Efficient Design with Interdependent Valuations

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Abstract

We study efficient, Bayes-Nash incentive compatible mechanisms in a social choice setting that allows for informational and allocative externalities. We show that such mechanisms exist only if a congruence condition relating private and social rates of information substitution is satisfied. If signals are multi-dimensional, the congruence condition is determined by an integrability constraint, and it can hold only in non-generic cases where values are private or a certain symmetry assumption holds. If signals are one-dimensional, the congruence condition reduces to a monotonicity constraint and it can be generically satisfied. We apply the results to the study of multi-object auctions, and we discuss why such auctions cannot be reduced to one-dimensional models without loss of generality.

1. Introduction

There exists an extensive literature on efficient auctions and mechanism design. A lot of attention has been devoted to the case where each agent i has a quasi-linear utility function that depends on the chosen social alternative, on information (or signal) privately known to i, and on a monetary transfer, but does not depend on information available to other agents. In this framework, a prominent role

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is played by the Clarke-Groves-Vickrey (CGV) mechanisms (see Clarke, 1971, Groves, 1973, Vickrey, 1961). These are mechanisms that ensure both that an efficient decision is taken and that truthful revelation of privately held information is a dominant strategy for each agent. The result holds for arbitrary dimensions of signal spaces and for arbitrary signals' distributions[1].

In this paper we study the case where each agent has a quasi-linear utility function having as arguments signals received by *all* agents and the chosen social alternative. Hence, besides allocative externalities, we allow for informational externalities, and we speak of "interdependent valuations". Signals may be multi-dimensional, but we assume that they are independently drawn across agents. (Signal independence is the most seriously restrictive assumption; observe though that this assumption does not bite for the "principal-agent" framework of Example 4.4, and it is not required for the result in the one-dimensional case of Section 5.)

For an illustration, consider an auction where a set M of heterogenous objects is divided among $n+1$ agents (agent zero is the seller, the rest are potential buyers). An alternative is a partition u of M, $u = \{u_i\}_{i=0}^{N}$, where u_i is the set of objects allocated to bidder i, $i = 1, 2, ...N$ and u_0 is the set of unsold objects. Agent i receives a signal s_b^i for each possible bundle $b \in 2^M$, and has a valuation function V_u^i for each partition u. Different models are obtained by varying the dependence of valuations on partitions and signals. Consider the following examples: 1) V_u^i only depends on u_i and $s_{u_i}^i$. This is a pure "private values" model; 2) V_u^i depends on the entire partition u and on $s_{u_i}^i$. This is a "private values" model which allows for allocative externalities. 3) V_u^i depends on u and on $\{s_{u_i}^j\}_{j=0}^n$, or V_u^i depends on u and on $\{s_{u_j}^j\}_{j=0}^n$. These are models which allow for both allocative and informational externalities[2].

For our present purpose, the main common feature of the above examples is that the information available to each agent is multi-dimensional (one signal per bundle) and that different signals affect valuations in different alternatives.

There are many auction papers that go beyond the private values case (e.g., the literature following Milgrom and Weber, 1982), but almost all of them restrict attention to situations where there is one object (or there are several identical units), signals are one-dimensional, agents are ex-ante symmetric and do not care

[1] It is well known that, generally, CGV mechanisms cannot simultaneously satisfy conditions such as budget-balancedness and individual rationality (for example, Myerson and Satterthwaite's (1983) impossibility result can be obtained as a corollary of this fact).

[2] For example, consider an auction where the bidders are firms in an oligopoly. Independence of signals across bidders is plausible if i's private information concerns the modification of its cost structure (fixed and variable costs) induced by the acquisition of a bundle u_i. Together with the final allocation of objects (e.g., licenses, patents, plants), this information affects the profit of all firms through the oligopolistic equilibrium.

about what other agents receive at the auction[3].

Most of the works on mechanism design with informational interdependent valuations consider one-dimensional signals. Williams and Radner (1988) show that efficient, dominant-strategy incentive compatible mechanisms do not generally exist[4]. Dasgupta and Maskin (1999) offer a general study of multi-object auctions where agents have one-dimensional signals and where there are no allocative externalities[5]. They assume that the designer is not informed about the bidders' valuation functions, and hence these must also be reported. Dasgupta and Maskin construct a mechanism that does not depend on the functional form of the valuation functions and achieves efficient allocations under appropriate conditions on marginal valuations. Under similar informational assumptions, Perry and Reny (1999a) construct an efficient bidding procedure which is less complex than Dasgupta and Maskin's mechanism, but which works only for a one-dimensional model with m identical units, no allocative externalities and decreasing marginal valuations. In their procedure agents place many bids which depend on the unit and on the potential competitor on that unit. In the same framework with m identical units, Ausubel (1997, Appendix B) assumes that the valuation functions are known to the designer and describes an efficient revelation mechanism[6]. Under appropriate conditions on marginal valuations (such as those in Perry and Reny, 1999a) this mechanism is incentive compatible and it generalizes the revelation mechanisms for the one-unit case constructed in Maskin (1992) and Dasgupta and

[3] Auction models emphasizing the role of allocative externalities in a one-object setup are discussed in Jehiel and Moldovanu (1996) and Jehiel, Moldovanu and Stacchetti (1996, 1999).

[4] Crémer and McLean (1985,1988) and McAfee and Reny (1992) have given conditions under which a principal can extract the full surplus available when types are correlated. Full extraction mechanisms are, in particular, efficient. Neeman (1998) shows that these results do not hold in a model that can be interpreted as one where agents have multi-dimensional signals, and signals have some private and some common components. Aoyagi (1998) presents a general existence result of efficient, budget balanced and incentive compatible mechanisms when agents have finitely many correlated types. None of the above papers covers the present framework (i.e., a continuum of mutually payoff relevant multi-dimensional types), but we suspect that correlation among types allows some possibility results. On the other hand, the mechanisms displayed in the literature above are not very intuitive and require potentially unlimited transfers as correlations get small.

[5] Dasgupta an Maskin allow for heterogenous objects. But, if the units are not identical, the representation of preferences on various bundles generally requires at least one scalar signal per bundle - see the examples above.

[6] Ausubel (1997) also studies an indirect, ascending bidding procedure which is efficient for the case of interdependent valuations only if bidders are ex-ante symmetric and have constant marginal valuations up to a fixed capacity. Perry and Reny (1999b) show how to modify this procedure in order to get efficiency when agents are asymmetric and marginal valuations are decreasing.

Maskin (1997)[7].

Maskin (1992) observed that, in general, no efficient, incentive-compatible one-unit auction exists if a buyer's valuation for that unit depends on a multi-dimensional signal (see further comments on this result in Section 4 below). Dasgupta and Maskin (1999) show how to transform such a framework into one where valuations depend on a one-dimensional sufficient statistic[8]. The reduced one-dimensional model admits efficient, incentive compatible mechanisms which are also constrained efficient (i.e., second-best) for the original model.

This paper is organized as follows: In Section 2 we present the social choice model. In Section 3 we obtain a characterization theorem for Bayesian incentive compatible direct mechanisms. In Section 4 we exhibit impossibility results about efficient, Bayesian incentive compatible mechanisms. We only require value maximization, and we completely ignore budget-balancedness and any other constraints. Hence, we show that providing incentives for truthful revelation of privately held information is not compatible even with a very weak efficiency requirement.

Relatively simple results are obtained for situations where incentive compatible mechanisms cannot condition on some signal which is relevant for efficiency considerations. Theorem 4.1 shows impossibility for the case where there is at least one agent possessing information that affects other agents, but does not directly affect the owner of that information. A similar argument is used in Example 4.2 which shows that efficient, incentive compatible mechanisms may not exist if there are an alternative k and an agent i such that agent i's signal affecting her valuation in alternative k is multi-dimensional (this corresponds to Maskin's (1992) example). The basic intuition behind these results is that a one-dimensional instrument (agent $i's$ transfer in alternative k) is not sufficient to extract multi-dimensional information relevant for an efficient choice of alternative k.

Our main impossibility result is Theorem 4.3. We consider there a framework where each agent i has a $K-$dimensional signal s^i (K is the number of alternatives). The coordinate s^i_k is a *one-dimensional* signal affecting the valuations of *all* agents for alternative k. This framework is critical since, a-priori, incentive compatible mechanisms may condition on all signals, and since the one-dimensional transfer associated with alternative k should, in principle, be sufficient to extract the one-dimensional signal s^i_k.

