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EFFICIENT SOLUTION CONCEPTS AND THEIR RELATIONS IN STOCHASTIC
MULTIOBJECTIVE PROGRAMMING

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ABSTRACT

In this work different concepts of efficient solutions to problems of Stochastic Multiple Objective Programming are analyzed. We centre our interest on problems in which some of the objective functions depend on random parameters. The existence of different concepts of efficiency for one single stochastic problem, such as expected value efficiency, minimum-risk efficiency, etc., raise the question of their "quality". Starting from this idea we establish some relationships between the different concepts. Our study enables us to determine what type of efficient solutions are obtained by each of these concepts.

Key Words. Stochastic multiobjective programming, expected value efficiency, minimum variance efficiency, minimum-risk efficiency, efficiency in probability.

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1. Introduction

In many multicriteria decision problems some parameters take unknown values at the moment of making the decision. This uncertainty can be due to problems of observing the parameters themselves or that their values depend on such factors as nature, the decisions of other agents, etc. If these parameters are random variables, the resulting problem is called *Stochastic Multiobjective Programming*.

There is much research in the literature dealing with the study of such problems, among which we could mention the books by Goicoechea, Hansen and Duckstein (Ref. 1), Stancu-Minasian (Ref. 2), and Slowinski and Teghem (Ref. 3), and the articles by Teghem, Dufrane, Thauvoye and Kunsch (Ref. 4), Stancu-Minasian and Tigan (Ref. 5), Stancu-Minasian (Ref. 6), Urii and Nadeau (Ref. 7), and Ben Abdelaziz, Lang and Nadeau (Refs. 8 and 9).

The review of these works shows that the resolution of such problems always involves transforming the problem into a deterministic one, which is called the *equivalent deterministic problem*. This transformation is carried out by using some statistical characteristic of the random variables involved in the problem (expected value, variance, etc.). In the literature this has led to different definitions of the efficient solution concept for the same stochastic multiobjective programming problem, so that different efficient sets are obtained for the same stochastic problem. These sets, in principle, are *non-comparable* since they utilize different characteristics of the initial problem. In this work we consider five such sets: *expected value*, *minimum variance*, *expected value minimum standard deviation*, *minimum-risk efficiency*, and *efficiency in probability* and raise the following question: do these efficiency sets have some relationship to each other? An initial answer is given in Caballero, Cerdá, Muñoz and Rey (Ref.

10), in which relations between two of the efficient sets were established. Specifically, it is shown that, given certain conditions, the minimum-risk efficient solution set and the efficiency in probability set are *reciprocal*. This paper deals with the analysis of other relations between the previously mentioned sets. On the one hand we relate the concept of expected value-standard deviation efficiency to that of expected value efficiency and minimum standard deviation efficiency, and on the other hand, relate the concept of efficiency in probability to the expected value-standard deviation efficiency.

2. Efficient Solution Concepts for a Stochastic Multiobjective Programming Problem

Let us consider the stochastic multiobjective programming problem:

$$(SMP) \quad \min_{x \in D} \left(\tilde{z}_1(x, \tilde{c}), \tilde{z}_2(x, \tilde{c}), \dots, \tilde{z}_q(x, \tilde{c}) \right)$$

where:

- $x \in R^n$ is the vector of decision variables of the problem and \tilde{c} is a random vector whose components are random continuous variables, defined on the set $E \subset R^q$. Let us assume that given the family F of events, that is, subsets of E , and the distribution of probability P defined on F , so that for any subset of E , $A \subset E$, $A \in F$, the probability of A , $P(A)$, is known. Also, we assume that the distribution of probability, P , is independent of the decision variables x_1, \dots, x_n .
- the functions $\tilde{z}_1(x, \tilde{c}), \tilde{z}_2(x, \tilde{c}), \dots, \tilde{z}_q(x, \tilde{c})$ are defined on $R^n \times E$,
- the set of the problem opportunities, $D \subset R^n$, is non-empty, compact and convex.

Let $\bar{z}_k(\mathbf{x})$ denote the expected value of the k -th objective function and $\sigma_k(\mathbf{x})$ its standard deviation, $k \in \{1, 2, \dots, q\}$. Let us assume that for every $k \in \{1, 2, \dots, q\}$ and for every feasible vector \mathbf{x} of the (SMP) problem, the standard deviation, $\sigma_k(\mathbf{x})$, is finite.

As previously pointed out, in the literature different concepts of efficient solution exist for the (SMP) problem. In this section we present some which share the common feature that the efficient solution concept is defined from a multiobjective problem which is generated by applying the same criterion to all the stochastic objectives separately.

The first concept to look at is the expected value efficient solution. This concept is obtained from the construction of a multiobjective problem in which the objective functions vector is the expected value of the vector of stochastic objectives of the initial problem.

Definition 2.1: Expected value efficient solution. The point $\mathbf{x} \in D$ is an expected value efficient solution of the (SMP) problem if it is Pareto efficient to the following problem:

$$(E) \quad \text{Min}_{\mathbf{x} \in D} \left(\bar{z}_1(\mathbf{x}), \bar{z}_2(\mathbf{x}), \dots, \bar{z}_q(\mathbf{x}) \right)$$

Let E_E be the set of expected value efficient solutions of the (SMP) problem.

The next concept considered is that of the minimum variance efficient solution. In this case, the concept comes from obtaining the variance of each stochastic objective and outlining the multiobjective problem of minimizing such variances.

Definition 2.2. Minimum variance efficient solution. The point $\mathbf{x} \in D$ is a minimum variance efficient solution for the Stochastic Multiobjective Programming (SMP) problem if it is a Pareto efficient solution for the problem:

$$(\sigma^2) \quad \text{Min}_{\mathbf{x} \in D} \left(\sigma_1^2(\mathbf{x}), \sigma_2^2(\mathbf{x}), \dots, \sigma_q^2(\mathbf{x}) \right)$$

Let E_{σ^2} be the set of efficient solutions of the problem (σ^2) .

Next, we define the concept of expected value-standard deviation efficiency. In this case, the concept of efficiency arises from the construction of a problem with $2q$ objectives involving the expected value and the standard deviation of each stochastic objective.

