Efficient solution of a vibration equation involving fractional derivatives

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Abstract

Fractional order (or, shortly, fractional) derivatives are used in viscoelasticity since the late 1980s, and they grow more and more popular nowadays. However, their efficient numerical calculation is nontrivial, because, unlike integer-order derivatives, they require evaluation of history integrals in every time step. Several authors tried to overcome this difficulty, either by simplifying these integrals or by avoiding them. In this paper, the Adomian decomposition method is applied on a fractionally damped mechanical oscillator for a sine excitation, and the analytical solution of the problem is found. Also, a series expansion is derived which proves very efficient for calculations of transients of fractional vibration systems. Numerical examples are included.

Keywords: vibration , fractional derivative , fractional differential equation , Adomian decomposition

1 Introduction

It is common to date fractional calculus back to Leibniz, and several textbooks are available on the subject (e.g. [26]). Several authors have been examining the possibility of using fractional derivatives in material modelling in the last two-three decades [5; 6], and the field is of a growing interest nowadays [1; 15–17; 27; 30]. Finite element formulations are also in development [8; 12; 30]. However, the wide spread of these models is obstructed among others by the difficulty concerning their efficient numerical calculation. This comes from the fact that they need the evaluation of a time-history integral. There are several solution methods available in the literature.

The analytical solution is theoretically known [26]. However, it requires the computation of a convolution of the two-parameter Mittag-Leffler function, which is numerically challenging. Therefore, other calculation methods are sought and used.

The most commonly used method is based on the well-known Grünwald–Letnikov definition of fractional derivatives, used e.g. by Schmidt and Gaul [30]. These authors proceeded to reduce the volume of the time history integral [32], achieving an almost two-magnitude gain in calculation time on an example.

Suarez and Shokooh [35] solved a vibration equation with fractional damping of order 1/2 analytically, taking advantage of the derivative order. Therefore, their solution is restricted to cases where the order of damping is one half. Still, the presented numerical results are widely

used as reference solutions. Moreover, Suarez and Shokooh presented a solution for the initial value problem of the vibration system, which is uncommon in the literature.

Yuan and Agrawal [38] have rewritten the definition of a fractional derivative, and turned a fractional differential equation to a system of linear differential equations. However, Schmidt and Gaul [31] have shown that in some cases, this method loses the advantages of fractional calculus over integer-order calculus. Later, this has also been checked by the present author [25] for the field of interest of this paper.

The Adomian decomposition method (ADM) has been introduced by George Adomian in the late 1980s. Essentially, it approximates the solution of a non-linear differential equation with a series of functions. The method is getting into use for the solution of fractional differential equations [9–11; 22; 29; 33; 36]. For a special case, Saha Ray, Poddar and Bera [29] have proven that the Adomian solution converges to the analytical solution. Recently, Hu, Luo and Lu [19] have shown that this is true for any linear fractional differential equation. It will be shown in the present paper that this series expansion is very efficient in many cases.

There are some further methods (e.g. the variational iteration method [23; 24], the method of Atanackovic and Stankovic [2; 3] and the method of Singh and Chatterjee [34]) that will not be treated in this paper. Also, the Adomian decomposition method has been improved by some authors [20; 21; 28]. However, the convergence to the analytical solution has only been proven for the original method.

The present paper aims to provide a computationally efficient solution method for the fractionally damped vibration equation using the Adomian decomposition method and Taylor series. The obtained solution will be compared to analytical solutions, either known previously or developed here in a series form. To show the connection with classical methods, solutions using the Grünwald–Letnikov definition will also be calculated.

In this paper, Section 2 presents the fractionally damped vibration equation, also giving a brief introduction to fractional derivatives. It also describes the Adomian decomposition method which will be used afterwards. In Section 3, some solutions existing in the literature are shown. Next, in Section 4, new solutions are presented: in Section 4.1, the proposed method is developed, while in Section 4.2, an analytical solution is calculated for reference. Finally, Section 5 shows the numerical examples, and Section 6 concludes the paper.

2 Preliminaries

This section gives the equation to be solved, including a basic introduction to fractional derivatives. The Adomian decomposition method, which is extensively used in this paper, is also presented.

2.1 The equation to be solved

The differential equation to be solved is the vibration equation with fractional damping, with one degree of freedom:

$$\mathbf{D}^2 x(t) + \frac{c}{m}\mathbf{D}^\alpha x(t) + \frac{k}{m}x(t) = f(t)\,, \tag{1}$$

where $\mathbf{D} = \dfrac{\mathrm{d}}{\mathrm{d}t}$ is the differential operator. Another common form of Equation (1) is

$$\mathbf{D}^2 x(t) + 2\eta\omega_n^{2-\alpha}\mathbf{D}^\alpha x(t) + \omega_n^2 x(t) = f(t)\,, \tag{2}$$

with

$$2\eta\omega_n^{2-\alpha} = \frac{c}{m} \text{ and } \omega_n^2 = \frac{k}{m}.$$

2.2 Fractional integrals

Fractional integrals and derivatives are deduced from the generalisation of the integer-order operations. It is usual to define the integral operator $\mathbf{D}^{-q}$ as

$$ {}_a\mathbf{D}_t^{-q} x(t) = \frac{1}{\Gamma(q)} \int_a^t (t-\tau)^{q-1}\, x(\tau)\; d\tau, \tag{3} $$

where $q > 0$ and $\Gamma(x)$ is the Gamma function

$$ \Gamma(x) = \int_0^\infty \mathrm{e}^{-z} z^{x-1}\, dz. \tag{4} $$

For a continuous $x(t)$,

$$ \mathbf{D}^{-p}\mathbf{D}^{-q} x(t) = \mathbf{D}^{-(p+q)} x(t)\,, \tag{5} $$

as given in [26] (if both p and q are non-negative).

The fractional integral of a polynomial will be needed later so it is given below:

$$ \mathbf{D}^{-q} t^\nu = \frac{\Gamma(\nu+1)}{\Gamma(\nu+1+q)} t^{q+\nu}, \tag{6} $$

With the fractional integral operator, fractional derivatives are easily introduced.