To understand the insight behind Theorem 4.3, consider a situation where there are $K \geq 2$ alternatives and where only agent i obtains a private K-dimensional signal. Keep this signal constant in all but two coordinates k and k', and imagine the locus in the $(s^i_k, s^i_{k'})$ sub-space where alternatives k and k' yield

[7] This is an early version of Dasgupta and Maskin (1999).

[8] A similar reduction is perfomed in Jehiel, Moldovanu and Stacchetti (1996).

the same highest social welfare (see Figure 1 in Section 4). At each point, the slope of this curve equals the social (i.e., with respect to social welfare) marginal rate of substitution among i's signals in alternatives k and k'. In order to make i choose efficiently, we must ensure that i's types along that curve are indifferent between alternatives k and k'. This means that, along the curve, i's value in alternative k plus the transfer he obtains in this alternative must equal his value inlternative k' plus the transfer in k'. But, for any given transfers, the locus in the $(s_k^i, s_{k'}^i)$ sub-space where i is indifferent between k and k' is given by a different curve whose slope equals at each point the private (i.e, with respect to i's welfare function) marginal rate of substitution among i's signals in alternatives k and k'. Efficient, incentive compatible mechanisms exist only in the non-generic situation where the two curves coincide. Theorem 4.3 generalizes this intuition to the more complex setting where several agents obtain private signals. For the linear model detailed in the paper, we can exhibit a simple global necessary condition that needs to be satisfied by incentive-compatible, efficient mechanisms. The condition relates private and social rates of informational substitution, and it holds only for a closed, zero-measure set of parameters[9].

The proof of Theorem 4.3 is based on the following technical observation: an incentive compatible mechanism generates for each agent a vector field that associates to each type a vector of expected probabilities with which the various alternatives are chosen. A generalization of the standard one-dimensional envelope argument shows that this vector field is the gradient of the equilibrium expected utility function. Since it is a gradient, the vector field must satisfy an integrability condition involving its cross-derivatives[10]. The impossibility results follow by showing that the vector fields generated by efficient mechanisms satisfy the required conditions only under very restrictive conditions.

Since the integrability constraint bites in any multi-dimensional model, results similar to Theorem 4.3 hold as soon as there is at least one agent whose signal is of dimension $d \geq 2$.

In Section 5 we study the remaining case where signal spaces are one-dimensional. We construct a mechanism that is efficient and incentive compatible if several inequalities relating private and social marginal valuations are satisfied. The main idea of the construction is to make i's transfer equal to the cumulative effect of i's action (here a signal report) on all other agents[11]. Since i's effect on others

[9] We show that the congruence condition is satisfied in situations where either a certain symmetry condition, or the private values assumption hold.

[10] A similar condition appears in the classical demand theory for several goods (see Chapter 3 in Mas-Colell, Whinston and Green, 1995): the matrix of price derivatives for a demand function arising from utility maximization must be *symmetric*.

[11] The idea can be traced back to Pigou. It constitutes the basis of the Clarke-Groves-Vickrey approach.

depends here i's signal, incentive compatible transfers must neutralize this influence. The first illustration of this idea in an auction context with interdependent valuations appears in Maskin (1992).

To get an intuition for the result, consider again a situation where only one agent receives a private signal, and consider a type s^* of this agent where alternatives k and k' yield the same highest social welfare[12]. As above, in order to induce the agent to choose efficiently, the transfers in alternatives k and k' must make type s^* indifferent between the two[13]. This relation fixes the difference between the two transfers, and, given a condition on *private* marginal valuations, all types can be induced to correctly choose among k and k'. The final step is to find a condition (relating *private* and *social* marginal valuations) that allows to aggregate in a consistent way the transfer differences obtained for each pair of alternatives[14].

Concluding comments are gathered in Section 6. In particular, we comment on the difficulty of finding constrained efficient (i.e., second-best) mechanisms in the general multi-dimensional setup.

2. The Model

There are K social alternatives, indexed by $k = 1, ...K$ and there are N agents, indexed by $i = 1, .., N$.

Each agent i has a signal (or type) s^i which is drawn from a space $S^i \subseteq \Re^{K \times N}$ according to a continuous density $f_i(s^i) > 0$, independently of other agents' signals. Each agent i knows s^i, and the densities $\{f_j\}_{j=1}^N$ are common knowledge. The idea is that the coordinate s^i_{kj} of s^i influences the utility of agent j in alternative k[15].

We assume that the signal spaces S^i are bounded and convex[16], and that they have a non-empty interior (given the usual topology in $\Re^{K \times N}$) and a piecewise smooth boundary. Let S denote the Cartesian product $\prod_{i=1}^N S^i$, with generic element s. Denote by S^{-i} the type space of agents other than i, with s^{-i} as generic element.

[12] In contrast to the case of multi-dimensional signals, generically we cannot vary now this signal without violating the condition that both alternatives yield the same social welfare. Hence, the locus discussed above degenerates here to a single point.

[13] Without transfers, i is indifferent when he has, say, a type $s' \neq s^*$.

[14] This is necessary when there are more than two alternatives.

[15] We address below (see Example 4.2) situations where the signal of an agent i affecting the utility of agent j in alternative k is itself multi-dimensional.

[16] Convexity is assumed for convenience. If S^i is simply-connected all results go through unchanged.

If alternative k is chosen, and if i obtains a transfer x_i, then $i's$ utility is given by $V_k^i(s_{ki}^1,...,s_{ki}^n)+x_i$, where $V_k^i(s_{ki}^1,...,s_{ki}^n)=\sum_{j=1}^n a_{ki}^j s_{ki}^j$, and where the scalar parameters[17] $\{a_{ki}^j\}_{1\le k\le K,1\le j,i\le N}$ are common knowledge. We assume throughout the paper that $\forall i,\forall k,\ a_{ki}^i\ge 0$.

We now define direct social choice mechanisms. A function $p:S\to\Re^K$ such that $\forall k,s,\ 0\le p_k(s)\le 1$ and $\forall s,\ \sum_{k=1}^K p_k(s)=1$ is called a *social choice rule.* A social choice rule (SCR) is said to be *efficient* if

$$\forall s,\ p_h(s)\neq 0\ \Rightarrow h\in\arg\max_k\sum_{i=1}^N V_k^i(s^1,..s^N)=\arg\max_k\sum_{i=1}^N\sum_{j=1}^N a_{ki}^j s_{ki}^j.$$

A *direct revelation mechanism* (DRM) is defined by a pair (p,x) where p is a social choice rule, and $x:S\to\Re^N$ is a payment scheme. The term $p_k(s)$ is the probability that alternative k is chosen if the agents report signals $s=(s^1,...,s^N)$, and $x_i(s)$ is the transfer to agent i if the agents report signals s. A DRM is *efficient* if the associated social choice rule is efficient[18].

Given a payment scheme x and a social choice rule p, we now define for each agent i the conditional expected payment function $y_i:S^i\to\Re$ and the conditional expected probability assignment functions $q^i:S^i\to\Re^K$ associated with x and $p:$

$$y_i(t^i)=\int_{S^{-i}}x_i(t^i,s^{-i})f_{-i}(s^{-i})ds^{-i}$$

$$q_k^i(t^i)=\int_{S^{-i}}p_k(t^i,s^{-i})f_{-i}(s^{-i})ds^{-i}.$$

Assume that agent i believes that all other agents report truthfully and assume that i reports type t^i when his true type is s^i. Then, $i's$ expected utility is given by:

$$U_i(t^i,s^i)=\int_{S^{-i}}[\sum_k(p_k(t^i,s^{-i})\sum_{j=1}^N a_{ki}^j s_{ki}^j)]f_{-i}(s^{-i})ds^{-i}+y_i(t^i)=$$
$$\sum_k a_{ki}^i s_{ki}^i q_k^i(t^i)+\sum_k\int_{S^{-i}}[(p_k(t^i,s^{-i})\sum_{j\neq i}a_{ki}^j s_{ki}^j)]f_{-i}(s^{-i})ds^{-i}+y_i(t^i).\quad(2.1)$$

Define also

$$V_i(s^i)=U_i(s^i,s^i).\quad(2.2)$$

[17] The analysis directly extends to the case where the valuation functions include also a constant, i.e., $V_k^i(s_{ki}^1,...,s_{ki}^n)=\sum_{j=1}^n a_{ki}^j s_{ki}^j+b_k^i$ (because such constants do not affect incentives).

[18] We ignore here (as in the CGV approach) the (ex post) "budget balancedness" condition, which imposes $\sum_i x_i(s)\le 0,\ \forall s$. In other words, we abstract from efficiency losses due to potential external subsidies.

