EIGENVALUE ESTIMATES FOR SCHRÖDINGER OPERATORS WITH COMPLEX POTENTIALS

ARI LAPTEV AND OLEG SAFRONOV

Abstract. We discuss properties of eigenvalues of non-self-adjoint Schrödinger operators with complex-valued potential V. Among our results are estimates of the sum of powers of imaginary parts of eigenvalues by the L^p-norm of $\Im V$.

1. Introduction

Throughout the paper, $f_\pm$ denotes either the positive or the negative part of f, which is either a function or a self-adjoint operator. The symbols $\Re z$ and $\Im z$ denote the real and the imaginary part of z. If a is a function on $\mathbb{R}^d$, then $a(i\nabla)$ is the operator whose integral kernel is $(2\pi)^{-d}\int e^{i\xi(x-y)}a(\xi)d\xi$.

Let H be a non-self-adjoint Schrödinger operator in $L^2(\mathbb{R}^d)$

$$H = -\Delta + V(x)$$

with a complex-valued potential V. We call λ an eigenvalue of H if there is a solution of the equation $H\psi = \lambda\psi$ for some $\psi \in L^2$. A given number $\lambda \in \mathbb{C}$ may occur several times in this list according to the dimension of the generalized eigenspace $\{\psi : (H-\lambda)^k\psi = 0 \text{ for some } k \in \mathbb{N}\}$, which is called the algebraic multiplicity. In principle a generalized eigenspace could have infinite dimension, but, as we shall see, this will not occur in the situations considered in this paper. We deal with operators that have countably many eigenvalues lying in a bounded subset of the cut plane $\mathbb{C}\setminus[0,\infty)$. We denote them by $\lambda_j,\ j = 1,2,3,\dots$ listing them by decreasing values of their modules and repeated according to their algebraic multiplicities.

The main result of [8] tells us, that for any $t > 0$, the eigenvalues λ_j of H lying outside the sector $\{z : \ |\Im z| < t\, \Re z\}$ satisfy the estimate

$$\sum |\lambda_j|^\gamma \le C \int |V(x)|^{\gamma+d/2}dx, \qquad \gamma \ge 1,$$

---

[0]Key Words: Schrödinger operator, eigenvalues of non-self-adjoint operators, complex potential

[5]MSC: Primary, 35P15; Secondary, 81Q10

The authors would like to thank Grigori Rozenblioum, Rupert Frank, Stanislav Molchanov and Robert Seiringer for their remarks.

where the constant C may depend on t, γ and d.

In this paper we study inequalities on the eigenvalues lying inside the conical sector $\{z: \ |\Im z| < t\Re z\}$, so they might be close to the positive half-line. In particular, our results provide some information about the rate of accumulation of eigenvalues to the positive real half-line $\mathbb{R}_+ = [0,\infty)$.

Theorem 1. *Let $\Re V \geq 0$ be a bounded function. Assume that $\Im V \in L^p(\mathbb{R}^d)$, where $p > d/2$ if $d \geq 2$ and $p \geq 1$ if $d = 1$. Then the eigenvalues λ_j of the operator $H = -\Delta + V$ satisfy the estimate*

$$\sum_j \Big(\frac{\Im \lambda_j}{|\lambda_j + 1|^2 + 1} \Big)_+^p \leq C \int_{\mathbb{R}^d} (\Im V)_+^p(x)\, dx. \tag{1.1}$$

The constant C can be computed explicitly:

$$C = (2\pi)^{-d} \int_{\mathbb{R}^d} \frac{d\xi}{(\xi^2+1)^p}. \tag{1.2}$$

Note that the right hand side of (1.1) is independent of real part of the potential V and therefore the statement is true for arbitrary $\Re V \geq 0$. It is not the case when we try to obtain an estimate of the sum $\sum_j (\Im \lambda_j/(|\lambda_j + 1|^2+1))_+^p$ where we allow $p \leq d/2$ and where a certain regularity of $\Re V$ is required.

Theorem 2. *Let $\Re V \geq 0$ and $\Im V$ be two bounded real valued functions. Assume that $\Im V \in L^p(\mathbb{R}^d)$, where $p > d/4$ if $d \geq 4$ and $p \geq 1$ if $d \leq 3$. Then the eigenvalues λ_j of the operator $H = -\Delta + V$ satisfy the estimate*

$$\sum_j \Big(\frac{\Im \lambda_j}{|\lambda_j + 1|^2 + 1} \Big)_+^p \leq C\,(1 + ||V||_\infty)^{2p} \int_{\mathbb{R}^d} (\Im V)_+^p(x)\, dx, \tag{1.3}$$

where

$$C = (2\pi)^{-d} \int_{\mathbb{R}^d} \frac{d\xi}{((\xi^2+1)^2+1)^p}. \tag{1.4}$$

Next Theorems 3-4 give sufficient conditions on V that guarantee convergence of the sum

$$\sum_{a < \Re\lambda_j < b} |\Im \lambda_j|^\gamma < \infty$$

for $0 \leq a < b < \infty$.

Let us introduce $W = (|V|^2 + 4\Im V)_+$ and let

$$\Psi_b(W) = \int_{\mathbb{R}^d} W^{d/4-1/2+r} dx + b^{d/2-1} \int_{\mathbb{R}^d} W^r dx, \quad d \geq 2.$$

Theorem 3. *Assume that $\gamma > 3/2$ and $r \in (\gamma - \frac{1}{2}, \gamma)$ and let $V \in L^{d/2-1+2r}(\mathbb{R}^d) \cap L^r(\mathbb{R}^d)$, $d \geq 2$. Then the eigenvalues λ_j of the operator H lying inside the semi-infinite strip $\Pi_b = \{z : \ 0 < \Re z < b, \Im z > 0\}$ satisfy the inequality*

$$\sum_{\lambda_j \in \Pi_b} |\Im \lambda_j|^\gamma \leq C |\Psi_b(W)|^{\frac{2\gamma-1}{2r-1}} (b + |\Psi_b(W)|^{\frac{1}{2r-1}}). \tag{1.5}$$

The constant C in this inequality depends on d, γ and r.

Applying the same method we also prove:

Theorem 4. *Let λ_j be the eigenvalues of the operator H lying inside the semi-infinite strip $\Pi_{a,b} = \{z : \ a < \Re z < b, \Im z > 0\}$ with $a > 0$. Then for any $\gamma > 3/2$ and $r \in (\gamma - \frac{1}{2}, \gamma)$ the condition $V \in L^{2r}(\mathbb{R}) \cap L^r(\mathbb{R})$ implies*

$$\sum_{\lambda_j \in \Pi_{a,b}} |\Im \lambda_j|^\gamma \leq C |\Phi_a(W)|^{\frac{2\gamma-1}{2r-1}} (b + |\Phi_a(W)|^{\frac{1}{2r-1}}),$$

where

$$\Phi_a(W) = a^{-1/2} \int_{\mathbb{R}} W^r \, dx.$$

The constant in this inequality depends on γ and r.

Note that in Theorems 3 and 4 the inequality $\gamma \geq 3/2$ is required in any dimension while in Theorems 1 and 2 the values of p could be smaller in lower dimensions.

One should mention, that the paper [8] had been motivated by a question of E.B. Davies (see [1] and [7]), where he obtains that if $d = 1$ and $V \in L^1(\mathbb{R})$, then all eigenvalues λ of H which do not belong to $\mathbb{R}_+$ satisfy

$$|\lambda| \leq \frac{1}{4} \Big(\int |V(x)| dx \Big)^2.$$

The question was raised if a similar estimate holds in dimension $d \geq 2$. The following conjecture seems to be reasonable

Conjecture. Let $d \geq 2$, $0 < \gamma \leq d/2$ and let $V \in L^{d/2+\gamma}(\mathbb{R}^d)$ be a complex-valued potential. Then for any eigenvalue $\lambda \notin \mathbb{R}_+$ of the operator $H = -\Delta + V$

$$|\lambda|^\gamma \leq C \int_{\mathbb{R}^d} |V(x)|^{d/2+\gamma} dx, \tag{1.6}$$

for every complex valued potential and every eigenvalue $\lambda \notin \mathbb{R}_+$ of the operator $H = -\Delta + V$.

We carefully avoid the case $\gamma > d/2$, since the operator H in this case might have arbitrary large positive eigenvalues due to Wigner-Von Neumann example [17] (we are grateful to S. Molchanov for drawing our attention to this circumstance). So far, we are able to prove only the following result related to this conjecture:

Theorem 5. *Let V be a function from $L^p(\mathbb{R}^d)$, where $p \geq d/2$, if $d \geq 3$; $p > 1$, if $d = 2$, and $p \geq 1$, if $d = 1$. Then every eigenvalue λ of the operator $H = -\Delta + V$ with the property $\Re\lambda > 0$ satisfies the estimate*

$$|\Im\lambda|^{p-1} \leq |\lambda|^{d/2-1} C \int_{\mathbb{R}^d} |V|^p dx. \tag{1.7}$$

The constant C in this inequality depends only on d and p. Moreover, $C = 1/2$ for $p = d = 1$.

