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EFFICIENT RECTANGULAR MIXED FINITE ELEMENTS IN TWO AND THREE SPACE VARIABLES (*)

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Abstract. — *Two families of mixed finite elements for second order elliptic equations are introduced, one in two variables and the other in three. These rectangular elements are related to ones of Brezzi, Douglas and Marini in two space and Brezzi, Douglas, Durán and Fortin in three space. They give the same rates of convergence as the corresponding Raviart-Thomas elements with fewer parameters per rectangle. Hybridization of the mixed method for these elements is considered, and alternating-direction iterative techniques are discussed.*

Résumé. — *On introduit deux familles d'éléments finis mixtes pour des problèmes aux limites elliptiques d'ordre deux en dimension deux et trois. Il s'agit d'éléments rectangulaires liés à ceux de Brezzi, Douglas et Marini en dimension deux et de Brezzi, Douglas, Durán et Fortin en dimension trois. Ils donnent le même ordre de convergence que les éléments correspondants de Raviart-Thomas, mais avec moins de paramètres par rectangle. On considère la formulation hybridisée associée et on étudie des techniques itératives de directions alternées.*

1. INTRODUCTION

The object of this paper is to present families of rectangular mixed finite elements that are derived from the elements of Brezzi, Douglas and Marini [7, 8] in two space variables and of Brezzi, Douglas, Durán and Fortin [6] in three space variables. These modifications of the elements of [7, 8] and [6] have both advantages and disadvantages in comparison with those elements ; however, they compare very favorably with the spaces of Raviart and Thomas [21, 22] in that they provide greater algebraic simplicity than the Raviart-Thomas spaces giving the same rates of convergence. This

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results in a significantly reduced computational cost for comparable accuracy in the approximate solution.

We treat the two-dimensional case in Part I (Sections 2.5) and the three-dimensional case in Part II (Sections 6 and 7). The convergence analysis for the new elements is essentially identical to that carried out for the Raviart-Thomas elements (« RT » or « RT_j », where the subscript j indicates the index of the space, as defined below) by Douglas and Roberts in [16], for the Brezzi-Douglas-Marini elements (« BDM » or « BDM_j ») in [7], and for the Brezzi-Douglas-Durán-Fortin elements (« BDDF » or « $BDDF_j$ ») in [6]. Consequently, only a brief indication of the analysis will be given, though the results will be presented. We shall study several computational aspects related to our new elements and then draw comparisons with both the RT elements and either the BDM or BDDF elements.

Throughout this paper we shall consider the Dirichlet problem

$$-\operatorname{div}\,(a(x)\,\nabla u) = f \quad \text{in } \Omega\,, \tag{1.1a}$$

$$u + g = 0 \quad \text{on } \partial\Omega\,, \tag{1.1b}$$

where Ω is a bounded domain in either $\mathbb{R}^2$ or $\mathbb{R}^3$; ∇ indicates the gradient operator and div the divergence operator. As usual in the consideration of mixed methods, we introduce the flux variable

$$\psi = -\,a\,\nabla u\,. \tag{1.2}$$

Let $c(x) = a(x)^{-1}$. (Vectors will be represented by Greek letters.) Let $(\ ,\ )$ denote the inner product in $L^2(\Omega)$ or $L^2(\Omega)^m$, $m = 2$ or 3, and $\langle\ ,\ \rangle$ that in $L^2(\partial\Omega)$. Set $\Psi = H(\text{div}, \Omega)$ and $W = L^2(\Omega)$. The weak form of (1.1) that leads to the mixed finite element method is given by seeking a solution $\{\psi, u\} \in \Psi \times W$ satisfying the equations

$$(c\psi, \mu) - (\text{div}\,\mu, u) = \langle g, \mu \cdot \nu\rangle\,, \quad \mu \in \Psi\,, \tag{1.3a}$$

$$(\text{div}\,\psi, w) = (f, w)\,, \quad w \in W\,, \tag{1.3b}$$

where ν denotes the outer normal to $\partial\Omega$; (1.3a) results from testing (1.2), divided by $a(x)$, against Ψ and (1.3b) from (1.1a) tested against W.

We wish to discretize (1.3) in the following way. Let $\mathcal{K}_h$ be a decomposition of Ω into rectangles K of diameter not greater than h ; we suppose for analytical purposes that a sphere of diameter greater than ρ diam (K) can be inscribed in each $K \in \mathcal{K}_h$ for some $\rho \in (0, 1)$ which is independent of h. A boundary « rectangle » is permitted to have one curved face. We shall associate suitable subspaces $\Psi_h \times W_h$ of $\Psi \times W$ with $\mathcal{K}_h$ and seek an approximate solution $\{\psi_h, u_h\} \in \Psi_h \times W_h$ of (1.3) such that

$$(c\psi_h, \mu) - (\text{div}\,\mu, u_h) = \langle g, \mu \cdot \nu\rangle\,, \quad \mu \in \Psi_h\,, \tag{1.4a}$$

$$(\text{div}\,\psi_h, w) = (f, w)\,, \quad w \in W_h\,. \tag{1.4b}$$

The spaces $\Psi_h \times W_h$ to be treated below will be derived as special cases of the variable degree elements [8] related to the BDM elements and their analogues in three space.

We have chosen to treat the simple equation $-\operatorname{div}(a \nabla u) = f$ in place of the linear equation $-\operatorname{div}(a \nabla u + \beta u) + \gamma \cdot \nabla u + du = f$ primarily to reduce the technical detail in the presentation ; all of the convergence results extend to the more general case. See [13, 16] for such extensions. Some of the discussion of the algebraic equations associated with (1.4) would require modification to handle the general case.

We shall denote the restriction of polynomials of total degree not greater than j to the set K by $P_j(K)$ and the tensor products of polynomials of degree not greater than i_j in the j-th variable by $Q_{i_1, i_2}(K)$ or $Q_{i_1, i_2, i_3}(K)$.

PART I

2. THE REDUCED BDM ELEMENTS

Let us recall the BDM_j space over an ordinary rectangle K. Let j be a positive integer and set

$$\begin{aligned}(2.1)\quad \mathrm{BDM}_j(K) &= \Phi(j, K) \times W(j, K) \\ &= [P_j(K)^2 + \text{Span curl } \{x_1 x_2^{j+1}, x_1^{j+1} x_2\}] \times P_{j-1}(K),\end{aligned}$$

where curl (z) is the vector $(\partial z/\partial x_2, -\partial z/\partial x_1)$. An element ξ of $\Phi(j, K)$ is determined uniquely [7] by the following degrees of freedom :

$$(2.2a)\quad \langle \xi \cdot \nu, p \rangle_e, \quad p \in P_j(e), e \in \{e_1, \ldots, e_4\} \text{ (the edges of } K),$$

$$(2.2b)\quad (\xi, \chi)_K, \quad \chi \in P_{j-2}(K)^2 ;$$

the choice of the degrees of freedom for $W(j, K)$ is immaterial for our purposes at the moment.

Consider the following subspace of $\mathrm{BDM}_j(K)$:

$$(2.3a)\quad \Psi(j, K) = \{\xi \in \Phi(j, K) ; \xi \cdot \nu \in P_{j-1}(e_i), i = 1, \ldots, 4\},$$

$$(2.3b)\quad W(j, K) = P_{j-1}(K),$$

$$(2.3c)\quad \mathrm{BDFM}_j(K) = \Psi(j, K) \times W(j, K).$$

It is clear [8] that a set of degrees of freedom for $\Psi(j, K)$ can be given as

$$(2.4a)\quad \langle \xi \cdot \nu, p \rangle_e, \quad p \in P_{j-1}(e), \quad e \in \{e_1, \ldots, e_4\},$$

$$(2.4b)\quad (\xi, \chi)_K, \quad \chi \in P_{j-2}(K)^2.$$

A simple calculation shows that

$$(2.5)\quad \Psi(2, K) = \text{Span}\ \{1, x_1, x_2, x_1^2, x_1 x_2\} \times \text{Span}\ \{1, x_1, x_2, x_1 x_2, x_2^2\}$$
$$= [P_2 \setminus \{x_2^2\}] \times [P_2 \setminus \{x_1^2\}].$$

More generally, we can recharacterize $\Psi(j, K)$ as follows.