3. Incentive Compatible Mechanisms

By the revelation principle it is enough to restrict attention to direct, incentive compatible revelation mechanisms. A DRM is (Bayes-Nash) *incentive compatible* if:

$$\forall i, \forall s^i, t^i \in S^i, \; V_i(s^i) = U_i(s^i, s^i) \geq U_i(t^i, s^i).$$

For the characterization of incentive compatible mechanisms we need several definitions. A vector field $\Psi : S^i \to \Re^{K \times N}$ is *monotone* if:

$$\forall s^i, t^i \in S^i, \; (s^i - t^i) \cdot (\Psi(s^i) - \Psi(t^i)) \geq 0$$

A vector field $\Psi : S^i \to \Re^{K \times N}$ is *conservative* if there exists a differentiable function $\rho : S^i \to \Re$ such that $\Psi = \nabla \rho$ (where ∇ denotes the gradient). A function ρ with the above property is called a *potential function* for Ψ. For a convex (and hence simply-connected) domain S^i the existence of a potential function for Ψ is equivalent to the following condition: For any $s^i, t^i \in S^i$ the integral of Ψ from s^i to t^i is independent of the path of integration[19].

Theorem 3.1. *Let (p, x) be a DRM, and let $\{q^i\}_{i=1}^n$ be the associated conditional probability assignments. For each agent i, let $Q^i(s^i) : \Re^{K \times N} \to \Re^{K \times N}$ be the vector field, where, for each alternative k, the ki^{th} coordinate is given by $a^i_{ki} q^i_k(s^i)$ and the kj^{th} coordinate, $j \neq i$, is zero. Then (p, x) is incentive compatible if and only if the following conditions hold:*

1. $\forall i$, the vector field Q^i is monotone and conservative.

2. $\forall i, \; \forall s^i, t^i \in S^i, V_i(s^i) = V_i(t^i) + \int_{t^i}^{s^i} Q^i(\tau^i) d\tau^i$ [20],[21]

Proof. See Appendix.

[19] Path integrals and the equivalence result are discussed in any multivariate caculus textbook. For a particularly clear and simple exposition, see Chapter V in Lang (1973).

[20] The integral can be defined on any path connecting t^i and s^i since Q^i is conservative. For example, we can choose a straight line: $\int_{t^i}^{s^i} Q^i(\tau^i) d\tau^i = \int_0^1 Q^i((1-\alpha)t^i + \alpha s^i)) \cdot (s^i - t^i).d\alpha$

[21] The Theorem implies a "Revenue Equivalence" result. The conditional expected payment of agent i in any incentive compatible mechanism is solely a function of the associated expected probability assignment, and of the expected utility of an arbitrary type. Any two incentive compatible mechanisms with the same probability assignment yield, up to a constant, the same conditional expected payments. The characterization of incentive compatibility is *not valid* if signals are not independent.

4. Impossibility Results

In an incentive compatible mechanism (p, x) we have $V_i(s^i) = \max_{t^i} U_i(t^i, s^i)$. The function V_i is convex (see the proof of Theorem 3.1), and hence twice differentiable almost everywhere. Assuming that V_i is differentiable at s^i we obtain by the Envelope Theorem that:

$$\forall k,\ \frac{\partial V_i}{\partial s^i_{ki}}(s^i) = a^i_{ki} q^i_k(s^i) \tag{4.1}$$

$$\forall k, \forall j \neq i,\ \frac{\partial V_i}{\partial s^i_{kj}}(s^i) = 0 \tag{4.2}$$

Assuming that V_i is twice continuously differentiable at s^i, we obtain by Schwarz's Theorem that the cross-derivatives at s^i must be equal. This implies :

$$\forall k, k',\ a^i_{ki} \frac{\partial q^i_k(s^i)}{\partial s^i_{k'i}} = \frac{\partial^2 V_i}{\partial s^i_{k'i} \partial s^i_{ki}}(s^i) = \frac{\partial^2 V_i}{\partial s^i_{ki} \partial s^i_{k'i}}(s^i) = a^i_{k'i} \frac{\partial q^i_{k'}(s^i)}{\partial s^i_{ki}}\ ; \tag{4.3}$$

$$\forall k, k', \forall j \neq i,\ a^i_{ki} \frac{\partial q^i_k(s^i)}{\partial s^i_{k'j}} = \frac{\partial^2 V_i}{\partial s^i_{k'j} \partial s^i_{ki}}(s^i) = \frac{\partial^2 V_i}{\partial s^i_{ki} \partial s^i_{k'j}}(s^i) = 0\ . \tag{4.4}$$

The mathematical idea behind the following impossibility results is to check whether efficient mechanisms yield conditional expected probability assignment functions that satisfy conditions 4.3 and 4.4.

Note that an efficient SCR is piece-wise constant. Hence, for efficient mechanisms we obtain that the associated functions $\{q^i)\}_{i=1}^n$ are everywhere continuously differentiable by recalling that the (convex) type spaces have a non-empty interior and a piece-wise smooth boundary, and that for all i and all $s^i \in S^i$, $f_i(s^i) > 0$.

We first focus on the simpler condition 4.4.

Theorem 4.1. *Let (p, x) be an efficient DRM. Assume that the following are satisfied: 1) There exist i, j, k such that $i \neq j$, $a^i_{ki} \neq 0$ and $a^i_{kj} \neq 0$. 2) There exist[22] open neighborhoods $\Theta^i \in S^i$, $\Theta^{-i}_1, \Theta^{-i}_2 \in S^{-i}$ such that $p_k(s^i, s^{-i}) = 1$ for all $(s^i, s^{-i}) \in \Theta^i \times \Theta^{-i}_1$ and $p_k(s) = 0$ for all $(s^i, s^{-i}) \in \Theta^i \times \Theta^{-i}_2$. Then (p, x) cannot be incentive compatible.*

[22] This requirement ensures that the alternative k is not always welfare dominated or welfare dominant. Since an efficient SCR is uniquely defined almost everywhere, the choice of a specific efficient SCR p is immaterial.

Proof. Let $\{q^l\}_{l=1}^N$ be the conditional expected probability assignments associated with (p, x). Let $s^i \in \Theta^i$. By efficiency, we obtain

$$q_k^i(s^i) = \int_{\Delta_k(s^i)} f_{-i}(s^{-i}) ds^{-i} \tag{4.5}$$

where

$$\Delta_k(s^i) = \{s^{-i} \mid \sum_{g=1}^N \sum_{l=1}^N a_{kg}^l s_{kg}^l = \max_{k'} \sum_{g=1}^N \sum_{l=1}^N a_{k'g}^l s_{k'g}^l\} \tag{4.6}$$

By definition and by condition 2 in the statement of the Theorem, we obtain that $\Delta_k(s^i)$ is a non-empty closed set, strictly included in S^{-i}. Because $a_{kj}^i \neq 0$, we obtain that the areay $\Delta_k(s^i)$ changes when s_{kj}^i varies. By definition, we obtain that $\frac{\partial q_k^i(s^i)}{\partial s_{kj}^i} \neq 0$ for all $s^i \in \Theta^i$.

Suppose now that (p, x) is incentive compatible. Since the expected equilibrium utility V_i is twice differentiable almost everywhere, there exists $t^i \in \Theta^i$ where V_i satisfies this requirement. Since $a_{ki}^i \neq 0$, equation 4.4 yields $\frac{\partial q_k^i(t^i)}{\partial s_{kj}^i} = 0$. This is a contradiction. ■

So far we have assumed that s_{kj}^i, agent i' s signal affecting the utility of agent j in alternative k, is one-dimensional. We next look at an example where this requirement is not satisfied. An impossibility result in such situations has been observed by Maskin (1992).

Example 4.2. *There are two agents $i = 1, 2$ and two alternatives $k = A, B$. Signals are two-dimensional, $s^i = (s_1^i, s_2^i)$, $i = 1, 2$. Valuations are given by*[23]*:*

$$\begin{aligned} V_A^1(s^1, s^2) &= s_1^1 + a(s_2^1 + s_2^2);\ V_B^1(s^1, s^2) = 0 \\ V_B^2(s^1, s^2) &= s_1^2 + a(s_2^1 + s_2^2);\ V_A^2(s^1, s^2) = 0 \end{aligned}$$

Consider the change of variables:

$$t^i = (t_1^i, t_2^i) = (s_1^i + a s_2^i, s_2^i)$$

In the t^i type space we obtain:

$$\begin{aligned} V_A^1(t^1, t^2) &= t_1^1 + a t_2^2;\ V_B^1(t^1, t^2) = 0 \\ V_B^2(t^1, t^2) &= t_1^2 + a t_2^1;\ V_A^2(t^1, t^2) = 0. \end{aligned}$$

[23]Imagine an auction for an indivisible good where the components s_1^i , $i = 1, 2$, are the private parts of the signals while the components s_2^i are common parts.

