\mathcal{M}_{\mathbf{y}_{t+1}}\right]_{\beta_1}\left[\mathbf{g}_{\mathbf{x}}\right]_{\gamma_1}^{\beta_1}\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}+\left[\mathcal{M}_{\mathbf{x}_{t+1}}\right]_{\gamma_1}\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\right).$$

So

$$
\begin{aligned}
&2E_t\left(\left[\mathcal{M}_\sigma\right]R_{pp}\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_3}^{\gamma_3}\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right)\\
&=2E_t\left(\left[\mathcal{M}_{\mathbf{y}_{t+1}}\right]_{\beta_1}\left[\mathbf{g}_{\mathbf{x}}\right]_{\gamma_1}^{\beta_1}\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}R_{pp}\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_3}^{\gamma_3}\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right)\\
&+2E_t\left(\left[\mathcal{M}_{\mathbf{x}_{t+1}}\right]_{\gamma_1}\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}R_{pp}\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_3}^{\gamma_3}\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right)\\
&=2\left[\mathcal{M}_{\mathbf{y}_{t+1}}\right]_{\beta_1}\left[\mathbf{g}_{\mathbf{x}}\right]_{\gamma_1}^{\beta_1}\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}R_{pp}\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_3}^{\gamma_3}\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\left[\mathbf{I}\right]_{\phi_1}^{\phi_2}\\
&+2\left[\mathcal{M}_{\mathbf{x}_{t+1}}\right]_{\gamma_1}\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}R_{pp}\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_3}^{\gamma_3}\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\left[\mathbf{I}\right]_{\phi_1}^{\phi_2}
\end{aligned}
$$

and

$$
\begin{aligned}
&2E_t\left(\left[\mathcal{M}_{\sigma}\right]R_p\left(p^{k-1}\right)\left[p_{\mathbf{xx}}^{k-1}\right]_{\gamma_2\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_3}^{\gamma_3}\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right)\\
&=2E_t\left(\left[\mathcal{M}_{\mathbf{y}_{t+1}}\right]_{\beta_1}\left[\mathbf{g}_{\mathbf{x}}\right]_{\gamma_1}^{\beta_1}\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}R_p\left(p^{k-1}\right)\left[p_{\mathbf{xx}}^{k-1}\right]_{\gamma_2\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_3}^{\gamma_3}\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right)\\
&+2E_t\left(\left[\mathcal{M}_{\mathbf{x}_{t+1}}\right]_{\gamma_1}\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}R_p\left(p^{k-1}\right)\left[p_{\mathbf{xx}}^{k-1}\right]_{\gamma_2\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_3}^{\gamma_3}\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right)\\
&=2\left[\mathcal{M}_{\mathbf{y}_{t+1}}\right]_{\beta_1}\left[\mathbf{g}_{\mathbf{x}}\right]_{\gamma_1}^{\beta_1}\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}R_p\left(p^{k-1}\right)\left[p_{\mathbf{xx}}^{k-1}\right]_{\gamma_2\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_3}^{\gamma_3}\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\left[\mathbf{I}\right]_{\phi_1}^{\phi_2}\\
&+2E_t\left[\mathcal{M}_{\mathbf{x}_{t+1}}\right]_{\gamma_1}\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}R_p\left(p^{k-1}\right)\left[p_{\mathbf{xx}}^{k-1}\right]_{\gamma_2\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_3}^{\gamma_3}\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\left[\mathbf{I}\right]_{\phi_1}^{\phi_2}
\end{aligned}
$$

To compute the $[\mathcal{M}_{\sigma\mathbf{x}}]_{\alpha_3}$ term, we need to find an expression for $[\mathcal{M}_{\sigma\mathbf{x}}]_{\alpha_3}$. Hence

$$
\begin{aligned}
\left[\mathcal{M}_{\sigma\mathbf{x}}\right]_{\alpha_3}=&\left(\left[\mathcal{M}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]_{\beta_1\beta_3}\left[\mathbf{g}_{\mathbf{x}}\right]_{\gamma_3}^{\beta_3}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_3}^{\gamma_3}+\left[\mathcal{M}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]_{\beta_1\beta_3}\left[\mathbf{g}_{\mathbf{x}}\right]_{\alpha_3}^{\beta_3}+\left[\mathcal{M}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]_{\beta_1\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_3}^{\gamma_3}+\left[\mathcal{M}_{\mathbf{y}_{t+1}\mathbf{x}_t}\right]_{\beta_1\alpha_3}\right)\\
&\times\left[\mathbf{g}_{\mathbf{x}}\right]_{\gamma_1}^{\beta_1}\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\\
&+\left[\mathcal{M}_{\mathbf{y}_{t+1}}\right]_{\beta_1}\left[\mathbf{g}_{\mathbf{xx}}\right]_{\gamma_1\gamma_3}^{\beta_1}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_3}^{\gamma_3}\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\\
&+\left(\left[\mathcal{M}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]_{\gamma_1\beta_3}\left[\mathbf{g}_{\mathbf{x}}\right]_{\gamma_3}^{\beta_3}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_3}^{\gamma_3}+\left[\mathcal{M}_{\mathbf{x}_{t+1}\mathbf{y}_t}\right]_{\gamma_1\beta_3}\left[\mathbf{g}_{\mathbf{x}}\right]_{\alpha_3}^{\beta_3}+\left[\mathcal{M}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]_{\gamma_1\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_3}^{\gamma_3}+\left[\mathcal{M}_{\mathbf{x}_{t+1}\mathbf{x}_t}\right]_{\gamma_1\alpha_3}\right)\\
&\times\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}
\end{aligned}
$$

