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EIGENFORMS OF THE LAPLACIAN FOR RIEMANNIAN V-SUBMERSIONS

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Abstract. We study when the pull-back of an eigenform of the Laplacian on the base of a compact Riemannian V-submersion is an eigenform of the Laplacian on the total space of the submersion, and when the associated eigenvalue can change.

1. Introduction. We first review the situation in the smooth context. Let $\pi : Z \to Y$ be a Riemannian submersion, where Y and Z are compact *smooth* manifolds without boundary. Let Φ_p be an eigen p-form of the Laplacian on Y with eigenvalue λ. Suppose that $\pi^*\Phi_p$ is an eigen p-form of the Laplacian on Z with corresponding eigenvalue μ. This is, of course, a rather rare phenomena that greatly restricts the admissible geometry. We showed [6, 7] that $\lambda \le \mu$. If $p = 0$, then in fact the eigenvalue does not change, i.e., $\lambda = \mu$. For $p \ge 2$, we constructed examples in which the eigenvalue actually changes, i.e., where $\lambda < \mu$. The case $p = 1$ is still open.

In the present paper, we shall generalize these results to the case where Y and Z are Riemannian V-manifolds; we must deal with the complications introduced by the singular sets to extend the results known in the smooth setting to the situation at hand. Throughout the paper, we shall only deal with compact V-manifolds without boundary. We use the Friedrichs extension to define the p-form valued Laplacian Δ_M^p on a V-manifold M; let $E_\lambda^p(M)$ be the associated eigenspaces. We shall always suppose that the singular set has codimension at least 2; this has some important analytic consequences as we shall see in Section 2.

There is a rather elegant geometric characterization of the situation when the pull back of every eigen p-form on Y is an eigen p-form on Z; necessarily the eigenvalues do not change in this setting:

THEOREM 1.1. *Let $\pi : Z \to Y$ be a Riemannian V-submersion of closed V-manifolds, where the singular sets of Z and Y are of codimension at least* 2.

(1) *Let $p = 0$. The following conditions are equivalent*:

(a) $\Delta_Z^0 \pi^* = \pi^* \Delta_Y^0$.

(b) *For any $\lambda \ge 0$, $\pi^* E_\lambda^0(Y) \subset E_\lambda^0(Z)$.*

(c) *For any $\lambda \ge 0$, there exists $\mu(\lambda) \ge 0$ such that $\pi^* E_\lambda^0(Y) \subset E_{\mu(\lambda)}^0(Z)$.*

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(d) *The fibers of* π *are minimal.*

(2) *Let* $p > 0$. *The following conditions are equivalent*:

(a) $\Delta_Z^p \pi^* = \pi^* \Delta_Y^p$.

(b) *For any* $\lambda \geq 0$, $\pi^* E_\lambda^p(Y) \subset E_\lambda^p(Z)$.

(c) *For any* $\lambda \geq 0$, *there exists* $\mu(\lambda) \geq 0$ *such that* $\pi^* E_\lambda^p(Y) \subset E_{\mu(\lambda)}^p(Z)$.

(d) *The fibers of* π *are minimal and the horizontal distribution is integrable.*

Theorem 1.1 shows that if all the eigenspaces are preserved, then all the eigenvalues are preserved as well. We now focus on what happens if just a single eigenform is preserved. The eigenvalue can not change if $p = 0$; more generally, the eigenvalue can not decrease if $p > 0$. We remark that this fails in the context of manifolds with boundary; Neumann eigenvalues can decrease [14].

THEOREM 1.2. *Let* $\pi : Z \to Y$ *be a Riemannian V-submersion of closed* V*-manifolds, where the singular sets of* Z *and* Y *are of codimension at least* 2.

(1) *If* $0 \neq \Phi \in E_\lambda^0(Y)$ *and if* $\pi^*\Phi \in E_\mu^0(Z)$, *then* $\lambda = \mu$.

(2) *Let* $p > 0$. *If* $0 \neq \Phi \in E_\lambda^p(Y)$ *and if* $\pi^*\Phi \in E_\mu^p(Z)$, *then* $\lambda \leq \mu$.

Theorem 1.2 is sharp if $p \geq 2$. We refer to [6, 7] for the proof of the following result in the smooth setting; the result in the more general context is then immediate.

THEOREM 1.3. *Let* $p \geq 2$ *and let* $0 \leq \lambda < \mu < \infty$ *be given. There exists a Riemannian V-submersion* $\pi : Z \to Y$ *and there exists* $0 \neq \Phi \in E_\lambda^p(Y)$ *so that* $\pi^*\Phi \in E_\mu^p(Z)$.

Here is a brief outline of the paper. In Section 2, we review the definition of a V-manifold, the Friedrichs extension of the Laplacian, and a basic regularity result. In Section 3, we recall some useful formulae intertwining the coderivative on the base and on the total space of a Riemannian submersion. We also discuss submersions in the context of V-manifolds. In Section 4 we introduce the Hopf fibration. We then take the quotient of the Hopf fibration by suitably chosen cyclic group actions to construct a useful family of V-submersions. We conclude the paper in Section 5 by completing the proof of Theorems 1.1 and 1.2.

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2. *V*-manifolds and *V*-submersions. The notion of a V-manifold was introduced by Satake [16]; he used the symbol 'V' to indicate that one was dealing with a cone-like singularity. Such manifolds are also called *orbifolds*, see, for example, [18, 19]. Their spectral geometry has been studied by many authors; for example, see [8]. They also appear in the study of foliations [10, 20].

In this paper, we follow the notational conventions of [12, 16, 17]. Let $O(m)$ be the orthogonal group. Let B_δ be the ball of radius δ in $\boldsymbol{R}^m$ centered at the origin. If G is a finite subgroup of $O(m)$, then G acts by isometries on B_δ; let B_δ/G be the associated quotient space.

It is worth noting for further use the following fact. Let G be a finite group acting on an open neighborhood $\mathcal{O}$ of the origin in $\boldsymbol{R}^m$. If G preserves some Riemannian metric on $\mathcal{O}$ and if G fixes the origin, then in geodesic coordinates, the action of G is linearizable in the sense given above.