LEMMA 2.1 : $\Psi(j, K) = [P_j(K) \setminus \{x_2^j\}] \times [P_j(K) \setminus \{x_1^j\}]$.

Proof: Denote temporarily the spaces defined by (2.3*a*) and in the statement of the lemma by Ψ^1 and Ψ^2, respectively. If $\xi \in \Psi^1$, it has the representation

$$\xi = \xi^* + c_1(x_2^j, 0) + c_2(0, x_1^j) + c_3 \operatorname{curl}(x_1 x_2^{j+1}) + c_4 \operatorname{curl}(x_1^{j+1} x_2)$$

with $\xi^* \in \Psi^2$. The requirement that $\xi \cdot \nu$ belong to $P_{j-1}(e)$ on the left and right edges of K forces c_1 and c_3 to vanish ; similarly, the constraints on the top and bottom edges kill c_2 and c_4, so that the proof is done.

Each of the equivalent definitions of $\Psi(j, K)$ is useful ; the unisolvence of the degrees of freedom (2.4) is obvious from (2.3) and the important property that

$$(2.6a)\qquad \Psi(j, K) = \Psi_1(j, K) + \Psi_2(j, K),$$

where

$$(2.6b)\qquad \Psi_1(j, K) = (P_j(K) \setminus \{x_2^j\}) \times \{0\},$$
$$\Psi_2(j, K) = \{0\} \times (P_j(K) \setminus \{x_1^j\}),$$

follows easily from Lemma 2.1. This splitting of the basis will permit greater flexibility in defining iterative methods for solving the algebraic problem associated with (1.4).

Now, let K be a boundary rectangle with one curved edge. There can be two, three, or four linear edges, say, $e_1, \ldots, e_r$. (We do not admit the case of a single straight edge.) Let $\Psi(j, K)$ be exactly as above for the ordinary rectangle and associate the following degrees of freedom with $\Psi(j, K)$:

$$(2.7a)\quad \langle \psi \cdot \nu, p \rangle_e, \quad p \in P_{j-1}(e), \quad \text{for each straight edge},$$
$$(2.7b)\quad (\operatorname{div} \psi, w)_K, \quad w \in P_{j-1}(K),$$
$$(2.7c)\quad (\psi, \xi)_K, \quad \xi \in \{\gamma \in \Psi(j, K) : \operatorname{div} \gamma = 0 \text{ and } \gamma \cdot \nu = 0 \text{ on each straight edge}\}.$$

We can now define the global subspaces $\Psi_h \times W_h$:

$$(2.8a)\quad \Psi_h = \Psi(j, \mathcal{K}_h) =$$
$$= \{\psi \in \Psi = H(\operatorname{div}, \Omega) : \psi|_K \in \Psi(j, K), K \in \mathcal{K}_h\},$$

$$(2.8b)\quad W_h = W(j, \mathscr{K}_h) = \\ = \left\{ w \in W = L^2(\Omega) : w|_K \in P_{j-1}(K),\ K \in \mathscr{K}_h \right\} ;$$

as usual, the requirement that ψ in (2.8a) belong to Ψ is equivalent to asking that the normal component of the vector ψ be continuous across each internal edge of the decomposition. Our new mixed method for planar domains is determined by using the space of (2.8) in (1.4).

In order to carry out the analysis of the mixed method (1.4)-(2.8) by the technique of Douglas and Roberts [16], it is necessary to define a projection $\Pi_h \times Q_h : H^1(\Omega)^2 \times L^2(\Omega) \to \Psi_h \times W_h$ such that the following diagram commutes :

$$\begin{array}{ccc} H^1(\Omega)^2 & \xrightarrow{\ \text{div}\ } & L^2(\Omega) \\ \Pi_h \downarrow & & \downarrow Q_h \\ \Psi_h & \xrightarrow{\ \text{div}\ } & W_h \to 0 \end{array}$$

The component Q_h can be taken to be defined locally by means of an L^2 projection into $P_{j-1}(K)$. If $z \in L^2(\Omega)$, let

$$(2.9)\qquad (z - Q_h z, w) = 0 , \quad w \in W_h ;$$

i.e.,

$$(2.9')\qquad ((z - Q_h z)|_K, w)_K = 0 , \quad w \in P_{j-1}(K) , \quad K \in \mathscr{K}_h .$$

Then, define Π_h locally by using the degrees of freedom given by either (2.4) or (2.7). Thus, if K is an ordinary rectangle, let $\psi \in H^1(K)^2$ and set

$$(2.10a)\quad \left\langle ((\psi - \Pi_h \psi)|_K) \cdot \nu, p \right\rangle_e = 0 , \quad p \in P_{j-1}(e), \text{ all four edges} ,$$

$$(2.10b)\qquad ((\psi - \Pi_h \psi)|_K, \chi)_K = 0 , \quad \chi \in P_{j-2}(K)^2 ;$$

for boundary rectangles, use the three relations in (2.7) instead to define $\Pi_h \psi$ on K. Note that it follows from (2.10) that

$$(2.11)\qquad (\text{div } (\psi - \Pi_h \psi), w)_K = 0 , \quad w \in P_{j-1}(K) ,$$

for ordinary rectangles ; since this property was built into the degrees of freedom for boundary rectangles, it follows that

$$(2.12)\qquad (\text{div } (\psi - \Pi_h \psi), w) = 0 , \quad w \in W_h .$$

Also, it is obvious that

$$(2.13)\qquad (\text{div } \psi, z - Q_h z) = 0 , \quad \psi \in \Psi_h ,$$

for any $z \in L^2(\Omega)$. The form (2.6) of $\Psi(j, K)$ implies that $\operatorname{div} \Psi_h = W_h$; hence, the diagram has been established. The following approximation properties of the projection are easily seen from the local nature of its definition (the subscript s indicates the norm in $H^s(\Omega)$ for positive s and in $(H^s(\Omega))'$ for negative s) :

$$\|\psi - \Pi_h \psi\|_0 \leq M \|\psi\|_s h^s, \qquad 1 \leq s \leq j, \tag{2.14a}$$

$$\|z - Q_h z\|_{-r} \leq M \|z\|_s h^{r+s}, \quad 0 \leq r \leq j, \quad 0 \leq s \leq j. \tag{2.14b}$$

The properties of Π_h and Q_h are the same as those for the corresponding Raviart-Thomas space of index $j-1$, where $\mathrm{RT}_i(K)$ is given by $[Q_{i+1,i} \times Q_{i,i+1}] \times Q_{i,i}$ [21, 22]. Thus, the stability and convergence of this mixed method follow from theorems of Fortin [17]. The following global error estimates are also consequences of the diagram and the approximation properties (2.14) ; the proof of Theorem 6.1 of Douglas and Roberts [16] applies without modification. Let Ω be called $(s+2)$-regular if, when $f \in H^s(\Omega)$ and $g \in H^{s+3/2}(\partial\Omega)$, the solution of the Dirichlet problem (1.1) lies in $H^{s+2}(\Omega)$ and

$$\|u\|_{s+2,\Omega} \leq M(\|f\|_{s,\Omega} + \|g\|_{s+3/2,\partial\Omega}).$$

THEOREM 2.2 : *Let Ω be $(s+2)$-regular. Then, there exists a unique solution $\{\psi_h, u_h\}$ of* (1.4). *Moreover, the following error estimates are valid :*

$$\|u - u_h\|_{-s} \leq \begin{cases} M\|u\|_r h^{r+s}, & \text{for } 0 \leq s \leq j-2 \\ & \text{and } 2 \leq r \leq j, \\ M\|u\|_{r+1} h^{r+j-1}, & \text{for } s = j-1 \\ & \text{and } 1 \leq r \leq j, \\ M\|u\|_{r+2} h^{r+j}, & \text{for } s = j \\ & \text{and } 0 \leq r \leq j; \end{cases} \tag{2.15a}$$

$$\|\psi - \psi_h\|_{-s} \leq \begin{cases} M\|u\|_{r+1} h^{r+s}, & \text{for } 0 \leq s \leq j-1 \\ & \text{and } 1 \leq r \leq j, \\ M\|u\|_{r+2} h^{r+j}, & \text{for } s = j \\ & \text{and } 0 \leq r \leq j; \end{cases} \tag{2.15b}$$

$$\|\operatorname{div}(\psi - \psi_h)\|_{-s} \leq \|u\|_{r+2} h^{r+s}, \quad \text{for } 0 \leq s \leq j \text{ and } 0 \leq r \leq j. \tag{2.15c}$$