2.3 Fractional derivatives

For a real $\alpha > 0$, $\mathbf{D}^\alpha$ is defined by the Riemann–Liouville definition [26], using the above fractional integral operator:

$$ {}_a\mathbf{D}_t^{\alpha} x(t) = \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n {}_a\mathbf{D}_t^{-(n-\alpha)} x(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_a^t (t-\tau)^{n-1-\alpha}\, x(\tau)\, d\tau. \tag{7} $$

Another choice is the Caputo definition

$$ {}_a^C\mathbf{D}_t^{\alpha} x(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-1-\alpha} \left[\left(\frac{\mathrm{d}}{\mathrm{d}\tau}\right)^n x(\tau)\right] d\tau. \tag{8} $$

In both cases,

$$ (n-1) < \alpha < n. $$

Actually, the two definitions only differ in the consideration of conditions at the start of the interval:

$$ {}_a\mathbf{D}_t^{\alpha} x(t) = {}_a^C\mathbf{D}_t^{\alpha} x(t) + \frac{1}{\Gamma\,(n-\alpha)} \sum_{k=0}^{n-1} \frac{\Gamma(n-\alpha)}{\Gamma(k-\alpha+1)}\, (t-a)^{k-\alpha}\, x^{(k)}(a)\,. \tag{9} $$

In the applications, $\mathbf{D}$ practically always means ${}_0\mathbf{D}_t$, and most authors use the Riemann–Liouville, or the mathematically equivalent Grünwald–Letnikov definition (see [26] for precise conditions of equivalence). Also, since the Riemann–Liouville definition has a singularity for non-zero initial conditions, the initial conditions are often considered zero. For a physical interpretation of this singularity, see [18].

The composition (5) can be extended to an integral and a derivative:

$$ \mathbf{D}^{-p}\mathbf{D}^{q} x(t) = \mathbf{D}^{q}\mathbf{D}^{-p} x(t) = \mathbf{D}^{q-p} x(t)\,, \tag{10} $$

with p, $q \geq 0$ and a continuous $x(t)$. (However, the composition of two fractional order derivatives is not straightforward.)

2.4 Adomian decomposition method

The decomposition method has been elaborated by George Adomian in the late 1980s, originally for non-linear differential equations. However, as mentioned in the introduction, it has also been used by several authors for fractional differential equations [9–11; 22; 29; 33; 36]. As it will be also used in this paper, it is presented in the followings.

The method has been described very clearly in [7]. The non-linear differential equation is written as:

$$\mathrm{L}x(t) + \mathrm{R}x(t) + \mathrm{N}x(t) = f(t)\,, \tag{11}$$

where L is a linear operator which can be inverted easily, R is the remaining linear part and N is a non-linear operator. The function $x(t)$ is derived as a series expansion:

$$x(t) = x_0(t) + x_1(t) + x_2(t) + \ldots\,, \tag{12}$$

using

$$x_0(t) = \left[\mathrm{L}^{-1}\mathrm{L}x(t) - x(t)\right] + \mathrm{L}^{-1}f(t) \tag{13}$$

and

$$x_{n+1}(t) = -\mathrm{L}^{-1}\mathrm{R}x_n(t) - \mathrm{L}^{-1}A_n(t) \tag{14}$$

with

$$A_n(t) = \frac{1}{n!}\left[\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\mathrm{N}\left(\sum_{i=0}^{\infty}\lambda^i x_i(t)\right)\right]_{\lambda=0}. \tag{15}$$

It has been proven recently by Hu, Luo and Lu [19] that for any linear fractional differential equation, the solution given by the decomposition method converges to the analytical solution. Thus, Adomian decomposition is an adequate tool for solving such equations. In the followings, it will be shown that it can be very efficient, too.

3 Known solutions

This section presents solutions for the fractionally damped vibration equation (Section 2.1) which are well-established in the literature, and will be used here as reference solutions.

3.1 Analytical solution for $\alpha = 1/2$

Suarez and Shokooh [35] calculate the analytical solution of Equation (2) for the special case $\alpha = 1/2$, both for a free vibration and a step function excitation. For the former, they obtain

$$x^{\mathrm{IC}}(t) = \sum_{j=1}^{4}\Psi_{4j}R_j\frac{1}{\sqrt{\pi t}} + \sum_{j=1}^{4}\Psi_{4j}R_j\lambda_j g_j(t)\,, \tag{16}$$

where

$$g_j(t) = \mathrm{e}^{\lambda_j^2 t}\left(1 + \mathrm{erf}\left(\lambda_j\sqrt{t}\right)\right) \tag{17}$$

and λ_j and $\boldsymbol{\Psi}_j$ are the solution of the eigenproblem

$$\mathbf{A}\boldsymbol{\Psi}_j = \lambda_j\mathbf{B}\boldsymbol{\Psi}_j \tag{18}$$

with

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\omega_n^2 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 2\eta\omega_n^{3/2} \end{bmatrix},$$

and Ψ_{4j} is the 4th coordinate of the jth eigenvector. In the above,

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \mathrm{e}^{-t^2}\, dt$$

is the error function, and R_j comes from the initial conditions, by

$$\begin{bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{bmatrix} = \left[\ \boldsymbol{\Psi}_1 \ \middle|\ \boldsymbol{\Psi}_2 \ \middle|\ \boldsymbol{\Psi}_3 \ \middle|\ \boldsymbol{\Psi}_4\ \right]^{-1} \begin{bmatrix} v_0 \\ 0 \\ x_0 \\ 0 \end{bmatrix}. \tag{19}$$

The response to a step function force (using the Heaviside function $H(t)$)

$$f(t) = f_0 H(t)\,,$$

where X_0 and V_0 are considered zero, is

$$x^H(t) = f_0 \sum_{j=1}^{4} \frac{\Psi_{4j}^2}{\lambda_j} \left(g_j(t) - 1\right). \tag{20}$$

For an arbitrary function $f(t)$ and arbitrary initial conditions, the proposal of Suarez and Shokooh is to consider the initial conditions according to the above. As for the force, they suggest discretising the function and handling it as a series of step loads, one in each time step. However, with this method, the time history integral of the fractional derivative is replaced by a sum of displacement responses to step loads, which seems actually slower to calculate.