The inequality (1.7) was established in [1] in the case $d = p = 1$. We prove it in higher dimensions and in dimension $d = 1$ for $p > 1$.

We also show the elementary estimate (see Theorem 16)

$$|\Im\sqrt{\lambda}|^{2\gamma} \leq C \int_{\mathbb{R}^3} |V|^{3/2+\gamma} dx, \qquad \gamma > 0,\ d = 3,$$

however it is not quite the same as (1.6). While we are not able to prove Conjecture 1.1, we find some information about the location of eigenvalues of the operator $-\Delta + iV$ with a positive $V \geq 0$, see Thorem 13. In particular, in Theorem 15 we prove that if $d = 3$ and $\int V dx$ is small and $\lambda \notin \mathbb{R}_+$ is an eigenvalue of $-\Delta + iV$, then $|\lambda|$ must be large. It might seem that eigenvalues do not exist at all for small values of $\int V dx$, however their presence in such cases can be easily established using scaling.

Proposition 1. *Let $d \geq 3$. Then there is a sequence of positive functions $V_n \geq 0$ such that the "largest modulus" eigenvalue $\lambda_n \notin \mathbb{R}_+$ of the operator $-\Delta + iV_n$ satisfies $|\lambda_n| \to \infty$ as $n \to \infty$, while $\lim_{n\to\infty} \int V_n(x)dx = 0$.*

Proof. If λ is an eigenvalue of $-\Delta + iV(x)$, then $n^2\lambda$ is an eigenvalue of $-\Delta + n^2 iV(nx)$. It remains to note that $\int n^2 V(nx)dx = Cn^{2-d}$. The idea of the proof of existence of a non-real eigenvalue of $-\Delta + iV(x)$ at least for one $V \geq 0$ is to start with the one-dimensional case, when $V(x) = \delta(x) + \delta(x - \epsilon)$. In this case, there is an eigenvalue of H that behaves like $1 + i\alpha\epsilon + O(\epsilon^2)$ as $\epsilon \to 0$. If V is spherically symmetric, then the multi-dimensional case can be reduced to the one-dimensional case by separation of variables. $\square$

Remark. Note that our results also imply that the eigenvalues of $-\Delta + iV$ can not accumulate to zero in $d = 3$, if $V \geq 0$ is integrable (Corollary 5).

2. Preliminaries

In what follows, the inner products and the norms in various spaces are denoted by $(\cdot, \cdot)$ and $||\cdot||$ respectively.

1. Let $a[\cdot, \cdot]$ be a sesquilinear form in a Hilbert space $\mathfrak{H}$. We assume that its domain $d[a]$ is dense in $\mathfrak{H}$ and a is semibounded from below and closed on $d[a]$. The form a induces the selfadjoint operator A in $\mathfrak{H}$. Fix the value of $\gamma \in \mathbb{R}$, such that $a_\gamma := a + \gamma \geq 1$, i.e.

$$a_\gamma[x,x] = a[x,x] + \gamma||x||^2 = ||x||^2, \qquad x \in d[a],$$

and denote by $\mathfrak{H}_\gamma[a]$ the (complete) Hilbert space $d[a]$ with the metric form

$$a_\gamma[x,x] = ||(A+\gamma I)^{1/2}x||^2, \qquad x \in d[a].$$

Let $V : \mathfrak{H} \mapsto \mathfrak{H}$ be a selfadjoint linear operator, satisfying $D(|V|^{1/2}) \supset d[a]$ and

$$G := |V|^{1/2}(A+\gamma I)^{-1/2} \in \mathfrak{S}_\infty \tag{2.1}$$

where $\mathfrak{S}_\infty$ denotes the space of compact operators in $\mathfrak{H}$. Put

$$v[x,y] = (\frac{V}{|V|^{1/2}}x, |V|^{1/2}y). \tag{2.2}$$

Then the form v is compact on $d[a]$. This means that the form v is continuous on $\mathfrak{H}_\gamma[a]$ and the corresponding operator Q (determined by the relations $a_\gamma[Qx,y] = v[x,y]$ for $x, y \in d[a]$) is compact on $\mathfrak{H}_\gamma[a]$. Define the operator H by setting

$$H + \gamma I = (A+\gamma I)(I + iQ), \tag{2.3}$$

on the domain $D(H) = (I+iQ)^{-1}D(A)$. It is clear that the operator H can be interpreted as the sum

$$H = A + iV.$$

Proposition 2. *The operator H defined in* (2.3) *is densely defined and closed.*

Proof. Let us first prove that H is densely defined. Assume the opposite, that there is a non-zero vector $h \in d[a]$ such that $a_\gamma[(I+iQ)^{-1}u, h] = 0$ for all vectors $u \in D(A)$. Then $a_\gamma[u, (I-iQ)^{-1}h] = 0$ for all $u \in d[a]$, which implies that $(I-iQ)^{-1}h = 0$. The latter relation contradicts the assumption that $h \neq 0$.

In order to prove that H is closed, it is sufficient to observe that $H+\gamma I$ is invertible and prove that the inverse is bounded. But this follows from the relation

$$(H+\gamma I)^{-1} = (I+iQ)^{-1}(A+\gamma I)^{-1},$$

and the fact that $(A+\gamma I)^{-1}$ maps continuously $\mathfrak{H}$ to $\mathfrak{H}_\gamma[a]$. □

Remark. The condition that the sesquilinear form v is generated by a self-adjoint operator V is excessive. We can always define H by (2.3), as soon as we know that $v[u,u] = a_\gamma[Qu,u]$, where Q is compact in the space $\mathfrak{H}_\gamma[a]$. This remark allows one to consider the case when the elliptic operator $A = \Delta^2$ is perturbed by a differential operator of first order.

Under the above assumptions, the difference between the resolvents of the operators A and H is compact. Hence, the spectrum $\sigma(H)$ of the operator H is discrete in $\mathbb{C} \setminus \sigma(A)$.

2. Let H be as described above. In order to develop the perturbation theory suitable for non-selfadjoint operators, we consider a contour C which contains a finite number of igenvalues $\lambda_1, \lambda_2, \dots, \lambda_m$ of the operator H. Then the projection onto the span of the corresponding root vectors is given by the formula

$$P = \frac{1}{2\pi i} \int_C (H - z)^{-1} dz.$$

Lemma 1 (see, for example, [13])**.** *If P and P_0 are two projections such that* $\mathrm{rank} P \neq \mathrm{rank} P_0$*, then*

$$||P - P_0|| \geq 1.$$

Consequently, if H_n is a family of closed operators in $\mathfrak{H}$ having the property that $\sigma(H_n) \setminus \mathbb{R}$ is discrete and satisfying the condition

$$(H_n - z)^{-1} - (H - z)^{-1} \to 0,$$

as $n \to \infty$, for some point z, then non-real eigenvalues of H_n converge to non-real eigenvalues of H and, possibly, to the real part of $\sigma(H)$.

3. We already know that A is semibounded from below. Suppose also that the negative spectrum of A is discrete. Then the operator H has only discrete set of eigenvalues in the left half-plane $\mathbb{C}_{left} = \{z : \quad \Re z < 0\}$. Moreover, suppose that $\lambda_j \in \mathbb{C}_{left}$ are eigenvalues of the operator H, and τ_j are negative eigenvalues of A enumerated in the order of increasing real parts. Then

$$\Big| \Re \sum_1^n \lambda_j \Big| \leq \sum_1^n |\tau_j|$$

for all n. Indeed, let P be the orthogonal projection onto the span of eigenvectors x_j corresponding to λ_j, $1 \leq j \leq n$. Then

$$\mathrm{tr}\, HP = \sum_1^n \lambda_j.$$

Consequently,

$$\begin{aligned} \Re \sum_1^n \lambda_j &= \sum_1^n (Ax_j, x_j) \\ &\geq \min_P \mathrm{tr} \Big[((A+\gamma)^{1/2} P)^* A (A+\gamma)^{-1} (A+\gamma)^{1/2} P \Big], \quad (2.4) \end{aligned}$$

where the minimum is taken over all orthogonal projections P of rank n with the property $\mathrm{Ran}P \subset d[a]$. Thus,

$$\sum_1^n \Re\lambda_j \geq \sum_1^n \tau_j \tag{2.5}$$

since the minimum in the right hand side of (2.4) coincides with the sum in the right hand side of (2.5).