Definition 2.3. Expected value-standard deviation efficient solution or $E\sigma$ efficient solution.

The point $\mathbf{x} \in D$ is an expected value-standard deviation efficient solution for the stochastic multiobjective programming problem if it is a Pareto efficient solution to the problem:

$$(E\sigma) \quad \text{Min}_{\mathbf{x} \in D} \left(\bar{z}_1(\mathbf{x}), \dots, \bar{z}_q(\mathbf{x}), \sigma_1(\mathbf{x}), \dots, \sigma_q(\mathbf{x}) \right)$$

Let $E_{E\sigma}$ be the set of expected value-standard deviation efficient solutions of the (SMP) problem.

Finally, we give the concepts of efficiency for the criteria of maximum probability. As we will see next, in order to define these two concepts, the minimum-risk criterion (concept of minimum-risk efficiency) and the Kataoka criterion (efficiency in probability) are respectively applied to each stochastic objective.

Definition 2.4. Minimum-risk efficient solution for levels u_1, u_2, \dots, u_q (Stancu-Minasian and Tigan (Ref. 5)).

The point $\mathbf{x} \in D$ is a minimum-risk vectorial solution for levels u_1, u_2, \dots, u_q if it is a Pareto efficient solution to the problem

$$(MR(u)) \quad \text{Max}_{\mathbf{x} \in D} \left(P(\bar{z}_1(\mathbf{x}, \tilde{c}) \leq u_1), \dots, P(\bar{z}_q(\mathbf{x}, \tilde{c}) \leq u_q) \right)$$

Let $E_{MR}(u)$ be the set of efficient solutions for the problem $(MR(u))$.

Definition 2.5. Efficient solution with probabilities $\beta_1, \beta_2, \dots, \beta_q$ or β -efficient solution

The point $x \in D$ is an efficient solution with probabilities $\beta_1, \beta_2, \dots, \beta_q$ if there exists $u \in R^q$

such that (x^*, u^*) is a Pareto efficient solution to the problem

$$(K(\beta)) \quad \begin{aligned} &\text{Min}_{x, u} \quad (u_1, \dots, u_q) \\ &\text{s.t.} \quad P\{\tilde{z}_k(x, \tilde{c}) \leq u_k\} \geq \beta_k, \quad k = 1, 2, \dots, q \\ &\quad \quad x \in D \end{aligned}$$

$E_K(\beta) \subset R^n$ denotes the set of efficient solutions with probabilities $\beta_1, \beta_2, \dots, \beta_q$ for the

(SMP) problem.

It may be noted that these definitions of efficient solutions are obtained by applying the same transformation criterion to each one of the stochastic objectives separately (expected value, minimum variance, etc.), and afterwards building the resulting deterministic multiobjective problem. In this sense it is necessary to point out that:

- The concepts of expected value, minimum variance, etc. weak and properly efficient solution can be defined in a natural way.
- The concepts of minimum-risk efficiency and β -efficiency require setting, a priori, a vector of aspiration levels (u) or a probability vector (β). This implies that, in both cases, the efficient set obtained depends on the predetermined vectors, in such a way that, in general, different level and probability vectors give rise to different efficient sets:

$$u \neq u' \Rightarrow E_{AR}(u) \neq E_{AR}(u') \quad \beta \neq \beta' \Rightarrow E_K(\beta) \neq E_K(\beta')$$

- The concept of expected value-standard deviation efficient solution is no more than an extension to multiobjective cases of the concept of the mean variance efficient solution

that Markowitz defines (Ref. 11) for the stochastic mono-objective problem of the portfolio selection. Note that in the concept we have just defined, instead of using the variance of each stochastic objective, we take the standard deviation. In this way we have the two statistical moments corresponding to each stochastic objective in the same measuring units. Since the square root function is strictly increasing, the set of efficient solutions does not vary in the problem if we substitute standard deviation for variance (see White (Ref. 12)).

- The efficiency in probability criterion is a generalization of the one presented by Goicoechea, Hansen and Duckstein (Ref. 1) that defines the same concept taking the same probability, β , for all the stochastic objectives and with the probabilistic equality constraints taking the form: $P\{\tilde{z}_k(x, \tilde{c}) \leq u_k\} = \beta_k$. This notion was introduced by Stancu-Minasian (Ref. 6) considering Kataoka's problem in the case of multiple criteria.

Starting from the given definitions, we can obtain different sets of efficient solutions for the (SMP) problem. This fact can give rise to some confusion. Although to begin with, one could consider that the concepts previously defined are not connected, since they utilize different statistical characteristics of the stochastic objectives, it will be shown that they are closely related. We begin by studying the relationship between the expected value-standard deviation efficient solutions and the efficient solutions of probabilities β_1, \dots, β_q . Next, we analyze the existence of relationships between minimum expected value efficient solutions, minimum variance efficiency and the expected value-standard deviation efficient solutions, corresponding to the problem of $2q$ objectives that includes both the expected value and the standard deviation

of the objectives. We will first present some results of efficient sets for deterministic multiobjective programming problems that will be used to establish the relations in stochastic multiple objective problems.

3. Preliminary Results

We will now present some relations between efficient sets of several problems of deterministic multiobjective programming. These results will later be used for the analysis of concepts of efficient solutions for multiple objective stochastic problems.

Let f and g be vectorial functions defined on the same set $H \subset \mathbb{R}^n$, $f: H \subset \mathbb{R}^n \rightarrow \mathbb{R}^q$ and $g: H \subset \mathbb{R}^n \rightarrow \mathbb{R}^q$, and let α, γ be non-null vectors with q real components, that is, $\alpha, \gamma \in \mathbb{R}^q$, $\alpha, \gamma \neq 0$ and let us consider the following multiobjective problems:

$$\text{Min}_{x \in D} (f_1(x), \dots, f_q(x), \gamma_1 g_1(x), \dots, \gamma_q g_q(x)) \quad (1)$$

$$\text{Min}_{x \in D} (f_1(x), \dots, f_q(x)) \quad (2)$$

$$\text{Min}_{x \in D} (\gamma_1 g_1^2(x), \dots, \gamma_q g_q^2(x)) \quad (3)$$

with $\gamma \in \mathbb{R}^q$, $\gamma \neq 0$. Let $E(f, \gamma g)$, $E(f)$ and $E(\gamma g^2)$ be the sets of efficient points of problems (1), (2) and (3), respectively. The following theorem relates these problems to each other. With superscripts d and p we denote weak and proper efficiency, respectively.