3.2 Grünwald–Letnikov formulation

It is common in the literature, especially in applications, to use the Grünwald–Letnikov definition of a fractional derivative [26]:

$$\mathbf{D}^\alpha f(t) = \lim_{N\to\infty} \left[\left(\frac{t}{N}\right)^{-\alpha} \sum_{j=0}^{N-1} \frac{\Gamma(j-\alpha)}{\Gamma(-\alpha)\,\Gamma(j+1)} f\left(t - j\frac{t}{N}\right) \right]. \tag{21}$$

This, along with any time stepping scheme, allows to find a numerical solution for Equation (1). For simplicity, an explicit scheme has been used for time stepping in this paper:

$$v_{i+1} = v_i + a_i h \tag{22}$$

and

$$x_{i+1} = x_i + v_i h + \frac{a_i h^2}{2}, \tag{23}$$

where

$$a_i = f_i - \frac{k}{m} x_i - \frac{c}{m} \frac{h^{-\alpha}}{\Gamma(-\alpha)} \sum_{j=0}^{i-1} \frac{\Gamma(j-\alpha)}{\Gamma(j+1)} x_{i-j-1}, \tag{24}$$

h being the time step size and f_i being the force per unit mass at $t = ih$.

4 New solutions

In this section, new solutions will be presented: first, an accurate and efficient series expansion, which is proposed for use; second, an analytical solution using the Adomian decomposition method as reference solution.

4.1 Proposed method: Taylor–Adomian series

The main goal of this paper is to propose a method for efficient calculation of the numerical solution of Equation (1) with non-zero initial conditions. In the followings, this method will be presented.

4.1.1 Description

In the proposed calculation method, Adomian decomposition is used, separately for initial conditions (starting displacement and velocity) and for the excitation. This can be done, as the differential operator defined in Section 2.3 is linear. The excitation is written in terms of Taylor series, and the resulting method will be referred to as *Taylor–Adomian series*.

For our equation, a possible choice is

$$\mathrm{L} = \mathbf{D}^2\,,\ \ \mathrm{R} = \frac{c}{m}\mathbf{D}^{\alpha} + \frac{k}{m}\ \text{ and }\ \mathrm{N} = 0.$$

In this case, conforming to Equations (13) and (14),

$$x_0(t) = \left[\mathbf{D}^{-2}\mathbf{D}^2 x(t) - x(t)\right] + \mathbf{D}^{-2} f(t) \tag{25}$$

and

$$x_{n+1}(t) = -\frac{k}{m}\mathbf{D}^{-2} x_n(t) - \frac{c}{m}\mathbf{D}^{\alpha-2} x_n(t)\,, \tag{26}$$

the latter leading to

$$x_n(t) = \frac{(-1)^n}{m^n}\sum_{j=0}^{n}\binom{n}{j} c^{n-j} k^j \mathbf{D}^{-((2-\alpha)n+\alpha j)} x_0(t) \tag{27}$$

using the composition property (10) of an integral and a derivative operator.

In Equation (25), the first term describes the contribution of initial conditions:

$$x_0^{\mathrm{IC}}(t) = \left[\mathbf{D}^{-2}\mathbf{D}^2 x(t) - x(t)\right] = X_0 + V_0 t, \tag{28}$$

where X_0 and V_0 are the initial position and velocity, respectively. With Equations (27) and (6), one can write

$$x_n^{\mathrm{IC}}(t) = \frac{(-1)^n}{m^n} t^{(2-\alpha)n} \sum_{j=0}^{n}\binom{n}{j} c^{n-j} k^j t^{j\alpha}\left(\frac{X_0}{\Gamma(2n+1-(n-j)\,\alpha)} + \frac{V_0 t}{\Gamma(2n+2-(n-j)\,\alpha)}\right). \tag{29}$$

This result has also been obtained by Baclic and Atanackovic [4] using Laplace transformation.

The second term of Equation (25) gives the response of the system to the excitation:

$$x_0^f(t) = \mathbf{D}^{-2} f(t)\,. \tag{30}$$

Suppose an excitation of the form of the Taylor series

$$f(t) = \sum_{i=0}^{\infty} T_i t^i, \tag{31}$$

T_is being the coefficients of the polynomial. This, together with Equations (30) and (6), leads to

$$x_{0,i}^f(t) = T_i \frac{\Gamma(i+1)}{\Gamma(i+3)} t^{i+2} \tag{32}$$

and

$$x^f_{n,i}(t) = T_i \frac{(-1)^n}{m^n} \Gamma(i+1)\, t^{i+(2-\alpha)n+2} \sum_{j=0}^{n} \binom{n}{j} \frac{c^{n-j} k^j}{\Gamma(i+3+2n-(n-j)\,\alpha)} t^{j\alpha}, \tag{33}$$

$x^f_{n,i}(t)$ being the nth term of the Adomian series of the response to the ith term of the Taylor series of the excitation. The total response is

$$x^f(t) = \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} x^f_{n,i}(t)\,. \tag{34}$$

Thus, the solution of Equation (1) has been given.

4.1.2 Technical considerations

The implementation of the method raises several questions. First, this is a double series expansion, which means that both the number of terms in the Taylor series and the number of terms in the Adomian series have to be defined. Moreover, Equations (29) and (33) show that terms $t^{j\alpha}$ become large for a relatively small t, which means a need for computing the difference of large numbers. Thus, the evaluation of $x(t)$ requires a computational precision better than usual.

In the calculation code prepared for this paper, the number of terms is increased dynamically throughout the calculation, based on error estimations from the next n_{inc} terms of the series: if the sum of the next n_{inc} terms is larger than a small limit ('error goal'), then the number of terms is increased by n_{inc}, and the error estimation will be based on the following n_{inc} terms. This is applied separately for both series.