So

$$
\begin{aligned}
&2E_t\left(\left[\mathcal{M}_{\sigma\mathbf{x}}\right]_{\alpha_3}R_p\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right)\\
&=2E_t\Bigg\{\left(\left[\mathcal{M}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]_{\beta_1\beta_3}\left[\mathbf{g}_{\mathbf{x}}\right]_{\gamma_3}^{\beta_3}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_3}^{\gamma_3}+\left[\mathcal{M}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]_{\beta_1\beta_3}\left[\mathbf{g}_{\mathbf{x}}\right]_{\alpha_3}^{\beta_3}+\left[\mathcal{M}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]_{\beta_1\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_3}^{\gamma_3}+\left[\mathcal{M}_{\mathbf{y}_{t+1}\mathbf{x}_t}\right]_{\beta_1\alpha_3}\right)\\
&\quad\times\left[\mathbf{g}_{\mathbf{x}}\right]_{\gamma_1}^{\beta_1}\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}R_p\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\\
&\quad+\left[\mathcal{M}_{\mathbf{y}_{t+1}}\right]_{\beta_1}\left[\mathbf{g}_{\mathbf{xx}}\right]_{\gamma_1\gamma_3}^{\beta_1}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_3}^{\gamma_3}\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}R_p\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\\
&\quad+\left(\left[\mathcal{M}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]_{\gamma_1\beta_3}\left[\mathbf{g}_{\mathbf{x}}\right]_{\gamma_3}^{\beta_3}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_3}^{\gamma_3}+\left[\mathcal{M}_{\mathbf{x}_{t+1}\mathbf{y}_t}\right]_{\gamma_1\beta_3}\left[\mathbf{g}_{\mathbf{x}}\right]_{\alpha_3}^{\beta_3}+\left[\mathcal{M}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]_{\gamma_1\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_3}^{\gamma_3}+\left[\mathcal{M}_{\mathbf{x}_{t+1}\mathbf{x}_t}\right]_{\gamma_1\alpha_3}\right)\\
&\quad\times\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}R_p\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\Bigg\}\\
&=2\left(\left[\mathcal{M}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]_{\beta_1\beta_3}\left[\mathbf{g}_{\mathbf{x}}\right]_{\alpha_3}^{\beta_3}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_3}^{\gamma_3}+\left[\mathcal{M}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]_{\beta_1\beta_3}\left[\mathbf{g}_{\mathbf{x}}\right]_{\alpha_3}^{\beta_3}+\left[\mathcal{M}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]_{\beta_1\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_3}^{\gamma_3}+\left[\mathcal{M}_{\mathbf{y}_{t+1}\mathbf{x}_t}\right]_{\beta_1\alpha_3}\right)\\
&\quad\times\left[\mathbf{g}_{\mathbf{x}}\right]_{\gamma_1}^{\beta_1}\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}R_p\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\left[\mathbf{I}\right]_{\phi_1}^{\phi_2}\\
&\quad+2\left[\mathcal{M}_{\mathbf{y}_{t+1}}\right]_{\beta_1}\left[\mathbf{g}_{\mathbf{xx}}\right]_{\gamma_1\gamma_3}^{\beta_1}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_3}^{\gamma_3}\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}R_p\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\left[\mathbf{I}\right]_{\phi_1}^{\phi_2}\\
&\quad+2\left(\left[\mathcal{M}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]_{\gamma_1\beta_3}\left[\mathbf{g}_{\mathbf{x}}\right]_{\alpha_3}^{\beta_3}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_3}^{\gamma_3}+\left[\mathcal{M}_{\mathbf{x}_{t+1}\mathbf{y}_t}\right]_{\gamma_1\beta_3}\left[\mathbf{g}_{\mathbf{x}}\right]_{\alpha_3}^{\beta_3}+\left[\mathcal{M}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]_{\gamma_1\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_3}^{\gamma_3}+\left[\mathcal{M}_{\mathbf{x}_{t+1}\mathbf{x}_t}\right]_{\gamma_1\alpha_3}\right)\\
&\quad\times\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}R_p\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\left[\mathbf{I}\right]_{\phi_1}^{\phi_2}
\end{aligned}
$$