Theorem 3.1. Let us assume that $g(x) > 0$ for every $x \in D$, then :

$$(i) E(f) \cap E(\gamma g^2) \subset E(f, \gamma g)$$

$$(ii) E(f) \cup E(\gamma g^2) \subset E^d(f, \gamma g)$$

$$(iii) E^d(f) \cup E^d(\gamma g^2) \subset E^d(f, \gamma g).$$

Proof.

(i) Let $x \in E(f) \cap E(\gamma g^2)$. Let us show that $x \in E(f, \gamma g)$ by *reductio ad absurdum*. We assume that $x \notin E(f, \gamma g)$. Then, there exist a $x^* \in D$, such that:

$$f_k(x^*) \leq f_k(x) \text{ and } \gamma_k g_k(x^*) \leq \gamma_k g_k(x) \text{ for every } k \in \{1, 2, \dots, q\}$$

there being an $s \in \{1, 2, \dots, q\}$ for which the inequality is strict:

$$f_s(x^*) < f_s(x) \text{ or } \gamma_s g_s(x^*) < \gamma_s g_s(x)$$

Thus $x \notin E(f)$, or $x \notin E(\gamma g^2)$, due to $\gamma_k g_k(x^*) \leq \gamma_k g_k(x)$ implies $\gamma_k g_k^2(x^*) \leq \gamma_k g_k^2(x)$,

contrary to $x \in E(f) \cap E(\gamma g^2)$.

(ii) Let $x \in E(f) \cup E(\gamma g^2)$. Let us see that $x \in E^d(f, \gamma g)$ by *reductio ad absurdum*:

We assume that $x \notin E^d(f, \gamma g)$. Then, there exist a vector, $x^* \in D$ that weakly dominates x and so verifies:

$$f_k(x^*) < f_k(x) \text{ and } \gamma_k g_k(x^*) < \gamma_k g_k(x) \text{ for every } k \in \{1, 2, \dots, q\},$$

thus, $x \notin E(f)$ and, due to $\gamma_k g_k(x^*) < \gamma_k g_k(x)$ implies $\gamma_k g_k^2(x^*) < \gamma_k g_k^2(x)$, $x \notin E(\gamma g^2)$, contrary to $x \in E(f) \cup E(\gamma g^2)$.

(iii) Let $x \in E^d(f) \cup E^d(\gamma g^2)$. Let us see that $x \in E^d(f, \gamma g)$, by *reductio ad absurdum*. We assume that $x \notin E^d(f, \gamma g)$. Then, there exist a vector $x^* \in D$ that weakly dominates x and, therefore, verifies that:

$$f_k(x^*) < f_k(x) \text{ and } \gamma_k g_k(x^*) < \gamma_k g_k(x) \text{ for every } k \in \{1, 2, \dots, q\},$$

thus $x \notin E^d(f)$ and, due to $\gamma_k g_k(x^*) < \gamma_k g_k(x)$ implies $\gamma_k g_k^2(x^*) < \gamma_k g_k^2(x)$, $x \notin E^d(\gamma g^2)$, contrary to $x \in E^d(f) \cup E^d(\gamma g^2)$.

From (iii), it is obvious that $E^d(f) \cap E^d(\gamma g^2) \subset E^d(f, \gamma g)$ is also verified. Furthermore, as $E(f) \subset E^d(f)$ and $E(\gamma g^2) \subset E^d(\gamma g^2)$, then, $E(f) \cup E(\gamma g^2) \subset E^d(f) \cup E^d(\gamma g^2)$, thus (ii) can be deduced from (iii).

Let us again consider the functions f and g . Let the problem be:

$$\text{Min}_{x \in D} (f_1(x) + \alpha_1 g_1(x), \dots, f_q(x) + \alpha_q g_q(x)) \quad (4)$$

Let $E(\alpha)$ and $E^p(\alpha)$ denote the efficient solutions set and the properly efficient set, respectively, for problem (4). We will now present some relations between these sets and the sets of efficient solutions and properly efficient sets for problem (1).

Theorem 3.2. For every $\alpha, \gamma \in R^q$ with $\alpha_k, \gamma_k \neq 0$ and $\text{sgn}(\alpha_k) = \text{sgn}(\gamma_k)$, $k = 1, 2, \dots, q$,

the following relation holds

$$E(\alpha) \subset E(f, \gamma g).$$

Proof. Let $x \in E(\alpha)$. Let us assume that $x \notin E(f, \gamma g)$. In this case there is a solution x^* that dominates x , that is:

$$f_k(x^*) \leq f_k(x) \quad \text{and} \quad \gamma_k g_k(x^*) \leq \gamma_k g_k(x) \quad \text{for every } k \in \{1, 2, \dots, q\}$$

and there exist at least one $s \in \{1, 2, \dots, q\}$, that is:

$$f_s(x^*) < f_s(x) \quad \text{or} \quad \gamma_s g_s(x^*) < \gamma_s g_s(x).$$

From this point onwards, since $f_k(x^*) \leq f_k(x)$, $\gamma_k g_k(x^*) \leq \gamma_k g_k(x)$ and $\text{sgn}(\alpha_k) = \text{sgn}(\gamma_k)$, the following inequalities are verified:

$$f_k(x^*) + \alpha_k g_k(x^*) \leq f_k(x) + \alpha_k g_k(x) \quad \text{for every } k \in \{1, 2, \dots, q\} \quad (5)$$

$$f_k(x) + \alpha_k g_k(x^*) \leq f_k(x) + \alpha_k g_k(x) \quad \text{for every } k \in \{1, 2, \dots, q\} \quad (6)$$