Corollary 1. *Let* $\gamma > 0$. *Then*

$$\sum_1^n (\Re\lambda_j + \gamma)_- \leq \sum_1^n (\tau_j + \gamma)_- \, .$$

4. Let T be a bounded operator in a Hilbert space, whose spectrum outside the unit circle $\{z : \ |z| > 1\}$ is discrete. Suppose also that the essential spectrum of the operator $(T^*T)^{1/2}$ is contained in $[0,1]$. Let λ_j be the eigenvalues of the operator T lying outside of the unit circle, and let $s_j > 1$ be the eigenvalues of $(T^*T)^{1/2}$. If we enumerate the sequences $|\lambda_j|$ and s_j in the decreasing order, then

$$\prod_1^n |\lambda_j| \leq \prod_1^n s_j \tag{2.6}$$

for all values of n. One should mention also that, if one of the sequences ends at $j = j_0$, we extend it by setting it equal to 1 for $j > j_0$.

This inequality has been discovered for compact operators by H. Weyl (see [18]). Weyl's proof is carried over to the case of bounded. Indeed, let P be the orthogonal projection onto the span of eigenvectors corresponding to λ_j, $1 \leq j \leq n$. Then for any $\alpha > 0$

$$\det\left(I + P(\alpha T^*T - I)P\right) = \alpha^n \prod_1^n |\lambda_j|^2.$$

Consequently,

$$\alpha^n \prod_1^n |\lambda_j|^2 \leq \det\left(I + P(\alpha T^*T - I)_+ P\right)$$
$$\leq \det\left(I + (\alpha T^*T - I)_+^{1/2} P (\alpha T^*T - I)_+^{1/2}\right).$$

Since $(\alpha T^*T - I)_+^{1/2} P (\alpha T^*T - I)_+^{1/2}) \leq (\alpha T^*T - I)_+)$ we can remove the orthogonal projection P in the right hand side and obtain

$$\alpha^n \prod_1^n |\lambda_j|^2 \leq \det\left(I + (\alpha T^*T - I)_+\right) = \prod_{\alpha s_j^2 > 1} \alpha s_j^2.$$

It remains to choose $\alpha = s_n^{-2}$. Note that if the number of $s_j > 1$ is finite, we can take $\alpha = 1$ to obtain that

$$\prod_1^n |\lambda_j|^2 \le \prod_{s_j^2>1} s_j^2.$$

for all n.

Corollary 2. *Let $\gamma \ge 1$. Then*

$$\sum_1^n (|\lambda_j|^2 - 1)^\gamma \le \sum_1^n (s_j^2 - 1)^\gamma$$

for all n.

Proof. Our arguments are quite standard and probably can be compared with the ones in the book by Birman and Solomyak [4], which contains a survey on different inequalities for compact operators. It is sufficient to consider the case $\gamma > 1$, because the proof in the case $\gamma = 1$ is obtained by passing to the limit as $\gamma \to 1$.

As a consequence of (2.6), we obtain that

$$\sum_1^n \log |\lambda_j| \le \sum_1^n \log s_j. \tag{2.7}$$

Moreover,

$$\sum_{j=1}^n (\log |\lambda_j| - \eta)_+ \le \sum_{j=1}^n (\log s_j - \eta)_+ \tag{2.8}$$

for any $-\infty < \eta < \infty$. Note now that the function $\phi(t) = (e^{2t} - 1)^\gamma$ is representable in the form

$$\phi(\lambda) = \int_0^\infty (\lambda - t)_+ \phi''(t)\, dt \quad \text{and} \quad \phi''(t) \ge 0 \ \text{ for } \ t \ge 0.$$

Since $\phi(\log|\lambda|) = (|\lambda|^2 - 1)^\gamma$, the statement of Corollary 2 for $\gamma > 1$ follows from (2.8). $\square$

5. Let T be a compact operator in a Hilbert space and let $n(s, T)$ be the counting function of its s-numbers (eigenvalues of $\sqrt{T^*T}$)

$$n(s, T) = \operatorname{card}\{j : \quad s_j > s\}, \quad s > 0.$$

Then by Ky Fan inequality (see [10]) for any pair of compact operators T_1 and T_2 and $s_1, s_2 > 0$

$$n(s_1 + s_2, T_1 + T_2) \le n(s_1, T_1) + n(s_2, T_2).$$

The class of operators T for which

$$[T]_p^p := \sup_{s>0} s^p n(s, T) < \infty$$

is called the weak Neumann-Schatten class Σ_p.

Let $\mathcal{F}$ Fourier transform

$$\mathcal{F}f(\xi) = \int_{\mathbb{R}^d} e^{-ix\xi} f(x)\, dx.$$

Theorem 6 (M.Cwikel [5]). *Let α and β be the operators of multiplication by the functions $\alpha(\xi)$ and $\beta(x)$. Suppose that $\beta \in L^q(\mathbb{R}^d)$, $q > 2$, and let*

$$[\alpha]_q^q = \sup_{t>0} t^q \text{meas}\{\xi \in (\mathbb{R}^d : \quad |\alpha(\xi)| > t\} < \infty.$$

Then the operator $T = \beta\mathcal{F}^*\alpha$ as (well as the operator $\alpha\mathcal{F}\beta$) is in Σ_p and

$$[T]_q^q \leq C[\alpha]_q^q \int |\beta(x)|^q \; dx. \tag{2.9}$$

Proposition 3 (Birman-Schwinger principle [3], [14]). *Let A and V be two positive self-adjoint operators acting in the same Hilbert space. Suppose that V is bounded and the operator $\sqrt{V}(A+I)^{-1/2}$ is compact. Then for $E > 0$, the number $N(E)$ of eigenvalues of the operator $A - V$ lying to the left of $-E$ satisfies the relation*

$$N(E) = n(1, \sqrt{V}(A+E)^{-1}\sqrt{V}).$$

In applications, A is a differential operator with constant coefficients and V is the operator of multiplication by a function. Then applying Theorem 6 to the operator $T = \sqrt{V}(A+I)^{-1/2}$ one obtains sharp inequalities for $N(E)$.

6. In order to state the next result we need to introduce one more Neumann-Schatten class $\mathfrak{S}_p$ of compact operators. Namely, we say that $T \in \mathfrak{S}_p$, $p \geq 1$, if

$$||T||_p^p := \text{tr}\; (T^*T)^{p/2} = \sum_j s_j^p < \infty.$$

It is easy to see that $\mathfrak{S}_p$ is a Banach space.

The next theorem gives us a sufficient condition guaranteeing that an operator of the form $\beta(x)\alpha(i\nabla)$ belongs to the class $\mathfrak{S}_p$.

Theorem 7. *Let α and β be the operators of multiplication by $\alpha(\xi)$ and $\beta(x)$. Suppose that $\alpha, \beta \in L^p(\mathbb{R}^d)$, where $p \geq 2$. Then $T = \beta\mathcal{F}^*\alpha \in \mathfrak{S}_p$ and*

$$||T||_p^p \leq (2\pi)^{-d} \int |\alpha(\xi)|^p \; d\xi \int |\beta(x)|^p \; dx. \tag{2.10}$$

This theorem can be found in [15]. See also [11] and [16].

7. We now formulate a statement about eigenvalue estimates for a certain operator with constant coefficients perturbed by a potential V. It is one of the consequences of the inequality (2.9).

Proposition 4. *Let $\alpha(\xi) = (|\xi|^2 - \mu)^2$, $V(x) \geq 0$ and $p > 1/2$. Suppose that $V \in L^{p+d/4}(\mathbb{R}^d) \cap L^{p+1/2}(\mathbb{R}^d)$ if $d \geq 2$, or $V \in L^{p+1/2}(\mathbb{R})$ if $d = 1$. Let $N(E)$ be the number of eigenvalues of the operator $\alpha(i\nabla) - V(x)$ lying to the left of the point $-E$, where $E > 0$. Then*

$$N(E) \leq \frac{C}{E^p}\Big(\int_{\mathbb{R}^d} V^{p+d/4} dx + \mu^{d/2-1} \int_{\mathbb{R}^d} V^{p+1/2} dx\Big), \qquad \text{if} \quad d \geq 2; \tag{2.11}$$

$$N(E) \leq \frac{C}{E^p \mu^{1/2}} \int_{\mathbb{R}^d} V^{p+1/2} dx, \qquad \text{if} \quad d = 1. \tag{2.12}$$