Let M be a compact metric space. We say that M is a *V-manifold* if every point $P \in M$ has an open neighborhood U_P which is homeomorphic to $B_{\delta(P)}/G_P$ for some $\delta(P) > 0$ and some finite subgroup $G_P \subset O(m)$. Let $\tilde{U}_P = B_{\delta(P)}$ and let

$$\rho_P : \tilde{U}_P \to \tilde{U}_P/G_P = U_P$$

be the natural projection. Let

$$\tilde{S}_P := \{\tilde{Q} \in \tilde{U}_P \,;\ \text{there exists } \gamma \in G_P \text{ such that } \gamma \neq \mathrm{Id} \text{ and } \gamma\tilde{Q} = \tilde{Q}\}$$

be the *fixed point set* of G_P. We then have that

$$\rho_P : \tilde{U}_P \setminus \tilde{S}_P \to U_P \setminus \rho_P(\tilde{S}_P)$$

is a covering projection. The *singular set* of M is defined to be

$$S_M := \bigcup_{P \in M} \{\rho_P \tilde{S}_P\}\,.$$

Note that $\tilde{S}_P$ is the union of a finite number of linear subspaces of $\tilde{U}_P$. We suppose that these subspaces all have codimension at least 2. We shall suppose that $M \setminus S_M$ has the structure of a smooth manifold and that the maps ρ_P from $\tilde{U}_P \setminus \tilde{S}_P$ to $U_P \setminus \rho_P(\tilde{S}_P)$ are local diffeomorphisms. The V-manifold is a smooth manifold if S_M is empty or, equivalently, if $G_P = \{\mathrm{Id}\}$ for every $P \in M$.

We assume that the metric on M is induced from a Riemannian metric on $M \setminus S_M$. We also assume that there is a Riemannian metric $\tilde{g}_P$ on each $\tilde{U}_P$, which is invariant under the action of the group G_P such that ρ_P is a local isometry from $\tilde{U}_P \setminus \tilde{S}_P$ to $U_P \setminus S_P$.

Let M be a Riemannian V-manifold. Let dx be the associated Riemannian measure on $M \setminus S_M$. We shall take the Friedrichs extension of the Laplacian from $M \setminus S_M$ to define the Laplacian Δ_M^p on $L^2(\Lambda^p(M))$. Let $C_0^\infty(\Lambda^p(M \setminus S_M))$ be the space of smooth p-forms which are compactly supported in $M \setminus S_M$. Then the L^2 space $L^2(\Lambda^p(M))$ and the Sobolev space $H_1(\Lambda^p(M))$ are defined as the completion of $C_0^\infty(\Lambda^p(M \setminus S_M))$ with respect to the inner products

$$(\phi, \psi)_0 := \int_{M \setminus S_M} (\phi, \psi)dx \quad \text{and}$$
$$(\phi, \psi)_1 := \int_{M \setminus S_M} \{(d\phi, d\psi) + (\delta\phi, \delta\psi) + (\phi, \psi)\}dx\,,$$

respectively. Introduce the quadratic form

$$I^p(\phi, \psi) := \int_{M \setminus S_M} \{(d\phi, d\psi) + (\delta\phi, \delta\psi)\}dx\,.$$

The *Friedrichs extension* Δ_M^p is then defined [15] by the identity:

$$(\Delta_M^p \phi, \psi)_0 = I^p(\phi, \psi) \quad \text{for} \quad \phi, \psi \in H_1(\Lambda^p(M)) .$$

We remark that if we removed a set S of codimension at least 2 from a smooth manifold M, then this definition of the space $H_1(\Lambda^p(M))$ and Δ_M^p would agree with the usual definition. We set

$$E_\lambda^p(M) = \{\phi \in H_1(\Lambda^p(M))\ ;\ \Delta_M^p \phi = \lambda\phi\} .$$

The following regularity result is a central one in the subject — we shall derive it from results of Harvey and Polking [9] but there are many other proofs see, for example, the discussion in [1, 2] for $p = 0$. It identifies the eigenforms on M with the smooth equivariant eigenforms on the desingularization.

THEOREM 2.1. *Let M be a closed Riemannian V-manifold, where the singular set has codimension at least* 2. *Let $\phi \in L^2(\Lambda^p(M))$. Then the following conditions are equivalent*:

(1) $\phi \in E_\lambda^p(M)$.

(2) *For any $P \in M$, there exists $\tilde{\phi}_P \in C^\infty(\Lambda^p(\tilde{U}_P))$ so that*

  (a) $\tilde{\phi}_P|_{(\tilde{U}_P \setminus \tilde{S}_P)} = \rho_P^*(\phi_P|_{(U_P \setminus \rho_P(\tilde{S}_P))})$,

  (b) $\tilde{\phi}_P \in E_\lambda^p(\tilde{U}_P)$, *and*

  (c) $\gamma^* \tilde{\phi}_P = \tilde{\phi}_P$ *for any* $\gamma \in G_P$.

Since S_P has codimension at least 2, $C_0^\infty(\tilde{U}_P \setminus \tilde{S}_P)$ is dense in $H_1(\tilde{U}_P)$. It is now immediate from the discussion given above that Condition (2) implies Condition (1). The converse is a smoothness result which shows that the pull-back eigenforms extend smoothly across the singular set. Before establishing this implication, we first recall a technical Lemma.

LEMMA 2.2. *Let Ω be an open subset of $\boldsymbol{R}^m$ and A a closed subset of Ω. Let $P(x, D)$ be a vector valued partial differential operator on Ω.*

(1) *Assume that* $\nu := m - 2 \cdot \text{order}(P) \geq 0$ *and that the lower Minkowski content $M_\nu(K)$ is finite for every compact set $K \subset A$. If $\phi \in L^2_{\text{loc}}(\Omega)$ and if $P\phi = 0$ on $\Omega - A$, then $P\phi = 0$ on Ω.*

(2) *If P is elliptic, if $\phi \in L^2_{\text{loc}}(\Omega)$, and if $P\phi$ is smooth on Ω, then ϕ is smooth on Ω.*

PROOF. Assertion (1) follows from Corollary 2.4 (a) of Harvey and Polking [9], who generalized earlier work of Littman [13]. Assertion (2) is a standard elliptic regularity result, see, for example, Gilkey [3]. □

PROOF OF THEOREM 2.1. Let ϕ be an L^2 eigenform of the Friedrichs extension of Δ corresponding to the eigenvalue λ; we omit p from the notation. Note that

$$\|\phi\|_1^2 := \|\phi\|_{L^2}^2 + \|(d+\delta)\phi\|_{L^2}^2 .$$

Let $\rho_P : \tilde{U}_P \to U_P = \tilde{U}_P / G_P$ be a local desingularization. Let $\tilde{d}$ and $\tilde{\delta}$ denote the exterior derivative and co-derivative on $\tilde{U}_P$, respectively. We then have $\rho_P^* d = \tilde{d}\rho_P^*$ and $\rho_P^*\delta = \tilde{\delta}\rho_P^*$, since ρ_P is an isometry off the singular set. We set

$$\tilde{\phi}_P := \rho_P^* \phi .$$

As the singularity set has codimension at least 2 and as $\phi \in H_1(M)$, $\tilde\phi_P \in H_1(\tilde U_P)$. The equivariance property is immediate, since $\gamma^*\rho_P^* = \rho_P^*$. To complete the proof, we must show that $\tilde\phi_P$ extends smoothly to all of $\tilde U_P$.