We finally note that $E_t([\mathcal{M}_{\sigma\sigma\mathbf{x}}]_{\alpha_3})$ can be solved for and then substituted out by exploiting the fact that for k=1 we have P^0=1 for all values of $(\mathbf{x}_t, \sigma)$ and so all derivatives have to equal zero. Thus

$$R_p(p^1)[p^1_{\sigma\sigma\mathbf{x}}]_{\alpha_3}=-R_{pp}(p^1)[p^1_{\mathbf{x}}]_{\alpha_3}[p^1_{\sigma\sigma}]+E_t([\mathcal{M}_{\sigma\sigma\mathbf{x}}]_{\alpha_3})R(p^0)$$

$$\Updownarrow$$

$$(R_p(p^1)[p^1_{\sigma\sigma\mathbf{x}}]_{\alpha_3}+R_{pp}(p^1)[p^1_{\mathbf{x}}]_{\alpha_3}[p^1_{\sigma\sigma}])/R(p^0)=E_t([\mathcal{M}_{\sigma\sigma\mathbf{x}}]_{\alpha_3})$$

So for $k>1$ we get

$$\begin{aligned}
R_p\left(p^k\right)\left[p^k_{\sigma\sigma\mathbf{x}}\right]_{\alpha_3} =& -R_{pp}\left(p^k\right)\left[p^k_{\mathbf{x}}\right]_{\alpha_3}\left[p^k_{\sigma\sigma}\right] \\
&+\left(R_p\left(p^1\right)\left[p^1_{\sigma\sigma\mathbf{x}}\right]_{\alpha_3}+R_{pp}\left(p^1\right)\left[p^1_{\mathbf{x}}\right]_{\alpha_3}\left[p^1_{\sigma\sigma}\right]\right)R\left(p^{k-1}\right)/R\left(p^0\right) \\
&+\left[p^1_{\sigma\sigma}\right]R_p\left(p^1\right)/R\left(p^0\right)R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} \\
&+2\left(\left[\mathcal{M}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]_{\beta_1\beta_3}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathcal{M}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]_{\beta_1\beta_3}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}+\left[\mathcal{M}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]_{\beta_1\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathcal{M}_{\mathbf{y}_{t+1}\mathbf{x}_t}\right]_{\beta_1\alpha_3}\right) \\
&\times\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\mathbf{I}\right]^{\phi_2}_{\varphi_1} \\
&+2\left[\mathcal{M}_{\mathbf{y}_{t+1}}\right]_{\beta_1}\left[\mathbf{g}_{\mathbf{xx}}\right]^{\beta_1}_{\gamma_1\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\mathbf{I}\right]^{\phi_2}_{\phi_1} \\
&+2\left(\left[\mathcal{M}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]_{\gamma_1\beta_3}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathcal{M}_{\mathbf{x}_{t+1}\mathbf{y}_t}\right]_{\gamma_1\beta_3}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_3}_{\alpha_3}+\left[\mathcal{M}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]_{\gamma_1\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}+\left[\mathcal{M}_{\mathbf{x}_{t+1}\mathbf{x}_t}\right]_{\gamma_1\alpha_3}\right) \\
&\times\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\mathbf{I}\right]^{\phi_2}_{\phi_1} \\
&+2\left[\mathcal{M}_{\mathbf{y}_{t+1}}\right]_{\beta_1}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}R_{pp}\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\mathbf{I}\right]^{\phi_2}_{\phi_1} \\
&+2\left[\mathcal{M}_{\mathbf{x}_{t+1}}\right]_{\gamma_1}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}R_{pp}\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\mathbf{I}\right]^{\phi_2}_{\phi_1} \\
&+2\left[\mathcal{M}_{\mathbf{y}_{t+1}}\right]_{\beta_1}\left[\mathbf{g}_{\mathbf{x}}\right]^{\beta_1}_{\gamma_1}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{xx}}\right]_{\gamma_2\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\mathbf{I}\right]^{\phi_2}_{\phi_1} \\
&+2E_t\left[\mathcal{M}_{\mathbf{x}_{t+1}}\right]_{\gamma_1}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{xx}}\right]_{\gamma_2\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\mathbf{I}\right]^{\phi_2}_{\phi_1} \\
&+\left[p^1_{\mathbf{x}}\right]_{\alpha_3}R_p\left(p^1\right)/R\left(p^0\right)R_{pp}\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\mathbf{I}\right]^{\phi_1}_{\phi_2} \\
&+R\left(p^1\right)R_{ppp}\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\mathbf{I}\right]^{\phi_1}_{\phi_2} \\
&+R\left(p^1\right)R_{pp}\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{xx}}\right]_{\gamma_2\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\mathbf{I}\right]^{\phi_1}_{\phi_2} \\
&+R\left(p^1\right)R_{pp}\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[p^{k-1}_{\mathbf{xx}}\right]_{\gamma_1\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\mathbf{I}\right]^{\phi_1}_{\phi_2} \\
&+\left[p^1_{\mathbf{x}}\right]_{\alpha_3}R_p\left(p^1\right)/R\left(p^0\right)R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{xx}}\right]_{\gamma_1\gamma_2}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\mathbf{I}\right]^{\phi_1}_{\phi_2} \\
&+R\left(p^1\right)R_{pp}\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[p^{k-1}_{\mathbf{xx}}\right]_{\gamma_1\gamma_2}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\mathbf{I}\right]^{\phi_1}_{\phi_2} \\
&+R\left(p^1\right)R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{xxx}}\right]_{\gamma_1\gamma_2\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\mathbf{I}\right]^{\phi_1}_{\phi_2} \\
&+\left[p^1_{\mathbf{x}}\right]_{\alpha_3}R_p\left(p^1\right)/R\left(p^0\right)R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}\left[\mathbf{h}_{\sigma\sigma}\right]^{\gamma_1} \\
&+R\left(p^1\right)R_{pp}\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}\left[\mathbf{h}_{\sigma\sigma}\right]^{\gamma_1} \\
&+R\left(p^1\right)R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{xx}}\right]_{\gamma_1\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[\mathbf{h}_{\sigma\sigma}\right]^{\gamma_1} \\
&+R\left(p^1\right)R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}\left[\mathbf{h}_{\sigma\sigma\mathbf{x}}\right]^{\gamma_1}_{\alpha_3}
\end{aligned}$$