Proof. It is an elementary application of the Cwikel estimate. Indeed, according to the Birman-Schwinger principle

$$N(E) = n(1, X),$$

where X is the compact operator defined by the equality

$$X = \sqrt{V}(\alpha(i\nabla) + E)^{-1}\sqrt{V}.$$

Let χ be the characteristic function of the ball $\{|\xi| \in \mathbb{R}^d : |\xi|^2 \leq \mu\}$. Let us split X such that $X = X_1 + X_2$, where

$$X_1 = \sqrt{V}(\alpha(i\nabla) + E)^{-1}\chi(i\nabla)\sqrt{V}.$$

According to the Ky Fan inequality,

$$n(1, X) \leq n(1, 2X_1) + n(1, 2X_2). \tag{2.13}$$

Therefore it is sufficient to estimate each term in the right hand side of (2.13) separately. We begin with the first term. Set $q_1 = p + d/4$. Then according to (2.9)

$$\begin{aligned} n(1, 2X_1) &\leq C_0 \int V^{q_1}\, dx \int_{|\xi|^2 > \mu} \frac{d\xi}{((|\xi|^2 - \mu)^2 + E)^{q_1}} \leq \\ &\leq C_1 \int V^{q_1}\, dx \int_\mu^\infty \frac{s^{d/2-1} ds}{((s-\mu)^2 + E)^{q_1}} \leq C_2 \int V^{q_1}\, dx \int_0^\infty \frac{s^{d/2-1} ds}{(s^2 + E)^{q_1}} \\ &= \frac{C}{E^p} \int_{\mathbb{R}^d} V^{p+d/4}\, dx. \end{aligned}$$

In order to estimate the second term in (2.13) we set $q_2 = p + 1/2$. Using (2.9) again we find

$$\begin{aligned} n(1, 2X_2) &\leq C_3 \int V^{q_2}\, dx \int_{|\xi|^2 < \mu} \frac{d\xi}{((|\xi|^2 - \mu)^2 + E)^{q_2}} \leq \\ &\leq C_4 \int V^{q_2}\, dx \int_0^\mu \frac{s^{d/2-1} ds}{((s-\mu)^2 + E)^{q_2}} \leq C_5 \int V^{q_2}\, dx \int_{-\infty}^\infty \frac{\mu^{d/2-1} ds}{(s^2 + E)^{q_2}} = \\ &= \frac{C\mu^{d/2-1}}{E^p} \int_{\mathbb{R}^d} V^{p+1/2}\, dx, \end{aligned}$$

which completes the proof of (2.11).

In order to proof (2.12) we note that $(|\xi|^2-\mu)^2 = (|\xi|-\sqrt{\mu})^2(|\xi|+\sqrt{\mu})^2 \geq (|\xi|-\sqrt{\mu})^2\mu$. Consequently, for any $q>1$,

$$\int_{\infty}^{\infty} \frac{d\xi}{((\xi^2-\mu)^2+E)^q} \leq \int_{\infty}^{\infty} \frac{d\xi}{((|\xi|-\sqrt{\mu})^2\mu+E)^q} = \frac{C}{\sqrt{\mu}E^{q-1/2}},$$

where

$$C = \int_{-\infty}^{\infty} \frac{ds}{(s^2+1)^q}.$$

If now $q = p+1/2$, then by using (2.9) we arrive at

$$N(E) \leq \frac{C\int V^q\,dx}{\sqrt{\mu}\,E^{q-1/2}} = \frac{C\int V^{p+1/2}\,dx}{\sqrt{\mu}\,E^p},$$

which means that (2.12) is also proven. $\square$

3. Proof of Theorem 1

The main tool of the proof is the linear fractional mapping that takes the upper half-plane $\{z: \ \Im z>0\}$ into the compliment of the unit disk $\{z: \ |z|>1\}$ given by the formula

$$z \mapsto \frac{z+i+1}{z-i+1}.$$

Insert the operator $H = -\Delta + V$ instead of z into this formula, i.e. consider the operator

$$U = (H+I+i)(H+I-i)^{-1} = I + 2i(H+I-i)^{-1}.$$

Obviously $z \notin \mathbb{R}$ is an eigenvalue of the operator H if and only if $(z+i+1)/(z-i+1)$ is an eigenvalue of U. Clearly

$$U^* = I - 2i(H^*+I+i)^{-1}$$

and therefore

$$U^*U = I + 2i(H+I-i)^{-1} - 2i(H^*+I+i)^{-1} + 4(H^*+I+i)^{-1}(H+I-i)^{-1}.$$

Using the Hilbert identity, we obtain

$$U^*U = I + 2i(H^*+I+i)^{-1}(H^*-H)(H+I-i)^{-1}$$

and since $H^*-H = -iV$

$$U^*U = I + 4(H^*+I+i)^{-1}\Im V(H+I-i)^{-1}.$$

In particular, this implies

$$U^*U - I \leq 4Y^*Y,$$

where $Y = \sqrt{\Im V_+}(H + I - i)^{-1}$. By using Corollary 2 the eigenvalues λ_j of the operator H satisfy the inequality

$$\sum_j \Big(\Big| \frac{\lambda_j + 1 + i}{\lambda_j + 1 - i} \Big|^2 - 1 \Big)_+^p \leq \mathrm{tr}\,(U^*U - I)_+^p \leq 4^p \mathrm{tr}\,(Y^*Y)^p = 4^p ||Y||_{2p}^{2p}.$$

It follows from this inequality that

$$\sum_j \Big(\frac{\Im \lambda_j}{|\lambda_j + 1|^2 + 1} \Big)_+^p \leq ||Y||_{2p}^{2p} \tag{3.1}$$

Indeed, denote $a = 2\Im \lambda_j / (|\lambda_j + 1|^2 + 1)$ and suppose that $\Im \lambda_j > 0$. Then

$$\Big| \frac{\lambda_j + 1 + i}{\lambda_j + 1 - i} \Big|^2 - 1 = \Big(\frac{1 + a}{1 - a} \Big) - 1 \geq 2a.$$

We come to the conclusion that one needs to estimate the norm of the operator

$$Y = \sqrt{\Im V_+}(H + I - i)^{-1}$$

in the class $\mathfrak{S}_{2p}$. Let us represent this operator in the form

$$Y = \sqrt{\Im V_+}(-\Delta + I)^{-1/2} B, \qquad \text{where} \ \ B = (-\Delta + I)^{1/2}(H + I - i)^{-1}.$$

We will show that the operator B is bounded and its norm does not exceed 1. In other words, we will show that

$$||(-\Delta + I)^{1/2}(H + I - i)^{-1} f||^2 \leq ||f||^2, \tag{3.2}$$

for all $f \in L^2$.

Denote $u = (H + I - i)^{-1} f$. It is obvious that

$$\int_{\mathbb{R}^d} (|\nabla u|^2 + (1 + \Re V(x))|u|^2)\, dx = \Re \int_{\mathbb{R}^d} f \bar{u}\, dx.$$

Due to the condition $\Re V \geq 0$, we obtain from this relation that

$$\int_{\mathbb{R}^d} (|\nabla u|^2 + |u|^2)\, dx \leq \frac{1}{2} \int_{\mathbb{R}^d} (|f|^2 + |u|^2)\, dx.$$

The latter inequality can be written in the form

$$\int_{\mathbb{R}^d} (2|\nabla u|^2 + |u|^2)\, dx \leq \int_{\mathbb{R}^d} |f|^2\, dx.$$

Replacing 2 by a smaller number we will make the inequality weaker. As a result we obtain the estimate

$$||(-\Delta + I)^{1/2} u||^2 \leq ||f||^2. \tag{3.3}$$

It remains to note that (3.3) is equivalent to (3.2).

Let us summarize the results. Since

$$Y = \sqrt{\Im V_+}(H + I - i)^{-1} = \sqrt{\Im V_+}(-\Delta + I)^{-1/2} B \quad \text{and} \quad ||B|| = 1,$$

we obtain

$$||Y||_{2p} \le ||\sqrt{\Im V_+}(-\Delta + I)^{-1/2}||_{2p}. \tag{3.4}$$

On the other side, according to Theorem 7,

$$||\sqrt{\Im V_+}(-\Delta + I)^{-1/2}||_{2p}^{2p} \le (2\pi)^{-d} C_0 \int \Im V_+^p \, dx,$$

where $C_0 = \int_{\mathbb{R}^d} (\xi^2 + 1)^{-p} \, d\xi$. Combining (3.4) with (3.1), we complete the proof of Theorem 1.

4. Proof of Theorem 2

The main arguments in the proof of this result remain the same apart from the estimate of the norm $||Y||_{2p}$ of the operator Y. Recall that

$$\sum_j \Big(\frac{\Im \lambda_j}{|\lambda_j + 1|^2 + 1} \Big)_+^p \le ||Y||_{2p}^{2p}, \tag{4.1}$$

where $Y = \sqrt{\Im V_+}(H + I - i)^{-1}$.