From (5) and (6) we obtain:

$$f_k(x^*) + \alpha_k g_k(x^*) \leq f_k(x) + \alpha_k g_k(x) \quad \text{for every } k \in \{1, 2, \dots, q\}$$

In particular, for $k = s$:

- if $f_s(x^*) < f_s(x)$ the following is verified:

$$f_s(x^*) + \alpha_s g_s(x^*) < f_s(x) + \alpha_s g_s(x)$$

and, the following inequality is obtained from (6):

$$f_s(x^*) + \alpha_s g_s(x^*) < f_s(x) + \alpha_s g_s(x)$$

- if $\gamma_s g_s(x^*) < \gamma_s g_s(x)$ the following is verified:

$$f_s(x^*) + \alpha_s g_s(x^*) < f_s(x^*) + \alpha_s g_s(x)$$

and since $f_s(x^*) \leq f_s(x)$ we obtain:

$$f_s(x^*) + \alpha_s g_s(x^*) < f_s(x) + \alpha_s g_s(x)$$

Therefore, for every $k \in \{1, 2, \dots, q\}$ the following is verified:

$$f_k(x^*) + \alpha_k g_k(x^*) \leq f_k(x) + \alpha_k g_k(x)$$

and there is at least a subscript $s \in \{1, 2, \dots, q\}$ for which the inequality is strict:

$$f_s(x^*) + \alpha_s g_s(x^*) < f_s(x) + \alpha_s g_s(x)$$

which implies that solution x^* dominates solution x and, therefore, we reach a contradiction with the hypothesis of x being the efficient solution to problem (4).

A natural question is whether Theorem 3.1 is true for the set of properly efficient solutions.

Next we prove that, given certain conditions, this relationship is preserved for properly efficient solutions. For this purpose we define problems $P_{f, \gamma g}(\lambda, \mu)$ and $P_\alpha(\omega)$, obtained from applying the weighting method to problems (1) and (4) respectively, as follows:

$$P_{f, \gamma g}(\lambda, \mu) \quad \text{Min}_{x \in D} \quad \lambda^t f(x) + \sum_{k=1}^q \mu_k \gamma_k g_k(x)$$

$$P_{\alpha}(\omega) \quad \text{Min}_{x \in D} \quad \sum_{k=1}^q \omega_k (f_k(x) + \alpha_k g_k(x))$$

We use the results available in the literature about the relationships between optimal solutions to the weighting problem and the efficient solutions to the multiobjective problem. Some results (see, for example, Chankong and Haimes (Ref. 13)) applied to problem (1) and its associated weighted problem, $P_{f, \gamma g}(\lambda, \mu)$ are as follows:

- If f and $(\gamma_1 g_1, \dots, \gamma_q g_q)^t$ are convex functions, D is convex and x^* is a properly efficient solution for the multiobjective problem (1), there exist some weight vectors λ , μ with strictly positive components such that x^* is the optimal solution for the weighted problem $P_{f, \gamma g}(\lambda, \mu)$.
- The optimal solution to the weighted problem $P_{f, \gamma g}(\lambda, \mu)$, for each vector of weights with strictly positive components, is properly efficient for the multiobjective problem (1).

Proposition 3.1. If f and $(\gamma_1 g_1, \dots, \gamma_q g_q)^t$ are convex functions, D is a convex set and $\text{sgn}(\alpha_k) = \text{sgn}(\gamma_k)$ for every $k \in \{1, 2, \dots, q\}$, then, $E^p(\alpha) \subset E^p(f, \gamma g)$.

Proof. If f and $(\gamma_1 g_1, \dots, \gamma_q g_q)^t$ are convex functions and D is a convex set then the set of properly efficient solutions to problems (1) and (4) are obtained from the associated weighted problems for strictly positive weight vectors. We will prove that any solution to the optimisation problem $P_{\alpha}(\omega)$, with $\omega > 0$ is the solution to problem $P_{f, \gamma g}(\lambda, \mu)$ for some vector $(\lambda^t, \mu^t)^t > 0$.

Let $x \in E^p(\alpha)$. Then, given the established hypotheses, there exists a vector $\omega > 0$ for which x is the solution for the problem $P_{\alpha}(\omega)$.

Let us assume that for every $k \in \{1, 2, \dots, q\}$ $\alpha_k, \gamma_k \neq 0$. Then, by making $\lambda_k = \omega_k$ and

$\mu_k = \omega_k \alpha_k / \gamma_k$, $\lambda_k, \mu_k > 0$, since $\omega > 0$, we obtain that x is the optimal solution to $P_{f, \gamma g}(\lambda, \mu)$.

If for some $i \in \{1, 2, \dots, q\}$ $\alpha_i = \gamma_i = 0$, then, the proof would be the same, since in problem (1)

function g_i is not involved and in problem (4) the i -th objective would be f_i .

The inverse inclusion does not hold, as it is shown by the following.

Example 3.1.

Let us consider the following problem:

$$\begin{aligned} & \text{Max}_{(x,y)} \quad (x, y) \\ & \text{s.t.} \quad x^2 + y^2 \leq 1 \\ & \quad \quad x, y \geq 0 \end{aligned}$$

with $f(x, y) = x$, $g(x, y) = y$ and $\gamma = 1$.

The set of efficient points for this problem is:

$$\{(x, y)^t \in \mathbb{R}^2 / x^2 + y^2 = 1, x, y > 0\}$$

represented in the following figure:

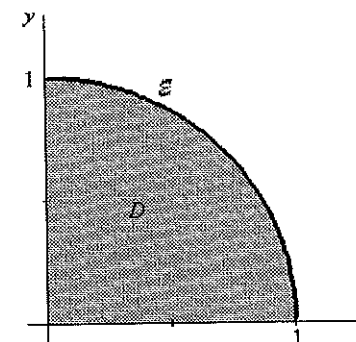


Figure 1

If we outline the resolution of the problem:

$$\begin{aligned} & \text{Max}_{(x,y)} \quad x + \alpha y \\ & \text{s.t.} \quad x^2 + y^2 \leq 1 \\ & \quad \quad x, y \geq 0 \end{aligned}$$

with $\alpha > 0$, for each $\alpha > 0$ we fix, the optimal solution of the resulting problem is one of the properly efficient solutions to the original bi-criteria problem.