$$+\left[p_{\mathbf{x}}^{1}\right]_{\alpha_3} R_p\left(p^1\right)/R\left(p^0\right)R_p\left(p^{k-1}\right)\left[p_{\sigma\sigma}^{k-1}\right]$$
$$+R\left(p^1\right)R_{pp}\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_3}^{\gamma_3}\left[p_{\sigma\sigma}^{k-1}\right]$$
$$+R\left(p^1\right)R_p\left(p^{k-1}\right)\left[p_{\sigma\sigma\mathbf{x}}^{k-1}\right]_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_3}^{\gamma_3}$$

For a logarithm transformation $R(p^{t,k})=M^k$, $R_p(p^{t,k})=M^k$, $R_{pp}(p^{t,k})=M^k$, and $R_{ppp}(p^{t,k})=M^k$. Using the expressions for first and second order derivatives of bond prices derived above, we get, after simplifying,

$$p_{\sigma\sigma\mathbf{x}}^{k}\left(1,\alpha_3\right)=$$
$$-2\mathcal{M}^{-1}\mathcal{M}_{\mathbf{y}_{t+1}}\mathbf{g}_{\mathbf{x}}\boldsymbol{\eta}\boldsymbol{\eta}'\left(p_{\mathbf{x}}^{k-1}\right)' p_{\mathbf{x}}^{1}\left(1,\alpha_3\right)$$
$$-2\mathcal{M}^{-1}\mathcal{M}_{\mathbf{x}_{t+1}}\boldsymbol{\eta}\boldsymbol{\eta}'\left(p_{\mathbf{x}}^{k-1}\right)' p_{\mathbf{x}:}^{1}\left(1,\alpha_3\right)$$
$$+p_{\sigma\sigma\mathbf{x}}^{1}\left(1,\alpha_3\right)$$
$$+2\mathcal{M}^{-1}p_{\mathbf{x}}^{k-1}\boldsymbol{\eta}\boldsymbol{\eta}'\left(\mathbf{g}_{\mathbf{x}}\right)'$$
$$\times\left(\mathcal{M}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\mathbf{g}_{\mathbf{x}}\mathbf{h}_{\mathbf{x}}\left(:,\alpha_3\right)+\mathcal{M}_{\mathbf{y}_{t+1}\mathbf{y}_t}\mathbf{g}_{\mathbf{x}}\left(:,\alpha_3\right)+\mathcal{M}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\mathbf{h}_{\mathbf{x}}\left(:,\alpha_3\right)+\mathcal{M}_{\mathbf{y}_{t+1}\mathbf{x}_t}\left(:,\alpha_3\right)\right)$$
$$+\sum_{\beta_1=1}^{n_y}2\mathcal{M}^{-1}\mathcal{M}_{\mathbf{y}_{t+1}}\left(1,\beta_1\right)p_{\mathbf{x}}^{k-1}\boldsymbol{\eta}\boldsymbol{\eta}'\mathbf{g}_{\mathbf{xx}}\left(\beta_1,:,:\right)\mathbf{h}_{\mathbf{x}}\left(:,\alpha_3\right)$$
$$+2\mathcal{M}^{-1}p_{\mathbf{x}}^{k-1}\boldsymbol{\eta}\boldsymbol{\eta}'$$
$$\times\left(\mathcal{M}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\mathbf{g}_{\mathbf{x}}\mathbf{h}_{\mathbf{x}}\left(:,\alpha_3\right)+\mathcal{M}_{\mathbf{x}_{t+1}\mathbf{y}_t}\mathbf{g}_{\mathbf{x}}\left(:,\alpha_3\right)+\mathcal{M}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\mathbf{h}_{\mathbf{x}}\left(:,\alpha_3\right)+\mathcal{M}_{\mathbf{x}_{t+1}\mathbf{x}_t}\left(:,\alpha_3\right)\right)$$
$$+2\mathcal{M}^{-1}\mathcal{M}_{\mathbf{y}_{t+1}}\mathbf{g}_{\mathbf{x}}\boldsymbol{\eta}\boldsymbol{\eta}'p_{\mathbf{xx}}^{k-1}\mathbf{h}_{\mathbf{x}}\left(:,\alpha_3\right)$$
$$+2\mathcal{M}^{-1}\mathcal{M}_{\mathbf{x}_{t+1}}\boldsymbol{\eta}\boldsymbol{\eta}'p_{\mathbf{xx}}^{k-1}\mathbf{h}_{\mathbf{x}}\left(:,\alpha_3\right)$$
$$+p_{\mathbf{x}}^{k-1}\boldsymbol{\eta}\boldsymbol{\eta}'p_{\mathbf{xx}}^{k-1}\mathbf{h}_{\mathbf{x}}\left(:,\alpha_3\right)$$
$$+p_{\mathbf{x}}^{k-1}\boldsymbol{\eta}\boldsymbol{\eta}'p_{\mathbf{xx}}^{k-1}\mathbf{h}_{\mathbf{x}}\left(:,\alpha_3\right)$$
$$+\sum_{\gamma_1=1}^{n_x}\boldsymbol{\eta}'\left(\gamma_1,:\right)\eta'p_{\mathbf{xxx}}^{k-1}\left(\gamma_1,:,:\right)\mathbf{h}_{\mathbf{x}}\left(:,\alpha_3\right)$$
$$+\left(\mathbf{h}_{\sigma\sigma}\right)' p_{\mathbf{xx}}^{k-1}\mathbf{h}_{\mathbf{x}}\left(:,\alpha_k\right)$$
$$+p_{\mathbf{x}}^{k-1}\mathbf{h}_{\sigma\sigma\mathbf{x}}\left(:,\alpha_3\right)$$
$$+p_{\sigma\sigma\mathbf{x}}^{k-1}\mathbf{h}_{\mathbf{x}}\left(:,\alpha_3\right)$$