In order to find a bound for the s-numbers of the operator Y we represent it in the form

$$Y = \sqrt{\Im V_+}(-\Delta + I - i)^{-1}(I - V(H + I - i)^{-1}).$$

In the previous Section we have found that

$$(H + I - i)^{-1} = (-\Delta + I)^{-1/2} B \qquad \text{and} \quad ||B|| \le 1.$$

Consequently,

$$||(H + I - i)^{-1}|| \le 1,$$

and this means that

$$||Y||_{2p} \le ||\sqrt{\Im V_+}(-\Delta + I - i)^{-1}||_{2p}(1 + ||V||_\infty).$$

By using Theorem 7 we obtain that for any $p > d/4$

$$||\sqrt{\Im V_+}(-\Delta + I - i)^{-1}||_{2p}^{2p} \le (2\pi)^{-d} C_0 \int \Im V_+^p \, dx,$$

where

$$C_0 = \int \frac{d\xi}{((\xi^2 + 1)^2 + 1)^p}.$$

Consequently,

$$||Y||_{2p}^{2p} \le (2\pi)^{-d} (1 + ||V||_\infty)^{2p} C_0 \int \Im V_+^p \, dx. \tag{4.2}$$

Consequently (4.1) and (4.2) imply (1.3).

5. Proof of Theorem 3 and some related results

Proof of Theorem 3. Assume that $\lambda_j \in \Pi_b$ are enumerated in the order of decreasing imaginary parts. Note, that the theorem would be proven, if instead of the infinite sum in the left hand side of (1.5), we estimated a partial sum

$$\sum_{j=1}^{m} |\Im\lambda_j|^\gamma \leq C|\Psi_b(W)|^{\frac{2\gamma-1}{2r-1}}(b+|\Psi_b(W)|^{\frac{1}{2r-1}}), \qquad \lambda_j \in \Pi_b. \tag{5.1}$$

On the other hand, it is sufficient to prove the estimate (5.1) for the case when

$$V \in C_0^\infty(\mathbb{R}^d).$$

Indeed, if $V \notin C_0^\infty(\mathbb{R}^d)$, then we can always find a sequence V_n of C_0^∞-functions that converges to V in $L^{d/2-1+2r}(\mathbb{R}^d) \cap L^r(\mathbb{R}^d)$. Obviously the corresponding sequence of quantities $\Psi_b(W_n)$ (here $W_n = |V_n|^2 + 4\Im V_n$) will converge to $\Psi_b(W)$. Moreover, the non-real eigenvalues λ_j of H will be the limits of the sequences of non-real eigenvalues $\lambda_j(n)$ of $H_n = -\Delta + V_n$, which implies that $\sum_{j=1}^m |\Im\lambda_j|^\gamma = \lim_{n\to\infty}\sum_{j=1}^m |\Im\lambda_j(n)|^\gamma$.

Essential role in the proof plays Corollary 1 as well as a trick relating the eigenvalues of the operator $H = -\Delta + V$ and the eigenvalues of the operator $(-\Delta + 2i - \mu + V)^2$, $\mu > 0$, lying to the left of $\Re z = -4$. Indeed, let λ_j be eigenvalues of the operator $-\Delta + V$ lying in the hyperbolic domain $D_\mu = \{z : \ (\Im z + 2)^2 - (\Re z - \mu)^2 \geq 4, \ \ \Im z > 0\}$, then $(\lambda_j - \mu + 2i)^2$ are eigenvalues of the operator $(-\Delta - \mu + 2i + V)^2$, and it is easy to see, that

$$\Re(\lambda_j - \mu + 2i)^2 = (\Re\lambda_j - \mu)^2 - (\Im\lambda_j + 2)^2 \leq -4, \qquad \forall \lambda_j \in D_\mu.$$

Consequently, due to Corollary 1,

$$\sum_1^n \Big|\Re(\lambda_j - \mu + 2i)^2 + 4\Big| \leq \Big|\sum_1^n s_j\Big|, \tag{5.2}$$

where s_j are eigenvalues of the operator

$$T_1 = (-\Delta - \mu)^2 + V_1(-\Delta - \mu) + (-\Delta - \mu)V_1 + V_1^2 - V_2^2 - 4V_2$$

where $V_1 = \Re V$ and $V_2 = \Im V$ are the real and the imaginary parts of the potential. The inequality (5.2) takes care of all eigenvalues from the domain D_μ. It turns out that we do not need all of them, but only the eigenvalues λ_j lying inside the domain $\Omega_\mu = \{z : \ (\Im z + 1)^2 - (\Re z - \mu)^2 \geq 1, \ \ \Im z > 0\}$. Note that the boundaries of both domains D_μ and Ω_μ touch the real line at the point $z = \mu$. Note also that $\Omega_\mu \subset D_\mu$ and therefore this might imply that bounds on eigenvalues lying in Ω_μ are better than those in D_μ.

It turns out that the imaginary parts of eigenvalues in Ω_μ can be estimated in terms of real parts of eigenvalues of the operator $(H - \mu + 2i)^2 + 4$. A similar trick was used by Davies in [6] to obtain individual inequalities for the eigenvalues of the operator$H$.

Let us study the relation between the spectra of the operators H and $(H-\mu+2i)^2$ in more detail. Assume that $\lambda_j \in \Omega_\mu$ and $\Im\lambda_j > s$. Then

$$\begin{aligned} 2(\Im\lambda_j - s) &\le (\Im\lambda_j + 1)^2 - (\Re\lambda_j - \mu)^2 - 1 + 2(\Im\lambda_j - s) \\ &= (\Im\lambda_j + 2)^2 - (\Re\lambda_j - \mu)^2 - 4 - 2s = -\Re(\lambda_j - \mu + 2i)^2 - 4 - 2s. \end{aligned}$$

Due to Corollary 1 it means that

$$2\sum_{\lambda_j\in\Omega_\mu}(\Im\lambda_j - s)_+ \le \operatorname{tr}\Big(\Re(H-\mu+2i)^2 + 4 + 2s\Big)_- \le \operatorname{tr}\Big(T_1 + 2s\Big)_- .$$

Now, we represent the operator T_1 in the form

$$T_1 = \frac{1}{2}(-\Delta-\mu)^2 + \Big(\frac{1}{\sqrt{2}}(-\Delta-\mu) + \sqrt{2}V_1\Big)^2 - 4V_2 - V_1^2 - V_2^2.$$

Since the operator

$$\Big(\frac{1}{\sqrt{2}}(-\Delta-\mu) + \sqrt{2}V_1\Big)^2 \ge 0$$

is positive, we obtain that the spectrum of the operator T_1 can be estimated by the spectrum of the operator

$$T_2 = \frac{1}{2}(-\Delta-\mu)^2 - |V|^2 - 4V_2.$$

Thus,

$$2\sum_{\lambda_j\in\Omega_\mu}(\Im\lambda_j - s)_+ \le \operatorname{tr}\Big(T_2 + 2s\Big)_- . \tag{5.3}$$

Let τ_j be negative eigenvalues of T_2. In order to estimate the right hand side of (5.3) we apply Proposition 4 according to which the number $N(E)$ of eigenvalues of T_2 lying to the left of the point $-E$ satisfies the inequality

$$N(E) \le \frac{C}{E^p}\Big(\int_{\mathbb{R}^d} W^{d/4+p}dx + \mu^{d/2-1}\int_{\mathbb{R}^d} W^{1/2+p}dx\Big) \tag{5.4}$$

with $p > 1/2$ and $d \ge 2$. If now $q > p > 1/2$ then

$$\begin{aligned} \sum_j |\tau_j|^q &= q\int_0^\infty E^{q-1}N(E)dE \\ &\le C\Big(\int_{\mathbb{R}^d} W^{d/4+p}dx + \mu^{d/2-1}\int_{\mathbb{R}^d} W^{1/2+p}dx\Big)|\lambda_1|^{q-p}. \end{aligned}$$

From (5.4) it follows that the lowest eigenvalue τ_1 satisfies the inequality

$$|\tau_1|^{r-1/2} \le C\Big(\int_{\mathbb{R}^d} W^{d/4+r-1/2}dx + \mu^{d/2-1}\int_{\mathbb{R}^d} W^r dx\Big) = C\Psi_\mu(W).$$

Hence for $q>p>1/2$ and $r>1$ we arrive at

$$\sum_j |\tau_j|^q \le C\Big(\int_{\mathbb{R}^d} W^{d/4+p}dx + \mu^{d/2-1}\int_{\mathbb{R}^d} W^{1/2+p}dx\Big)|\Psi_\mu(W)|^{2(q-p)/(2r-1)}.$$

Recall that

$$2\sum_{\lambda_j\in\Omega_\mu}(\Im\lambda_j - s)_+ \le \sum_j(\tau_j+2s)_-$$

and therefore

$$\sum_{\lambda_j\in\Omega_\mu}(\Im\lambda_j - s)_+ \le C\Big(\int_{\mathbb{R}^d}(W-2s)_+^{d/4+p}dx$$
$$+\mu^{d/2-1}\int_{\mathbb{R}^d}(W-2s)_+^{1/2+p}dx\Big)|\Psi_\mu(W)|^{\frac{2(1-p)}{2r-1}} =: F(s,\mu) \quad (5.5)$$

with $1/2<p<1$ and $r>1$.