Hence, agent 1 has a signal t_2^1 which does not affect her utility (in particular it does not affect her utility in alternative A), but affects the utility of agent 2 in alternative B. In incentive compatible mechanisms we obtain by condition 4.4 that agent 1's interim expected probability for alternative A cannot depend on t_2^1 , while t_2^1 is clearly matters for the determination of ex-post efficiency. Hence, incentive-compatible, efficient mechanisms do not exist.

The example[24] can be extended to the case where $V_A^1(s^1, s^2) = s_1^1 + as_2^1 + bs_2^2$ and $V_B^2(s^1, s^2) = s_1^2 + as_2^2 + bs_2^1$. Even when the dependence of an agent's valuation on the signal of another agent is very small (i.e., b is very close to zero), efficiency cannot be attained. ■

In Theorem 4.1 and Example 4.2, the intuition behind the impossibility results is that a (one-dimensional) payment associated to each alternative is not sufficient to elicit multi-dimensional information whose various components are all important for efficiency considerations.

The natural next step is to inquire the existence of efficient, incentive compatible mechanisms in a framework where, intuitively, K payments - one for each possible alternative- should suffice to elicit the entire information: consider then $K-$dimensional type-spaces[25], where s_k^i is agent i's *one-dimensional* piece of information affecting (possibly in different ways) the utility of *all* agents in alternative k. In this setup, the remaining question is whether the conditional expected probability assignment functions generated by efficient mechanisms satisfy condition 4.3[26].

Recall that we have derived conditions 4.3 and 4.4 for signals of dimension $K \times N$. For each K-dimensional signal $\widetilde{t^i}$, define $\widetilde{V}_i(\widetilde{t^i}) \equiv V_i(t^i)$ and $\widetilde{q}_k^i(\widetilde{t^i}) \equiv q_k^i(t^i)$, where t^i is the $K \times N-$dimensional signal such that $t_{kj}^i = \widetilde{t}_k^i$ for all k, j. Assuming that V_i is differentiable at t^i, we obtain by conditions 4.3 and 4.4 that:

$$\forall k, \frac{\partial \widetilde{V}_i}{\partial \widetilde{t}_k^i}(\widetilde{t^i}) = \sum_{j=1}^{N} \frac{\partial V_i}{\partial t_{kj}^i}(t^i) = a_{ki}^i q_k^i(t^i) = a_{ki}^i \widetilde{q}_k^i(\widetilde{t^i}).$$

The equality of cross-derivatives implies that :

[24] Compte and Jehiel (1998) look at related examples in order to study the value of competition in standard auctions.

[25] We assume below that the repective spaces $\{S^j\}_{j=1}^N$ and densities $\{f_j\}_{j=1}^N$ satisfy all conditions imposed in Section 2 (relative to $\Re^K$).

[26] An affirmative answer would imply an affirmative answer also for frameworks where $\forall i, j, i \neq j$, $\forall k$, s_{kj}^i is a linear function of the signals $s_{k'i}^i$, $k' = 1, ...K$, and where each $s_{k'i}^i$ is one-dimensional. The situation treated in the text corresponds to the case where $\forall i, j, i \neq j, \forall k, s_{kj}^i = s_{ki}^i$.

$$a^i_{ki}\frac{\partial\widetilde{q}^i_k(\widetilde{t}^i)}{\partial\widetilde{t}^i_{k'}}=a^i_{k'i}\frac{\partial\widetilde{q}^i_{k'}(\widetilde{t}^i)}{\partial\widetilde{t}^i_k} \tag{4.7}$$

In order to simplify notation, we drop from now on the "tilde" and denote by $s^i=(s^i_1,...s^i_K)$ a $K-$dimensional signal of agent i, yielding expected probability assignments $\{q^i_k\}^K_{k=1}$, and equilibrium expected utility V_i.

Theorem 4.3. *Assume that (p,x) is an efficient DRM ant that (p,x) is incentive compatible for agent i. Let k,k' be any pair of alternatives such that: 1) $a^i_{k'i}\neq 0$; 2) There exists a type t^i such that $q^i_k(s^i)\neq 0$, $q^i_{k'}(s^i)\neq 0$ for all s^i in a neighborhood of t^i [27]. Then it must be the case that*

$$\frac{a^i_{ki}}{a^i_{k'i}}=\frac{\sum_{j=1}^N a^i_{kj}}{\sum_{j=1}^N a^i_{k'j}}. \tag{4.8}$$

Proof. See Appendix[28].

Condition 4.8 is a congruence requirement between private and social rates of information substitution (see the Examples below for more intuition about these terms). The implied algebraic relations among parameters cannot be generically satisfied[29]. Note that condition 4.8 is trivially satisfied in two interesting cases: the private values case where $\forall i,j,i\neq j,\forall k$, $a^i_{kj}=0$, and the case where $\forall i,j,k,$ $a^i_{kj}=a^i_{ki}$. In the next two Examples we provide the intuition for Theorem 4.3. The first example (which was sketched in the Introduction) is very simple since only one agent receives a private signal[30].

Example 4.4. *: There are two agents $i=1,2$ and three alternatives $k=A,B,C$. Suppose that only agent 1 receives a signal, denoted by $s=(s_A,s_B,s_C)$[31].*

A mechanism can be defined here by a triple of transfers to agent 1, $x=(x_A,x_B,x_C)$. Given x, agent 1 chooses an alternative $k\in argmax_{k'}(V^1_{k'}(s)+x_{k'})$. In contrast, an efficient rule chooses an alternative $k\in argmax_{k'}(V^1_{k'}(s)+V^2_{k'}(s))$.

[27] Note that $q^i_k(t^i)\neq 0$, $q^i_{k'}(t^i)\neq 0$ imply that $\sum_{j=1}^N a^i_{k'j}\neq 0$ and that $\sum_{j=1}^N a^i_{kj}\neq 0$.

[28] The Theorem has also converse: If condition 4.8 is satisfied, and if an efficient SCR p yields, for each agent i a monotone vector field Q^i then there exist payment schedules x such that (p,x) is incentive compatible.

[29] i.e., the set of parameters satisfying the condition is closed and has Lebesgue-measure zero.

[30] This implies that the conditional expected probability assignment function is deterministic and coincides with the social choice rule.

[31] For ease of notation we omit here the superscript 1.

Consider an incentive compatible, efficient mechanism[32], i.e., a mechanism where x is such that the two choice rules coincide. Keeping fixed the signal affecting payoffs in alternative C, consider two types $s^* = (s_A^*, s_B^*, s_C^*)$ and $s^{**} = (s_A^{**}, s_B^{**}, s_C^*)$ such that:

$$\begin{aligned} \sum_{i=1}^{2} V_A^i(s^*) &= \sum_{i=1}^{2} V_B^i(s^*) > \sum_{i=1}^{2} V_C^i(s^*) \\ \sum_{i=1}^{2} V_A^i(s^{**}) &= \sum_{i=1}^{2} V_B^i(s^{**}) > \sum_{i=1}^{2} V_C^i(s^{**}) \end{aligned} \tag{4.9}$$

Together with the continuity of the valuation functions, efficiency and incentive compatibility imply that[33]:

$$\begin{aligned} V_A^1(s^*) + x_A &= V_B^1(s^*) + x_B \\ V_A^1(s^{**}) + x_A &= V_B^1(s^{**}) + x_B \end{aligned} \tag{4.10}$$

The above equalities yield

$$V_A^1(s^{**}) - V_B^1(s^{**}) = V_A^1(s^*) - V_B^1(s^*) \tag{4.11}$$

Insert Figure 1 around here.

Equations 4.9 and 4.11 show that, as we move in the (s_A, s_B) sub-space from s^* to s^{**} along the curve defined by $\sum_{i=1}^{2} V_A^i(s) = \sum_{i=1}^{2} V_B^i(s)$ we must also keep the difference $V_A^1(s) - V_B^1(s)$ constant (and equal to $V_A^1(s^*) - V_B^1(s^*)$). But it is obvious that the locus in the (s_A, s_B) sub-space where this difference is constant need not coincide with the locus defined by the society's indifference between alternatives k and k' (i.e., the curve from s^* to s**) In particular, for the two curves to coincide around s^* it is necessary that

$$\frac{\frac{\partial V_A^1}{\partial s_A}(s^*)}{\frac{\partial V_B^1}{\partial s_B}(s^*)} = \frac{\frac{\partial}{\partial s_A}(\sum_{i=1}^{2} V_A^i(s^*))}{\frac{\partial}{\partial s_B}(\sum_{i=1}^{2} V_B^i(s^*))} \tag{4.12}$$

■

The next example shows how the above intuition generalizes to the more complex case where several agents are privately informed.

