TABLE II
CHANGE IN APERTURE DISTRIBUTION NEEDED TO MODIFY A 30 dB TAYLOR PATTERN SO THAT THE SECOND RIGHT-HAND LOBE IS AT −40 dB

$p = \pi \frac{x}{a}$	Incremental Normalized Amplitude	Percent Change in Amplitude	Phase Change in Degrees
0	−0.0072	−2.39	0
$\frac{\pi}{12}$	−0.0120	−4.06	−0.1
$\frac{\pi}{6}$	−0.0187	−6.04	−0.4
$\frac{\pi}{4}$	−0.0175	−4.60	−0.8
$\frac{\pi}{3}$	−0.0096	−1.99	−1.5
$\frac{5\pi}{12}$	−0.0004	−0.07	−2.5
$\frac{\pi}{2}$	+0.0080	1.19	−3.4
$\frac{7\pi}{12}$	+0.0135	1.77	−4.3
$\frac{2\pi}{3}$	+0.0141	1.67	−4.8
$\frac{3\pi}{4}$	+0.0111	1.22	−4.7
$\frac{5\pi}{6}$	+0.0064	0.67	−3.4
$\frac{11\pi}{12}$	+0.0035	0.35	−1.3
π	0	0	0

rapid. The corresponding aperture distributions can be found routinely and computer costs are extremely reasonable.

Perhaps one of the most fruitful areas of application of this perturbation technique is diagnostics. It is now possible to take an experimental pattern (providing it has deep nulls) and analyze why it does not achieve the design goal merely by perturbing the design pattern until it is transformed to the experimental pattern. The resulting aperture distribution, when compared to the design aperture distribution, reveals exactly what changes need to be made in aperture excitation in order to correct the experimental pattern.

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Efficient Numerical Techniques for Solving Pocklington's Equation and Their Relationships to Other Methods

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***Abstract*—It is shown that testing Pocklington's equation with piecewise sinusoidal functions yields an integro-difference equation whose numerical solution is identical to that of the point-matched Hallen's equation when a common set of basis functions is used with each. For any choice of basis functions, the integro-difference equation has the simple kernel, the fast convergence, the simplicity of point-matching, and the adequate treatment of rapidly varying incident fields, but none of the additional unknowns normally associated with Hallen's equation. Furthermore, for the special choice of piecewise sinusoids as the basis functions, the method reduces to Richmond's piecewise sinusoidal reaction matching technique, or Galerkin's method. It is also shown that testing with piecewise linear (triangle) functions yields an integro-difference equation whose solution converges asymptotically at the same rate as that of Hallen's equation. The resulting equation is essentially that obtained by approximating the second derivative in Pocklington's equation by its finite difference equivalent. The authors suggest a simple and highly efficient method for solving Pocklington's equation. This approach is contrasted to the point-matched solution of Pocklington's equation and the reasons for the poor convergence of the latter are examined.**

INTRODUCTION

In order to handle complicated problems using moment methods it is necessary to optimize numerical solution procedures from the point of view of speed and convergence. This leads one to a study of the properties of various integral equation formulations and of the choice of basis and testing functions [1] in solution methods, both with an end toward improving the numerical efficiency of given computations. Also desirable are techniques which are conceptually simple to apply (so as to minimize programming time) and which have a wide range of applicability.

One difficulty which frequently arises in the numerical solution of an integral equation is the appearance of derivatives outside the vector potential integrals on the induced currents. For thin wires, this problem, encountered in Pocklington's equation, is usually handled in one of three ways. First, the *E*-field integro-differential equation may be converted to a Hallen type equation plus boundary conditions on the current. This procedure has the disadvantages of introducing additional unknowns into the problem (associated with the homogeneous solutions of the differential operator) and of producing a new integral equation which does not incorporate the boundary conditions on the unknown current. However, the Hallen-type equation offers good convergence for almost all commonly used basis functions. In the second scheme, the kernel of the *E*-field integral equation is made regular by approximations which result in the so-called reduced kernel, and the differentiation is brought inside the integral and onto the unknown current by integration by parts. When collocation (point-matching) is used with this technique and a basis representation for current is chosen which permits slope discontinuities in current, e.g., piecewise constant or piecewise linear representation, convergence is relatively slow.