Proposition 3.2.

If f and $(\gamma_1 g_1, \dots, \gamma_q g_q)^t$ are convex functions, then $E^p(f, \gamma g) \subset \bigcup_{\alpha \in \Omega} E^p(\alpha)$, with

$$\Omega = \left\{ \alpha \in \mathbb{R}^q / \text{sgn}(\alpha_k) = \text{sgn}(\gamma_k), k = 1, 2, \dots, q \right\}.$$

Proof. As in the previous case, the proof of the proposition is carried out by demonstrating that any solution to the problem $P_{f, \gamma g}(\lambda, \mu)$ is a solution to the problem $P_\alpha(\omega)$ for some vector $\alpha \in \mathbb{R}^q$, with $\text{sgn}(\alpha_k) = \text{sgn}(\gamma_k)$, $k \in \{1, 2, \dots, q\}$, and for some $\omega > 0$.

Let $x \in E^p(f, \gamma g)$. Then, as f and $(\gamma_1 g_1, \dots, \gamma_q g_q)^t$ are convex functions, there exist the vectors $\lambda, \mu > 0$ for which x is the solution to the problem $P_{f, \gamma g}(\lambda, \mu)$. By making $\omega_k = \lambda_k$ and $\alpha_k = \mu_k \gamma_k / \omega_k$, since $\omega, \mu > 0$, we obtain that x is also the solution to the problem $P_\alpha(\omega)$.

Note that from propositions 3.1 and 3.2, if f and $(\gamma_1 g_1, \dots, \gamma_q g_q)^t$ are convex functions and $\text{sgn}(\alpha_k) = \text{sgn}(\gamma_k)$ for every $k \in \{1, 2, \dots, q\}$, the sets of properly efficient solutions to problems (1) and (4) verify the following:

- Every properly efficient solution to problem (4) is properly efficient for problem (1).
- Setting a $\gamma \in \mathbb{R}^q$ with non-null components, the set of properly efficient solutions to problem (1) is a subset of the union in α of the sets of properly efficient solutions for problem (4).

By combining both results the following corollary can be stated:

Corollary 3.1. If f and $(\gamma_1 g_1, \dots, \gamma_q g_q)^t$ are convex functions, then for every $\gamma \in \mathbb{R}^q$:

$$E^p(f, \gamma g) = \bigcup_{\alpha \in \Omega} E^p(\alpha),$$

with $\Omega = \left\{ \alpha \in \mathbb{R}^q / \text{sgn}(\alpha_k) = \text{sgn}(\gamma_k), k = 1, 2, \dots, q \right\}$.

4. Relations between Expected Value Efficient Solutions, Minimum Variance Efficient Solutions and Expected Value-Standard Deviation Efficient Solutions

Let us consider again the (SMP) problem and the sets of efficient solutions for expected value (E), minimum variance (E_σ) and expected value-standard deviation ($E_{E\sigma}$) associated with the problem. Let E^d , E_σ^d and $E_{E\sigma}^d$ be the set of weakly efficient solutions associated with problems (E), (σ^2) and ($E\sigma$), respectively.

If we consider $f_k(x) = \bar{z}_k(x)$, $g_k(x) = \sigma_k(x)$ and $\gamma_k = 1$ are taken, given that for every $k \in \{1, 2, \dots, q\}$ it is verified that $\sigma_k: \mathbb{R}^n \rightarrow \mathbb{R}^+$, then the relations between these efficient sets are directly deduced from Theorem 3.1 in Sect. 3 as follows:

- (i) $E \cap E_\sigma \subset E_{E\sigma}$: Every solution which is both expected value efficient and minimum variance efficient is also an expected value-standard deviation efficient solution.
- (ii) $E \cup E_\sigma \subset E_{E\sigma}^d$: Every expected value solution or minimum variance efficient solution is an expected value-standard deviation weakly efficient solution.
- (iii) $E^d \cup E_\sigma^d \subset E_{E\sigma}^d$: The set of expected value-standard deviation weakly efficient solutions includes the union of the set of expected value weakly efficient solutions and the set of minimum variance weakly efficient solutions.

In Sect. 6 we present an example which illustrates these results. We now move on to study the relations between expected value-standard deviation efficient solutions and the efficient solutions with probabilities $\beta_1, \beta_2, \dots, \beta_q$.

5. Relations between Expected Value-Standard Deviation Efficient Solutions and Efficient Solutions with Probabilities $\beta_1, \beta_2, \dots, \beta_q$

Next, we analyse the existing relationships between expected value-standard deviation efficient solutions and efficient solutions with probabilities $\beta_1, \beta_2, \dots, \beta_q$. Given the stochastic multiobjective programming problem (SMP), we consider its associated problems (E σ) and (K(β)).

In order to determine the set of efficient solutions E σ of the (SMP) problem we only need to know the expected value and the standard deviation of each objective function of the stochastic problem. However, obtaining the β -efficient solution set for the (SMP) is more complex because the distribution functions of the stochastic objectives are involved in this definition of efficient solution, and so it is necessary to specify additional hypotheses about the stochastic objective functions and the probability distributions of the random parameters involved.

The studies carried out up to now have mainly focussed on linear objective functions (see, for example, Ref. 14) and linear fractional objective functions (see Ref. 6). As for the type of probability distribution normally used for the stochastic vectors, multivariate normal distribution is generally considered, or it is assumed that the vector of random parameters depends linearly on a single random variable (hypothesis of simple randomness). In these cases, and others, it is possible to obtain the distribution function of the stochastic objective, although in general this is a complex task.