Derivative of p^k with respect to (σ, σ, σ)

It is possible to show that $F_{\sigma\sigma\sigma}(\mathbf{x}_{ss}, 0)=0$ implies

$$\left[F_{\sigma\sigma\sigma}\left(\mathbf{x}_{ss},0\right)\right]=E_t\Big\{-R_p\left(p^k\right)\left[p_{\sigma\sigma\sigma}^{k}\right]$$
$$+\left[\mathcal{M}_{\sigma\sigma\sigma}\right]R\left(p^{k-1}\right)$$
$$+3\left[\mathcal{M}_{\sigma\sigma}\right]R_p\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}$$
$$+3\left[\mathcal{M}_{\sigma}\right]R_{pp}\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_3}\left[\boldsymbol{\eta}\right]_{\phi_3}^{\gamma_3}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_3}\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}$$
$$+3\left[\mathcal{M}_{\sigma}\right]R_p\left(p^{k-1}\right)\left[p_{\mathbf{xx}}^{k-1}\right]_{\gamma_2\gamma_3}\left[\boldsymbol{\eta}\right]_{\phi_3}^{\gamma_3}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_3}\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}$$
$$+R\left(p^1\right)R_{ppp}\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_3}\left[\boldsymbol{\eta}\right]_{\phi_3}^{\gamma_3}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_3}\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_1}\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}$$
$$+3R\left(p^1\right)R_{pp}\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\left[p_{\mathbf{xx}}^{k-1}\right]_{\gamma_1\gamma_3}\left[\boldsymbol{\eta}\right]_{\phi_3}^{\gamma_3}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_3}\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}$$

$$
\begin{aligned}
&+R\left(p^{1}\right)R_{p}\left(p^{k-1}\right)\left[p_{\mathbf{xxx}}^{k-1}\right]_{\gamma_1\gamma_2\gamma_3}\left[\boldsymbol{\eta}\right]_{\phi_3}^{\gamma_3}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_3}\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\\
&+R\left(p^{1}\right)R_{p}\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_1}\left[\mathbf{h}_{\sigma\sigma\sigma}\right]^{\gamma_1}\\
&+R\left(p^{1}\right)R_{p}\left(p^{k-1}\right)\left[p_{\sigma\sigma\sigma}^{k-1}\right]\Big\}=0
\end{aligned}
$$

We next use the expression for $[\mathcal{M}_{\sigma}]$ found previously. We also have from differentiation of $\mathcal{M}$ that