We decompose $\tilde\Delta - \lambda$ as the product of two first order operators:

$$\tilde\Delta - \lambda = (\tilde d + \tilde\delta + \sqrt{\lambda})(\tilde d + \tilde\delta - \sqrt{\lambda})\,.$$

We set

$$\psi := (d + \delta - \sqrt{\lambda})\phi \quad \text{and} \quad \tilde\psi_P := \rho_P^*\psi = (\tilde d + \tilde\delta - \sqrt{\lambda})\tilde\phi_P\,.$$

We then have that $\tilde\psi_P$ is in L^2. We may express:

$$\begin{aligned}(\tilde d + \tilde\delta + \sqrt{\lambda})\tilde\psi_P &= (\tilde d + \tilde\delta + \sqrt{\lambda})(\tilde d + \tilde\delta - \sqrt{\lambda})\tilde\phi_P = (\tilde\Delta - \lambda)\tilde\phi_P\\ &= \rho_P^*(\Delta - \lambda)\phi = 0 \quad \text{on} \quad \tilde U_P \setminus \tilde S_P\,.\end{aligned}$$

By assumption, $\tilde S_P$ is the finite union of finite number of linear subspaces of codimension at least 2 intersected with $\tilde U_P$. Thus the $m-2$ dimensional lower Minkowski measure of any compact subset of $\tilde S_P$ is finite. Lemma 2.2 (1) shows that

$$(\tilde d + \tilde\delta + \sqrt{\lambda})\tilde\psi_P = 0 \quad \text{on} \quad \tilde U_P\,.$$

Since $\tilde d + \tilde\delta + \sqrt{\lambda}$ is elliptic, Lemma 2.2 (2) implies that $\tilde\psi_P$ is smooth on $\tilde U_P$.

Since $\tilde\phi_P$ is in $H_1(\tilde U_P)$, $(\tilde d + \tilde\delta - \sqrt{\lambda})\tilde\phi_P = \tilde\psi_P$ in $L^2(\tilde U_P)$. Since $\tilde\psi_P$ is smooth on $\tilde U_P$ and since this operator is elliptic, another application of Lemma 2.2 (2) yields that $\tilde\phi_P$ is smooth on $\tilde U_P$ as desired. □

Theorem 2.1 shows the pull-back eigenforms of the Friedrichs Laplacian on M are ordinary eigenforms of the Laplacian on $\tilde U_P$ for any $P \in M$ which are invariant under the groups G_P. Conversely, if we are given a collection of eigenforms $\tilde\phi_P$ in $E_\lambda^p(\tilde U_P)$ which are invariant under the action of the groups G_P and which patch together, then they define an eigenform of Δ_M^p on M.

We can construct the associated spectral resolution using Theorem 2.1 and Rellich compactness. We refer to [2, 11, 18] for additional details. Let

$$\lambda_1 := \inf_{0\neq\phi\in H_1(\Lambda^p(M))} \frac{I^p(\phi,\phi)}{(\phi,\phi)_0}\,.$$

The infimum is attained by a function $\phi_1 \in \text{Domain}(\Delta_M^p)$ so that $\Delta_M^p\phi_1 = \lambda_1\phi_1$. The second eigenvalue λ_2 is then defined by setting:

$$\lambda_2 := \inf_{0\neq\phi\in H_1(\Lambda^p(M)),\ \phi\perp\phi_1} \frac{I^p(\phi,\phi)}{(\phi,\phi)_0}\,;$$

where '$\perp$' is with respect to the L^2 inner product. Again, the infimum is attained by a function $\phi_2 \in \text{Domain}(\Delta_M^p)$. One proceeds in this fashion to construct a complete orthonormal basis $\{\phi_i\}_{i=1}^\infty$ for $L^2(\Lambda^p(M))$ so that $\Delta_M^p\phi_i = \lambda_i\phi_i$. The collection $\{\lambda_i, \phi_i\}$ is called a *discrete spectral resolution* of Δ_M^p and we have that

$$E_\lambda^p(M) = \text{span}_{\{\lambda=\lambda_i\}}\{\phi_i\}\,.$$

Again, this definition coincides with the usual definition in the smooth setting. We summarize the discussion given above in the following result.

LEMMA 2.3. *Let M be a closed Riemannian V-manifold, where the singular set has codimension at least* 2. *Then the following hold.*

(1) Δ_M^p *is self-adjoint and non-negative.*

(2) *There exists a discrete spectral resolution* $\{\lambda_i, \phi_i\}$ *for* Δ_M^p, *where* $\lambda_i \to \infty$.

(3) *We have a complete orthogonal direct sum decomposition*

$$L^2(\Lambda^p(M)) = \bigoplus_\lambda E_\lambda^p(M)\,.$$

3. Submersions in the context of *V*-manifolds. We begin by reviewing some of the geometry of a Riemannian submersion in the smooth setting. Let $\pi : Z \to Y$ be a Riemannian submersion of closed smooth manifolds. For $z \in Z$, we decompose $T_zZ = V_z \oplus H_z$, where

$$V_z := \ker(\pi_{*z}) \quad \text{and} \quad H_z := V_z^\perp$$

are the *vertical* and *horizontal* spaces, respectively; by assumption π_* is an isometry from H_z to $T_{\pi z}Y$. We introduce the following notational conventions. Let indices i, j and k index local orthonormal frames $\{e_i\}$ and $\{e^i\}$ for the vertical distributions and co-distributions $\mathcal{V}$ and $\mathcal{V}^*$ of π. Let indices a, b, and c index local orthonormal frames $\{f_a\}$ and $\{f^a\}$ for the horizontal distributions and co-distributions $\mathcal{H}$ and $\mathcal{H}^*$ of π. If ξ is a covector, then let $\operatorname{ext}(\xi)$ and $\operatorname{int}(\xi)$ denote *exterior multiplication* and the dual, *interior multiplication*, respectively. We define tensors θ and ω and endomorphisms $\mathcal{E}$ and Ξ by setting

$$\tag{3.a} \begin{aligned} &\theta := -\sum_{i,a} g_Z([e_i, f_a], e_i) f^a, \quad \omega_{abi} := \frac{1}{2} g_Z(e_i, [f_a, f_b])\,, \\ &\mathcal{E} := \sum_{a,b,i} \omega_{abi} \operatorname{ext}_Z(e^i) \operatorname{int}_Z(f^a) \operatorname{int}_Z(f^b), \quad \Xi := \operatorname{int}_Z(\theta) + \mathcal{E}\,. \end{aligned}$$

The tensor θ is the unnormalized mean curvature co-vector of the fibers of π and ω is the curvature of the horizontal distribution. We say that the fibers are *minimal* if $\theta = 0$. We say that the horizontal distribution $\mathcal{H}$ is *integrable* if $\omega = 0$.