Another point is the required computational effort. The first two terms of the sum in Equation (29) give

$$x^{\text{IC}}(t) = X_0 + V_0 t - \frac{c}{m} \frac{X_0}{\Gamma(3-\alpha)} t^{2-\alpha} - \frac{k}{m} \frac{X_0}{\Gamma(3)} t^2 - \frac{c}{m} \frac{V_0}{\Gamma(4-\alpha)} t^{3-\alpha} - \frac{k}{m} \frac{V_0}{\Gamma(4)} t^3 \pm \dots \tag{35}$$

This shows that the exponent of t in the series expansion is either an integer or a real number, the latter being computationally expensive. However, if α is restricted to be rational, as

$$\alpha = \frac{p}{q}$$

with p and q both integers, the right-hand side of Equation (35) becomes a polynomial for $\left(t^{1/q}\right)$:

$$\begin{aligned} x^{\text{IC}}(t) = X_0 + V_0 \left(t^{1/q}\right)^q - \frac{c}{m} \frac{X_0}{\Gamma(3-\alpha)} \left(t^{1/q}\right)^{2q-p} - \frac{k}{m} \frac{X_0}{\Gamma(3)} \left(t^{1/q}\right)^{2q} \\ - \frac{c}{m} \frac{V_0}{\Gamma(4-\alpha)} \left(t^{1/q}\right)^{3q-p} - \frac{k}{m} \frac{V_0}{\Gamma(4)} \left(t^{1/q}\right)^{3q} \pm \dots \end{aligned} \tag{36}$$

This can be computed efficiently.

4.1.3 Examples

Step function excitation For an excitation with the Heaviside function $H(t)$ of the form

$$f(t) = f_0 H(t)\,,$$

solved by Suarez and Shokooh [35] for $\alpha = 1/2$, only T_0 is non-zero of the T_is, resulting in

$$x^f_n(t) = f_0 \frac{(-1)^n}{m^n} t^{(2-\alpha)n+2} \sum_{j=0}^{n} \binom{n}{j} \frac{c^{n-j} k^j}{\Gamma(3+2n-(n-j)\,\alpha)} t^{j\alpha}. \tag{37}$$

This has also been obtained by Saha Ray, Poddar and Bera [29] for $\alpha = 1/2$ using Adomian decomposition (with a slightly different selection for L, R, and N).

Sine excitation For a sine excitation of the form

$$f(t) = f_0 \sin(\omega_e t)\,,$$

the Taylor coefficients are

$$T_i = \begin{cases} 0 & \text{if } i \text{ is even} \\ (-1)^{(i-1)/2} f_0 \dfrac{\omega_e^i}{i!} & \text{if } i \text{ is odd} \end{cases}.$$

Using $i = 2h + 1$ gives

$$T_h = (-1)^h \frac{\omega_e^{2h+1}}{(2h+1)!},$$

which, inserted into Equation (33), leads to

$$x_{n,h}^f(t) = f_0 \frac{(-1)^{n+h}}{m^n} \omega_e^{2h+1} t^{(2-\alpha)n+3+2h} \sum_{j=0}^{n} \binom{n}{j} \frac{c^{n-j} k^j}{\Gamma(4 + 2\,(n+h) - (n-j)\,\alpha)} t^{j\alpha}, \tag{38}$$

also using the property

$$\Gamma(n+1) = n!$$

of the Gamma function.

4.2 Analytical solution by Adomian decomposition

Equation (27) gives the nth term of the Adomian series of $x(t)$ if the original differential equation is Equation (1). For a sine excitation

$$f(t) = f_0 \sin(\omega_e t)\,,$$

this yields to

$$x_n(t) = \frac{(-1)^n}{m^n} \sum_{j=0}^{n} \binom{n}{j} c^{n-j} k^j \mathbf{D}^{-((2-\alpha)n+\alpha j)} \left(D^{-2} f_0 \sin(\omega_e t)\right).$$

Using the definition (3) of the integral operator, one obtains

$$x_0(t) = \int_0^t (t-\tau) \sin(\omega_e \tau)\; d\tau = \frac{1}{\omega_e^2} \left(\omega_e t - \sin(\omega_e t)\right). \tag{39}$$

This leads to

$$x_n(t) = \frac{(-1)^n}{m^n} \frac{f_0}{\omega_e^2} \sum_{j=0}^{n} \binom{n}{j} c^{n-j} k^j \frac{1}{\Gamma(q)} \int_0^t (t-\tau)^{q-1} \left(\omega_e \tau - \sin(\omega_e \tau)\right)\, d\tau, \tag{40}$$

where

$$q = (2-\alpha)\, n + \alpha j.$$

This results in

$$x_n(t) = \frac{(-1)^n}{m^n} \frac{f_0}{\omega_e^2} \sum_{j=0}^{n} \binom{n}{j} c^{n-j} k^j \frac{1}{\Gamma(q+2)} \sqrt{t}\, \omega_e^{1/2-q}\, s_{3/2+q,1/2}(\omega_e t)\,, \tag{41}$$

for $n > 0$, where $s_{\mu,\nu}(t)$ is the Lommel-function [37]

$$s_{\mu,\nu}(t) = {}_1F_2\left(1; \frac{1}{2}\,(\mu - \nu + 3)\,, \frac{1}{2}\,(\mu + \nu + 3)\,; -\frac{1}{4} t^2\right) \frac{t^{\mu+1}}{(\mu+1)^2 - \nu^2}$$

with ${}_1F_2(a_1; b_1, b_2; t)$ being the hypergeometric function

$$ {}_1F_2(a_1; b_1, b_2; t) = \sum_{k=0}^{\infty} \frac{(a_1)_k}{(b_1)_k (b_2)_k} \frac{t^k}{k!}, $$

where $(a)_k$ is the Pochhammer symbol

$$ (a)_k = \frac{\Gamma(a+k)}{\Gamma(a)}. $$

This can be calculated numerically by any computer algebra system.

To the author's knowledge, this solution has not been calculated in the literature before.