We will now analyse the existing relationship between the set of expected value-standard deviation properly efficient solutions and the set of properly efficient solutions with probabilities $\beta_1, \beta_2, \dots, \beta_q$ when the objective functions are linear and the random parameter

vector follows a normal distribution or it verifies the hypothesis of simple randomness. Before going any further, we outline the two cases we are going to analyse.

a) Normal linear case

Let us assume that the k -th objective function takes the form $\tilde{z}_k(x, \tilde{c}) = \tilde{c}_k^t x$, where \tilde{c}_k is a random vector multinormal with expected value \bar{c}_k and with positive definite variance and covariance matrix, V_k . Let us also assume that $0 \notin D$.

Under such hypotheses the expected value of the random variable $\tilde{z}_k(x, \tilde{c}) = \tilde{c}_k^t x$ is $\bar{z}_k(x) = \bar{c}_k^t x$, its standard deviation is $\sigma_k(x) = \sqrt{x^t V_k x}$ and the distribution function of the random variable calculated in u_k is $P(\tilde{c}_k^t x \leq u_k) = \Phi((u_k - \bar{c}_k^t x) / \sqrt{x^t V_k x})$, where Φ is the standardised normal distribution function.

From now on, the probabilistic constraint $P(\tilde{c}_k^t x \leq u_k) \geq \beta_k$ is equivalent to $\bar{c}_k^t x + \Phi^{-1}(\beta_k) \sqrt{x^t V_k x} \leq u_k$, an inequality that can be expressed as $\bar{z}_k(x) + \alpha_k \sigma_k(x) \leq u_k$, with $\alpha_k = \Phi^{-1}(\beta_k)$. Since the expected value is linear and the function $\sqrt{x^t V_k x}$ is convex (see Stancu-Minasian (Ref. 2)), this inequality defines a convex set if $\beta_k \geq 0.5$ since, in this case, $\alpha_k = \Phi^{-1}(\beta_k) \geq 0$.

b) Simple linear randomization case.

Let us assume that the k -th objective function takes the form $\tilde{z}_k(x, \tilde{c}) = \tilde{c}_k^t x$, where \tilde{c}_k is a random vector, which linearly depends on the random variable \tilde{t}_k , in such a way that $\tilde{c}_k = c_k^1 + \tilde{t}_k c_k^2$. Let \bar{t}_k be the expected value of \tilde{t}_k , v_k its standard deviation, $v_k < \infty$ and F_k its distribution function, which we assume is strictly increasing. We also assume that for every $x \in D$ it is verified that $c_k^{2t} x > 0$.

If these hypotheses are verified we obtain that:

- the expected value of $\tilde{c}_k^1 x$ is $\bar{z}_k(x) = c_k^1 x + \bar{t}_k c_k^2 x$ and its standard deviation is

$$\sigma_k(x) = v_k c_k^2 x$$

- the distribution function of the random variable $\tilde{c}_k^1 x$ valued in u_k is

$$P(\tilde{c}_k^1 x \leq u_k) = F_k((u_k - c_k^1 x) / c_k^2 x), \text{ and therefore, } P(\tilde{c}_k^1 x \leq u_k) \geq \beta_k \text{ is equivalent to}$$

$$c_k^1 x + F_k^{-1}(\beta_k) c_k^2 x \leq u_k, \text{ an inequality that defines a convex set for every } \beta_k, \text{ and which}$$

$$\text{we can also express as } \bar{z}_k(x) + \alpha_k \sigma_k(x) \leq u_k, \text{ with } \alpha_k = (F_k^{-1}(\beta_k) - \bar{t}_k) / v_k.$$

Having analysed these two cases, we see that the probabilistic constraint

$$P(\tilde{z}_k(x, \tilde{c}) \leq u_k) \geq \beta_k \text{ is equivalent to } \bar{z}_k(x) + \alpha_k \sigma_k(x) \leq u_k, \text{ and therefore, the set of } \beta\text{-efficient}$$

solutions for the problem (SMP) coincides in both cases with that of the following

multiobjective problem:

$$K_\alpha \quad \min_{x \in D} (\bar{z}_1(x) + \alpha_1 \sigma_1(x), \dots, \bar{z}_q(x) + \alpha_q \sigma_q(x))$$

that is, the β -efficient set is obtained via a problem of q objective functions in which each

objective function takes the expected value of the stochastic problem plus its standard deviation

weighted by a coefficient that depends on the pre-determined or fixed probability.

Naturally emerging from this idea is the comparison between the set of efficient solution of

the problem K_α , and the set of expected value-standard deviation efficient solutions of the

original stochastic multiobjective problem. If we make $f_k(x) = \bar{z}_k(x)$, $g_k(x) = \sigma_k(x)$ and

$\gamma_k = 1$, from the results obtained in Sect. 3, we can relate the efficient solutions sets of both

problems. Since γ_k is the parameter that weights the standard deviation of the stochastic

objective, and in Sect. 3 the hypothesis of $\text{sgn}(\gamma_k) = \text{sgn}(\alpha_k)$ holds, then if we restrict to

$\gamma_k = 1$, in order to maintain the relations obtained in previous results, it will be necessary that

$\alpha_k > 0$. Let us see what this involves in each case:

a) Normal linear case: $\alpha_k = \Phi^{-1}(\beta_k)$, and so $\alpha_k > (\leq) 0$, if $\beta_k > (\leq) 0.5$

b) Linear simple randomness case: $\alpha_k = (F_k^{-1}(\beta_k) - \bar{t}_k) / v_k$, and so $\alpha_k > (\leq) 0$ if $\beta_k > (\leq)$

$$F_k(\bar{t}_k)$$

Therefore, in both cases the fact that parameter α_k takes a strictly positive value implies

that the fixed probability must be "high".

Therefore, from Theorem 3.2, Propositions 3.1 and 3.2, and the Corollary 3.1 in Sect. 3, we

can assert that if the stochastic objectives of the (SMP) problem fulfil the hypotheses in (a) or

(b), then it is verified that:

1. Given a fixed vector of probabilities β_1, \dots, β_q , such that the associated $\alpha_k, k = 1, 2, \dots,$

q , are strictly positive, the set of efficient solutions with probabilities β_1, \dots, β_q is a

subset of the set of expected value-standard deviation efficient solutions: $E_{K_\alpha} \subset E_{\beta\sigma}$,

where E_{K_α} will denote the set of efficient solutions for problem K_α .