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Convergence can usually be improved by a somewhat more complicated choice of basis functions having no slope discontinuities. Finally, a relatively expensive testing procedure, such as Galerkin's method, may be used to treat the derivatives and to accelerate convergence.

The desirable feature of Hallen's equation that the convergence is almost *independent of the basis functions* is an overwhelming advantage from the standpoint of computational simplicity because the simple pulse or triangle expansions can be used for almost all calculations. It is shown in this paper, however, that by the proper choice of the testing functions, Pocklington's equation is cast into a form which has *exactly the same solution* (and hence the same convergence rate) *as the point-matched Hallen's equation* but without the constants of integration of the latter. Furthermore, it is shown that the well-known piecewise sinusoidal reaction matching technique of Richmond [2] is a special case of the present method wherein the basis functions are chosen to be piecewise sinusoids. Given the equivalence between the present method and Hallen's equation with its relative independence of the basis set used, this technique may be viewed as an expansion of the class of basis functions which enjoy the good convergence properties of Richmond's method. Finally, we demonstrate that approximation of the derivative operator in Pocklington's equation by a finite difference operator is closely akin to testing with piecewise linear (triangle) testing functions and that the resultant equation is asymptotically convergent at the same rate as Hallen's equation with point-matching.

Testing with Piecewise Sinusoids

We consider a thin cylindrical dipole of radius a formed by a perfectly conducting tube of length L subject to an impressed field $E_z^i(z)$ tangent to the wire. We include antennas as well as scatterers by permitting $E_z^i(z)$ to be a delta function to model a delta-gap generator. Requiring the z component of the tangential electric field to vanish on the conductor results in Pocklington's equation

$$\left[\frac{d^2}{dz^2} + k^2\right] A_z(z) = -j\omega\mu\varepsilon E_z^i(z) \tag{1}$$

where

$$A_z(z) = \frac{\mu}{4\pi}\int_{-L/2}^{+L/2} I(z')G(z-z')\,dz'$$

and where the kernel is given by

$$G(w) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{-jk\sqrt{w^2+4a^2\sin^2\phi'/2}}}{\sqrt{w^2+4a^2\sin^2\phi'/2}}\,d\phi'.$$

In the preceding expressions $I(z)$ is the unknown total axial current, μ and ε are the permeability and permittivity, respectively, of the medium surrounding the tube, k is the wavenumber or 2π/(wavelength in the medium), and ω is the angular frequency of the suppressed time dependence $\exp(j\omega t)$. If we define the piecewise sinusoidal testing functions by

$$S_m(z) = \begin{cases} \dfrac{\sin k(\Delta z - |z - z_m|)}{\sin k\Delta z}, & z_{m-1} \le z \le z_{m+1} \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

where $z_m = m\Delta z$, $\Delta z = L/(2N+2)$, and the inner product [1] (or more accurately, the reaction) between quantities g and h as

$$\langle g,h\rangle = \int_{-L/2}^{L/2} g(z)h(z)\,dz$$

then testing (1) and (2) results in

$$\left\langle\left[\frac{d^2}{dz^2} + k^2\right] A_z(z), S_m(z)\right\rangle = -j\omega\mu\varepsilon\langle E_z^i(z), S_m(z)\rangle. \tag{3}$$

Integrating by parts twice, one may write the left side of (3) as

$$\frac{k}{\sin k\Delta z}\int_{-L/2}^{L/2} [\delta(z - z_{m+1}) - 2\cos k\Delta z\,\delta(z - z_m) + \delta(z - z_{m-1})]A_z(z)\,dz$$

where $\delta(z)$ is the familiar delta function, from which we easily obtain the *integro-difference* equation

$$A_z(z_{m+1}) - 2\cos k\Delta z A_z(z_m) + A_z(z_{m-1}) = -\frac{j\omega\mu\varepsilon}{k}\int_{z_{m-1}}^{z_{m+1}} E_z^i(z)\sin k(\Delta z - |z - z_m|)\,dz, \quad m = 0, \pm 1, \cdots, \pm N. \tag{4}$$