5 Numerical examples

So far, existing (Section 3) and new (Section 4) methods for the solution of the fractionally damped vibration equation have been presented. In this section, they will be tested and compared for some parameter sets.

5.1 Software

First, some details are given on software used for the calculations.

The code for the Taylor–Adomian series is written in C. The computational precision is assured by the GNU Multiple precision library [14] and one of its derivatives, MPFR [13].

The Grünwald–Letnikov formulation is also implemented in C, but it uses the conventional double-precision arithmetics.

The analytical solution of Suarez and Shokooh (described in Section 3.1) has been calculated using the Maple computer algebra system.

Also, for the analytical solution described in Section 4.2, the Maple computer algebra system has been used with Equations (12), (29) and (41). Precision was higher than double precision, usually 50 or 75 decimal digits have been set. The run-time increase of the number of Adomian terms, described for the Taylor–Adomian method, has also been used here, with an error goal of 10^{-25} for ten terms.

5.2 Problems and results

In the followings, numerical examples and their results will be presented and compared. For the latter purpose, let us introduce an absolute error indicator ε as

$$ \varepsilon = \frac{1}{N} \sum_{i=1}^{N} \left| x_i^{\mathrm{num}} - x_i^{\mathrm{ref}} \right|. $$

Six problems have been analysed, two using a unit step force and four with a harmonic excitation. Table 1 resumes the parameters of the different examples. They lead to steady-state amplitudes between 0.012 and 0.021. The treated time interval is from 0 to 5 seconds. With the Taylor–Adomian method, 10 000 values of $x(t)$ have been calculated for each problem. The floating-point precision was adjusted by hand as the required error level needed it; it was either 128 or 192 bits for all calculations on this time interval.

Parameters	1	1u	2	2u	3	4
m	1	1	1	1	1	1
α	1/2	1/2	1/2	1/2	1/5	1/5
ω_n	10	10	10	10	10	10
η	0.5	0.5	0.05	0.05	0.5	0.05
f_0	1	1	1	1	1	1
ω_e	4π	—	4π	—	4π	4π
X_0	0.25	0.25	0.25	0.25	0.25	0.25
V_0	0	0	0	0	0	0

Table 1: Numerical parameters of the examples. Problems 1u and 2u use a step function excitation.

5.2.1 Derivative of order 1/2, step function excitation (problems 1u and 2u)

Here, the reference solution is from Suarez and Shokooh (Section 3.1), and the same numerical parameters have been used as in [35] to reproduce the same results (Figure 1). Here, and in the followings, only the reference solutions are plotted; other solutions are very near when not practically identical to them.

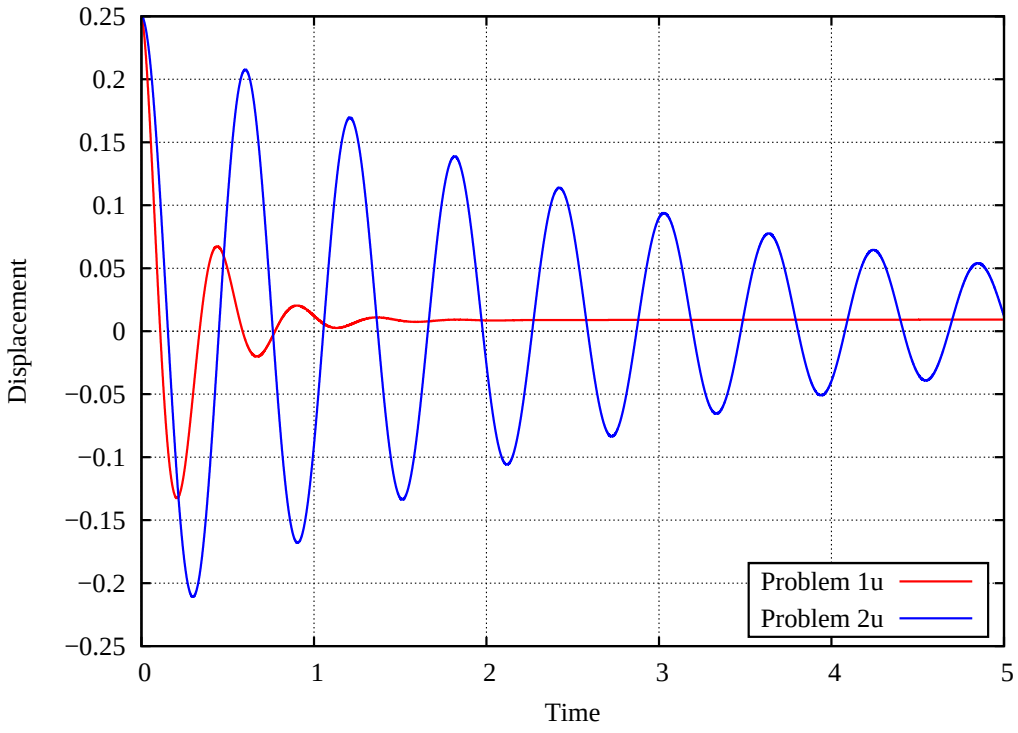

Figure 1: Solution curves for problems 1u and 2u.

Figure 2 shows the error indicator ε and the elapsed CPU time for the two problems. It is immediate to see that, for the same computational effort, the Taylor–Adomian method offers a much greater precision. (However, an engineering precision is assured by any of the methods.)

The difference of the calculation times for the Taylor–Adomian method comes from the difference of numerical conditions (mostly, the dissipation factor η), which require different numbers of terms in the Adomian series. As one would expect, the calculation times of the method based on the Grünwald–Letnikov formulation were not affected by the numerical parameters.