The pull-back π^* is a linear map from $C^\infty(\Lambda^p(Y))$ to $C^\infty(\Lambda^p(Z))$, which commutes with the exterior derivative, i.e., $\pi^* d_Y = d_Z \pi^*$. However, π^* does *not* in general commute with the coderivative. We refer to [7] for the proof of the following result; what is crucial for our present considerations is that the result in question is purely local — it does not rely on any compactness considerations.

LEMMA 3.1. *Let $\pi : Z \to Y$ be a Riemannian submersion of Riemannian manifolds. Then $\delta_Z \pi^* - \pi^* \delta_Y = \Xi \pi^*$ and $\Delta_Z \pi^* - \pi^* \Delta_Y = (d_Z \Xi + \Xi d_Z)\pi^*$.*

If $p = 0$, the situation is simpler. If Φ is a 0-form, i.e., a function, then

$$\sum_{a,b,i} \omega_{abi} \operatorname{ext}_Z(e^i) \operatorname{int}_Z(f^a) \operatorname{int}_Z(f^b) \pi^* \Phi = 0\,,$$
$$\sum_{a,b,i} \omega_{abi} \operatorname{ext}_Z(e^i) \operatorname{int}_Z(f^a) \operatorname{int}_Z(f^b) d_Z \pi^* \Phi = 0\,, \quad \text{and}$$
$$\operatorname{int}_Z(\theta)\pi^*\Phi = 0\,.$$

The following Corollary is now immediate.

COROLLARY 3.2. *Let $\pi : Z \to Y$ be a Riemannian submersion of Riemannian manifolds. Then $\Delta^0_Z \pi^* - \pi^* \Delta^0_Y = \operatorname{int}_Z(\theta) d_Z \pi^*$ on $C^\infty(Y)$.*

We say that the horizontal distribution is integrable if $\omega = 0$. We refer to [5] for the proof of the following result which relates θ to the local volume element in this setting.

LEMMA 3.3. *Let X be the fiber of a Riemannian submersion $\pi : Z \to Y$. Assume that the horizontal distribution of π is integrable. Then there exist local coordinates $z = (x, y)$ on Z so $\pi(x, y) = y$,*

$$ds^2_Y = \sum_{a,b} h_{ab}(y) dy^a \circ dy^b\,, \quad \textit{and}$$
$$ds^2_Z = \sum_{i,j} g_{ij}(x, y) dx^i \circ dx^j + \sum_{ab} h_{ab}(y) dy^a \circ dy^b\,.$$

If we set $g_X := \det(g_{ij})^{1/2}$, then $\theta = -d_Y \ln(g_X)$.

We now extend these notions to the singular setting. Let X be a closed smooth manifold. We enlarge slightly the notion of admissible charts and consider an action

$$\gamma \cdot (\tilde{u}, x) = (\gamma\tilde{u}, \gamma(\tilde{u})x) \quad \text{on} \quad \tilde{U}_P \times X \quad \text{for} \quad \gamma \in G_P\,. \tag{3.b}$$

DEFINITION 3.4. Let Y and Z be V-manifolds and let X be a smooth manifold. We say that $\pi : Z \to Y$ is a *V-manifold fiber bundle with fiber X* if we can choose charts $\tilde{U}_y/G_y$ over Y and charts $\{\tilde{U}_y \times X\}/H_y$ over Z so that we have a diagram

$$\begin{array}{ccc} \tilde{U}_y \times X & \xrightarrow{\rho^Z_y} & \{\tilde{U}_y \times X\}/H_y \\ \downarrow \tilde{\pi} & & \downarrow \pi \\ \tilde{U}_y & \xrightarrow{\rho_y} & U_y := \tilde{U}_y/G_y \end{array} \qquad \text{with} \quad \pi \circ \rho^Z_y = \rho_y \circ \bar{\pi}\,. \tag{3.c}$$

Here H_y is a subgroup of G_y, $\tilde{\pi}$ and π are projection on the first factors, the action of H_y on $\tilde{U}_y \times X$ is as discussed above in Equation (3.b), and the projections ρ^Z_y and ρ_y are the associated quotient maps. We say that π is a Riemannian V-submersion if additionally $\tilde{\pi}$ is a Riemannian submersion.

We remark that the Riemannian metric on $\tilde{U}_y \times X$ is not in general a product metric and that the decomposition in question is only local; in general, of course, Z is not $Y \times X$.

Let $\pi : Z \to Y$ be a Riemannian V-submersion. Let $Y_r := Y \setminus S_Y$ and $Z_r := Z \setminus \pi^{-1}\{S_Y\}$; these are open Riemannian manifolds. Although Y_r is the regular set of Y, the regular set of Z may be larger than Z_r. Let π_r be the induced Riemannian submersion from Z_r to Y_r.

LEMMA 3.5. *Let $\pi : Z \to Y$ be a Riemannian V-submersion of closed V-manifolds, where the singular sets of Z and Y are codimension at least 2. If $\Phi \in E^p_\lambda(Y)$ and if $\pi_r^*\Phi \in E^p_\mu(Z_r)$, then $\pi^*\Phi \in E^p_\mu(Z)$.*

PROOF. Let $y \in Y$. Let $\tilde{U}_y/G_y$ and $\{\tilde{U}_y \times X\}/H_y$ be desingularizing local charts on Y and on Z, respectively. By Theorem 2.1, $\tilde{\Phi}_y := \rho_y^*\Phi$ extends smoothly from $\tilde{U}_y \setminus \tilde{S}_y$ to $\tilde{U}_y$ with $\Delta^p_{\tilde{U}_y}\tilde{\Phi}_y = \lambda\tilde{\Phi}_y$. Pulling back then yields that $\tilde{\pi}^*\tilde{\Phi}_y$ is smooth on $\tilde{U}_y \times X$. Since

$$\Delta^p_{\tilde{U}_y \times X}\tilde{\pi}^*\tilde{\Phi}_y = \mu\pi^*\tilde{\Phi}_y \quad \text{on} \quad \{\tilde{U}_y \setminus \tilde{S}_y\} \times X$$

and since $\tilde{S}_y \times X$ has co-dimension at least 2 in $\tilde{U}_y \times X$, we have by continuity that

$$\Delta^p_{\tilde{U}_y \times X}\tilde{\pi}^*\tilde{\Phi}_y = \mu\pi^*\tilde{\Phi}_y \quad \text{on} \quad \tilde{U}_y \times X\,.$$

As the equivariance property is immediate, Theorem 2.1 implies $\pi^*\Phi \in E^p_\mu(Z)$. □

We shall need to generalize the *fiber product* from the smooth setting to the singular one. Let $\pi_i : Z_i \to Y$ be Riemannian V-submersions with fibers X_i for $i = 1, 2$. The fiber product is defined by setting

$$Z = Z(Z_1, Z_2) := \{z = (z_1, z_2) \in Z_1 \times Z_2\,;\ \pi_1(z_1) = \pi_2(z_2)\}\,.$$