Moreover,

$$\|u_h - Q_h u\|_0 \leq \begin{cases} M\|u\|_3 h^2, & \text{for } j = 1, \\ M\|u\|_{r+1} h^{r+1}, & \text{for } 1 \leq r \leq j \\ & \text{and } j > 1. \end{cases} \tag{2.15d}$$

Before turning to interior estimates and then to specific questions related to the evaluation of the solution of (1.4), let us compare the local dimensions of the RT and BDM spaces with those of the new spaces for similar optimal global rates of convergence in the approximation of the vector variable. First, note that $\mathrm{BDFM}_1(K) = \mathrm{RT}_0(K)$ and recall that there is no corresponding BDM space for which linear convergence is optimal for the vector variable. So, consider $j > 1$. In order to achieve an $O(h^j)$-rate in $L^2(\Omega)$, then we compare $\mathrm{RT}_{j-1}(K)$, $\mathrm{BDM}_{j-1}(K)$, and $\mathrm{BDFM}_j(K)$:

$$\dim\,(\mathrm{RT}_{j-1}(K)) = 3j^2 + 2j\,, \tag{2.16a}$$

$$\dim\,(\mathrm{BDM}_{j-1}(K)) = 1.5j^2 + 0.5j + 2\,, \tag{2.16b}$$

$$\dim\,(\mathrm{BDFM}_j(K)) = 1.5j^2 + 3.5j\,; \tag{2.16c}$$

thus, the new spaces require significantly fewer parameters than the Raviart-Thomas spaces for this measure of accuracy, but slightly more than the BDM spaces. In order to obtain an $O(h^{2j})$-rate in the Sobolev space of negative index of most rapid convergence for the vector variable, we must compare $\mathrm{RT}_{j-1}(K)$, $\mathrm{BDM}_j(K)$, and $\mathrm{BDFM}_j(K)$:

$$\dim\,(\mathrm{RT}_{j-1}(K)) = 3j^2 + 2j\,, \tag{2.17a}$$

$$\dim\,(\mathrm{BDM}_j(K)) = 1.5j^2 + 3.5j + 4\,, \tag{2.17b}$$

$$\dim\,(\mathrm{BDFM}_j(K)) = 1.5j^2 + 3.5j\,, \tag{2.17c}$$

so that on the basis of this criterion the new spaces retain a significant advantage over the RT spaces and have a small advantage over the BDM spaces.

The interior estimates derived by Douglas and Milner [13] for the Raviart-Thomas spaces, as the global estimates, depended only on the properties of the projection $\Pi_h \times Q_h$; hence, the corresponding estimates hold for the new elements. Let G be an open subset of Ω, and set

$$\Psi_h(G) = \{\xi \in \Psi_h : \mathrm{supp}\ (\xi) \subset G\}\,, \tag{2.18a}$$

$$W_h(G) = \{w \in W_h : \mathrm{supp}\ (w) \subset G\}\,. \tag{2.18b}$$

The pair $\{\psi_h, u_h\} \in \Psi_h \times W_h$ is said to be an interior solution of the mixed method equations (1.4) on $G \Subset \Omega$ if

$$(c\psi_h, \mu) - (\mathrm{div}\ \mu, u_h) = 0\,, \qquad \mu \in \Psi_h(G)\,, \tag{2.19a}$$

$$(\mathrm{div}\ \psi_h, w) = (f, w)\,, \qquad w \in W_h(G)\,; \tag{2.19b}$$

note that the boundary values g are not involved in the equations for an interior solution, since $\mathrm{supp}\ (\mu) \subset G$. The following theorem is analogous to Theorem 2.2 of [13].

THEOREM 2.3 : *Let* $\Omega_1 \Subset \Omega_2 \Subset \Omega$, *and assume that the solution* u *of* (1.1) *belongs to* $H^q(\Omega) \cap H^r(\Omega_2)$, *where* $2 \leq q \leq j+2$. *Let* $\{\psi_h, u_h\}$ *be the solution of* (1.4)-(2.8). *For* h *sufficiently small and* $0 \leq s \leq j$,

$$(2.20a) \quad \|u - u_h\|_{-s,\Omega_1} \leq M[\|u\|_{r,\Omega_2} h^{r+s-(s-j+2)^+} + \|u\|_{q,\Omega} h^{q+j-2}], \quad 2 \leq r \leq j + (s-j+2)^+ ,$$

$$(2.20b) \quad \|\psi - \psi_h\|_{-s,\Omega_1} \leq M[\|u\|_{r,\Omega_2} h^{r+s-1-(s-j+1)^+} + \|u\|_{q,\Omega} h^{q+j-2}], \quad 2 \leq r \leq j+1+(s-j+1)^+ ,$$

$$(2.20c) \quad \|\operatorname{div}(\psi - \psi_h)\|_{-s,\Omega_1} \leq M_\varepsilon[\|u\|_{r,\Omega_2} h^{r+s-2} + \|u\|_{q,\Omega} h^{j-2+\min(q, s+2-\varepsilon)}], \quad 2 \leq r \leq j+2 ,$$

for any positive ε.

The negative norm estimates of this theorem and an analogous one for difference quotients, corresponding to Theorem 4.1 of [13] when the decomposition has a translation invariance over an interior subdomain can be used to derive superconvergence via Bramble-Schatz postprocessing of the approximate solution. Presentation of these and the superconvergence results that are related to the hybridization procedure to be discussed in the next section will be delayed to Section 5.

3. HYBRIDIZATION

Some years ago Fraeijs de Veubeke [18, 19] introduced a hybridization of the mixed method with his object being the simplification of the algebraic problem that must be solved in order to evaluate the solution of the procedure. Recently, Arnold and Brezzi [1] analyzed this modification of the mixed method for Raviart-Thomas spaces ; the hybridization was also introduced and analyzed for the BDM [7] and BDDF [6] spaces. We shall consider the same concept here. Let $\mathscr{E}_h$ denote the set of all internal edges associated with the rectangles of the partition $\mathscr{K}_h$, and let

$$(3.1) \quad \Lambda_h = \Lambda(j, \mathscr{K}_h) = \{m \in L^2(\mathscr{E}_h) : m|_e \in P_{j-1}(e), e \in \mathscr{E}_h\} .$$

We shall interpret $m \in \Lambda_h$ to be zero on $\partial\Omega$ while it is being computed ; it will be convenient to give it another interpretation later when it is being used in a postprocessing. Next, release the continuity constraint of the normal component of the vector variable across the internal edges by replacing Ψ_h by

$$(3.2) \quad \Upsilon_h = \Upsilon(j, \mathscr{K}_h) = \{\xi \in L^2(\Omega)^2 : \xi|_K \in \Psi(j, K), K \in \mathscr{K}_h\} .$$

Reimpose this continuity by introducing a Lagrange multiplier $m_h \in \Lambda_h$, and look for a solution $\{\psi_h, u_h, m_h\} \in \Upsilon_h \times W_h \times \Lambda_h$ of the equations

$$(3.3a) \quad (c\psi_h, \chi) - \sum_K (\operatorname{div} \chi, u_h)_K + \sum_K \langle \chi \cdot \nu_K, m_h \rangle_{\partial K} = \langle g, \chi \cdot \nu \rangle, \chi \in \Upsilon_h,$$

$$(3.3b) \quad \sum_K (\operatorname{div} \psi_h, w)_K = (f, w), \quad w \in W_h,$$

$$(3.3c) \quad \sum_K \langle \psi_h \cdot \nu_K, v \rangle_{\partial K} = 0, \quad v \in \Lambda_h.$$

The relations (3.3c) force ψ_h to lie not only in Υ_h but also in Ψ_h; consequently, the pair $\{\psi_h, u_h\}$ obtained from (3.3) coincides with the solution of (1.4)-(2.8) as functions, though the degrees of freedom of ψ_h as determined from (3.3) differ from those of the original method. Clearly, the error estimates of the previous section remain valid for u_h and ψ_h; error estimates related to m_h can be derived as in [1, 6, 7] and will be stated and used in the section on superconvergence.