2. Regarding properly efficient solutions, we have to point out that, given that the

functions $\bar{z}_k(x)$ and $\sigma_k(x)$ are convex in both cases, the results from Propositions 3.1

and 3.2 and the Corollary 3.1 are verified, with which, for each vector of probabilities

β_1, \dots, β_q such that the associated α_k are strictly positive, it is verified that:

- the set of properly efficient solutions with probabilities β_1, \dots, β_q is a subset of the set of expected value-standard deviation efficient solutions: $E_{\alpha\sigma}^p \subset E_{\beta\sigma}^p$.
- The union of sets properly β -efficient corresponding to probabilities such that the associated α_k are strictly positive, gives a set that coincides with the set of expected value-standard deviation properly efficient solutions: $\bigcup_{\alpha > 0} E_{\alpha\sigma}^p = E_{\beta\sigma}^p$.

Therefore, the criteria of expected value-standard deviation efficiency and efficiency in probability are closely related, at least in the cases analysed. Now we analyse the results for the cases studied.

In both instances, our study enables us to see how the application of the efficiency in probability criteria gives an entire range of solutions, as a function of the fixed probabilities that include some of the expected value-standard deviation efficiencies and more, all those corresponding to "low" probabilities, efficient solutions that, in general, are impossible to obtain by means of the expected value-standard deviation efficiency criteria. This fact corroborates the idea that the expected value-standard deviation efficiency is appropriate when the decision maker is risk averse (the hypothesis which is held, among others, in the models of portfolio selection in order to consider this criterion of efficiency as an appropriate one).

All this leads to the following question: would it be possible to obtain β -efficient solutions via some other criteria? When we fix a "low" probability for some stochastic objective in cases (a) and (b), its standard deviation is weighted negatively; and furthermore, the lower the probability, the smaller it is. If in such cases we consider the following problem:

$$\text{Min}_{x \in D} (\bar{z}_1(x), \dots, \bar{z}_q(x), \gamma_1 \sigma_1(x), \dots, \gamma_q \sigma_q(x)) \quad (7)$$

where $\gamma_k = 1$ if the fixed probability is "high" and $\gamma_k = -1$ if the fixed probability is "low", from Theorem 3.2 we can assert that the set of β -efficient solutions is a subset of the set of efficient solutions for problem (7). Regarding the relations for the properly efficient sets shown in Proposition 3.1 and the Corollary 3.1, these only hold if low probabilities correspond to stochastic objectives that verify the hypothesis in case (b), simple randomness, since in this case the function $-\sigma_k(x)$ is linear and thus convex.

Finally, following the paper of Caballero, Cerdá, Muñoz and Rey (Ref. 10) - which analyses the conditions under which the analysis of efficiency with probabilities $\beta_1, \beta_2, \dots, \beta_q$ and minimum-risk efficiency with aspiration levels of u_1, u_2, \dots, u_q of the (SMP) problem are equivalent- it can be asserted that given that the necessary conditions for reciprocity between efficient solutions with probabilities $\beta_1, \beta_2, \dots, \beta_q$ and efficiency of minimum-risk with aspiration levels u_1, u_2, \dots, u_q of the (SMP) problem are verified, the relationship established between the set of efficient solutions with probabilities $\beta_1, \beta_2, \dots, \beta_q$ and the set of expected value-standard deviation properly efficient solutions is also verified between the latter and the minimum-risk efficient solutions with aspiration levels u_1, u_2, \dots, u_q .

In order to illustrate these results we present an example.

6. Example

Let us consider the following stochastic bi-objective problem:

$$\begin{aligned} & \text{"Min"} (\tilde{c}_1^T x, \tilde{c}_2^T x) \\ & \text{s. t. } x_1 + x_2 \geq 1 \\ & \quad x_1 + 3x_2 \leq 10 \\ & \quad -2 \leq -x_1 + x_2 \leq 2 \\ & \quad x_1, x_2 \geq 0 \end{aligned}$$

where $\tilde{c}_i = c_i^1 + \tilde{t}_i c_i^2$, $i = 1, 2$, with $c_1^1 = (-7, -12)^t$, $c_1^2 = (6, 5)^t$, $c_2^1 = (3, -5)^t$, $c_2^2 = (4, 8)^t$, \tilde{t}_i follows a normal distribution of expected value 1 and variance 4, $\tilde{t}_i \sim N(1, 4)$, and \tilde{t}_2 follows an exponential distribution of the parameter $\lambda = 2$.

From this data, the expected value-standard deviation efficiency problem of the above mentioned stochastic problem is:

$$\begin{aligned} \text{Min}_x \quad & (-x_1 - 7x_2, 12x_1 + 10x_2, 3x_1 - 5x_2, 2x_1 + 4x_2) \\ \text{s. t.} \quad & x_1 + x_2 \geq 1 \\ & x_1 + 3x_2 \leq 10 \\ & -2 \leq -x_1 + x_2 \leq 2 \\ & x_1, x_2 \geq 0 \end{aligned}$$

The following figure (Figure 2) shows the set of expected value-standard deviation efficient solutions for the problem. Points $E1 = (1, 3)$, $E2 = (0, 2)$, $S1 = (0, 1)$ and $S2 = (1, 0)$ are optimal solutions for the expected value problem of the first and second objective functions, and for the problem of minimum variance of stochastic objectives 1 and 2, respectively. This implies that the segment joining $E1$ and $E2$, $E1E2$, is the set of expected value efficient points, $E_E = E1E2$, and segment $S1S2$ is the minimum variance efficient set, $E_{\sigma} = S1S2$. The expected value-standard deviation efficient solutions are the points of the segments $S1S2$, $S1E2$ and $E2E1$. Note that $E_E = E_E^d$, $E_{\sigma} = E_{\sigma}^d$ and $E_{E\sigma} = E_{E\sigma}^d$.

As it can be observed, the expected value-standard deviation efficient set includes the expected value and minimum variance efficient sets, which is shown in the results obtained in Sect. 4.