We now study the local geometry. Choose charts on Y, associated charts on Z_i, and local projections as given above. We shall assume that the associated groups on Z_i are the same, i.e., $H_{y,1} = H_{y,2}$ for all $y \in Y$; this is a crucial point. The assumption $H_{y,1} = H_{y,2}$ causes no difficulty as we shall be taking $Z_1 = Z_2$ subsequently. Then local charts for the fiber product and the fiber product action which generalize those given in Equation (3.b) are defined by taking the action:

$$\gamma \cdot (\tilde{y}, x_1, x_2) = (\gamma\tilde{y}, \gamma(\tilde{y})x_1, \gamma(\tilde{y})x_2) \quad \text{on} \quad \tilde{U}_y \times X_1 \times X_2 \quad \text{for } \gamma \in H_{y,1} = H_{y,2}\,.$$

This shows that $\pi : Z \to Y$ is a V-manifold fiber bundle with fiber $X_1 \times X_2$. A similar argument can be used to show that π is a Riemannian V-submersion, where a suitable rescaling of the metric on the horizontal distribution is used, exactly as was done in the non-singular setting [6]. Let $\sigma_1(z_1, z_2) := z_1$ and $\sigma_2(z_1, z_2) := z_2$ define maps $\sigma_i : Z \to Z_i$. These are Riemannian V-submersions as well.

LEMMA 3.6. *Adopt the notation given above. If $\Phi \in E^p_\lambda(Y)$, if $\pi_1^*\Phi \in E^p_{\lambda+\varepsilon_1}(Z_1)$, and if $\pi_2^*\Phi \in E^p_{\lambda+\varepsilon_2}(Z_2)$, then $\pi^*\Phi \in E^p_{\lambda+\varepsilon_1+\varepsilon_2}(Z)$.*

PROOF. Off the singular locus, we use the computations given in [6] to see

$$\theta_Z = \sigma_1^*\theta_{Z_1} + \sigma_2^*\theta_{Z_2} \quad \text{and} \quad \mathcal{E}_Z\pi^* = \sigma_1^*\mathcal{E}_{Z_1}\pi_1^* + \sigma_2^*\mathcal{E}_{Z_2}\pi_2^*\,.$$

A straight forward application of Lemma 3.1 then shows that

$$\Delta_Z^p \pi^* \Phi = (\lambda + \varepsilon_1 + \varepsilon_2)\pi^* \Phi \quad \text{on} \quad Z_r \,.$$

The desired conclusion now follows from Lemma 3.5. □

4. The Hopf fibration. There is a useful family of examples which we can describe as follows. We give the unit sphere S^n in $\boldsymbol{R}^{n+1}$ the standard metric g_n of constant sectional curvature $+1$. We identify $\boldsymbol{R}^4 = \boldsymbol{C}^2$ to regard $S^3 \subset \boldsymbol{C}^2$ and we identify $\boldsymbol{R}^3 = \boldsymbol{C} \oplus \boldsymbol{R}$ to regard $S^2 \subset \boldsymbol{C} \oplus \boldsymbol{R}$. The *Hopf fibration* $\tilde{h} : S^3 \to S^2$ is then defined by setting

$$\tilde{h}(z_1, z_2) := (2z_1\bar{z}_2, |z_1|^2 - |z_2|^2) \,.$$

One then has that $\tilde{h} : (S^3, g_3) \to (S^2, (1/4)g_2)$ is a Riemannian submersion; $(S^2, (1/4)g_2)$ is, of course, just the sphere of radius $1/2$ in $\boldsymbol{R}^3$.

EXAMPLE 4.1. Let $\boldsymbol{Z}_n$ be the group of n-th roots of unity in $\boldsymbol{C}$. We define actions ρ_2 and ρ_3 of $\boldsymbol{Z}_n$ on S^2 and on S^3, respectively, by setting

$$\rho_2(\gamma)(w, t) := (\gamma w, t) \quad \text{and} \quad \rho_3(\gamma)(z_1, z_2) = (\gamma z_1, z_2) \quad \text{for} \quad \gamma \in \boldsymbol{Z}_n \,.$$

Since the actions in question are by isometries, this gives the quotient spaces

$$M_n^2 := S^2/\rho_2(\boldsymbol{Z}_n) \quad \text{and} \quad M_n^3 := S^3/\rho_3(\boldsymbol{Z}_n)$$

the structure of Riemannian V-manifolds. Furthermore, as the group actions are compatible with the Hopf fibration, we have a diagram

$$\begin{array}{ccc} S^3 & \xrightarrow{\pi_3} & M_n^3 \\ \downarrow \tilde{h} & & \downarrow h \\ S^2 & \xrightarrow{\pi_2} & M_n^2 \end{array} \quad \text{with} \quad h \circ \pi_3 = \pi_2 \circ \tilde{h} \,. \tag{4.a}$$

The induced Hopf map h is a Riemannian V-submersion.

Let $N = (0, 1)$ and $S = (0, -1)$ be the north and south poles of $S^2 \subset \boldsymbol{C} \oplus \boldsymbol{R}$. Then $\tilde{h}^{-1}(N)$ is the circle $(z_1, 0)$, while $\tilde{h}^{-1}(S)$ is the circle $(0, z_2)$. The action of $\boldsymbol{Z}_n$ on $\tilde{h}^{-1}(N)$ is without fixed points; thus the image of this circle in M_n^3 is non-singular; the singular set of M_n^3 is $\tilde{h}^{-1}(S)$. Thus this example illustrates that the singular set of the total space is not simply the inverse image of the singular set of the base. Note that the singular sets of M_n^3 and of M_n^2 have codimension 2.

Let $\tilde{\nu}_2$ be the volume form on $(S^2, (1/4)g_2)$. Since $\tilde{\nu}_2$ is invariant under the action of $\boldsymbol{Z}_n$, it descends, by Theorem 2.1, to define a harmonic 2-form $\nu_2 \in E_0^2(M_n^2)$. One computes that $\tilde{h}^*\tilde{\nu}_2 \in E_4^2(S^3)$, see [4] for details. Thus similarly $h^*\nu_2 \in E_4^2(M_n^3)$. This illustrates Theorem 1.3 by providing an example where the eigenvalue changes.

EXAMPLE 4.2. Let (p, q) be coprime integers and let $n = pq$. Choose integers a and b so that $ap - bq = 1$. We define actions ρ_2 and ρ_3 of $\boldsymbol{Z}_n$ on S^2 and on S^3, respectively, by setting

$$\rho_{2,p,q}(\gamma)(w, t) := (\gamma w, t) \quad \text{and} \quad \rho_{3,p,q}(\gamma)(z_1, z_2) := (\gamma^{ap} z_1, \gamma^{bq} z_2) \quad \text{for} \ \gamma \in \boldsymbol{Z}_n \,.$$

Since the actions in question are compatible with the Hopf map, we once again have a commutative diagram of the form given above in Diagram (4.a). The action of $\boldsymbol{Z}_n$ on the fiber circles $\tilde{h}^{-1}(N) = (z_1, 0)$ and $\tilde{h}^{-1}(S) = (0, z_2)$ in S^3 is not faithful. The isotropy group for these fiber circles is $\boldsymbol{Z}_q$ and $\boldsymbol{Z}_p$, respectively. This example illustrates that the structure groups can be different over different components of the singular set.