Let now Π_b be the strip $\{z:\ 0<\Re z<b, \Im z>0\}$. Since the boundary of Ω_μ touches the real line in the parabolically, it is obvious, that for small values of $s<\varepsilon_0$, the set of all points $z\in\Pi_b$ whose $\Im z>s>0$ can be covered by not more than $m(b)=[Cb/\sqrt{s}]+1$ sets of the form Ω_μ. Since Ω_μ contains the sector $\Im z>|\Re z-\mu|$, we obtain that the number of domains Ω_μ covering the strip Π_b can be also estimated by $[b/s]+1$ for any $s>0$. If $s\ge\varepsilon_0$ then $1/\sqrt{s}\ge\sqrt{\varepsilon_0}/s$ and therefore without loss of generality one can assume that

$$m(b)=[Cb/\sqrt{s}]+1, \qquad \forall s>0.$$

Since $\Im\lambda_j\le C|\Psi_b(W)|^{\frac{2}{2r-1}}$ for any $\lambda_j\in\Pi_b$, we obtain

$$\sum_{\lambda_j\in\Pi_b}(\Im\lambda_j-s)_+ \le \sum_{l=1}^{m(b)}\sum_{\lambda_j\in\Omega_{\mu_l}}(\Im\lambda_j-s)_+ \le C\frac{(b+|\Psi_b(W)|^{\frac{1}{2r-1}})}{\sqrt{s}}F(s,b).$$

Obviously

$$\sum_{\lambda_j\in\Pi_b}|\Im\lambda_j|^\gamma = \gamma(\gamma-1)\sum_{\lambda_j\in\Pi_b}\int_0^\infty(\Im\lambda_j-s)_+ s^{\gamma-2}ds,$$

which leads to

$$\sum_{\lambda_j\in\Pi_b}|\Im\lambda_j|^\gamma \le (b+|\Psi_b(W)|^{\frac{1}{2r-1}})C\int_0^\infty s^{\gamma-5/2}F(s,b)ds$$

The integral in the right hand side converges only if $\gamma > 3/2$ and by using the notation inyroduced in (5.5) we finally obtain

$$\sum_{\lambda_j \in \Pi_b} |\Im \lambda_j|^\gamma \le C |\Psi_b(W)|^{\frac{2\delta}{2r-1}} (b + |\Psi_b(W)|^{\frac{1}{2r-1}})$$
$$\times \Big(\int_{\mathbb{R}^d} |W|^{d/4-1/2+\gamma-\delta} dx + + b^{d/2-1} \int_{\mathbb{R}^d} |W|^{\gamma-\delta} dx \Big),$$

where $0 < \delta < 1/2$. It remains to set $r = \gamma - \delta$ to complete the proof. □

We have proved inequalities for $\sum |\Im \lambda_j|^\gamma$ with $\gamma > 3/2$. However, (5.5) allows us to obtain a bound on eigenvalues belonging to Ω_μ with $\gamma = 1$.

Corollary 3. *Let λ_j be the eigenvalues of the operator $-\Delta + V$ lying inside $\Omega_\mu = \{z : \ (\Im z + 1)^2 - (\Re z - \mu)^2 \ge 1, \quad \Im z > 0\}$ and let $d \ge 2$. Then*

$$\sum_j |\Im \lambda_j| \le C \Big(\int_{\mathbb{R}^d} W^{d/4+p} dx + \mu^{d/2-1} \int_{\mathbb{R}^d} W^{1/2+p} dx \Big) ||W||_\infty^{1-p}$$

for any $1/2 < p < 1$.

Similarly we can show

Corollary 4. *Let $d = 1$ and let λ_j be the eigenvalues of the operator $-d^2/dx^2 + V$ lying inside $\Omega_\mu = \{z : \ (\Im z + 1)^2 - (\Re z - \mu)^2 \ge 1, \quad \Im z > 0\}$. Then*

$$\sum_j |\Im \lambda_j| \le C ||W||_\infty^{1-p} \mu^{-1/2} \int_{\mathbb{R}^d} W^{1/2+p} dx$$

for any $1/2 < p < 1$.

Unfortunately if $d = 1$ then in order to obtain similar results we have to avoid the point $z = 0$ and in this case we deal with the strip $a < \Re z < b$, $a > 0$. However, this is no longer true if $\gamma > 7/4$. Indeed:

Theorem 8. *Let λ_j be the eigenvalues of the operator $H = -d^2/dx^2 + V$ lying inside the semi-infinite strip $\Pi_b = \{z : \ 0 < \Re z < b, \, \Im z > 0\}$. Then for any $\gamma > 7/4$, $r \in (\gamma - \frac{1}{2}, \gamma)$ and $V \in L^\infty(\mathbb{R}^d)$ such that $W = (|V|^2 + 4\Im V)_+ \in L^{r-1/4}$ we have*

$$\sum_{\lambda_j \in \Pi_b} |\Im \lambda_j|^\gamma \le C ||W||_\infty^{\gamma - r} (b + \|W\|_\infty^{1/2}) \Big(\int_{\mathbb{R}^d} |W|^{r-1/4} dx \Big).$$

Proof. The inequality (5.5) could be easily modified and we can obtain that for $1/2 < p < 1$

$$\sum_{\lambda_j \in \Omega_\mu} (\Im \lambda_j - s) \le C \Big(\mu^{-1/2} \int_{\mathbb{R}^d} (W - 2s)_+^{1/2+p} dx \Big) \|W\|_\infty^{1-p} =: F(s, \mu), \quad (5.6)$$

where $\|W\|_\infty^{1-p}$ appears when we estimate the lowest eigenvalue λ_1 of the operator $T_2 = \frac{1}{2}(-d^2/dx^2 - \mu)^2 - W$.

Now consider the part of the strip $\Pi_b = \{z : 0 < \Re z < b, \Im z > 0\}$ satisfying $\Im z > s >$. We cover it by sets Ω_μ, $\mu \in \mathbb{R}_+$. While doing this, we avoid the value $\mu = 0$ taking μ as large as possible. The optimal choice of such μ would be $mu = \mu_0 = \sqrt{s^2 + 2s}$, where μ_0 satisfies the equation $(s+1)^2 - \mu_0^2 = 1$. Thus, without loss of generality, we can assume that $\mu \geq \sqrt{s^2 + 2s}$.

Arguing as in the proof of Theorem 3, we find that set of all points $z \in \Pi_b$ whose $\Im z > s$ can be covered by not more than $m(b) = [Cb/\sqrt{s}] + 1$ sets of the form Ω_μ.

Since there is no $\lambda_j \in \Pi_b$ satisfying $\Im \lambda_j > \|W\|_\infty$, we obtain

$$\sum_{\lambda_j \in \Pi_b} (\Im \lambda_j - s)_+ \leq \sum_{l=1}^{m(b)} \sum_{\lambda_j \in \Omega_{\mu_l}} (\Im \lambda_j - s)_+ \leq C \, \frac{b + ||W||_\infty^{1/2}}{\sqrt{s}} F(s, \mu_0).$$

Therefore

$$\sum_{\lambda_j \in \Pi_b} |\Im \lambda_j|^\gamma = \gamma(\gamma - 1) \sum_{\lambda_j \in \Pi_b} \int_0^\infty (\Im \lambda_j - s)_+ s^{\gamma - 2} ds$$

$$\leq (b + ||W||_\infty^{1/2}) C \int_0^\infty s^{\gamma - 5/2} F(s, \sqrt{s}) \, ds.$$

The integral in the right hand side converges only if $\gamma > 7/4$ and using (5.6) we arrive at

$$\sum_{\lambda_j \in \Pi_b} |\Im \lambda_j|^\gamma \leq C ||W||_\infty^{1-p} (b + ||W||_\infty^{1/2}) \Big(\int_{\mathbb{R}^d} |W|^{\gamma - 5/4 + p} dx \Big)$$

with $1/2 < p < 1$. It remains to set $r = \gamma + p - 1$ to complete the proof. $\square$