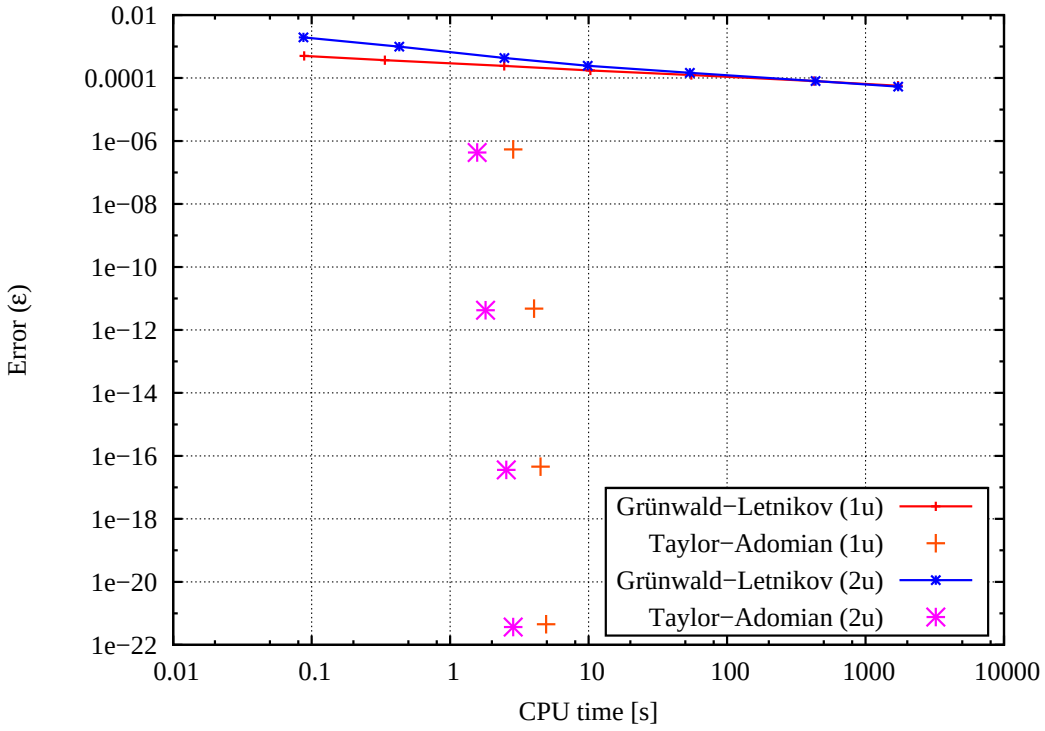

Figure 2: Comparison of methods for problems 1u and 2u ($\alpha = 1/2$, step function excitation).

5.2.2 Derivative of order $1/2$, harmonic excitation (problems 1 and 2)

Here, the reference solution is the solution by Adomian decomposition (Section 4.2). Results are shown in Figure 3.

Error indicators and CPU times are shown in Figure 4. Observations are the same as above: for a calculation time of the 10-second order, the Taylor–Adomian method is much more precise than the scheme with the Grünwald–Letnikov derivative.

5.2.3 Derivative of order $1/5$, harmonic excitation (problems 3 and 4)

To leave derivatives of order $1/2$, another value has been chosen for α. The reference solution for these problems is the solution by Adomian decomposition (Section 4.2).

Results are shown in Figure 5, while Figure 6 shows the values of the error indicator and CPU times. It is immediate to see the same as before: the Taylor–Adomian method is clearly much more precise while keeping the calculation time low.

5.3 Simulating a longer time interval

As seen above, the terms of Equations (29) and (33) become very large as t increases. It is easy to see that a major limitation of the Taylor–Adomian method will be this property: beyond a certain interval length, a huge number of terms are required, which increases the calculation time horrendously.

To check the extents of this phenomenon, calculations have been run with the Taylor–Adomian method up to the end of the transient, but at least to $t = 25$, with 200 calculated points per time unit. The error goal was 10^{-10}. Figure 7 shows elapsed CPU time versus simulated time. The points are at the end of the transient period, which is considered to be the end

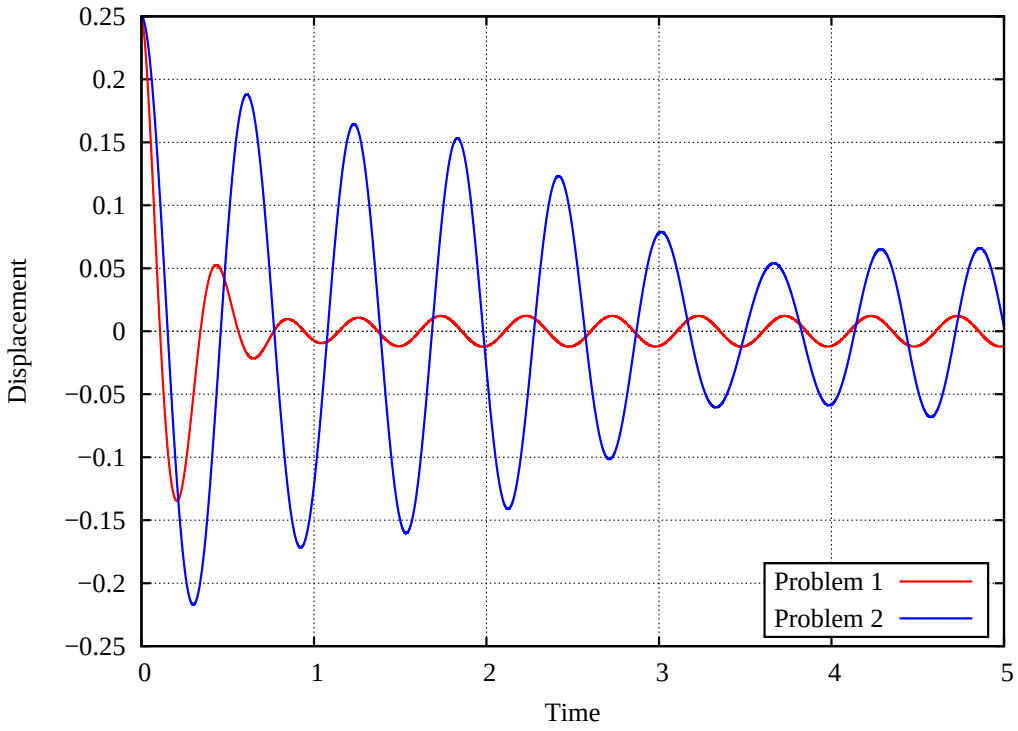

Figure 3: Solution of problems 1 and 2.