By direct substitution, it may be verified that a solution to the difference equation (4) is

$$A_z(z_m) = B\cos kz_m + C\sin kz_m - \frac{j\omega\mu\varepsilon}{k}\int_0^{z_m} E_z^i(z)\sin k(z_m - z)\,dz, \quad m = 0, \pm 1, \cdots, \pm(N+1) \tag{5}$$

which is precisely the point-matched Hallen's equation for arbitrary excitation. The current may be expanded in terms of a set of basis functions $f_n(z)$,

$$I(z) = \sum_{n=-n}^{N} I_n f_n(z)$$

which are assumed to be chosen so as to satisfy the current boundary conditions

$$I(\pm L/2) = 0.$$

When this expansion is substituted into (4) and (5), a matrix equation results for the unique determination of the unknown current expansion coefficients I_n. Since (4) and (5) are obtainable one from the other, the solution of either is identical. Hence, we reach the rather surprising but important conclusion that testing with piecewise sinusoids results in an integro-difference equation whose solution is identical to that of Hallen's equation with point-matching, *independent of the basis set chosen to represent* the current. Comparison of (4) and (5) shows that no extra calculations or integrations are needed in the integro-difference equation formulation compared to Hallen's formulation and, in addition, the two unknown constants B and C no longer appear. Furthermore, it is not necessary to regularize the kernel in Pocklington's equation by means of the reduced kernel approximation [3]

$$G(w) \simeq K(w) = \frac{e^{-jk\sqrt{w^2+a^2}}}{\sqrt{w^2+a^2}}$$

because the vector potential kernel is integrable. The fact that the equivalence holds for arbitrary excitation is also important because Hallen's equation, as does now the integro-difference equation (4), has the advantage that the incident field appears under an integral. Thus the influence of a rapidly varying incident field, even that of the widely-used delta-function generator, is properly reflected in the solution. With any of the usual point-matching procedures and Pocklington's equation, often the excitation is not sampled adequately [4].

It is also of interest to note that the use of piecewise sinusoids, (2), as a basis set in (4) leads directly to Richmond's so-called piecewise sinusoidal reaction matching, and one can obtain his formulas for the reaction between two sinusoidal dipoles [5] from the left side of (4) using the integral

$$\int_{z_1}^{z_2} \frac{e^{\pm jkz'} e^{-jk\sqrt{a^2+(z-z')^2}}}{\sqrt{a^2+(z-z')^2}}\, dz' = \mp e^{\pm jkz}[\mathrm{Ci}\,(ku_2) + j\,\mathrm{Si}\,(ku_2) - \mathrm{Ci}\,(ku_1) - j\,\mathrm{Si}\,(ku_1)]$$

$$u_i = \mp(z - z_i) - \sqrt{a^2 + (z - z_i)^2}, \qquad i = 1,2$$

where Ci and Si are cosine and sine integrals, respectively. The procedure leading to (4) has the characteristics of testing with functions other than delta functions and, indeed, is Galerkin's method if piecewise sinusoids are used to represent the current; yet (4) possesses all the computational simplicity usually attributed only to point-matching.

Testing with Piecewise Linear Functions and the Finite Difference Interpretation

We define the piecewise linear testing functions as

$$T_m(z) = \begin{cases} \dfrac{\Delta z - |z - z_m|}{\Delta z}, & z_{m-1} \le z \le z_{m+1} \\ 0, & \text{otherwise.} \end{cases} \qquad (6)$$

These functions are seen to approach the piecewise sinusoids (2) as $k\Delta z \to 0$ and one may expect that their use in numerical solutions closely approximates the use of piecewise sinusoids. To see this, we test (1) with (6) and arrive at an equation analogous to (4):

$$A_z(z_{m+1}) - 2A_z(z_m) + A_z(z_{m-1}) + k^2\Delta z\langle A_z(z),T_m(z)\rangle = -j\omega\mu\varepsilon\Delta z\langle E_z^i(z),T_m(z)\rangle.$$