$$
\begin{aligned}
\left[\mathcal{M}_{\sigma\sigma}\right]=&\left(\left[\mathcal{M}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]_{\beta_1\beta_3}\left[\mathbf{g}_{\mathbf{x}}\right]_{\gamma_3}^{\beta_3}\left[\boldsymbol{\eta}\right]_{\phi_3}^{\gamma_3}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_3}+\left[\mathcal{M}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]_{\beta_1\gamma_3}\left[\boldsymbol{\eta}\right]_{\phi_3}^{\gamma_3}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_3}\right)\left[\mathbf{g}_{\mathbf{x}}\right]_{\gamma_1}^{\beta_1}\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\\
&+\left[\mathcal{M}_{\mathbf{y}_{t+1}}\right]_{\beta_1}\left[\mathbf{g}_{\mathbf{xx}}\right]_{\gamma_1\gamma_3}^{\beta_1}\left[\boldsymbol{\eta}\right]_{\phi_3}^{\gamma_3}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_3}\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\\
&+\left[\mathcal{M}_{\mathbf{y}_{t+1}}\right]_{\beta_1}\left[\mathbf{g}_{\mathbf{x}}\right]_{\gamma_1}^{\beta_1}\left[\mathbf{h}_{\sigma\sigma}\right]^{\gamma_1}\\
&+\left[\mathcal{M}_{\mathbf{y}_{t+1}}\right]_{\beta_1}\left[\mathbf{g}_{\sigma\sigma}\right]^{\beta_1}\\
&+\left[\mathcal{M}_{\mathbf{y}_{t}}\right]_{\beta_1}\left[\mathbf{g}_{\sigma\sigma}\right]^{\beta_1}\\
&+\left(\left[\mathcal{M}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]_{\gamma_1\beta_3}\left[\mathbf{g}_{\mathbf{x}}\right]_{\gamma_3}^{\beta_3}\left[\boldsymbol{\eta}\right]_{\phi_3}^{\gamma_3}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_3}+\left[\mathcal{M}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]_{\gamma_1\gamma_3}\left[\boldsymbol{\eta}\right]_{\phi_3}^{\gamma_3}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_3}\right)\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}\\
&+\left[\mathcal{M}_{\mathbf{x}_{t+1}}\right]_{\gamma_1}\left[\mathbf{h}_{\sigma\sigma}\right]^{\gamma_1}
\end{aligned}
$$

For $[\mathcal{M}_{\sigma\sigma\sigma}]$, we exploit the fact $P^0=1$ for all values of $(\mathbf{x}_t, \sigma)$ and so all derivatives have to equal zero. Thus $R_p(p^1)[p^1_{\sigma\sigma\sigma}]=E_t\{[\mathcal{M}_{\sigma\sigma\sigma}]\}$.

To evaluate the expectations in the term for $[F_{\sigma\sigma\sigma}(\mathbf{x}_{ss}, 0)]$, we define

$$
\left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]_{\phi_2\phi_3}^{\phi_1}=\begin{cases} m^3\left(\boldsymbol{\epsilon}_{t+1}\left(\phi_1\right)\right) & \text{if } \phi_1=\phi_2=\phi_3\\ 0 & \text{otherwise}\end{cases}
$$

where $m^3(\boldsymbol{\epsilon}_{t+1})$ denotes the third moment of $\boldsymbol{\epsilon}_{t+1}(\phi_1)$ for $\phi_1=1, 2, \dots, n_{\boldsymbol{\epsilon}}$. Notice that $\mathbf{m}^3(\boldsymbol{\epsilon}_{t+1})$ is a $n_{\boldsymbol{\epsilon}}\times n_{\boldsymbol{\epsilon}}\times n_{\boldsymbol{\epsilon}}$ matrix. Following some simplifications we finally get

$$
\begin{aligned}
&R_p\left(p^k\right)\left[p_{\sigma\sigma\sigma}^{k}\right]=\\
&\quad+R_p\left(p^1\right)\left[p_{\sigma\sigma\sigma}^{1}\right]R\left(p^{k-1}\right)\\
&\quad+3\left[\mathcal{M}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]_{\beta_1\beta_3}\left[\mathbf{g}_{\mathbf{x}}\right]_{\gamma_3}^{\beta_3}\left[\boldsymbol{\eta}\right]_{\phi_3}^{\gamma_3}\left[\mathbf{g}_{\mathbf{x}}\right]_{\gamma_1}^{\beta_1}\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}R_p\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]_{\phi_2\phi_3}^{\phi_1}\\
&\quad+6\left[\mathcal{M}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]_{\beta_1\gamma_3}\left[\boldsymbol{\eta}\right]_{\phi_3}^{\gamma_3}\left[\mathbf{g}_{\mathbf{x}}\right]_{\gamma_1}^{\beta_1}\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}R_p\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]_{\phi_2\phi_3}^{\phi_1}\\
&\quad+3\left[\mathcal{M}_{\mathbf{y}_{t+1}}\right]_{\beta_1}\left[\mathbf{g}_{\mathbf{xx}}\right]_{\gamma_1\gamma_3}^{\beta_1}\left[\boldsymbol{\eta}\right]_{\phi_3}^{\gamma_3}\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}R_p\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]_{\phi_2\phi_3}^{\phi_1}\\
&\quad+3\left[\mathcal{M}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]_{\gamma_1\gamma_3}\left[\boldsymbol{\eta}\right]_{\phi_3}^{\gamma_3}\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}R_p\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]_{\phi_2\phi_3}^{\phi_1}\\
&\quad+3\left(\left[\mathcal{M}_{\mathbf{y}_{t+1}}\right]_{\beta_1}\left[\mathbf{g}_{\mathbf{x}}\right]_{\gamma_1}^{\beta_1}\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}+\left[\mathcal{M}_{\mathbf{x}_{t+1}}\right]_{\gamma_1}\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}\right)\left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]_{\phi_2\phi_3}^{\phi_1}R_{pp}\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_3}\left[\boldsymbol{\eta}\right]_{\phi_3}^{\gamma_3}\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\\
&\quad+3\left(\left[\mathcal{M}_{\mathbf{y}_{t+1}}\right]_{\beta_1}\left[\mathbf{g}_{\mathbf{x}}\right]_{\gamma_1}^{\beta_1}\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}+\left[\mathcal{M}_{\mathbf{x}_{t+1}}\right]_{\gamma_1}\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}\right)\left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]_{\phi_2\phi_3}^{\phi_1}R_{p}\left(p^{k-1}\right)\left[p_{\mathbf{xx}}^{k-1}\right]_{\gamma_2\gamma_3}\left[\boldsymbol{\eta}\right]_{\phi_3}^{\gamma_3}\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\\
&\quad+R\left(p^1\right)R_{ppp}\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_3}\left[\boldsymbol{\eta}\right]_{\phi_3}^{\gamma_3}\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_1}\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}\left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]_{\phi_2\phi_3}^{\phi_1}\\
&\quad+3R\left(p^1\right)R_{pp}\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\left[p_{\mathbf{xx}}^{k-1}\right]_{\gamma_1\gamma_3}\left[\boldsymbol{\eta}\right]_{\phi_3}^{\gamma_3}\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}\left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]_{\phi_2\phi_3}^{\phi_1}
\end{aligned}
$$