EXAMPLE 4.3. Let N be an arbitrary closed Riemannian manifold. We extend the actions of $\boldsymbol{Z}_n$ defined in Example 4.2 to act trivially on N to define a diagram

$$\text{(4.b)}\quad \begin{array}{ccc} S^3 \times N & \xrightarrow{\pi_3 \times \mathrm{Id}} & M_n^3 \times N \\ \downarrow \tilde{h} \times \mathrm{Id} & & \downarrow h \times \mathrm{Id} \\ S^2 \times N & \xrightarrow{\pi_2 \times \mathrm{Id}} & M_n^2 \times N \end{array} \quad \text{with} \quad (h \times \mathrm{Id}) \circ (\pi_3 \times \mathrm{Id}) = (\pi_2 \times \mathrm{Id}) \circ (\tilde{h} \times \mathrm{Id})\,.$$

By replacing ν_2 by $\nu_2 \wedge \mu_{p-2}$ for a suitably chosen eigen-form μ_{p-2} on N, and by rescaling the metrics appropriately, a family of examples can be constructed illustrating Theorem 1.3 in full generality. We omit details in the interests of brevity and refer to [6, 7] for further details.

EXAMPLE 4.4. One can also consider the higher dimensional Hopf fibration $\tilde{h} : S^{2n+1} \to \boldsymbol{CP}^n$, where the sphere S^{2n+1} is given the standard metric and where complex projective space $\boldsymbol{CP}^n$ is given a suitably scaled Fubini-Study metric. Let $\mu_2 \in E_0^2(\boldsymbol{CP}^n)$ be the Kaehler form; this restricts to a multiple of the volume form on $S^2 = \boldsymbol{CP}^1$. If $1 \leq p \leq n$, then

$$\mu_2^p \in E_0^{2p}(\boldsymbol{CP}^n) \quad \text{and} \quad \tilde{h}^* \mu_2^p \in E_{4p(n+1-p)}^{2p}(S^{2n+1})\,.$$

We refer to Remark 3.6 [4] for further details. Again, suitable cyclic group actions yield appropriate V-manifold examples.

5. Proof of theorems.

PROOF OF THEOREM 1.2 (1). Let $\pi : Z \to Y$ be a Riemannian V-submersion, where Z and Y are closed V-manifolds. We assume that the singular sets of both Y and Z have codimension at least 2. Let $0 \neq \Phi \in E_\lambda^0(Y)$. Assume that $\Psi := \pi^* \Phi \in E_\mu^0(Z)$. Let $\rho_y : \tilde{U}_y \to U_y$ be an V-manifold chart on Y. We apply Theorem 2.1 to see that $\tilde{\Phi}_y := \rho_y^* \Phi$ is smooth on $\tilde{U}_y$. Since $\tilde{\Phi}_y$ is invariant under the action of the group G_y, Φ is continuous on the quotient $U_y = \tilde{U}_y / G_y$. Since Φ is continuous on a compact space Y, we may choose $y_0 \in Y$ so $\Phi(y_0)$ is maximal. By replacing Φ by $-\Phi$ if necessary, we may assume without loss of generality that $\Phi(y_0) > 0$.

Let $\tilde{Z}_{y_0} := \tilde{U}_{y_0} \times X$ and let $\tilde{\Phi}_{y_0} := \rho_{y_0}^* \Phi$. By a slight generalization of Theorem 2.1,

$$\tilde{\Phi}_{y_0} \in E_\lambda^0(\tilde{U}_{y_0}) \quad \text{and} \quad \tilde{\pi}^* \tilde{\Phi}_{y_0} \in E_\mu^0(\tilde{Z}_{y_0})\,.$$

We apply Corollary 3.2 to the Riemannian submersion $\tilde{\pi} : \tilde{Z}_{y_0} \to \tilde{U}_{y_0}$ to see

$$\text{(5.a)}\qquad (\mu - \lambda)\tilde{\pi}^* \tilde{\Phi}_{y_0} = \{\Delta_{\tilde{Z}}^0 \pi^* - \pi^* \Delta_{\tilde{Y}}^0\} \tilde{\Phi}_{y_0} = \mathrm{int}(\tilde{\theta}) d_{\tilde{Z}} \tilde{\pi}^* \tilde{\Phi}_{y_0} = \mathrm{int}(\tilde{\theta}) \tilde{\pi}^* d_{\tilde{Y}} \tilde{\Phi}_{y_0}\,.$$

Choose $\tilde{z}_0$ so $\tilde{\pi}\tilde{z}_0 = \tilde{y}_0$. Since $\tilde{\Phi}_{y_0}$ has a maximum at $\tilde{y}_0$, $(\tilde{\pi}^* d_{\tilde{Y}} \tilde{\Phi}_{y_0})(\tilde{z}_0) = 0$. Since $(\pi^* \tilde{\Phi}_{y_0})(\tilde{z}_0) > 0$, evaluating Equation (5.a) at $\tilde{z}_0$ implies $\mu = \lambda$. □

PROOF OF THEOREM 1.2 (2). Let $\pi : Z \to Y$ be a Riemannian V-submersion, where Z and Y are closed V-manifolds. We assume that the singular sets of both Y and Z have codimension at least 2. Let $p > 0$, and let $0 \neq \Phi \in E^p_\lambda(Y)$. Assume that $\pi^*\Phi \in E^p_\mu(Z)$. We wish to show $\lambda \leq \mu$.

We generalize the argument given in [6]. Let $Z_0 := Z$ and let $Z_1 := Z(Z_0, Z_0)$ be the fiber product. Let $\varepsilon_0 := \mu - \lambda$. Then by Lemma 3.6,

$$\pi_1^* \Phi \in E^p_{\lambda + 2\varepsilon_0}(Z_1)\,.$$

We now inductively set $Z_n := Z(Z_{n-1}, Z_{n-1})$ and apply the same argument to see

$$\pi_n^* \Phi \in E^p_{\lambda + 2^n \varepsilon_0}(Z_n)\,.$$

Since $\Delta^p_{Z_n}$ is a non-negative operator by Theorem 2.3, we have $\lambda + 2^n\varepsilon_0 \geq 0$ for all n. This implies that $\varepsilon_0 \geq 0$ and hence $\mu \geq \lambda$. □

PROOF OF THEOREM 1.1. We extend the arguments given in [7]. Let $\pi : Z \to Y$ be a Riemannian V-submersion, where Z and Y are closed V-manifolds. We assume that the singular sets of both Y and Z have codimension at least 2.