[32] Note that setting $x_{k'} = V_{k'}^2(s)$ is not incentive compatible.

[33] To see this, consider types $s' = s^* + \varepsilon e_A$, and $s'' = s^* - \varepsilon e_A$ where $\varepsilon > 0$ is small and where $e_A = (1, 0, 0)$. At s' efficiency requires that alternative A is chosen, so that, by incentive compatibility, $V_B^1(s') + x_B \leq V_A^1(s') + x_A$. At s'' efficiency requires that alternative B is chosen so that $V_A^1(s') + x_A \leq V_B^1(s') + x_B$. The assertion follows by letting ε go to zero in the two inequalities above. The argument for s^{**} is analogous.

Example 4.5. *There are two agents $i = 1, 2$ and two alternatives $k = A, B$. Signals are two dimensional, $s^i = (s^i_A, s^i_B)$, $i = 1, 2$. For $i = 1, 2$ let $-i$ denote the agent other than i. Valuations are given by:*

$$V^i_k(s^i, s^{-i}) = a^i_{ki} s^i_k + a^{-i}_{ki} s^{-i}_k, \ i = 1, 2, ; k = A, B$$

Assume that an efficient, incentive compatible DRM exists, and denote it by (p, x). Let q^i_k denote i's interim expected probability that the mechanism chooses alternative k.

We will first show that, as a consequence of equation 4.7, incentive compatible mechanisms must yield the same vector of conditional expected probability assignments for types of agent $i, i = 1, 2$, lying on lines with slope $\frac{a^i_{Ai}}{a^i_{Bi}}$. We next show that efficient mechanism yield the same vector of conditional expected probability assignments for types lying on lines with slope $\frac{a^i_{Ai}+a^i_{A-i}}{a^i_{Bi}+a^i_{B-i}}$. Hence, incentive compatibility can be consistent with efficiency only if these two slopes are equal.

We know that

$$\forall i, \forall s^i, q^i_A(s^i) + q^i_B(s^i) = 1. \tag{4.13}$$

Consider agent 1. Equation 4.7 yields

$$a^1_{A1} \frac{\partial q^1_A(s^1)}{\partial s^1_B} = a^1_{B1} \frac{\partial q^1_B(s^1)}{\partial s^1_A}. \tag{4.14}$$

By taking the derivative with respect to s^1_A in identity 4.13, we get

$$\frac{\partial q^1_B(s^1)}{\partial s^1_A} = -\frac{\partial q^1_A(s^1)}{\partial s^1_A}$$

By equation 4.14, we get:

$$a^1_{A1} \frac{\partial q^1_A(s^1)}{\partial s^1_B} + a^1_{B1} \frac{\partial q^1_A(s^1)}{\partial s^1_A} = 0. \tag{4.15}$$

Fix now $t^1 = (t^1_A, t^1_B)$ such that the assumptions in the Theorem are satisfied, and consider a line in the type space of agent 1 having the form $s^1 = s^1(r) = (t^1_A + r, t^1_B + \frac{a^1_{A1}}{a^1_{B1}} r)$. By equation 4.15 we have:

$$\frac{dq^1_A(t^1_A + r, t^1_B + \frac{a^1_{A1}}{a^1_{B1}} r)}{dr} = \frac{\partial q^1_A(s^1)}{\partial s^1_A} + \frac{a^1_{A1}}{a^1_{B1}} \frac{\partial q^1_A(s^1)}{\partial s^1_B} = 0. \tag{4.16}$$

Hence, in incentive compatible mechanisms the function q_A^1 is constant along lines having the form $(t_A^1 + r, t_B^1 + \frac{a_{A1}^1}{a_{B1}^1} r)$ (by equation 4.13 the same is of course true for q_B^1).

We now turn to the consequences of efficiency. Alternative A is chosen by an efficient DRM at reports (s^1, s^2) iff

$$\sum_{i=1}^{2}\sum_{j=1}^{2} a_{Ai}^j s_A^j \geq \sum_{i=1}^{2}\sum_{j=1}^{2} a_{Bi}^j s_B^j$$

This is equivalent to:

$$(a_{A1}^1 + a_{A2}^1)s_A^1 - (a_{B1}^1 + a_{B2}^1)s_B^1 \geq (a_{B1}^2 + a_{B2}^2)s_B^2 - (a_{A1}^2 + a_{A2}^2)s_A^2 \tag{4.17}$$

Efficiency implies that:

$$q_A^1(s^1) = \int_{\Delta(s^1)} f_2(s^2)ds^2$$

where $\Delta(s^1) = \{s^2 \text{ such that condition 4.17 is satisfied}\}$.

Consider a line in agent 1's type space having the form $s^1 = s^1(r) = (t_A^1 + r, t_B^1 + \frac{a_{A1}^1 + a_{A2}^1}{a_{B1}^1 + a_{B2}^1} r)$. For any two signals θ^1, τ^1, on this line, we have $\Delta(\theta^1) = \Delta(\tau^1)$. Therefore $q_A^1(s^1(r))$ does not depend on r. Taking the derivative with respect to r, and multiplying by $(a_{B1}^1 + a_{B2}^1) \neq 0$, this yields :

$$(a_{B1}^1 + a_{B2}^1)\frac{\partial q_A^1(s^1)}{\partial s_A^1} + (a_{A1}^1 + a_{A2}^1)\frac{\partial q_A^1(s^1)}{\partial s_B^1} = 0 \tag{4.18}$$

Equations 4.16 and 4.18 yield together:

$$\frac{a_{A1}^1}{a_{B1}^1} = \frac{a_{A1}^1 + a_{A2}^1}{a_{B1}^1 + a_{B2}^1}. \tag{4.19}$$

The same logic yields an analogous condition for $i = 2$. ■

Several remarks regarding Theorem 4.3 follow.

Remark 1: The impossibility of incentive compatible, efficient mechanisms is a general phenomenon, and it is not confined to our linear setting. Indeed, recall that the *only* crucial property of valuation functions used in Example 4.4 is continuity[34]. Linearity (which implies that marginal valuations are constant) was used to get the simple global formula 4.8. Without linearity, one obtains congruence conditions that locally equate average private and social rates of substitution

[34]Differentiability was used there only to quantify the condition relating the two loci.

(where the average is taken over the area in which two alternatives are equally efficient).

Remark 2: Theorem 4.3 applies as stated to the case where the dimensionality of signal spaces coincides with the number of alternatives $K \geq 2$. But Example 4.4 makes it clear that the same type of result holds whenever, for at least one agent, the dimension of the signal space is greater than one. In particular, we obtain impossibility results for auctions of several heterogenous objects, where the dimension of signal spaces is usually greater than 1, but smaller than the number of alternatives (which equals the number of possible partitions of the set of auctioned objects).

Remark 3: Dasgupta and Maskin (1999) suggest that the difficulties appearing in multi-dimensional models can be circumvented by performing a reduction to a one-dimensional model for which efficient, incentive compatible mechanisms can be constructed under less restrictive assumptions (see next Section). The efficient mechanism for the reduced model is then constrained efficient (i.e., second-best) for the original multi-dimensional model. Dimension reductions are indeed readily available in two cases: 1) If a variable $\hat{s}^i_{kj}$, $j \neq i$, moves independently of $(\hat{s}^i_{k'i})_{k'}$, condition 4.4 shows that incentive compatible mechanisms cannot condition on it. Hence, such variables can be eliminated without affecting the maximum efficiency performance obtainable by incentive compatible mechanisms. 2) In Example 4.5 there were only two social alternatives A and B, and we have shown that incentive compatible mechanisms have the property that the conditional expected probability assignment vector field q^i is constant along lines in the type space S^i with the slope $\frac{a^i_{Ai}}{a^i_{Bi}}$. A parameterization of this family of parallel lines yields a one-dimensional type space[35] for which an efficient, incentive compatible mechanism can be constructed[36]. This mechanism is necessarily second-best for the original model where first-best efficiency was impossible.

We wish to stress here that this insight does not hold anymore if at least one agent perceives more than two payoff relevant alternatives[37]. Recall Example 4.4 where we had to keep one coordinate constant while operating on the other two. For each pair of alternatives we obtain a one-dimensional family of lines as above, but, since there are at least three different pairs, it is not a-priori clear how to

[35] Instead of reporting a type (s^i_A, s^i_B), agent i reports, say, the intercept of a line with slope $\frac{a^i_{Ai}}{a^i_{Bi}}$.