$$
\begin{aligned}
&+R\left(p^{1}\right)R_{p}\left(p^{k-1}\right)\left[p_{\mathbf{xxx}}^{k-1}\right]_{\gamma_1\gamma_2\gamma_3}\left[\boldsymbol{\eta}\right]_{\phi_3}^{\gamma_3}\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}\left[\mathbf{m}^{3}\left(\boldsymbol{\epsilon}_{t+1}\right)\right]_{\phi_2\phi_3}^{\phi_1}\\
&+R\left(p^{1}\right)R_{p}\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_1}\left[\mathbf{h}_{\sigma\sigma\sigma}\right]^{\gamma_1}\\
&+R\left(p^{1}\right)R_{p}\left(p^{k-1}\right)p_{\sigma\sigma\sigma}^{k-1}
\end{aligned}
$$

For a logarithm transformation, it is straightforward to show that

$$
\begin{aligned}
p_{\sigma\sigma\sigma}^{k}=p_{\sigma\sigma\sigma}^{1}&+3\mathcal{M}^{-1}\left[\mathcal{M}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]_{\beta_1\beta_3}\left[\mathbf{g}_{\mathbf{x}}\right]_{\gamma_3}^{\beta_3}\left[\boldsymbol{\eta}\right]_{\phi_3}^{\gamma_3}\left[\mathbf{g}_{\mathbf{x}}\right]_{\gamma_1}^{\beta_1}\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\left[\mathbf{m}^{3}\left(\boldsymbol{\epsilon}_{t+1}\right)\right]_{\phi_2\phi_3}^{\phi_1}\\
&+6\mathcal{M}^{-1}\left[\mathcal{M}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]_{\beta_1\gamma_3}\left[\boldsymbol{\eta}\right]_{\phi_3}^{\gamma_3}\left[\mathbf{g}_{\mathbf{x}}\right]_{\gamma_1}^{\beta_1}\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\left[\mathbf{m}^{3}\left(\boldsymbol{\epsilon}_{t+1}\right)\right]_{\phi_2\phi_3}^{\phi_1}\\
&+3\mathcal{M}^{-1}\left[\mathcal{M}_{\mathbf{y}_{t+1}}\right]_{\beta_1}\left[\mathbf{g}_{\mathbf{xx}}\right]_{\gamma_1\gamma_3}^{\beta_1}\left[\boldsymbol{\eta}\right]_{\phi_3}^{\gamma_3}\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\left[\mathbf{m}^{3}\left(\boldsymbol{\epsilon}_{t+1}\right)\right]_{\phi_2\phi_3}^{\phi_1}\\
&+3\mathcal{M}^{-1}\left[\mathcal{M}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]_{\gamma_1\gamma_3}\left[\boldsymbol{\eta}\right]_{\phi_3}^{\gamma_3}\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\left[\mathbf{m}^{3}\left(\boldsymbol{\epsilon}_{t+1}\right)\right]_{\phi_2\phi_3}^{\phi_1}\\
&+3\mathcal{M}^{-1}\left(\left[\mathcal{M}_{\mathbf{y}_{t+1}}\right]_{\beta_1}\left[\mathbf{g}_{\mathbf{x}}\right]_{\gamma_1}^{\beta_1}\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}+\left[\mathcal{M}_{\mathbf{x}_{t+1}}\right]_{\gamma_1}\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}\right)\left[\mathbf{m}^{3}\left(\boldsymbol{\epsilon}_{t+1}\right)\right]_{\phi_2\phi_3}^{\phi_1}\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_3}\left[\boldsymbol{\eta}\right]_{\phi_3}^{\gamma_3}\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\\
&+3\mathcal{M}^{-1}\left(\left[\mathcal{M}_{\mathbf{y}_{t+1}}\right]_{\beta_1}\left[\mathbf{g}_{\mathbf{x}}\right]_{\gamma_1}^{\beta_1}\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}+\left[\mathcal{M}_{\mathbf{x}_{t+1}}\right]_{\gamma_1}\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}\right)\left[\mathbf{m}^{3}\left(\boldsymbol{\epsilon}_{t+1}\right)\right]_{\phi_2\phi_3}^{\phi_1}\left[p_{\mathbf{xx}}^{k-1}\right]_{\gamma_2\gamma_3}\left[\boldsymbol{\eta}\right]_{\phi_3}^{\gamma_3}\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\\
&+\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_3}\left[\boldsymbol{\eta}\right]_{\phi_3}^{\gamma_3}\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_1}\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}\left[\mathbf{m}^{3}\left(\boldsymbol{\epsilon}_{t+1}\right)\right]_{\phi_2\phi_3}^{\phi_1}\\
&+3\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\left[p_{\mathbf{xx}}^{k-1}\right]_{\gamma_1\gamma_3}\left[\boldsymbol{\eta}\right]_{\phi_3}^{\gamma_3}\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}\left[\mathbf{m}^{3}\left(\boldsymbol{\epsilon}_{t+1}\right)\right]_{\phi_2\phi_3}^{\phi_1}\\
&+\left[p_{\mathbf{xxx}}^{k-1}\right]_{\gamma_1\gamma_2\gamma_3}\left[\boldsymbol{\eta}\right]_{\phi_3}^{\gamma_3}\left[\eta\right]_{\phi_2}^{\gamma_2}\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}\left[\mathbf{m}^{3}\left(\boldsymbol{\epsilon}_{t+1}\right)\right]_{\phi_2\phi_3}^{\phi_1}\\
&+\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_1}\left[\mathbf{h}_{\sigma\sigma\sigma}\right]^{\gamma_1}+p_{\sigma\sigma\sigma}^{k-1}
\end{aligned}
$$