To simplify the following we treat, without loss of generality, an antenna driven at a delta-gap located at the center of the wire. Hence, we set $E_z^i(z) = V\delta(z)$. We also note that the vector potential is generally a very smooth function regardless of the basis set and, hence, the integral in the preceding can be approximated as

$$\langle A_z(z),T_m(z)\rangle = \int_{z=z_{m-1}}^{z=z_{m+1}} A_z(z)T_m(z)\, dz \simeq \Delta z A_z(z_m) \qquad (7)$$

so that we finally obtain, analogous to (4),

$$A_z(z_{m+1}) - 2(1 - k^2\Delta z^2/2)A_z(z_m) + A_z(z_{m-1}) = -j\omega\mu\varepsilon V\Delta z\delta_{mo}, \qquad m = 0,\pm 1,\cdots,\pm N \qquad (8)$$

where δ_{mo} is the Kronecker delta function. A solution of this difference equation is easily verified to be

$$A_z(z_m) = B\cos m\theta + C\sin m\theta - j\frac{\omega\mu\varepsilon V}{2k}\cdot\frac{\sin|m\theta|}{\sqrt{1-(k\Delta z/2)^2}}, \qquad m = 0,\pm 1,\cdots,\pm(N+1) \qquad (9)$$

where $\theta = \cos^{-1}(1 - k^2\Delta z^2/2)$. Equation (9) corresponds closely to Hallen's equation (5) as can be seen by noting that, for N large, $k\Delta z \ll 1$ and we may write

$$\cos\theta = 1 - \frac{k^2\Delta z^2}{2} \simeq 1 - \frac{\theta^2}{2!}$$

TABLE I
Illustration of Convergence of Values of θ to $k\Delta z$ with Decreasing Subdomain Size

$k\Delta z$	$\theta = \cos^{-1}\left[1 - \frac{k^2\Delta z^2}{2}\right]$
1.5000	1.6961
1.3000	1.4152
1.0000	1.0472
0.8000	0.8230
0.6000	0.6094
0.5000	0.5054
0.4000	0.4027
0.2500	0.2507
0.1250	0.1251
$0.6000\cdot 10^{-1}$	$0.6001\cdot 10^{-1}$
$0.6000\cdot 10^{-2}$	$0.6000\cdot 10^{-2}$

from which we conclude that

$$\lim_{N\to\infty} \theta = k\Delta z.$$

In a practical sense, the limit is approached rather quickly as illustrated in Table I. When this limit is substituted into (9), Hallen's (point-matched) equation (4) specialized to the antenna problem is obtained. We conclude that a solution of (8) will very quickly converge to the corresponding solution of Hallen's equation with point-matching.

An alternative interpretation of (8) is also illuminating. Consider the point-matched Pocklington's equation for the center-driven antenna,

$$\left[\frac{d^2}{dz^2} + k^2\right] A_z(z)|_{z=z_m} = \frac{-j\omega\mu\varepsilon V}{\Delta z}\delta_{m0}, \qquad m = 0,\pm 1,\cdots,\pm N. \qquad (10)$$

To obtain the right side of (10), the delta-function drive is approximated by a rectangular pulse of unit area distributed over the subdomain $m = 0$, i.e., over the interval $[-\Delta z/2, \Delta z/2]$, and the resulting driving field is sampled at the center of the region. In view of the difficulties mentioned earlier in connection with the differential operator in (10), one might choose to replace the derivative by the finite difference approximation

$$\left[\frac{d^2}{dz^2} + k^2\right] A_z(z)|_{z=z_m} \simeq \left[\frac{\Delta^2}{\Delta z^2} + k^2\right] A_z(z)|_{z=z_m} = \frac{A_z(z_{m+1}) - 2(1 - k^2\Delta z^2/2)A_z(z_m) + A_z(z_{m-1})}{\Delta z^2}$$

which, when substituted into (10), also yields (8). This approach has been used by Tesche [6] and others with good success. We are led to conclude that testing by piecewise linear functions is approximately equivalent to replacing the derivative operator by a finite difference approximation. This same interpretation also applies to testing with piecewise sinusoids when it is realized that the left side of (4) is a *weighted* finite difference approximation to the harmonic operator.