[36] Similar reductions can be performed in models where there are possibly more than two alternatives, but each agent perceives only two outcomes as payoff relevant. For example, in an auction for one unit of an indivisible good without allocative externalities, an agent cares only about "winning" or "losing".

[37] This is the general case in auctions of several heterogenous objects or in auctions of one object with allocative externalities.

consistently combine the pair-wise information in order to reduce the dimension of the signal spaces.

5. One-Dimensional Signals

We now assume that agents have one-dimensional signals. Agent i's payoff in alternative k is given by

$$V_k^i(s^i, s^{-i}) = \sum_{j=1}^{N} a_{ki}^j s^j$$

where $s^j \in [\underline{s}^j, \overline{s}^j]$ denotes the one-dimensional signal of agent j. Signals need not be independently distributed, and the result below does not depend on the signals' distribution functions.

In order to avoid a tedious case differentiation, we assume that, for each agent i, there are no alternatives k , k', $k' \neq k$, such that $a_{ki}^i = a_{k'i}^i$. Our result will rely on the following assumption:

$$\forall i, \forall k, k',\ a_{ki}^i > a_{k'i}^i \Rightarrow \sum_{j=1}^{N} a_{kj}^i > \sum_{j=1}^{N} a_{k'j}^i \tag{5.1}$$

Condition 5.1 (referred below as the weak congruence condition) requires that the sequence of alternatives obtained by ordering (in terms of magnitude) the impacts of i's signal on i's payoff is the same as the sequence obtained by ordering the impacts of i's signal on social welfare. By rewriting this condition as

$$\forall i, \forall k, k',\ \frac{a_{ki}^i}{a_{k'i}^i} > 1 \Rightarrow \frac{\sum_{j=1}^{N} a_{kj}^i}{\sum_{j=1}^{N} a_{k'j}^i} > 1$$

we note a certain (formal) analogy with condition 4.8, but also the gained slack in the one-dimensional framework. This slack (i.e., required inequalities instead of equalities) allows the condition to be satisfied for an open set of parameters' values.

Theorem 5.1. *Assume that the weak congruence condition 5.1 is satisfied. Then there exists an efficient, Bayesian incentive compatible mechanism. Moreover, the associated transfers do not depend on the distribution of signals*[38].

[38] Technically, this result is not a special case of Dasgupta and Maskin (1999) because they study multi-object auctions (without allocative externalities), while we study a general social choice problem. Dasgupta and Maskin's mechanism is more complex since it also elicits reports about valuation functions, which, in their model, are not known to the designer. This allows them to construct a mechanism whose rules do not depend on valuation functions. Building on the insight in Dasgupta and Maskin (1997), the condition allowing implementation (condition 5.1) was first identified in an earlier version of this paper.

Proof. See Appendix.

We note here that the logic of the efficient revelation mechanism constructed in the proof of the Theorem works in *any* quasi-linear framework under appropriate conditions on marginal valuations (which, as shown by Dasgupta and Maskin (1999), are generally more complex than condition 5.1)

6. Conclusions

We have shown that efficient, Bayesian incentive compatible mechanisms can exist only if a congruence condition relating private and social rates of information substitution is satisfied. If signals are multi-dimensional, the congruence condition is determined by an integrability constraint, and it can be satisfied only in non-generic cases. If signals are one-dimensional, the congruence condition reduces to a monotonicity constraint and it can be generically satisfied.

The impossibility results in the multi-dimensional case suggest a quest for the second-best (or constrained efficient) mechanisms. It is straightforward to construct second-best mechanisms if the inefficiency is purely due to the fact that some informational variables must have a zero marginal effect on the expected probability assignment in incentive compatible mechanisms. It is then possible to reduce the dimensionality of the model (without loss of efficiency) by eliminating such variables. If, after performing these reductions, it is still the case that the payoff-relevant information depends in a non-trivial way on the chosen alternative (as it is the case, say, in a general multi-object auction), we are left in a framework covered by Theorem 4.3 and further dimension reductions become endogenous. The construction of a second-best mechanism is then equivalent to the difficult problem of finding a monotone and conservative vector field that maximizes the (expected) welfare functional[39]. This will be the subject of future work.

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Appendix

Proof of Theorem 3.1

a) Assume first that a DRM (p,x) satisfies the conditions in the Theorem. Choose any agent i. We must show that $\forall s^i, t^i$, $U_i(s^i,s^i) - U_i(t^i,s^i) \geq 0$. We obtain the following chain of equalities:

$$\begin{aligned} U_i(s^i,s^i) - U_i(t^i,s^i) &= V_i(s^i) - V_i(t^i) - Q^i(t^i)\cdot(s^i - t^i) \\ &= \int_{t^i}^{s^i} Q^i(\tau^i)\cdot d\tau^i - Q^i(t^i)\cdot(s^i - t^i) \\ &= \int_0^1 [Q^i((1-\alpha)t^i + \alpha s^i)) - Q^i(t^i)]\cdot(s^i - t^i)d\alpha \end{aligned}$$

The first equality follows by equation 2.1 and by the definition of V_i. The second equality follows by assumption. The last equality follows by choosing to perform the integration on the straight line connecting t^i and s^i.

The condition $\forall s^i, t^i$, $U_i(s^i,s^i) - U_i(t^i,s^i) \geq 0$ is therefore equivalent to the condition

$$\forall s^i, t^i,\ \int_0^1 [Q^i((1-\alpha)t^i + \alpha s^i)) - Q^i(t^i)]\cdot(s^i - t^i)d\alpha \geq 0.$$

It is enough to show that the integrand is non-negative for any α, $0 \leq \alpha \leq 1$. For $\alpha = 0$, the claim is obvious. Assume that $\alpha > 0$. Noting that $s^i - t^i = \frac{1}{\alpha}((1-\alpha)t^i + \alpha s^i - t^i)$, we obtain:

$$\begin{aligned} [Q^i((1-\alpha)t^i + \alpha s^i)) - Q^i(t^i)]\cdot(s^i - t^i) &= \\ \frac{1}{\alpha}[Q^i((1-\alpha)t^i + \alpha s^i)) - Q^i(t^i)]\cdot((1-\alpha)t^i + \alpha s^i - t^i) &\geq 0 \end{aligned}$$

The last inequality follows from the monotonicity of Q^i.

b) For the converse, assume that the DRM (p,x) is incentive compatible. This implies that $V_i(s^i) = U_i(s^i,s^i) = \max_{t^i} U_i(t^i,s^i)$. The function V_i is the supremum of a collection of affine functions and it must be convex. Convex functions are twice differentiable almost everywhere[40]. The convexity of V_i implies the monotonicity of the sub-differential map $\partial V_i(s^i)$. At all points where V_i is differentiable (i.e., a.e.) the sub-differential ∂V_i consists of a unique point, the gradient ∇V_i. Hence,

[40] This and all following properties of convex functions are listed in the classical text of Rockafellar, 1997.

the function ∇V_i is a.e. well-defined, monotone and differentiable. Assuming that V_i is differentiable at s^i we obtain by expression 3.1 and by the Envelope Theorem that:

$$\forall k,\ \frac{\partial V_i}{\partial s^i_{ki}}(s^i) = \frac{\partial U_i}{\partial s^i_{ki}}(t^i, s^i)\ |_{t^i=s^i} = a^i_{ki} q^i_k(s^i) \tag{7.1}$$

$$\forall k, \forall j \neq i,\ \frac{\partial V_i}{\partial s^i_{kj}}(s^i) = \frac{\partial U_i}{\partial s^i_{kj}}(t^i, s^i)\ |_{t^i=s^i} = 0 \tag{7.2}$$

Hence, we obtain $\nabla V_i(s^i) = Q^i(s^i)$ whenever the gradient is well-defined (a.e.). The integral representation is immediately obtained from the Fundamental Theorem of Calculus if V_i is everywhere differentiable. Otherwise, the result follows by noting that a convex function is (up to a constant) uniquely determined by its sub-differential (see Rockafellar 1997, Theorem 24.9), and that it can be recovered (up to a constant) by integrating any measurable selection from its sub-differential map (see Krishna and Maenner, 1999). ■

Proof of Theorem 4.3: Let (p, x) be an efficient, incentive compatible DRM, and let $(q^i_k)_{k=1}^K$ be the associated vector field of interim expected probabilities for agent i. Consider a type t^i and two alternatives k and k' such that $q^i_k(s^i) \neq 0$ and $q^i_{k'}(s^i) \neq 0$ for all s^i in a neighborhood of t^i. We consider below signals s^i in that neighborhood.