Let us consider some computational aspects of the hybridized procedure. The matrix associated with (3.3) takes the form

$$(3.4) \quad \begin{bmatrix} A & B & C \\ B^* & 0 & 0 \\ C^* & 0 & 0 \end{bmatrix},$$

where A is block diagonal with the blocks being $N_1 \times N_1$, $N_1 = \dim(\Psi(j, K))$, and B has a block structure dependent on the dimension of $W(j, K)$. Thus, the parameters defining ψ_h-parameters can be eliminated simply, with the matrix for the remaining degrees of freedom taking the form

$$(3.5) \quad \begin{bmatrix} B^* A^{-1} B & B^* A^{-1} C \\ C^* A^{-1} B & C^* A^{-1} C \end{bmatrix},$$

where now $B^* A^{-1} B$ is block diagonal with $N_2 \times N_2$ blocks, $N_2 = \dim(P_{j-1}(K))$, so that the u_h-parameters can be eliminated, again at the element level. The resulting matrix, say D, for the m_h-parameters is symmetric and positive definite in the case that no lower order terms are present in the differential equation (1.1a). If lower order terms are present, then the elimination procedure can be carried out almost as outlined above, at least for sufficiently small h; however, the form of (3.5) and D will be somewhat different, with D not retaining symmetry. It will have a dominant part that is symmetric, positive-definite.

The graph structure of D is quite reasonable. Each set of j parameters associated with the polynomial of degree $j-1$ on an edge e that is the restriction of m_h to e sees the sets of parameters associated with the edges of the two rectangles for which e is an edge. (Thus, the matrix has essentially the same sparsity structure as the matrix for a nonconforming Galerkin procedure based on polynomials of degree j after the internal degrees of freedom have been eliminated.) The equations split naturally into sets of the following form :

$$(3.6a) \qquad EL_{i-1,j}\,\gamma_{i-1,j} + E_{ij}\,\gamma_{ij} + ER_{i+1,j}\,\gamma_{i+1,j} + F_{ij}\,\delta = \varphi_{ij}\,,$$

$$(3.6b) \qquad GB_{i,j-1}\,\delta_{i,j-1} + G_{ij}\,\delta_{ij} + GA_{i,j+1}\,\delta_{i,j+1} + H_{ij}\,\gamma = \omega_{ij}\,,$$

where γ_{ij} denotes the parameters associated with the left edge of the rectangle K_{ij} and δ_{ij} those with the bottom edge. The matrix F_{ij} operates on $\delta_{i-1,j}$, $\delta_{i-1,j+1}$, δ_{ij}, and $\delta_{i,j+1}$; H_{ij} operates on the neighboring γ's. The symbols L, R, B, and A refer to left, right, below, and above, respectively. If a preconditioned conjugate gradient iteration is used to solve (3.6), a clear choice of a preconditioner comes from this splitting. The equations (3.6a) are tridiagonal in γ for an x_1-ordering, while (3.6b) is x_2-ordered. We shall note in the superconvergence section that m_h gives an approximation to u ; consequently, the equations of both (3.6a) and (3.6b) represent generalized difference approximations to the differential operator $-\operatorname{div}(a\nabla)$. As a result, the obvious choice of preconditioner for solving for γ using (3.6a) and for δ using (3.6b) does not reduce the $O(h^{-2})$ bound on the condition mumber of the effective matrix in the iteration to some lower order in h^{-1}. This preconditioner is not the equivalent of an alternating-direction preconditioner, and the equations (3.6) retain the difficulty of solution associated with Galerkin and finite difference methods ; however, it is possible to associate an alternating-direction iterative procedure with (3.3) and one will be indicated in the next section.

The Raviart-Thomas space of index $j-1$ leads to equations having exactly the same graph as those of (3.6) when the Fraeijs de Veubeke hybridization is applied to it. Since the dimensions of both the vector and scalar components of $\mathrm{RT}_{j-1}(K)$ exceed those of $\mathrm{BDFM}_j(K)$, the elimination of the ψ_h and u_h parameters is cheaper for the new elements than for the RT elements. On the other hand, if no postprocessing of the solution to obtain a better approximation of the scalar variable is intended, then the $\mathrm{BDM}_{j-1}(K)$ space has fewer parameters than $\mathrm{BDFM}_j(K)$ and leads again to equations of the same form as (3.6), so that the first two steps are less expensive for it than for the new elements.

4. ALTERNATING-DIRECTION ITERATION

Lemma 2.1 implies that the linear equations generated by (1.4)-(2.8) can be written in the form

$$(4.1a) \quad (c\psi_h^1, \chi^1) - (\partial\chi^1/\partial x, u_h) = \langle g, (\chi^1, 0) \cdot \nu \rangle , \quad \chi^1 \in \Psi_h^1 ,$$

$$(4.1b) \quad (c\psi_h^2, \chi^2) - (\partial\chi^2/\partial y, u_h) = \langle g, (0, \chi^2) \cdot \nu \rangle , \quad \chi^2 \in \Psi_h^2 ,$$

$$(4.1c) \quad (\operatorname{div} \psi_h, w) = (f, w) , \quad w \in W_h ,$$

where $\Psi_h = \Psi_h^1 \times \Psi_h^2$, as results from (2.6). These equations can be written in matricial form as

$$(4.2a) \quad A_1 \psi_1 + B_1 u = \gamma_1 ,$$

$$(4.2b) \quad A_2 \psi_2 + B_2 u = \gamma_2 ,$$

$$(4.2c) \quad B_1^* \psi_1 + B_2^* \psi_2 = \varphi .$$

The symbols ψ_1, ψ_2, and u now represent the vectors of parameters giving the degrees of freedom associated with Ψ_h^1, Ψ_h^2, and W_h, respectively. The vector ψ_1 should be ordered in blocks of parameters associated with the rectangular elements ordered in an x-orientation, while ψ_2 should be ordered in a y-orientation. In (4.2a) u should be interpreted to have an x-orientation, and in (4.2b) it should be interpreted to be ordered in a y-orientation. Then (4.2) can be considered to be the steady-state equations associated with the transient problem given by either

$$(4.3a) \quad A_1 \psi_1 + B_1 u = \gamma_1 ,$$

$$(4.3b) \quad A_2 \psi_2 + B_2 u = \gamma_2 ,$$

$$(4.3c) \quad D\, \partial u/\partial t_2 + B_1^* \psi_1 + B_2^* \psi_2 = \varphi ,$$

$$(4.3d) \quad u(t_2 = 0) \text{ arbitrary} ,$$

or

$$(4.4a) \quad A_1 \partial\psi_1/\partial t_1 + A_1 \psi_1 + B_1 u = \gamma_1 ,$$

$$(4.4b) \quad A_2 \partial\psi_2/\partial t_1 + A_2 \psi_2 + B_2 u = \gamma_2 ,$$

$$(4.4c) \quad D\, \partial u/\partial t_2 + B_1^* \psi_1 + B_2^* \psi_2 = \gamma ,$$

$$(4.4d) \quad \psi_1(t_1 = 0) , \quad \psi_2(t_1 = 0) , \quad u(t_2 = 0) \text{ arbitrary} .$$

The equations (4.3) represent a parabolic problem whenever the matrix D is defined as the Grammian associated with a positive weight function on Ω ; the usual choice for the weight function is $a(x)$. Equations (4.4) represent a damped wave equation if the two pseudo-time variables t_1 and t_2 are the same ; otherwise, they are related to a different damped hyperbolic

problem. Note that initial values must be specified for both the vector and scalar variables in (4.4), whereas only the scalar can be initialized in (4.3). In many applications of mixed methods to physical problems a better estimate of the vector is available than of the scalar ; consequently, it is often advantageous to be able to initialize the vector, so that (4.4) can provide the foundation for a better iterative procedure than (4.3). It should also be noted that the splitting of the basis for Ψ_h allows the formulation of a practical iteration based on (4.3) ; this is not really the case if the basis does not split, as happens with the elements of Brezzi-Douglas-Marini [7] and Brezzi-Douglas-Durán-Fortin [6].