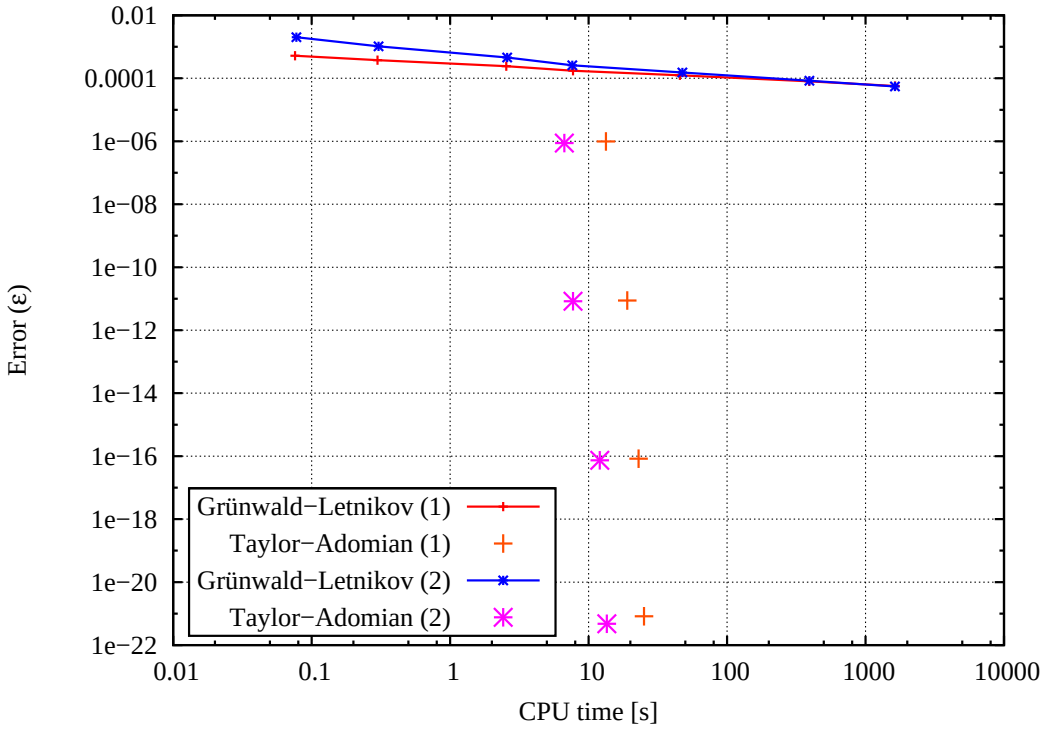

Figure 4: Comparison of methods for problems 1 and 2 ($\alpha = 1/2$, harmonic excitation).

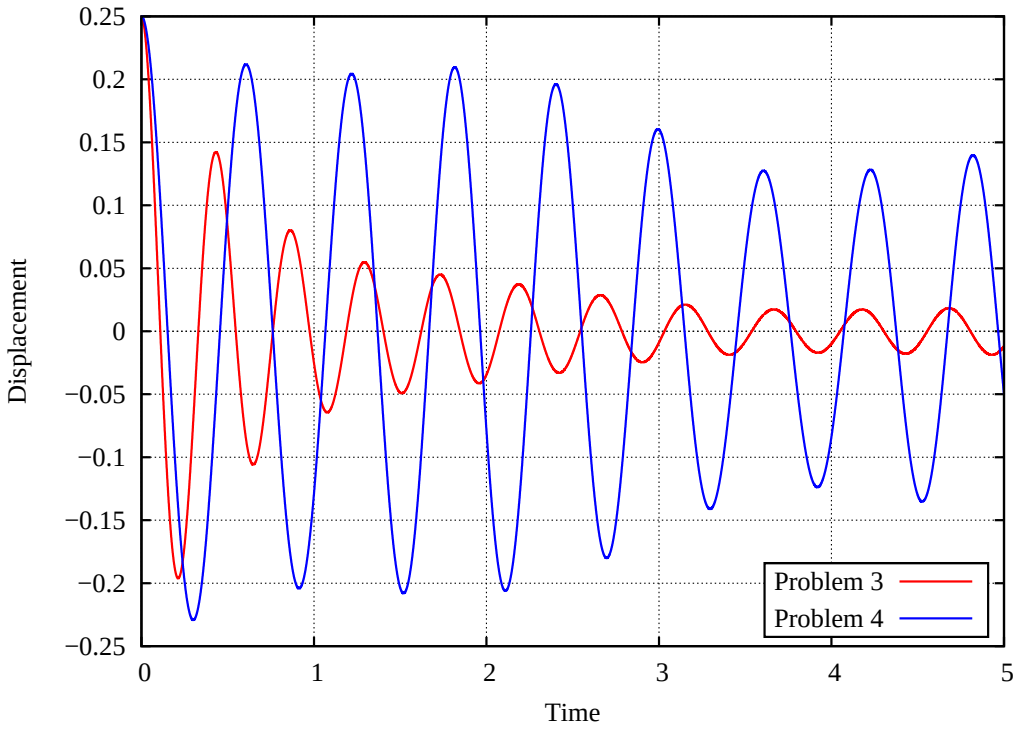

Figure 5: Solution of problems 3 and 4.