We first show that Assertion (1d) implies Assertion (1a) and that Assertion (2d) implies Assertion (2a). Assume that off the singular set the fibers of π are minimal, and that if $p > 0$, then the horizontal distribution is integrable. Let $0 \neq \Phi \in E^p_\lambda(Y)$. Then Lemma 3.1 and Corollary 3.2 imply that

$$\Delta^p_Z \pi^* \Phi = \lambda \pi^* \Phi \quad \text{on} \quad Z_r\,.$$

Thus by Lemma 3.5, $\pi^*\Phi \in E^p_\lambda(Z)$. This shows that

$$\pi^* E^p_\lambda(Y) \subset E^p_\lambda(Z)$$

for all λ; the intertwining relations of Assertions (1a) and (2a) now follow, as the span of the eigenspaces is dense in the appropriate topology.

It is immediate that Assertion (1a) implies Assertion (1b) and that Assertion (2a) implies Assertion (2b). Similarly, Assertion (1b) implies Assertion (1c) and Assertion (2b) implies Assertion (2c). We complete the proof by showing that Assertion (1c) implies Assertion (1d) and that Assertion (2c) implies Assertion (2d).

Suppose that $\pi^* E^p_\lambda(Y) \subset E^p_{\mu(\lambda)}(Z)$ for all λ. Let $\Phi_\lambda \in E^p_\lambda(Y)$. By Lemma 3.1 we have

$$(5.b) \qquad (\mu - \lambda)\pi^* \Phi_\lambda = \{d_Z(\mathrm{int}_Z(\theta) + \mathcal{E}) + (\mathrm{int}_Z(\theta) + \mathcal{E})d_Z\}\pi^*\Phi_\lambda \quad \text{on} \quad Z_r\,.$$

Suppose first that $p = 0$. By Theorem 1.2 (1), we have $\mu(\lambda) = \lambda$. We use Corollary 3.2 to rewrite Equation (5.b) in the form:

$$\mathrm{int}_Z(\theta)\pi^* d_Y \Phi_\lambda = 0 \quad \text{on} \quad Z_r\,.$$

Let Φ be any smooth function which is compactly supported near a regular point y of Y. We can approximate Φ in the Sobolev-H_1 topology as a finite sum of eigenfunctions. Thus

$$\mathrm{int}_Z(\theta)\pi^* d_Y \Phi = 0\,.$$

Since θ is a horizontal co-vector and since $\Phi(y)$ is arbitrary, this implies $\theta = 0$ at any point $z \in \pi^{-1}(y)$. Thus θ vanishes on Z_r; passing to a local desingularization, we conclude $\tilde{\theta}$ vanishes everywhere by continuity. This completes the proof of Theorem 1.1 (1).

We now suppose that $p > 0$. Let $\pi_{\mathcal{H}}$ be orthogonal projection from $\Lambda^p(Z_r)$ to $\Lambda^p(\mathcal{H})$. Let $\Phi_\lambda \in E_\lambda^p(Y)$. We apply $(1 - \pi_{\mathcal{H}})$ to Equation (5.b) to see that

$$\begin{aligned} 0 &= (\mu - \lambda)(1 - \pi_{\mathcal{H}})\pi^* \Phi_\lambda \\ &= (1 - \pi_{\mathcal{H}})\{d_Z(\mathrm{int}_Z(\theta) + \mathcal{E}) + (\mathrm{int}_Z(\theta) + \mathcal{E})d_Z\}\pi^* \Phi_\lambda \quad \text{on} \quad Z_r\,. \end{aligned}$$

The natural domain of this identity is the Sobolev space $H_1(Y_r)$ and, as the eigenfunctions are dense in $H_1(Y_r)$, by continuity we then have

$$0 = (1 - \pi_{\mathcal{H}})\{d_Z(\mathrm{int}_Z(\theta) + \mathcal{E}) + (\mathrm{int}_Z(\theta) + \mathcal{E})d_Z\}\pi^* \Phi \quad \text{for} \quad \Phi \in H_1(Y_r)\,. \tag{5.c}$$

Let $\pi z_0 = y_0 \in Y_r$. Choose $F \in C_0^\infty(Y_r)$ so that $F(y_0) = 0$. Let $\xi := dF(y_0)$. Since $\mathrm{int}_Z(\theta) + \mathcal{E}$ is a 0^{th} order operator, we apply Equation (5.c) to the product $F\Phi$ and evaluate at z_0 to see

$$0 = (1 - \pi_{\mathcal{H}})\{\mathrm{ext}_Z(\pi^*\xi)(\mathrm{int}_Z(\theta) + \mathcal{E}) + (\mathrm{int}_Z(\theta) + \mathcal{E})\,\mathrm{ext}_Z(\pi^*\xi)\}\pi^* \Phi(y_0)\,.$$

Since

$$0 = (1 - \pi_{\mathcal{H}})\{\mathrm{ext}_Z(\pi^*\xi)\,\mathrm{int}_Z(\theta) + \mathrm{int}_Z(\theta)\,\mathrm{ext}_Z(\pi^*\xi)\}\pi^*\,,$$

and since $\mathcal{E}$ always introduces a vertical covector, we conclude that

$$\begin{aligned} 0 &= (1 - \pi_{\mathcal{H}})\{\mathrm{ext}_Z(\pi^*\xi)\mathcal{E} + \mathcal{E}\,\mathrm{ext}_Z(\pi^*\xi)\}\pi^* \\ &= \{\mathrm{ext}_Z(\pi^*\xi)\mathcal{E} + \mathcal{E}\,\mathrm{ext}_Z(\pi^*\xi)\}\pi^*\,. \end{aligned} \tag{5.d}$$

Let $\{f^a, e^i\}$ be the orthonormal frames of the dual distributions $\mathcal{H}^*$ and $\mathcal{V}^*$. Adopt the notation of Equation (3.a). To simplify the notation, set

$$\mathfrak{e}_i := \mathrm{ext}_Z(e^i)\,, \quad \mathfrak{e}_a := \mathrm{ext}_Z(f^a)\,, \quad \text{and} \quad \mathfrak{i}_a := \mathrm{int}_Z(f^a)\,.$$

We then have the Clifford commutation relations:

$$\mathfrak{e}_a \mathfrak{i}_b + \mathfrak{i}_b \mathfrak{e}_a = \delta_{ab}\,.$$