Alternating-direction iterative procedures can be defined by discretizing (4.3) and (4.4) in time. The virtual time step associated with the variable t_1 will be taken fixed, while a cycle of steps will be used for t_2. If the step τ_1 is infinite, then the iteration below is the Uzawa alternating-direction associated with (4.3) ; initial values for ψ_1 and ψ_2 should be ignored. If $0 < \tau_1 < \infty$, then the iteration requires initial values for ψ_1 and ψ_2 ; this iteration is the Arrow-Hurwitz alternating-direction associated with (4.4). In either case initial values are required for u. Let $0 < \tau_2^1 < \tau_2^2 < \cdots < \tau_2^N$ and $0 < \tau_1 \leqslant \infty$. The general step of the alternating-direction iteration is as follows :

$$A_1[(\psi_1^{\#} - \psi_1^n)/\tau_1 + \psi_1^{\#}] + B_1 u^{\#} = \gamma_1 , \tag{4.5a}$$

$$D(u^{\#} - u^n)/\tau_2^n + B_1^* \psi_1^{\#} + B_2^* \psi_2^n = \varphi , \tag{4.5b}$$

$$A_2[(\psi_2^{n+1} - \psi_2^n)/\tau_1 + \psi_2^{n+1}] + B_2 u^{n+1} = \gamma_2 , \tag{4.5c}$$

$$D(u^{n+1} - u^{\#})/\tau_2^n + B_1^* \psi_1^{\#} + B_2^* \psi_2^{n+1} = \varphi , \tag{4.5d}$$

and either

$$A_1[(\psi_1^{n+1} - \psi_1^n)/\tau_1 + \psi_1^{n+1}] + B_1 u^{n+1} = \gamma_1 , \tag{4.5e'}$$

or

$$\psi_1^{n+1} = \psi_1^{\#} . \tag{4.5e''}$$

The x_1-sweep is given by (4.5*a*) and (4.5*b*), and the x_2-sweep by (4.5*c*) and (4.5*d*) ; (4.5*e'*) is a corrector step, while (4.5*e''*) omits the corrector and accepts $\psi_1^{\#}$ as the new approximation to ψ_1.

For the lowest index case, $j = 1$, recall that the space BDFM_1 coincides with RT_0. Thus, all of the results of [14] apply directly to the procedure (4.5), which is the one introduced in [14]. In particular, the spectral analysis analogous to that given years ago by one of the authors for alternating-direction iteration for finite difference equations remains valid in the special case of the Laplace equation on a rectangle for the Uzawa procedure, and the heuristic application of these results to the selection of iteration

parameters was found valuable in the experiments reported in [12] and [14]. The leading term error relation derived in [14] for the Arrow-Hurwitz method also is unchanged for $BDFM_1$. A similar spectral analysis should be constructable for $BDFM_j$ for $j > 1$ in the model problem case. Experiments will be run shortly on $BDFM_2$.

The technique described above can be adapted to the hybridized form of the mixed method [15]. Let the equations associated with (3.3) be written in the form

$$A_1 \psi_1 + B_1 u + C_1 m_1 = \gamma_1 , \tag{4.6a}$$

$$A_2 \psi_2 + B_2 u + C_2 m_2 = \gamma_2 , \tag{4.6b}$$

$$B_1^* \psi_1 + B_2^* \psi_2 = \varphi , \tag{4.6c}$$

$$C_1^* \psi_1 = 0 , \tag{4.6d}$$

$$C_2^* \psi_2 = 0 . \tag{4.6e}$$

Here, the parameters m_1 relate to the Lagrange multiplier m_h restricted to vertical edges and m_2 to horizontal edges. The matrices $A_1, \ldots, B_2$ are not the same as in (4.2), since the vector space is no longer in $H(\mathrm{div}, \Omega)$; however, A_1 and A_2 are block diagonal with block size $\dim(\Psi^1(j, K)) = \dim(\Psi^2(j, K))$, etc. Form the alternating-direction iteration as follows:

$$A_1[(\psi_1^{\#} - \psi_1^n)/\tau_1 + \psi_1^{\#}] + B_1 u^{\#} + C_1 m_1^{\#} = \gamma_1 , \tag{4.7a}$$

$$D(u^{\#} - u^n)/\tau_2^n + B_1^* \psi_1^{\#} + B_2^* \psi_2^n = \varphi , \tag{4.7b}$$

$$C_1^* \psi_1^{\#} = 0 , \tag{4.7c}$$

$$A_2[(\psi_2^{n+1} - \psi_2^n)/\tau_1 + \psi_2^{n+1}] + B_2 u^{n+1} + C_2 m_2^{n+1} = \gamma_2 , \tag{4.7d}$$

$$D(u^{n+1} - u^{\#})/\tau_2^n + B_1^* \psi_1^{\#} + B_2^* \psi_2^{n+1} = \varphi , \tag{4.7e}$$

$$C_2^* \psi_2^{n+1} = 0 , \tag{4.7f}$$

and either

$$A_1[(\psi_1^{n+1} - \psi_1^n)/\tau_1 + \psi_1^{n+1}] + B_1 u^{n+1} + C_1 m_1^{n+1} = \gamma_1 , \tag{4.7g'}$$

$$C_1^* \psi_1^{n+1} = 0 , \tag{4.7h'}$$

or

$$\psi_1^{n+1} = \psi_1^{\#} , \tag{4.7g''}$$

$$m_1^{n+1} = m_1^{\#} . \tag{4.7h''}$$

It is necessary that the initial values of ψ_1 and ψ_2 satisfy (4.6*d*) and (4.6*e*), respectively, since no relaxation of these conditions through the addition of a pseudo-time derivative of the Lagrange multiplier to (4.7*c*) and (4.7*f*) was introduced.

It is clear from the inner product form of (4.7) that the functions represented by ψ_1^n, ψ_2^n, and u^n are identical to those coming from (4.5), assuming of course the same initial data and the same time steps. Thus, (4.7) provides only an alternative for the evaluation of the iterate, except that the Lagrange multiplier is computed as well in (4.7). The solution process for (4.7) is as follows. Using (4.7*a*), eliminate $\psi_1^\#$ rectangle by rectangle. Then use (4.7*b*) to eliminate $u^\#$, so that $m_1^\#$ can be evaluated by solving block tridiagonal systems over each horizontal set of rectangles ; $u^\#$ and $\psi_1^\#$ can be evaluated from $m_1^\#$. Next, use (4.7*d*)-(4.7*f*) to find m_2^{n+1}, u^{n+1}, and ψ_2^{n+1}. Correct ψ_1 and m_1 if this is desired.

The first cycle of time steps will take somewhat longer with (4.7) than with (4.5). If the eliminations of ψ_1, $u^\#$, ψ_2, and u^{n+1} are retained for each step in the first cycle, subsequent cycles will require much less calculation than for the corresponding cycles for (4.5). If as many as three cycles are needed, (4.7) will result in a significant saving in work. This observation leads to the expectation that the optimum cycle length for the alternating-direction procedure in form (4.7) is shorter than that for (4.5) ; i.e., for (4.7) it is better to reduce the amount of work associated with the elimination of the vector and scalar variables and to increase the number of cycles to account for the lowered rate of convergence that comes from the shorter cycle. Experimentation will be carried out in the near future.

5. SUPERCONVERGENCE

Two varieties of superconvergence can be associated with the mixed method. One depends on the hybridized form of the method and is applicable for quasiregular polygonalizations of Ω ; the other makes no use of the Lagrange multiplier m_h and can provide very rapid convergence, but a translation invariant partition, at least locally, is required.