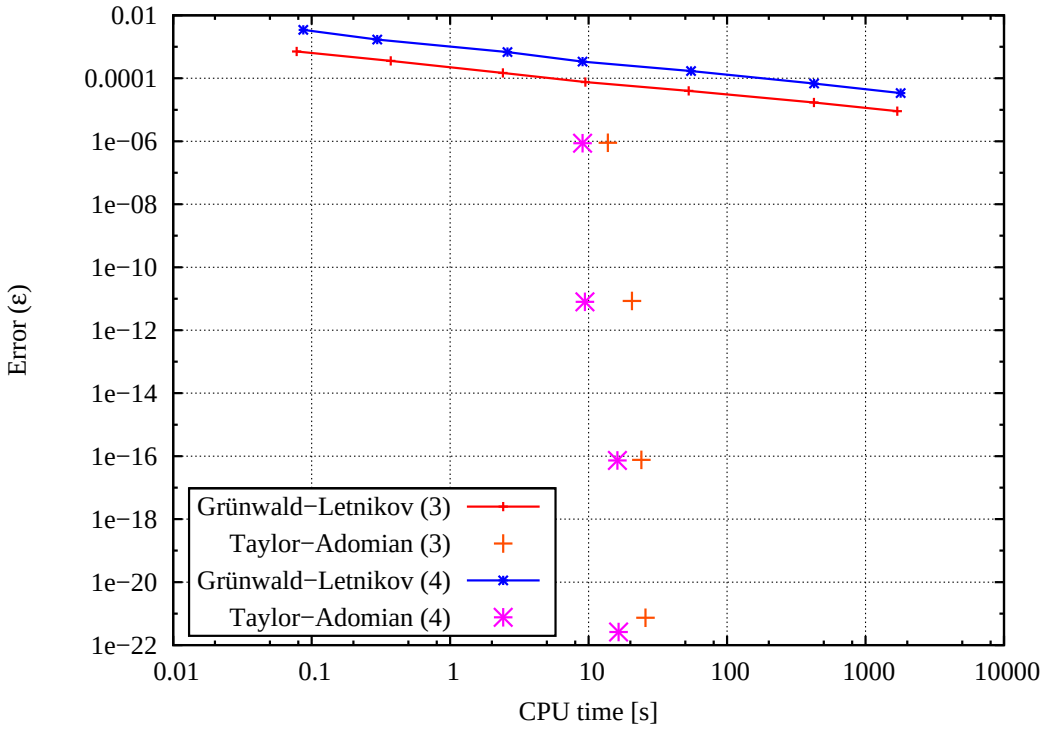

Figure 6: Comparison of methods for problems 3 and 4 ($\alpha = 1/5$, harmonic excitation).

of the last period of excitation for which the error indicator

$$\frac{1}{N}\sum_{i=1}^{N}\left|x_i - x_i^{\text{steady}}\right|$$

is above 1% of the steady-state amplitude.

It is immediate to see that problems 1 and 3, i.e. the highly damped equations settle very fast, and their CPU time for a very high precision is in the order of some or some 10 seconds. Even problem 2 settles within a CPU time of less than 10 minutes on an average 64-bit system. Contrarily, the transient of problem 4 required somewhat more than 6 hours to calculate on the same computer. This shows that the presented Taylor–Adomian method may be very efficient for the accurate solution of the fractionally damped vibration equation, especially when damping is not very weak.

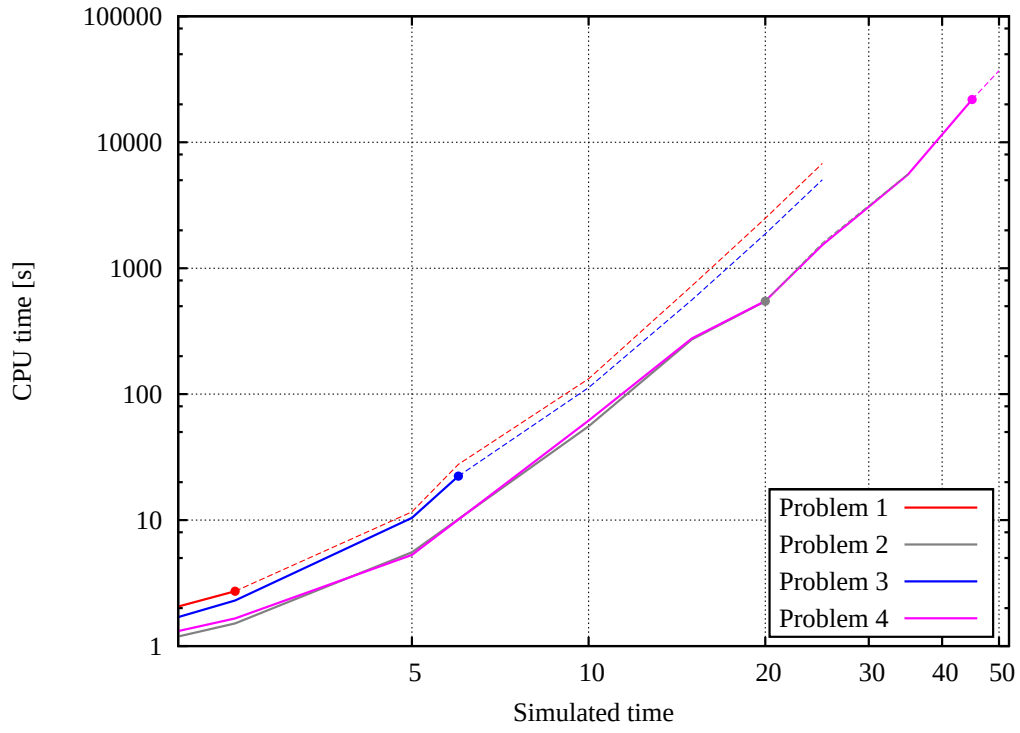

Figure 7: Calculation times for longer time intervals.

6 Conclusion

In this paper, the main goal was to present a method to solve the fractionally damped, inhomogeneous 1-DOF vibration equation with initial conditions. The Taylor–Adomian method has been proposed and tested on step-function excited and harmonically excited cases. It has proven very efficient in calculating the solution for a reasonably long time interval, practically with an arbitrarily low error. This makes the method immediately usable in providing quick reference results for other methods. The engineering application may be limited, however, due to the necessity of high-precision arithmetics and the fast increase of calculation time as the simulated time interval gets longer.

Moreover, the analytical solution of the same equation (with a harmonic excitation) has been calculated by the Adomian decomposition method. This solution served as a reference for the

Taylor–Adomian method.

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