Appendix B

Matlab implementation of the POP method

The approximation method presented in the body of the text is implemented in Matlab. For the first perturbation step, we apply the codes accompanying Schmitt-Grohé and Uribe (2004) to compute first and second order derivatives, while the routines underlying Andreasen (2012b) are used for all third-order terms. For the second perturbation step, the user only needs to specify the stochastic discount factor in `Anal_PricingKernel_derivatives.m` and the position of the one-period bond price in $\mathbf{y}_t$. Analytical derivatives of the pricing kernel are then computed based on symbolic differentiation, and these derivatives are evaluated in the steady state by `num_eval_PricingKernel.m`. Bond prices are then computed in `Get_Bond_Prices_3rd.m` up to third order, either for the level of bond prices or for a log-transformation.

Appendix C

Closed-form solution to the endowment model with habits

For the considered habit model, we have

$$\mathcal{M}_{t+1} := \beta \frac{(C_{t+1} - hZ_{t+1})^{-\gamma}}{(C_t - hZ_t)^{-\gamma}} = \beta \frac{(1 - h\exp\{z_{t+1}\})^{-\gamma}}{(1 - h\exp\{z_t\})^{-\gamma}} \exp\{-\gamma x_{t+1}\}.$$

A closed-form solution for zero-coupon bond prices is given by (see Zabczyk 2014)

$$\begin{aligned} P_t^k &= (1 - h\exp\{z_t\})^{\gamma} \beta^k \exp\left\{-\gamma\left(k\mu + (x_t - \mu)\frac{\rho(1-\rho^k)}{(1-\rho)}\right)\right\} \\ &\times \sum_{n=0}^{+\infty} \binom{-\gamma}{n} \left[(-h)\exp\left\{(1-\phi)^k z_t - \mu\frac{1-(1-\phi)^k}{\phi} - (x_t - \mu)\rho\frac{\rho^k - (1-\phi)^k}{\rho - (1-\phi)}\right\}\right]^n \\ &\times \prod_{j=1}^{k} \mathcal{L}_{\xi}\left(\gamma\frac{(1-\rho^j)}{(1-\rho)} + n\frac{\rho^j - (1-\phi)^j}{\rho - (1-\phi)}\right). \end{aligned}$$

Here, $\mathcal{L}_{\xi}$ is the Laplace transform of ξ, and $\binom{\alpha}{n}$ denotes a generalized binomial coefficient, i.e.,

$$\binom{\alpha}{n} := \prod_{k=1}^{n} (\alpha - k + 1)/k, \text{ for } n > 0 \text{ and } \binom{\alpha}{0} := 1,$$

where $\alpha \in \mathbb{R}$; and $n \in \mathcal{N}$. The condition for convergence of this solution is $\beta \exp\left\{\frac{Var(\xi)}{2}\left(\frac{\gamma}{1-\rho}\right)^2\right\} < 1.$

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