Since (p, x) is incentive compatible, the associated indirect utility function V_i is twice-differentiable a.e. Since (p, x) is efficient, the associated functions $(q^i_k)_{k=1}^K$ are continuously differentiable everywhere.

By equation 4.7 we obtain for almost all s^i:

$$\forall k, k',\ a^i_{ki} \frac{\partial q^i_k(s^i)}{\partial s^i_{k'}} = a^i_{k'i} \frac{\partial q^i_{k'}(s^i)}{\partial s^i_k} \tag{7.3}$$

Since p is efficient, we obtain:

$$q^i_k(s^i) = \int_{\Delta_k(s^i)} f_{-i}(s^{-i}) ds^{-i} \tag{7.4}$$

where

$$\Delta_k(s^i) = \{s^{-i} \mid \sum_{j=1}^N \sum_{g=1}^N a^j_{kg} s^j_k = \max_{k^*} \sum_{j=1}^N \sum_{g=1}^N a^j_{k^*g} s^j_{k^*}\} \tag{7.5}$$

An analogous expression holds for $q^i_{k'}(s^i)$. For a fixed s^i define now the locus in S^{-i} where alternatives k and k' achieve the same highest utility:

$$\Omega_{k,k'}(s^i) = \{s^{-i} \mid \sum_{j=1}^N \sum_{g=1}^N a^j_{kg} s^j_k = \sum_{j=1}^N \sum_{g=1}^N a^j_{k'g} s^j_{k'} = \max_{k^*} \sum_{j=1}^N \sum_{g=1}^N a^j_{k^*g} s^j_{k^*}\} \tag{7.6}$$

We now want to calculate the derivative $\frac{\partial q_k^i(s^i)}{\partial s^i_{k'}}$ using expressions 7.4, 7.5 and 7.6: a marginal variation of $s^i_{k'}$ affects $q^i_k(s^i)$ only by marginally shifting the boundary $\Omega_{k,k'}(s^i) \subset \Delta_k(s^i)$ where k and k' are equally efficient. Hence $\frac{\partial q_k^i(s^i)}{\partial s^i_{k'}}$ is equal to an integral over the boundary multiplied by the marginal shift, which is proportional to $(\sum_{g=1}^N a^i_{k'g})$, the constant coefficient of $s^i_{k'}$ in the equation defining $\Omega_{k,k'}(s^i)$[41].

To make this observation precise, define:

$$\begin{aligned} z_0 &= \sum_{j\neq i}\sum_{g=1}^{N} a^j_{kg}s^j_k - \sum_{j\neq i}\sum_{g=1}^{N} a^j_{k'g}s^j_{k'} \\ c &= -(\sum_{g=1}^{N} a^i_{kg})s^i_k + (\sum_{g=1}^{N} a^i_{k'g})s^i_{k'} \end{aligned} \tag{7.7}$$

Note that:

$$\begin{aligned} \Delta_k(s^i) &= \{s^{-i} \mid z_0 \geq c \wedge \sum_{j=1}^{N}\sum_{g=1}^{N} a^j_{kg}s^j_k \geq \sum_{j=1}^{N}\sum_{g=1}^{N} a^j_{k''g}s^j_{k''} \text{ for } k'' \neq k'\} \\ \Omega_{k,k'}(s^i) &= \Delta_k(s^i) \cap \{s^{-i} \mid z_0 = c\} \end{aligned} \tag{7.8}$$

Consider an affine, bijective change of variables in the space S^{-i}, where z_0 is one of the new variables, and where $\mathbf{z}$ denotes the set of the remaining variables (with differential element $d\mathbf{z}$)[42]. Such a bijective change of variables exists because z_0 is not identically equal to zero (since $q^i_k(t^i) \neq 0$ and $q^i_{k'}(t^i) \neq 0$). To fix ideas, suppose that the coefficients are such that for all alternatives k'' there exists an agent $j(k'') \neq i$, such that $a^{j(k'')}_{k''j(k'')} \neq 0$. Consider the mapping $G : \{s^j_{k''}\}_{j\neq i,k''} \rightarrow \{z^j_{k''}\}_{j\neq i,k''}$ where: 1) For $k'' \neq k$, $j = j(k'')$, $z^{j(k'')}_{k''} = \sum_{j\neq i}\sum_{g=1}^N a^j_{kg}s^j_k - \sum_{j\neq i}\sum_{g=1}^N a^j_{k''g}s^j_{k''}$ (observe that $z^{j(k')}_{k'} = z_0$); 2) For all (j,k'') such that $k'' = k$ or $j \neq j(k'')$, $z^j_{k''} = s^j_{k''}$.

[41] This is the generalization to several dimensions of a standard one-dimensional result: define $H(y) = \int_{d(y)}^{c(y)} g(x)dx$ where g is continuous and where c, d are continuously differentiable. By the Fundamental Theorem of Calculus (FTC), $H'(y) = g(c(y))c'(y) - g(d(y))d'(y)$. A general proof of the multi-dimensional analog uses a multi-dimensional version of the FTC, called the *Divergence Theorem.* (see Lang, 1973).

[42] If $\mathbf{z} = (z_1, ..., z_m)$, then $d\mathbf{z} = dz_1 dz_2 \cdots dz_m$. The purpose of the change of variables is to concentrate the entire dependence on $s^i_k$and $s^i_{k'}$ in a single dimension, z_0. This allows us to use the one-dimensional argument of the previous footnote in the derivation of expression 7.9 below. The choice of variables $\mathbf{z}$ is entirely arbitrary as long as they are well defined (we need to take care about possible zero coefficients).

Let G^{-1} be the inverse of G, and let $\mid \Delta_{G^{-1}} \mid \neq 0$ denote the absolute value of the Jacobian determinant associated with G^{-1}. Note that $\mid \Delta_{G^{-1}} \mid$ is a constant (i.e., it does not depend on $(z_0, \mathbf{z})$) because G is affine. We obtain now :

$$\begin{aligned}
\frac{\partial q_k^i(s^i)}{\partial s_{k'}^i} &= \frac{\partial}{\partial s_{k'}^i}\left(\int_{\Delta_k(s^i)} f_{-i}(s^{-i})ds^{-i}\right) \\
&= \frac{\partial}{\partial s_{k'}^i}\left(\mid \Delta_{G^{-1}} \mid \int_{G(\Delta_k(s^i))} f_{-i}(G^{-1}(z_0,\mathbf{z}))d\mathbf{z}dz_0\right) \\
&= -\frac{\partial c}{ds_{k'}^i} \mid \Delta_{G^{-1}} \mid \int_{G(\Omega_{k,k'}(s^i))} f_{-i}(G^{-1}(c,\mathbf{z}))d\mathbf{z} \\
&= -(\sum_{g=1}^{N} a_{k'g}^i) \mid \Delta_{G^{-1}} \mid \int_{G(\Omega_{k,k'}(s^i))} f_{-i}(G^{-1}(c,\mathbf{z}))d\mathbf{z}. \qquad (7.9)
\end{aligned}$$

The first equality in 7.9 follows by the definition of $q_k^i(s^i)$; the second equality follows by the multi-dimensional *change of variables formula* (see Lang, 1973); the third follows by expressions 7.8 and by the argument following expression 7.6; the last equality follows by the definition of c in 7.7.

Using the same change of variables as above, the term $\frac{\partial q_{k'}^i(s^i)}{\partial s_k^i}$ is analogously computed[43]:

$$\begin{aligned}
\frac{\partial q_{k'}^i(s^i)}{\partial s_k^i} &= \frac{\partial c}{ds_k^i} \mid \Delta_{G^{-1}} \mid \int_{G(\Omega_{k,k'}(s^i))} f_{-i}(G^{-1}(c,\mathbf{z}))d\mathbf{z} \\
&= -(\sum_{g=1}^{N} a_{kg}^i) \mid \Delta_{G^{-1}} \mid \int_{G(\Omega_{k,k'}(s^i))} f_{-i}(G^{-1}(c,\mathbf{z}))d\mathbf{z}. \qquad (7.10)
\end{aligned}$$

Combining equations 7.9 and 7.10 , we obtain that:

$$\frac{\partial q_k^i(s^i)}{\partial s_{k'}^i}(\sum_{g=1}^{N} a_{kg}^i) = \frac{\partial q_{k'}^i(s^i)}{\partial s_k^i}(\sum_{g=1}^{N} a_{k'g}^i) \qquad (7.11)$$

Equations 7.3 and 7.11 yield together the wished result. ■

Proof of Theorem 5.1: Since all a_{ki}^i are assumed to be different, we can re-order the alternatives so that the sequence $(a_{ki}^i)_k$ is strictly increasing, i.e.