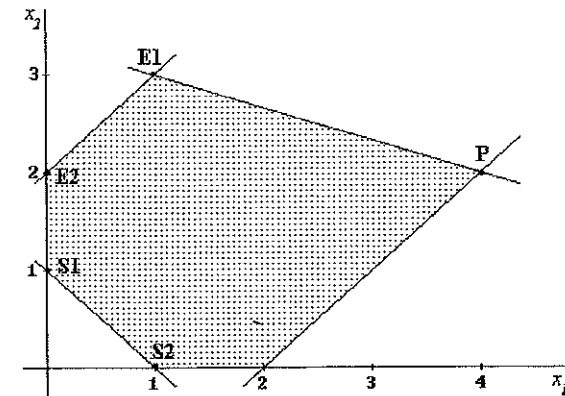


Figure 2

On the other hand, the problem of the efficiency with probabilities β_1, β_2 is:

$$\begin{aligned} \text{Min}_x \quad & (-x_1 - 7x_2 + \Phi^{-1}(\beta_1)(12x_1 + 10x_2), 3x_1 - 5x_2 + F^{-1}(\beta_2)(4x_1 + 8x_2)) \\ \text{s. t.} \quad & x_1 + x_2 \geq 1 \\ & x_1 + 3x_2 \leq 10 \\ & 2 \leq -x_1 + x_2 \leq 2 \\ & x_1, x_2 \geq 0 \end{aligned}$$

The sets of efficient solutions to this problem for different probability vectors is shown in the Table 1, where column *efficient set* contains the efficient set that is obtain for the probability vector given in the second column:

	Probability Vector	Efficient Set
1.	$(\beta_1 = 0.9, \beta_2 = 0.99)$	Minimum variance efficient set
2.	$(\beta_1 = 0.75, \beta_2 = 0.99)$	Minimum variance efficient set and segment $S1E2$
3.	$(\beta_1 = 0.6, \beta_2 = 0.99)$	$E\sigma$ efficient set
4.	$(\beta_1 = 0.3, \beta_2 = 0.99)$	$E\sigma$ efficient set and segment $E1P$
5.	$(\beta_1 = 0.9, \beta_2 = 0.8)$	Solution $(x_1 = 0, x_2 = 1)$
6.	$(\beta_1 = 0.75, \beta_2 = 0.8)$	Segment $S1E2$

7.	$(\beta_1 = 0.6, \beta_2 = 0.8)$	Expected value efficient set and segment S1E2
8.	$(\beta_1 = 0.3, \beta_2 = 0.8)$	Expected value efficient set and segments S1E2 and E1P
9.	$(\beta_1 = 0.9, \beta_2 = 0.4)$	Segment S1E2
10.	$(\beta_1 = 0.75, \beta_2 = 0.4)$	Solution $(x_1 = 0, x_2 = 2)$
11.	$(\beta_1 = 0.6, \beta_2 = 0.4)$	Expected value efficient set
12.	$(\beta_1 = 0.3, \beta_2 = 0.4)$	Expected value efficient set and segment E1P
13.	$(\beta_1 = 0.9, \beta_2 = 0.1)$	Expected value efficient set and segment S1E2
14.	$(\beta_1 = 0.75, \beta_2 = 0.1)$	Expected value efficient set
15.	$(\beta_1 = 0.6, \beta_2 = 0.1)$	Solution $(x_1 = 1, x_2 = 3)$
16.	$(\beta_1 = 0.3, \beta_2 = 0.1)$	Expected value efficient set and segment E1P

Table 1

The results in the table show that the set of efficient solutions with probabilities β_1, β_2 go on changing according to the fixed probabilities and help to illustrate the theoretical results obtained in Sect. 5. The probability vectors for which the standard deviation has an associate positive weightare such that $\beta_1 > 0.5$ and $\beta_2 > 0.6321205$. From this the following can be outlined:

- In general terms, it is necessary to point out that the "dimension" of the efficiency in probability set carries on varying according to the "disparity" existing between the fixed probabilities, as is shown in lines 4, 8 or 13 in the table.
- For probability vectors such that $\beta_1 > 0.5$ and $\beta_2 > 0.6321205$ (which in problem K_α implies that $\alpha > 0$) the solutions obtained are subsets of the expected value-standard deviation efficient set. For example, for the probability vector $(\beta_1 = 0.75, \beta_2 = 0.8)$ the efficient set obtained is S1E2, but if probability β_1 is lowered to 0.6, keeping the

second value the same, the new set includes the previous one and all the expected value efficient solutions. Furthermore, in some cases, for example for vector $(\beta_1 = 0.6, \beta_2 = 0.99)$, the efficient set that give us the efficiency in probability criterion coincides with the efficient set $E\sigma$.

- When the fixed or predetermined probabilities are such that $\beta_1 \leq 0.5$ or $\beta_2 \leq 0.6321205$ (the case in row 4 and from the seventh row onwards of the table), the resulting efficient set can include points which are not part of the efficient set $E\sigma$. In all instances, these points are those of segment E1P.

7. Conclusions

From the results obtained, we can assert that the concepts of efficient solution considered in this work for a single problem of stochastic multiobjective programming are closely related, under certain conditions. We consider that the relations established can help to obtain efficient solutions to a problem with the characteristics described here, since these concepts include different statistical characteristics of the stochastic objectives and, apparently, do not have to have any relationship to each other.

Based on our results, it is possible to deal with attaining efficient solutions from a different perspective since, given a particular problem, we will be able to see from the established relationships what concept of efficiency is the most appropriate or the one that best fits the preferences of the decision maker.

Therefore, this study helps to choose between different efficiency criteria for solving stochastic multiple object problems. In this sense, the "richness" of the efficiency in probability criterion may be highlighted, since by varying the fixed probability for each stochastic objective, different efficient sets are obtained. Also, the fact that the decision-maker has to fix a probability for each stochastic objective is more of an advantage than a hindrance, since, in a certain way, it determines the risk he or she is willing to take in each of the stochastic objectives.

Finally, it should also be noted that the preliminary results obtained in Sect. 3 are applicable to any deterministic multiobjective problem.

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