While it seems reasonable that the smoothing resulting from testing by piecewise linear or sinusoidal functions might enhance the convergence of the solution, it may at first appear surprising that the same effect results from replacing an exact derivative

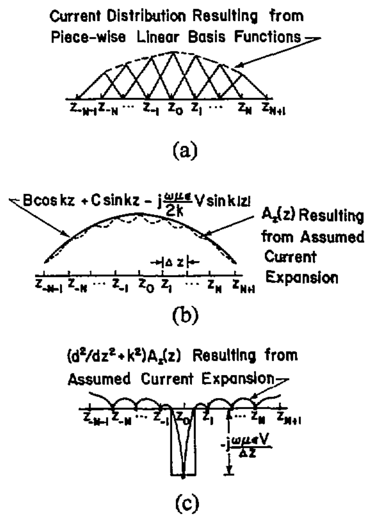

Fig. 1. (a) Piecewise linear current representation. (b) Point-matching solution of Hallen's equation. (c) Point-matching solution of Pocklington's equation.

by an approximate one. An explanation of this follows from a graphical comparison of (5) and (10). In Fig. 1(a) is depicted a current represented by piecewise linear functions, and in Fig. 1(b) we represent *qualitatively* both sides of (5) specialized to an antenna. The resulting vector potential $A_z(z)$ ripples somewhat, but at precisely the match points z_m the potential is required to be equal to the right side of (5) which is a smooth function. For the purposes of illustration we regard the constants B and C as already determined by the two equations $m = \pm(N + 1)$ in (5). Ideally, of course, the left and right sides should be equal for all z, but this is unlikely in an approximate solution. We note further that because of the rippled form of the vector potential, as represented by the left side of (5) with subdomain basis functions, derivatives of A_z will be completely inaccurate (compared to those of the "correct" vector potential) unless the subdomain size is extremely small. Yet, (10) may be interpreted as a constraint on a weighted sum of the vector potential and its second derivative at the match point. The inaccuracy of the second derivative of the potential produces resultant fields which are extremely peaked at the match points. This is shown by Fig. 1(c) which illustrates qualitatively both sides of (1) with the reduced kernel and with the pulse approximation for the delta function and a point-matching solution. Because of the unphysical behavior of the left side of (10) resulting from the slope discontinuities in current, the solution of the point-matched Pocklington's equation (10) tends to converge slowly as compared to that of Hallen's equation (5) in which the unnatural slope discontinuities are effectively smoothed. The choice of the finite difference approximation to the harmonic operator might be expected to result in improved convergence for Pocklington's equation since the derivative is calculated by "numerical differentiation" which, because of its gross sampling, is insensitive to the small-scale ripples introduced by the unphysical representation of the current (subdomain basis functions).

Conclusions

It is shown above that testing Pocklington's integro-differential equation with piecewise sinusoids results in a system of linear integro-difference equations whose numerical solution is identical to the collocation solution of Hallen's equation for any choice of basis functions. This formulation in terms of integro-difference equations (4) enjoys the advantages normally associated with collocation solutions of Hallen's equation, which are listed here.

1) The method exhibits the same rapid convergence rate associated with solutions of Hallen's equation.

2) Only well-behaved kernels (exact) need be calculated numerically.

3) The method admits the use with equal ease of any form of excitation, e.g., delta-gap voltage source and incident field illumination, and assures that the forcing function is adequately sensed [4].

4) The simplicity of collocation is retained.

On the other hand, the method does not suffer the major disadvantage of Hallen-type equations; specifically, in the technique, no complicating arbitrary constants of integration are introduced—the system of difference equations retains the boundary conditions of the problem. We also mention that, when one uses piecewise sinusoids for basis functions as well as for testing, the method suggested in the preceding readily specializes to Richmond's piecewise sinusoidal reaction matching technique.

One draws an equivalence between piecewise linear testing of Pocklington's equation and the approximation of its derivative operator by the corresponding difference operator. Furthermore, piecewise linear and piecewise sinusoidal testing yield systems of integro-difference equations which approach a common limit as the number of testing functions is increased. Hence, in this limit, observations pertinent to one hold for the other testing set.

Aside from the comparisons and interrelationships discussed here, it should be emphasized that direct use of (4), or a similar form obtained via piecewise linear testing of (1), in a numerical solution procedure yields an efficient and computationally simple scheme for solving equations like (1). The efficiency and simplicity of these schemes, when compared to, for example, the numerically equivalent Hallen formulation, become greater factors in problems more complex than the simple straight wire discussed here.

The authors have found observations presented here to be not only interesting, but also extremely useful in analyzing or predicting the success or failure of various numerical approaches [7]. We also add that the equivalences and techniques described here have been generalized to include arbitrary skew wires as well as plates. It is, of course, in these much more difficult problems where the techniques described will be most useful. These extensions will be described in a later paper.

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