Choose F so that $\pi^*\xi(z_0) = f^c(z_0)$ and apply Equation (5.d) to compute at z_0 that

$$\begin{aligned} 0 &= \sum_{a,b,i} \omega_{abi}\,\{\mathfrak{e}_c \mathfrak{e}_i \mathfrak{i}_a \mathfrak{i}_b + \mathfrak{e}_i \mathfrak{i}_a \mathfrak{i}_b \mathfrak{e}_c\} = \sum_{a,b,i} \omega_{abi}\,\mathfrak{e}_i \{-\mathfrak{e}_c \mathfrak{i}_a \mathfrak{i}_b + \mathfrak{i}_a \mathfrak{i}_b \mathfrak{e}_c\} \\ &= \sum_{a,b,i} \omega_{abi}\,\mathfrak{e}_i \{\mathfrak{i}_a \mathfrak{e}_c \mathfrak{i}_b + \mathfrak{i}_a \mathfrak{i}_b \mathfrak{e}_c - \delta_{ac} \mathfrak{i}_b\} \\ &= \sum_{a,b,i} \omega_{abi}\,\mathfrak{e}_i \{-\mathfrak{i}_a \mathfrak{i}_b \mathfrak{e}_c + \mathfrak{i}_a \mathfrak{i}_b \mathfrak{e}_c - \delta_{ac} \mathfrak{i}_b + \delta_{bc} \mathfrak{i}_a\} = -2 \sum_{b,i} \omega_{cbi}\,\mathfrak{e}_i \mathfrak{i}_b\,. \end{aligned}$$

As $p \geq 1$, we may conclude that $\omega = 0$ on Z_r and consequently $\mathcal{H}$ is integrable.

We must now show that the fibers are minimal. Let d_X denote exterior differentiation along the fiber. We set $\mathcal{E} = 0$ and use Equation (5.c) to compute

$$\begin{aligned} 0 &= (1-\pi_{\mathcal{H}})\{d_Z \operatorname{int}_Z(\theta) + \operatorname{int}_Z(\theta) d_Z\}\pi^* \\ &= (1-\pi_{\mathcal{H}}) d_X \operatorname{int}_Z(\theta)\pi^* \quad \text{on} \quad C_0^\infty(\Lambda^p(Y_r))\,. \end{aligned}$$

This implies θ is constant on the fibers so that $\theta = \pi^*\Theta$ is the pull back of a globally defined 1-form away from the singular set on the base.

We apply Lemma 3.3. Let $d\nu_x^e$ be the Euclidean measure and let

$$\psi(y) := \int_X g_X(x,y) d\nu_x^e$$

be the volume of the fiber $\pi^{-1}(y)$ for $y \in Y_r$. Then

$$\begin{aligned} d_Y\psi(y) &= d_Y \int_X g_X(x,y) d\nu_x^e = \int_X (g_X g_X^{-1} d_Y g_X)(x,y) d\nu_x^e \\ &= -\int_X g_X(x,y)\theta(x,y) d\nu_x^e = -\Theta(y)\int_X g_X(x,y) d\nu_x^e \\ &= -\Theta(y)\psi(y)\,, \quad \text{so} \\ \theta &= -\pi^* d_Y \ln\psi \quad \text{on} \quad Y_r\,. \end{aligned}$$

If y is a singular point, we let $\tilde{U}_y$ be the desingularization of Y and $\tilde{U}_y \times X$ be the desingularization of Z. Since $\tilde{\mathcal{E}} = 0$ on $(\tilde{U}_y \setminus \tilde{S}_y) \times X$, we have $\tilde{\mathcal{E}} = 0$ on $\tilde{U}_y \times X$ by continuity, since the singular set has codimension at least 2. Thus we can apply exactly the same argument given above to see that $\tilde{\psi} := \rho_y^*\psi$ extends to a smooth function $\tilde{\psi}_y$ on all of $\tilde{U}_y$.

We define a conformal variation of the metric on the vertical distribution, which leaves the metric on the horizontal distribution unchanged by setting

$$g(t)_Z = \psi^{2t} ds_{\mathcal{V}}^2 + ds_{\mathcal{H}}^2 \quad \text{on} \quad Z_r\,.$$

The argument given that above shows that this variation extends to the desingularization to define a smooth one-parameter family of Riemannian V-submersions.

We have that $\pi : Z(t) \to Y$ is a Riemannian submersion with integrable horizontal distribution. We use Lemma 3.3 to see $\theta(t) = (1 + t\dim(X))\theta$ away from the singular set and thus

$$\begin{aligned} \Delta_Z^p\pi^* - \pi^*\Delta_Y^p &= (1+t\dim(X))(d_Z \operatorname{int}_Z(\theta) + \operatorname{int}_Z(\theta) d_Z)\pi^* \\ &= (1+t\dim(X))(\Delta_Z^p\pi^* - \pi^*\Delta_Y^p)\,. \end{aligned}$$

Let $\Phi \in E_\lambda^p(Y)$ and let $\pi^*\Phi \in E_\mu^p(Z)$. Set $\varepsilon = \mu - \lambda$. Then

$$\Delta_Z^p\pi^*\Phi = \{\lambda + (1+t\dim(X))\varepsilon\}\pi^*\Phi \quad \text{on} \quad Z_r\,.$$

Consequently, by Lemma 3.5,

$$\pi^* E_\lambda^p(Y) \subset E_{\lambda+(1+t\dim(X))\varepsilon(\lambda)}^p(Z(t))\,.$$

By Lemma 2.3, $\lambda + (1 + t\dim(X))\varepsilon(\lambda) \geq 0$. Since t is arbitrary, $\varepsilon(\lambda) = 0$. Thus

$$(d_Z \operatorname{int}_Z(\theta) + \operatorname{int}_Z(\theta) d_Z)\pi^* = 0 \quad \text{on} \quad E^p_\lambda(Y)\,. \tag{5.e}$$

Since these eigenspaces are dense in the Sobolev space H_1, Equation (5.e) continues to be valid on H_1. Let $y_0 \in Y_r$. Let $f \in \mathcal{H}^*(z_0)$, where $\pi(z_0) = y_0$. Choose $\Phi \in C_0^\infty(U_y)$ so that

$$\Phi(y_0) = 0 \quad \text{and} \quad \pi^* d\Phi(y_0) = f\,.$$

Let $\Psi \in C_0^\infty(Y_r)$ with $\Psi(y_0)$ arbitrary. We apply Equation (5.e) to the product $\Phi\Psi$ and evaluate at y_0 to see

$$(\operatorname{ext}_Z(f) \operatorname{int}_Z(\theta) + \operatorname{int}_Z(\theta) \operatorname{ext}_Z(f))\pi^*\{\Psi(y_0)\} = 0\,.$$

Since

$$\operatorname{ext}_Z(f) \operatorname{int}_Z(\theta) + \operatorname{int}_Z(\theta) \operatorname{ext}_Z(f) = g_Z(f,\theta)\,,$$

$g_Z(f,\theta)(z_0) = 0$. Since θ is horizontal and since f was an arbitrary horizontal covector, we conclude θ vanishes away from singular set; passing to a local desingularization, we complete the proof by checking that $\tilde{\theta} = 0$ everywhere by continuity. □

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