We begin with a discussion of the use of the Lagrange multiplier. The argument of Arnold and Brezzi [1] estimating the error $m_h - u|_e$ for the hybridized mixed method employing RT_{j-1}, $j > 0$, applies without alteration to (3.3). Let

$$|v|_{0,h}^2 = \sum_{e \in \mathscr{E}_h} \|v\|_{0,e}^2 , \tag{5.1a}$$

$$|v|_{-1/2,h}^2 = \sum_{e \in \mathscr{E}_h} |e| \, \|v\|_{0,e}^2 . \tag{5.1b}$$

Theorem 1.4 of [1], as interpreted for (1.4)-(2.8) for BDFM_j, states that

$$\|m_h - P_h u\|_{0,e} \leqslant M(h_K^{1/2} \|\psi - \psi_h\|_{0,K} + h_K^{1/2} \|u_h - Q_h u\|_{0,K}) , \tag{5.2a}$$

$$|m_h - P_h u|_{-1/2,h} \leqslant M(h \|\psi - \psi_h\|_0 + \|u_h - Q_h u\|_0) , \tag{5.2b}$$

where P_h is the L^2-projection onto $P_{j-1}(\mathscr{E}_h)$ and h_K is the diameter of the element K. It follows from Theorem 2.2 that

$$|m_h - P_h u|_{-1/2,h} \leq M \|u\|_{r+1} h^{r+1}, \quad 1 \leq r \leq j \tag{5.3}$$

for $j > 1$; for $j = 1$ it is necessary to replace $r + 1$ by $r + 2$ in the Sobolev index to obtain the same rate of convergence.

The error estimate (5.3) contains the key for constructing superconvergent approximations of the scalar function u. In order to show it we describe first a general way for postprocessing u_h and m_h in order to obtain, element by element, the desired superconvergent approximation u_h^*. We then indicate some particular choices that will make the general way easier in our case.

Let K be an element in K_h and introduce the space

$$\Lambda(j, K) = \{m \in L^2(\partial K), m_{|e} \in P_{j-1}(e), e \in \partial K\} .$$

Let us assume now that we are given a finite dimensional space $W^*(j, K)$; we shall indicate later on some convenient choices for $W^*(j, K)$; for the moment we just assume that the following property holds

$$\inf_{m \in \Lambda(j,K) - \{0\}} \sup_{v \in W^*(j,K) - \{0\}} \frac{\langle v, m \rangle_{\partial K}}{|||v||| \, |||m|||} \geq c > 0 \tag{5.4}$$

with c independent of K and h. The triple-bar norms that appear in (5.4) are defined as follows

$$|||v||| = h_k^{-1} |v|_{0,K} + |v|_{1,K} \tag{5.5}$$

$$|||m||| = h_K^{1/2} |m|_{0,\partial K} . \tag{5.6}$$

Note that (5.4), in some sense, requires that the space $W^*(j, K)$ is big enough. Now we can use in K a local version of Babuska's method [2] for the Dirichlet problem, and define $u_h^* \in W^*(j, K)$ and $z_h^* \in \Lambda(j, K)$ as the solution of the equations

$$(a \nabla u_h^*, \nabla v)_K - \langle z_h^*, v \rangle_{\partial K} = (f, v)_K, \; v \in W^*(j, K) \tag{5.7a}$$

$$\langle m, u_h^* \rangle_{\partial K} = \langle m, m_h \rangle_{\partial K}, \; m \in \Lambda(j, K) \tag{5.7b}$$

where, for a boundary element K, m_h on $\partial\Omega$ is replaced (here and in the following) by the boundary values $-g$. Standard results in the approximation of saddle-points (see e.g., Brezzi [5]) yield now

$$|||u - u_h^*||| + |||a \, \partial u/\partial n - z_h^*||| \leq M \left\{ \inf_{v \in W^*(j,K)} |||u - v||| + \right.$$

$$\left. + \inf_{m \in \Lambda(j,K)} |||a \, \partial u/\partial n - m||| + \sup_{m \in \Lambda(j,K)} |\langle m, m_h - u \rangle_{\partial K}| / |||m||| \right\} . \tag{5.8}$$

If we assume that $W^*(j, K)$ contains $P_j(K)$ then

$$\inf_{v \in W^*(j,K)} |||\, u - v \,||| \leq M \, \|u\|_{j+1,K} \, h_K^j \,. \tag{5.9}$$

On the other hand it is easy to see that

$$\inf_{m \in \Lambda(j,K)} |||\, a \, \partial u/\partial n - m \,||| \leq M \, \|u\|_{j+1,K} \, h_K^j \,. \tag{5.10}$$

Finally one has

$$\sup_{m \in \Lambda(j,K)} \left\{ \left| \langle m, u - m_h \rangle_{\partial K} \right| / |||\, m \,||| \right\} \leq h_K^{-1/2} |m_h - P_h u|_{0,\partial K} \,. \tag{5.11}$$

Collecting (5.8)-(5.11) and using (5.3) we finally obtain, for $j > 1$

$$\|u - u_h^*\|_0 \leq M(\|f\|_{j-1} + |g|_{j+1/2}) \, h^{j+1} \,; \tag{5.12}$$

for $j = 1$ the regularity indices on f and g must be increased by one, as in (5.3).

Note that, in general, (5.7) might be unconvenient due to the high number of degrees of freedom and to the fact that its matrix is indefinite. Moreover we still have to find examples in which (5.4) is satisfied. However a simple scaling procedure shows that (5.4) is satisfied if and only if we can use in $W^*(j, K)$ (among others) the degrees of freedom

$$\langle v, m \rangle \;, m \in \Lambda(j, K) \,. \tag{5.13}$$

If this is the case, we can set, for $\chi = m_h$ or $\chi = 0$

$$W_\lambda = \{v \in W^*(j, K) \,, \langle v - \chi, m \rangle = 0 \,, m \in \Lambda(j, K)\} \tag{5.14}$$

and compute u_h^* by means of : $u_h^* \in W_{m_h}$ and

$$(a \, \nabla u_h^*, \nabla v)_K = (f, v)_K \,, \quad v \in W_0 \tag{5.15}$$

which is much more convenient than (5.7). We have now a clear indication on the choice of $W^*(j, K)$: it must be a space containing $P_j(K)$ and such that the degres of freedom (5.13) are linearly independent in $W^*(j, K)$. For instance as in [7] we can set

$$W^*(j, K) = P_j(K) + \text{Span}\,\{x^j y, xy^j, q^j\} \tag{5.16a}$$

where

$$q^j(x, y) = \begin{cases} x^{j+1} y - x y^{j+1} \,, & j \text{ even} \geq 2 \\ x^{j+1} - y^{j+1} \,, & j \text{ odd} \geq 3 \,, \end{cases} \tag{5.16b}$$

while for $j = 1$ we take

$$W^*(1, K) = P_1 + \text{Span}\{x^2, y^2\} = P_2 \backslash \{xy\} \ . \tag{5.16c}$$

It is clear that $P_j(K) \subset W^*(j, K)$. Moreover in $W^*(1, K)$ we can take as degrees of freedom

$$\langle v, m \rangle_{\partial K} \ , \ m \in \Lambda(1, K) \tag{5.17a}$$

$$(v, 1)_K \ . \tag{5.17b}$$

For $j > 1$ degrees of freedom for $v \in W^*(j, K)$ can be taken as follows [5] :

$$\langle v, m \rangle_{\partial K} \ , \quad m \in \Lambda(1, K) \tag{5.18a}$$

$$(v, w)_K, \ w \in \{P_{j-4}(K) + \text{Span}\ (l_{j-2}(x)\, l_{j-2}(y))\} \tag{5.18b}$$

where $l_r(t)$ is a convenient affine image of the Laguerre or Gegenbauer polynomial $C_r^{(3/2)}$ of degree r [20] on $[-1, 1]$. Thus it is the r^{th} degree polynomial in the sequence orthogonal on $[-1, 1]$ with respect to the weight function $1 - t^2$:

$$\int_{-1}^{1} l_r(t)\, l_s(t)\ (1 - t^2)\, dt = \delta_s \ . \tag{5.18c}$$

Other more convenient *ad hoc* choices could be found for any fixed $j > 1$.