6. Proof of Theorem 5

Theorem 5 has been already proved before for $d = p = 1$ (see[1], [7]). Consider first the case when $p > \max\{1, d/2\}$. By using Birman-Schwinger principle, we find that the value $\lambda \notin \mathbb{R}_+$ is an eigenvalue of the operator H if and only if 1 is an eigenvalue of the operator

$$X = |V|^{1/2} (-\Delta - \lambda)^{-1} |V|^{-1/2} V,$$

and thus $\|X\| \geq 1$. Note now that

$$\|X\| \leq \|X\|_p \leq \|Q\|_{2p}^2$$

where

$$Q = |V|^{1/2} |-\Delta - \lambda|^{-1/2}.$$

Using Theorem 7 we obtain that

$$1 \le ||Q||_{2p}^{2p} \le (2\pi)^{-d} \int_{\mathbb{R}^d} |V|^p dx \int_{\mathbb{R}^d} \frac{d\xi}{||\xi|^2 - \lambda|^p}.$$

Assuming that $p > d/2$ there is a constant C such that

$$J = \int_{\mathbb{R}^d} \frac{d\xi}{||\xi|^2 - \lambda|^p} = |\lambda|^{d/2-p} \int_{\mathbb{R}^d} \frac{d\xi}{||\xi|^2 - e^{i\phi}|^p} \le C\, |\lambda|^{d/2-p}\, |\sin\phi|^{1-p}.$$

where $\phi = \arg\lambda$ and consequently,

$$J \le C\, |\Im\lambda|^{1-p}\, |\lambda|^{d/2-1}.$$

It remains to note that

$$1 \le (2\pi)^{-d} J \int_{\mathbb{R}^d} |V|^p dx.$$

In order to prove Theorem 5 for $p = d/2 > 1$ we use just Theorem 6 instead of Theorem 7. Indeed, let

$$a(\xi) = \frac{1}{||\xi|^2 - \lambda|} \qquad \text{and} \quad p = d/2 > 1.$$

Then, using homogeneity, we obtain

$$[a]_p^p = [a_0]_p^p, \qquad \text{where} \quad a_0 = \frac{1}{|\xi^2 - e^{i\phi}|}.$$

There is a constant $C > 0$ such that

$$[a]_p^p = [a_0]_p^p \le C|\sin\phi|^{1-p} = C\Big|\frac{\Im\lambda}{\lambda}\Big|^{1-p}.$$

It remains to note that, if λ is an eigenvalue of H, then

$$1 \le C[a]_p^p \int_{\mathbb{R}^d} |V|^p dx \qquad p = d/2.$$

The proof is complete. □

7. Individual eigenvalue estimates

Let us now consider a Schrödinger operator$H = -\Delta + iV(x)$ whose potential is pure imaginary. Besides we assume that $V \ge 0$ and $\lim_{|x|\to\infty} V(x) = 0$.

Our first statement concerns the case $d = 3$.

Theorem 9. *Let $V \in L^1(\mathbb{R}^3)$, $V \ge 0$ and let $z = k^2 \notin \mathbb{R}_+$ be an eigenvalue of $H = -\Delta + iV(x)$. Then*

$$\frac{\Re k}{4\pi} \int_{\mathbb{R}^3} V(x)\, dx \ge 1. \tag{7.1}$$

In particular, this shows that if $\int_{\mathbb{R}^3} V\,dx$ is small, then the real part of the square root of the eigenvalue of H is large. That implies that non-real eigenvalues of $-\Delta + itV$ escape any compact subset of $\mathbb{C}$, as $t \to 0$. It does not necessary imply that the eigenvalues tend to infinity as $t \to 0$, because they might simply reach the positive real semi-axis for some $t > 0$ (see Theorem 15).

Proof of Theorem 9. By using the Birman-Schwinger principle we find that $z = k^2 \notin \mathbb{R}_+$ is an eigenvalue of the operator $H = -\Delta + iV$ if and only if the operator

$$X = -i\sqrt{V}\,(-\Delta - z)^{-1}\sqrt{V} \tag{7.2}$$

has an eigenvalue 1.

Suppose that $\Im z > 0$. Then the real part of the operator X is positive and, consequently, the spectrum of this operator lies in the right half plane. Therefore if z is an eigenvalue of H, then

$$\sum_j \Re\,\zeta_j \geq 1,$$

where ζ_j are eigenvalues of X. On the other side,

$$\sum_j \Re\,\zeta_j \leq \operatorname{tr} \Re X = \int_{\mathbb{R}^3} \tau(x,x)\,dx,$$

where $\tau(x,y)$ is the integral kernel of the operator $\Re X$.

Since the kernel of the operator $(-\Delta - z)^{-1}$ equals

$$g(x,y) = \frac{e^{ik|x-y|}}{4\pi|x-y|},$$

we obtain that the kernel of the operator $\Im(-\Delta - z)^{-1}$ equals

$$g_0(x,y) = (2i)^{-1}(g(x,y) - \overline{g(y,x)})$$

whose diagonal values are

$$g_0(x,x) = \frac{k + \bar{k}}{8\pi} = \frac{\Re k}{4\pi}.$$

Finally

$$\operatorname{tr} \Re\,X = \int_{\mathbb{R}^3} V(x)\,g_0(x,x)\,dx = \frac{\Re\,k}{4\pi}\int_{\mathbb{R}^3} V(x)\,dx$$

implies (7.1). $\square$

Corollary 5. *Let $d = 3$ and let $V \in L^1(\mathbb{R}^3)$ be a positive function. Then non-real eigenvalues of $-\Delta + iV$ do not accumulate to zero.*

Using the same approach we obtain the following two results in dimensions $d = 1$ and $d = 2$.

Theorem 10. *Let $d = 1$, $z = k^2 \notin \mathbb{R}_+$ be an eigenvalue of the operator $H = -\Delta + iV$, $V \geq 0$, $V \in L^1(\mathbb{R})$. Then*

$$\frac{\Re k}{2|k|^2} \int_{\mathbb{R}} V(x)\, dx \geq 1,$$

which means that k lies inside the circle of radius $4^{-1} \int V(x)\, dx$ with the centre at $4^{-1} \int_{\mathbb{R}} V(x)\, dx$.

It is interesting to observe that if $d = 2$then the eigenvalues do not appear at all if the integral of V is small.

Theorem 11. *Let $d = 2$, $z \notin \mathbb{R}_+$ be an eigenvalue of $H = -\Delta + iV$, $V \geq 0$, $V \in L^1(\mathbb{R}^2)$. Then*

$$\frac{1}{2}\Big(\frac{\pi}{2} + \arctan(\Re z / \Im z)\Big) \int V(x)\, dx \geq 1.$$

In particular, the spectrum of H is real if

$$\frac{\pi}{2} \int V(x)\, dx < 1.$$

Proof. In order to prove this statement we just notice that if X is the Birman-Schwinger operator (7.2) defined in the proof of Theorem 9, then

$$\operatorname{tr} \Re X = \int V(x)\, dx \int_{\mathbb{R}^2} \Im\Big[\frac{1}{2\pi(|\xi|^2 - z)}\Big] d\xi.$$

□

The next result deals with some properties of complex eigenvalues of Schrödinger operators in higher dimensions $d \geq 4$.

Theorem 12. *Let $d \geq 4$ and let $z \notin \mathbb{R}_+$ be an eigenvalue of $H = -\Delta + iV$ with $V \geq 0$. Then*

$$(2\pi)^{-d+1} \omega_{d-1} \Big| \Re z + 2\|V\|_\infty \Big|^{(d-2)/2} \int V(x)\, dx \geq 2, \tag{7.3}$$

where ω_{d-1} is the area of the unit sphere $\mathbb{S}^{d-1}$.

Proof. If as before X is the Birman-Schwinger operator introduced in (7.2) and z is an eigenvalue of the operator H, then $1/2$ is an eigenvalue of the operator $X - 1/2$. Consequently,

$$\operatorname{tr}(\Re X - 1/2)_+ \geq 1/2. \tag{7.4}$$

Indeed, for the eigenvalues λ_j of the operator X we have

$$\sum (\Re \lambda_j - 1/2)_+ \leq \operatorname{tr}(\Re X - 1/2)_+.$$

Therefore the eigenvalue sum in the left hand side is not less than $1/2$.