[43]Note that the area in $\Delta_{k'}(s^i)$ where marginal variations of s_k^i are relevant is also $\Omega_{k,k'}(s^i)$. The resulting expressions in 7.9 and 7.10 contain the same integrand over the same boundary, but differ in terms of orientations (since the respective outward normal vectors have opposite signs) and shifts (since the variables $s_{k'}^i$ and s_k^i appear with different coefficients in the definition of c).

$a^i_{(k+1)i} > a^i_{ki}$ for $k = 1, .., K-1$. Condition 5.1 implies then that the sequence $(\sum_{j=1}^N a^i_{kj})_k$ is also strictly increasing.

We construct an efficient, incentive compatible, DRM. For any reported signals the mechanism chooses an efficient alternative given those reports. To specify transfers, we proceed as follows. For fixed reports s^{-i} and i's report t^i, denote by $k^*(t^i)$ the efficient alternative chosen as a function of t^i, i.e.

$$k^*(t^i) \in \arg\max_k \sum_{j=1}^N V_k^j(t^i, s^{-i}).$$

Because the sequence $(\sum_{j=1}^N a^i_{kj})_k$ is strictly increasing , we can define for every vector s^{-i}, a *non-decreasing* sequence of agent i's signals $(\overline{s}^{i,k}(s^{-i}))_k$ with the property that, for any $t^i \in (\overline{s}^{i,k}(s^{-i}), \overline{s}^{i,k+1}(s^{-i}))$, the efficient alternative is $k^*(t^i) = k$.

For each vector s^{-i} we inductively define a sequence of transfers, $\{\overline{x}_i^k(s^{-i})\}_k$, as follows: $\overline{x}_i^1(s^{-i}) \in \Re$ is an arbitrary constant, and for all k, $1 < k \leq K-1$,

$$\overline{x}_i^{k+1}(s^{-i}) - \overline{x}_i^k(s^{-i}) = \sum_{j, j\neq i} [V_{k+1}^j(\overline{s}^{i,k+1}(s^{-i}), s^{-i}) - V_k^j(\overline{s}^{i,k+1}(s^{-i}), s^{-i})] \tag{7.12}$$

If the vector of reports is (t^i, s^{-i}), then i's transfer is defined to be $x_i^*(t^i, s^{-i}) = \overline{x}_i^{k^*(t^i)}(s^{-i})$[44].

The logic underlying the specification of the transfers is as follows. Fix a vector of reports s^{-i}. Suppose that both intervals $(\overline{s}^{i,k}(s^{-i}), \overline{s}^{i,k+1}(s^{-i}))$ and $(\overline{s}^{i,k+1}(s^{-i}), \overline{s}^{i,k+2}(s^{-i}))$ are non-empty. For s^i slightly above $\overline{s}^{i,k+1}(s^{-i})$ the only efficient alternative is $k+1$. For s^i slightly below $\overline{s}^{i,k+1}(s^{-i})$ the only efficient alternative is k. At $s^i = \overline{s}^{i,k+1}(s^{-i})$ both alternatives are efficient. The transfers are adjusted so that, given s^{-i}, agent i with type $\overline{s}^{i,k+1}(s^{-i})$ is made indifferent between alternative k with transfer $\overline{x}_i^k(s^{-i})$ and alternative $k+1$ with transfer $\overline{x}_i^{k+1}(s^{-i})$.

We now show that it is optimal for agent i to report truthfully if all other agents report truthfully. Fix s^{-i} the (truthfully) reported signal of all agents other than i. In order to have a more transparent notation, we omit below the dependence of $\overline{s}^{i,k}$ and $\overline{x}_i^k$ on the fixed s^{-i}.

[44]To avoid a cumbersome case differentiation, we have assumed that, given s^{-i}, the set $\{k^*(t^i)\}_{t^i \in S^i}$ includes the entire set of alternatives. If this is not the case, then some of the intervals $(\overline{s}^{i,k}(s^{-i}), \overline{s}^{i,k+1}(s^{-i}))$ may be empty. Transfers are then defined up to the arbitrary value of the transfer in the first non-empty interval. Furthermore, if a signal $\overline{s}^{i,k+1}(s^{-i})$ hits the upper bound of agent i's signal interval, then the transfer for all reports $t^i > \overline{s}^{i,k}(s^{-i})$ is set to be equal to $\overline{x}_i^k(s^{-i})$.

Suppose without loss of generality that agent i's true type s^i lies in $\left[\overline{s}^{i,k}, \overline{s}^{i,k+1}\right)$. If agent i reports truthfully $t^i = s^i$, his payoff is

$$U_i(s^i, s^{-i}) = V^i_k(s^i, s^{-i}) + \overline{x}^k_i.$$

For any report $t^i \in \left[\overline{s}^{i,k}, \overline{s}^{i,k+1}\right)$, agent i gets the same payoff. Suppose that agent i makes a report $t^i \in \left[\overline{s}^{i,k+r}, \overline{s}^{i,k+r+1}\right)$ with $r > 0$. This non-truthful report yields for agent i a payoff of

$$U_i(t^i, s^{-i}) = V^i_{k+r}(s^i, s^{-i}) + \overline{x}^{k+r}_i.$$

Noting that $\overline{x}^{k+r}_i = \sum_{l=1}^{r}(\overline{x}^{k+l}_i - \overline{x}^{k+l-1}_i) + \overline{x}^k_i$ and using expression 7.12, we obtain:

$$\begin{aligned} U_i(s^i, s^{-i}) - U_i(t^i, s^{-i}) &= V^i_k(s^i, s^{-i}) - V^i_{k+r}(s^i, s^{-i}) \\ &\quad - \sum_{l=1}^{r}\left(\sum_{j,j\neq i}[V^j_{k+l}(\overline{s}^{i,k+l}, s^{-i}) - V^j_{k+l-1}(\overline{s}^{i,k+l}, s^{-i})]\right). \end{aligned}$$

By the definition of $\overline{s}^{i,k+l}$ (at which both alternatives $k+l-1$ and $k+l$ are efficient), we obtain:

$$\sum_{j,j\neq i}[V^j_{k+l}(\overline{s}^{i,k+l}, s^{-i}) - V^j_{k+l-1}(\overline{s}^{i,k+l}, s^{-i})] = -[V^i_{k+l}(\overline{s}^{i,k+l}, s^{-i}) - V^i_{k+l-1}(\overline{s}^{i,k+l}, s^{-i})]$$

Finally, we obtain that:

$$\begin{aligned} &U_i(s^i, s^{-i}) - U_i(t^i, s^{-i}) = V^i_k(s^i, s^{-i}) - V^i_k(\overline{s}^{i,k+1}, s^{-i}) \\ &+ \sum_{l=1}^{r-1}[V^i_{k+l}(\overline{s}^{i,k+l}, s^{-i}) - V^i_{k+l}(\overline{s}^{i,k+l+1}, s^{-i})] + V^i_{k+r}(\overline{s}^{i,k+r}, s^{-i}) - V^i_{k+r}(s^i, s^{-i}) \quad = \\ &a^i_{ki}\left(s^i - \overline{s}^{i,k+1}\right) + \sum_{l=1}^{r-1}[a^i_{(k+l)i}\left(\overline{s}^{i,k+l} - \overline{s}^{i,k+l+1}\right)] + a^i_{(k+r)i}\left(\overline{s}^{i,k+r} - s^i\right) \quad = \\ &\sum_{l=1}^{r}\left(a^i_{(k+l-1)i} - a^i_{(k+l)i}\right)\left(s^i - \overline{s}^{i,k+l}\right) \quad \geq \quad 0 \end{aligned}$$

The last inequality follows because each of the terms in the sum is non-negative[45]

The proof for a report $t^i \in \left[\overline{s}^{i,k+r}, \overline{s}^{i,k+r+1}\right)$ with $r < 0$ is completely analogous.

Note that the transfers defined above do not depend on the distribution of signals, and our mechanism implements the efficient social choice rule no matter how the signals of the various agents are distributed[46]. ■

[45] By the assumption on the sequence $(a^i_{ki})_k$, we have $a^i_{(k+l-1)i} - a^i_{(k+l)i} < 0$; because s^i lies in $\left[\overline{s}^{i,k}, \overline{s}^{i,k+1}\right)$, and because the sequence $\overline{s}^{i,k}$ is non-decreasing, we have $s^i - \overline{s}^{i,k+l} \leq 0$.

[46] In other words, truth-telling constitutes an *ex-post equilibrium*.