Note that, if in $W^*(j, K)$ we can use degrees of freedom of the form

$$\langle v, m \rangle \quad m \in \Lambda(j, K) \tag{5.19a}$$

$$(v, \tilde{w})_K \quad w \in Q(j, K) \tag{5.19b}$$

where $Q(j, K)$ is a subset of $P_{j-1}(K)$ (obviously with $dim\ (W^*(j, K)) = 3\, j + \dim\ (Q(j, K))$) then one can also consider in $W^*(j, K)$ a different superconvergent approximation u_h^* defined by

$$\langle u_h^* - m_h, m \rangle = 0 \ , \qquad m \in \Lambda(j, K) \ , \tag{5.20a}$$

$$(u_h^* - u_h, w)_K = 0 \ , \qquad w \in Q(j, K) \ . \tag{5.20b}$$

As in [1] it is easy to show, using (2.15*d*) and (5.3), that (5.12) still holds for this new choice of u_h^*.

Let us turn now to a different post-processing technique to obtain superconvergence. Let Ω_1 be a compact subdomain of Ω, and assume that $\mathcal{K}_h$ is translation invariant over Ω_1 ; i.e., assume that there exists $\mathbf{h} = (h_1, h_2)$, $h_i > 0$ and comparable with h, such that

$$K + (\theta_1 h_1, \theta_2 h_2) \in \mathcal{K}_h$$

for each $K \in \mathscr{K}_h$ such that $K \subset \Omega_1$ and $\theta_i = \pm 1$. Then let

$$K_h(x_1, x_2) = K_{h,j}^{2j}(x_1, x_2)$$

be the Bramble-Schatz kernel [3] ; [4], (5.12)] that results from allowing $h_1 \neq h_2$. Form the convolutions

$$\psi_h^* = K_h^* \psi_h , \quad u_h^* = K_h^* u_h . \tag{5.21}$$

The interior estimates of Theorem 2.3 and the analogous ones for difference quotients can be applied in a fashion very similar to the argument in Section 5 of [13] to show that, if $\Omega_2 \subset \Omega_1 \subset \Omega$,

$$\|u - u_h^*\|_{0,\Omega_2} \leq M[\|u\|_{r+s,\Omega_1} h^{r+s-(s-j+2)^+} + \|u\|_{q,\Omega} h^{q+j-2}] \tag{5.22a}$$

for $2 \leq r \leq j + (s - j + 2)^+$, $0 \leq s \leq j$, and $2 \leq q \leq j + 2$. Also,

$$\|\psi - \psi_h^*\|_{0,\Omega_2} \leq M[\|u\|_{r,\Omega_1} h^{r+s-1-(s-j+1)^+} + \|u\|_{q,\Omega} h^{q+j-2}] \tag{5.22b}$$

for $2 \leq r \leq j + 1 + (s - j + 1)^+$, $0 \leq s \leq j$, and $2 \leq q \leq j + 2$. Note that the maximum rate of convergence is $O(h^{2j})$ for both u_h^* and ψ_h^*. For $j > 1$, this rate of convergence for u_h^* exceeds that obtainable through the direct use of the Lagrange multiplier ; however, the additional constraint of a translational invariant partition, at least locally, has been induced.

PART II

6. THE REDUCED BDDF ELEMENTS

The three-dimensional rectangular elements formed by restriction analogous to that of (2.3) of the BDDF elements can be described more simply by proceeding directly to the type of characterization given in Lemma 2.1. First, let $x_i' = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_3)$ and denote by $P_j(\text{hom}, i)$ the homogeneous polynomials of degree j in the variables x_i'. Then, if K is a rectangle in three space, let

$$\Psi(j, K) = [P_j \backslash P_j(\text{hom}, 1)] \times [P_j \backslash P_j(\text{hom}, 2)] \times [P_j \backslash P_j(\text{hom}, 3)]|_K , \tag{6.1a}$$

$$W(j, K) = P_{j-1}(K) , \tag{6.1b}$$

$$\text{BDFM}_j(K) = \Psi(j, K) \times W(j, K) . \tag{6.1c}$$

It can be seen that $\Psi(j, K)$ consists exactly of those elements of $\mathbf{V}(j, K)$ of [[6], (3.1a)] having normal components lying in $P_{j-1}(e)$ for each of the six faces e of K. Moreover, it is clear that a set of degrees of freedom for $\Psi(j, K)$ is given by

$$\langle \psi \cdot \nu, p \rangle_e, \quad p \in P_{j-1}(e), \text{ for each face } e \text{ of } K, \tag{6.2a}$$

$$(\psi, \chi)_K, \quad \chi \in P_{j-2}(K)^3. \tag{6.2b}$$

Boundary elements are allowed to have a curved face, as well as possibly nonrectangular flat faces. The space $\mathrm{BDFM}_j(K)$ is taken to be the same as on an ordinary rectangular element. The degrees of freedom for $\Psi(j, K)$ can be assigned as

$$\langle \psi \cdot \nu, p \rangle_e, \quad p \in P_{j-1}(e) \text{ for each flat face } e, \tag{6.3a}$$

$$(\operatorname{div} \psi, w)_K, \quad w \in P_{j-1}(K), \tag{6.3b}$$

$$(\psi, \chi)_K, \quad \chi \in \{\gamma \in \psi(j, K) : \operatorname{div} \gamma = 0 \text{ and } \gamma \cdot \nu = 0 \text{ on each flat face of } K\}. \tag{6.3c}$$

The degrees of freedom associated with (6.2) or (6.3) allow the construction of a projection $\Pi_h \times Q_h$ with exactly the same properties as are valid for the corresponding projection in the two-dimensional case. Consequently, the error estimates of Theorems 2.2 and 2.3 hold without modification in form for the spaces derived from (6.1).

If $\mathscr{E}_h$ now denotes the set of faces of elements of $\mathscr{K}_h$ not lying in $\partial\Omega$ and if Λ_h consists of functions over $\mathscr{E}_h$ that restrict to polynomials of degree $j-1$ on each face in $\mathscr{E}_h$, then the hybridization procedure (3.3) carries over to the three-dimensional case. The local version of Babuska method as given above by (5.7) can be applied in the three space case to find a superconvergent approximation $u_h^* \in W^*(j, K)$. A general construction of suitable $W^*(j, K)$'s has not been considered for general j : however many *ad hoc* choices can be made ; for instance for $j = 1$ we can take

$$W^*(1, K) = P_1(K) + \mathit{Span}\,\{x^2, y^2, z^2\} \tag{6.4}$$

with the degrees of freedom

$$\langle v, m \rangle_{\partial K}, \quad m \in \Lambda(1, K) \ (= \textit{constant on each face}) \tag{6.5a}$$

$$(v, w)_K, \quad w \in P_0(K). \tag{6.5b}$$

Note that the degrees of freedom (6.5) have the particular (5.19) so that (5.20) can also be used as an alternative procedure for defining u_h^*. In both cases we have the error bound

$$\|u - u_h^*\|_0 \leq M(\|f\|_1 + |g|_{3/2})\, h^2. \tag{6.6}$$

Before turning to the solution of the algebraic problem arising from this mixed method, let us compare the elements of (6.1) with the RT_{j-1} and

BDDF_{j-1} elements that provide the same optimal order of accuracy asymptotically for the vector variable. The table below indicates the local dimensions of the spaces.

Local Dimensions

	Vector	Scalar	Hybridization
$\mathrm{BDDF}_{j-1}(K)$	$(j^3+3j^2+8j)/2$	$(j^3-j)/6$	$j(j+1)/2$
$\mathrm{BDFM}_j(K)$	$(j^3+6j^2+5j)/2$	$(j^3+3j^2+2j)/6$	$j(j+1)/2$
$\mathrm{RT}_{j-1}(K)$	$3(j^3+j^2)$	j^3	j^2

We shall discuss an alternating-direction iteration method in the next section ; the considerations below will be seen to be applicable there. Consider the hybridized form of the method. In the methods associated with $\mathrm{BDFM}_j(K)$ and $\mathrm{RT}_{j-1}(K)$, the equations can be written in the form (corresponding to (4.6))

$$A_i \psi_i + B_i u + C_i m_i = \gamma_i , \quad i = 1, 2, 3 , \tag{6.7a}$$

$$\sum_{i=1}^{3} B_i^* \psi_i = \varphi , \tag{6.7b}$$

$$C_i^* \psi_i = 0 , \quad i = 1, 2, 3 , \tag{6.7c}$$

where, e.g., A_i is a square matrix of size $\dim \Psi(j, K)/3$. Since the dimensions of both the vector and scalar spaces for $\mathrm{BDFM}_j(K)$ are significantly smaller than those for $\mathrm{RT}_{j-1}(K)$ for $j > 1$, it is clear that the elimination of the vector and scalar parameters is cheaper for $\mathrm{BDFM}_j(K)$ than for $\mathrm{RT}_{j-1}(K)$; asymptotically, the work ratio tends to $1/6^3 = 1/216$. The resulting equations

$$Dm = \beta \tag{6.8}$$

have the same block graph structure ; since the block size is much smaller for BDFM_j than for RT_{j-1}, it seems very likely that the mixed method based on $\mathrm{BDFM}_j(K)$ will be more efficient (i.e., same accuracy for less work) than that based on $\mathrm{RT}_{j-1}(K)$.

The comparison between $\mathrm{BDDF}_{j-1}(K)$ and $\mathrm{BDFM}_j(K)$ is not quite so clear. The dimension of the vector space for BDDF_{j-1} is smaller than that for BDFM_j, but its Grammian matrix is considerably less sparse than that for BDFM_j. On the other hand, the dimension of the scalar space for BDDF_j is smaller than that for BDFM_j, so that the second elimination step is cheaper for BDDF_j. The graph structures for the D-matrices for the two methods are the same. Thus, for the model Laplace equation there is no large difference in the work required by the two methods for the same

accuracy in L^2. If post-processing is feasible to increase accuracy, the BDFM_j method should lead to less work because of better superconvergence.

7. ALTERNATING-DIRECTION ITERATION

The algebraic equations (6.8) are very expensive to solve by elimination ; since their band width is $O((\dim \Lambda_h)^{2/3})$, they are much costlier than the corresponding ones for the same number of parameters in the two-dimensional case. Moreover, it should be expected that more are needed in the three-dimensional case. Assume that $\mathscr{K}_h$ contains only ordinary rectangles, and consider the transient problem induced from the hybridized form (6.7) given by

$$\theta A_i\, \partial\psi_i/\partial t_1 + A_i\, \psi_i + B_i\, u + C_i\, m_i = \gamma_i\,, \quad i = 1, 2, 3\,, \tag{7.1a}$$

$$D\, \partial u/\partial t_2 + \sum_{i=1}^{3} B_i^*\, \psi_i = \varphi\,, \tag{7.1b}$$

$$C_i^*\, \psi_i = 0\,, \quad i = 1, 2, 3\,. \tag{7.1c}$$

If $\theta = 0$, only u_h must be specified initially ; if $\theta = 1$, both u_h and ψ_h must be given, with the initial values for ψ_h restricted to satisfy (7.1*c*), since no time derivative of m_h was introduced and the requirement that the function ψ_h lie in $H(\mathrm{div}, \Omega)$ was retained.

The alternating-direction iteration scheme formulated below includes the concepts of both the Uzawa ($\theta = 0$) and Arrow-Hurwitz ($\theta = 1$) versions of the procedure introduced by Douglas, Durán, and Pietra [11] for other mixed finite element spaces. It is presented here so as to treat the hybridized form of the equations ; again we borrow from unpublished work of Douglas and Pietra [15]. As in [11], the method is based on an old alternating-direction method of Douglas [10] for finite difference methods for parabolic or elliptic problems in three space variables. An analysis of the Uzawa version was carried out in [11] for the RT_0-space, which is the space generated by $\mathrm{BDFM}_1(K)$, applied to the model problem of the Laplace operator on a unit cube. Other theoretical and several experimental results appear there and in [12].

The method is as follows. Let τ_1 be positive or infinite for $\theta = 1$ or $\theta = 0$, respectively. Then let $\{\tau_2^n, n = 1, \ldots, N\}$ be such that $0 < \tau_2^1 < \cdots < \tau_2^N$; take $\tau_2^{n+kN} = \tau_2^n$. Initialize u_h and, if $\theta = 1$, initialize ψ_h consistent with (7.1*c*). The general step of the iteration is defined by the following algorithm :

x_1-sweep

$$A_1[\theta(\psi_1^{\#} - \psi_1^n)/\tau_1 + \psi_1^{\#}] + B_1\, u^{\#} + C_1\, m_1^{\#} = \gamma_1\,, \tag{7.2a}$$

$$(7.2b)\quad D(u^{\#}-u^n)/\tau_2^n + 0.5\,B_1^*(\psi_1^{\#}+\psi_1^n) + B_2^*\,\psi_2^n + B_3^*\,\psi_3^n = \varphi\,,$$

$$(7.2c)\quad C_1^*\,\psi_1^{\#} = 0\,;$$

x_2-sweep

$$(7.2d)\quad A_2[\theta(\psi_2^{\#}-\psi_2^n)/\tau_1 + \psi_2^{\#}] + B_2\,u^{\#\#} + C_2\,m_2^{\#} = \gamma_2\,,$$

$$(7.2e)\quad D(u^{\#\#}-u^n)/\tau_2^n + 0.5\,B_1^*(\psi_1^{\#}+\psi_1^n) + 0.5\,B_2^*(\psi_2^{\#}+\psi_2^n) + B_3^*\,\psi_3^n = \varphi\,,$$

$$(7.2f)\quad C_2^*\,\psi_2^{\#} = 0\,;$$

x_3-sweep

$$(7.2g)\quad A_3[\theta(\psi_3^{n+1}-\psi_3^n)/\tau_1 + \psi_3^{n+1}] + B_3\,u^{n+1} + C_3\,m_3^{n+1} = \gamma_3\,,$$

$$(7.2h)\quad D(u^{n+1}-u^n)/\tau_2^n + 0.5\sum_{i=1}^{2} B_i^*(\psi_i^{\#}+\psi_i^n) + 0.5\,B_3^*(\psi_3^{n+1}+\psi_3^n) = \varphi\,,$$

$$(7.2i)\quad C_3^*\,\psi^{n+1} = 0\,;$$

Next, either accept $\psi_1^{\#}$, $m_1^{\#}$, $\psi_2^{\#}$, and $m_2^{\#}$ as the new values or perform a correction ; i.e., take either

$$(7.2j)\quad \psi_1^{n+1} = \psi_1^{\#}\,,\quad \psi_2^{n+1} = \psi_2^{\#}\,,\quad m_1^{n+1} = m_1^{\#}\,,\quad m_2^{n+1} = m_2^{\#}\,,$$

or

$$(7.2k)\quad A_1[\theta(\psi_1^{n+1}-\psi_1^n)/\tau_1 + \psi_1^{n+1}] + B_1\,u^{n+1} + C_1\,m_1^{n+1} = \gamma_1\,,$$

$$(7.2l)\quad C_1^*\,\psi_1^{n+1} = 0\,;$$

$$(7.2m)\quad A_2[\theta(\psi_2^{n+1}-\psi_2^n)/\tau_1 + \psi_2^{n+1}] + B_2\,u^{n+1} + C_2\,m_2^{n+1} = \gamma_2\,,$$

$$(7.2n)\quad C_2^*\,\psi_2^{n+1} = 0\,.$$

Computational experience has shown that a good choice for the matrix D is given by the Grammian over W_h with respect to the weight function $a(x)$, the coefficient in the differential equation, as remarked earlier for the somewhat different alternating-direction procedure in the two-dimensional case. See [11] for a discussion of the selection of the virtual timesteps τ_1 and $\{\tau_2^n\}$. Note that requiring (6.7*c*) to be satisfied forces the functions ψ^n and u^n to coincide with those obtained by the alternating-direction methods of [11] for the standard, rather than hybridized, form of the mixed method.

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