Obviously,

$$
\begin{aligned}
(\Re X - 1/2)_+ &\leq \Big(\Re X - \frac{V}{2||V||_\infty}\Big)_+ \\
&= \sqrt{V}\Big(\Im(-\Delta - z)^{-1} - \frac{1}{2||V||_\infty}\Big)_+\sqrt{V}.
\end{aligned}
$$

Concequently, using (7.4) we have

$$
1/2 \leq (2\pi)^{-d}\int_{\mathbb{R}^d} V(x)\,dx \int_{\mathbb{R}^d}\Big(\frac{\Im z}{(|\xi|^2 - \Re z)^2 + (\Im z)^2} - \frac{1}{2\|V\|_\infty}\Big)_+ d\xi.
$$

The integration in the last integral is carried out over the domain where

$$
|\xi|^2 \leq \Re z + \sqrt{(2\|V\| - \Im z)_+ \Im z} \leq \Re z + 2\|V\|_\infty.
$$

Therefore

$$
\begin{aligned}
&\omega_{d-1}^{-1}\int_{\mathbb{R}^d}\Big(\frac{\Im z}{(|\xi|^2 - \Re z)^2 + (\Im z)^2} - \frac{1}{2\|V\|_\infty}\Big)_+ d\xi \\
&\qquad\leq 2^{-1}\pi\Big|\Re z + 2\|V\|_\infty\Big|^{(d-2)/2}. \qquad (7.5)
\end{aligned}
$$

and we obtain (7.3). □

We now obtain some results involving L^p norms of potentials with $p > 1$.

Theorem 13. *Let $d \geq 3$ and let $V \geq 0$. Suppose that $z \notin \mathbb{R}$ is an eigenvalue of $H = -\Delta + iV$. Then there are positive constants C_1 and C_2 depending only on d and $\gamma \geq 0$ such that*

$$
|\Im z|^\gamma \leq \Big(C_1 + C_2\big(\frac{\Re z}{\Im z}\big)^{d/2-1}\Big)\int V^{d/2+\gamma}\,dx. \qquad (7.6)
$$

Proof. Let as before

$$
X = -i\sqrt{V}(-\Delta - z)^{-1}\sqrt{V}.
$$

If z is an eigenvalue of H, then there is at least one eigenvalue of the operator $\Re X$ that is not less than 1. If by s_j we denote the eigenvalues of the operator $\Re X$, then this implies

$$
\sup_{s>0} s^{-(d/2+\gamma)}\ \mathrm{card}\{j:\quad s_j > s\} \geq 1.
$$

This supremum is related to the norm in the weak Neumann-Schatten class $\Sigma_{d/2+\gamma}$ and, due to Theorem 6, it can be estimated by

$$
\int V^{d/2+\gamma}dx\int_{\mathbb{R}^d}\Big(\frac{\Im z}{(\xi^2 - \Re z)^2 + \Im z^2}\Big)^{d/2+\gamma}d\xi. \qquad (7.7)
$$

We conclude the proof by estimating the latter integral

$$\int_{\mathbb{R}^d} \Big(\frac{\Im z}{(\xi^2 - \Re z)^2 + \Im z^2}\Big)^{d/2+\gamma} d\xi$$
$$\leq C \int_{-\infty}^{\infty} \Big(\frac{\Im z}{s^2 + \Im z^2}\Big)^{d/2+\gamma} s^{d/2-1} ds + C \int_{-\infty}^{\infty} \Big(\frac{\Im z}{s^2 + \Im z^2}\Big)^{d/2+\gamma} |\Re z|^{d/2-1} ds$$
$$\leq \Big(C_1 + C_2 \Big|\frac{\Re z}{\Im z}\Big|^{d/2-1}\Big) |\Im z|^{-\gamma}.$$

□

Applying this result for the case $\gamma = 0$ we obtain:

Corollary 6. *Let $d \geq 3$ and let C_1 be the constant in (7.6). If $C_1 \int V^{d/2} dx < 1$, then the eigenvalues of $-\Delta + iV$ belong to the conical sector $\{z : \ 0 \leq \arg z \leq \alpha\}$, where α satisfies the equation*

$$(C_1 + C_2 (\cot \alpha)^{d/2-1}) \int V^{d/2} dx = 1.$$

If $\gamma > 0$ then in the proof of Theorem 13 one can apply Theorem 6 even if $d = 2$ and obtain

Theorem 14. *Let $d = 2$ and let $V \geq 0$. Suppose that $z \notin \mathbb{R}$ is an eigenvalue of $H = -\Delta + iV$. Then there is a positive constant C depending only on $\gamma > 0$ such that*

$$|\Im z|^\gamma \leq C \int V^{1+\gamma} dx, \quad \gamma > 0.$$

8. Additional remarks

Concluding this paper, we mention two rather obvious facts, that are valid for an arbitrary complex potential V. For the sake of simplicity, we restrict our study to the case $d = 3$. As before, $H = -\Delta + V$ is the Schrödinger operator and ω_2 is the area of the unit sphere $\mathbb{S}^2$.

Theorem 15. *Let $d = 3$. If $V \in L^\infty \cap L^1$ and let $\omega_2 \|V\|_\infty + 2\|V\|_1 < 8\pi$. Then the spectrum of the operator H is real.*
The same statement is true if

$$\sup_x \int_{\mathbb{R}^3} \frac{|V(y)|}{|x - y|} dy < 4\pi.$$

Theorem 16. *Let $d = 3$ and let $z = k^2 \notin \mathbb{R}_+$ be an eigenvalue of the operator $H = -\Delta + V$, $\Im k > 0$. Then there is a positive constant C depending only on $\gamma > 0$, such that*

$$(\Im k)^{2\gamma} \leq C \int_{\mathbb{R}^3} |V|^{3/2+\gamma} dx.$$

Proof of both theorems. Suppose that $z = k^2$ is an eigenvalue of the operator H. Then the norm of the operator $X = |V|^{1/2}(-\Delta - z)^{-1}|V|^{1/2}$ is not smaller then 1. By Schur's inequality if G is an integral operator with the kernel $g(x, y)$ then

$$\|G\| \leq m_1 m_2, \tag{8.1}$$

where

$$m_1 = \sup_x \int |g(x,y)| \frac{dy}{\rho(x,y)} \quad \text{and} \quad m_2 = \sup_y \int |g(x,y)|\rho(x,y)\, dx$$

and ρ is a positive weight. Since the kernel of the operator X equals

$$|V(x)|^{1/2} \frac{e^{ik|x-y|}}{4\pi|x-y|} |V(y)|^{1/2},$$

then applying (8.1) with the weight $\rho = \sqrt{V(x)/V(y)}$, we obtain that

$$||X|| \leq \frac{1}{4\pi} \sup_x \int \frac{e^{-\Im k|x-y|}}{|x-y|} |V(y)|\, dy.$$

The statement of Theorem 15 follows from the trivial estimate

$$1 \leq \|X\| \leq \frac{1}{4\pi} \sup_x \int_{\mathbb{R}^3} \frac{|V(y)|}{|x-y|}\, dy \leq \frac{1}{8\pi}(\omega_2\|V\|_\infty + 2\|V\|_1).$$

We obtain the statement of Theorem 16 using the Hölder inequality

$$\begin{aligned} 1 \leq \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{|V(y)|}{|x-y|} e^{-\Im k|x-y|} dy \\ \leq C_0 ||V||_p \Big(\int_{\mathbb{R}^3} \frac{e^{-q\Im k|y|}}{|y|^q} dy \Big)^{1/q} = C \frac{||V||_p}{(\Im k)^{2\gamma/p}}, \end{aligned}$$

where $p = 3/2 + \gamma$ and $q = p/(p-1)$. $\Box$

Remark. By using similar arguments one can show that

$$\Big| \frac{\sqrt{z}}{\Im\sqrt{z}} \Big|^{\gamma+1/2} |\Im\sqrt{z}|^{2\gamma} \leq C \int_{\mathbb{R}} |V|^{1/2+\gamma} dx, \quad \gamma \geq 1/2,$$

for eigenvalues $z \notin \mathbb{R}_+$ of the one-dimensional Schrödinger operator $H = -d^2/dx^2 + V$. The constant C in this inequality can be computed explicitly

$$C = \frac{1}{2} \Big(\frac{2\gamma - 1}{2\gamma + 1} \Big)^{\gamma - 1/2}.$$

References

[1] Abramov, A.A., Aslanyan, A., Davies, E.B.: *Bounds on complex eigenvalues and resonances.* J. Phys. A **34**, 57–72 (2001)

[2] N. Austern: *The use of complex potentials in nuclear physics*, Annals of Physics **45**, Issue 1, 113-131

[3] Birman, M.: *On the spectrum of singular boundary value problems* Matem. sb. **55** (1961), 125-174.

[4] Birman, M. Sh. and Solomjak, M. Z.: *Spectral theory of selfadjoint operators in Hilbert space.* Mathematics and its Applications (Soviet Series). D. Reidel Publishing Co., Dordrecht, 1987.

[5] Cwikel, M.: *Weak type estimates for singular values and the number bound states of Schrodinger operators*, Ann. of Math. **106**, (1977)

[6] Davies, E.B.: *Linear operators and their spectra.* Cambridge Studies in Advanced Mathematics, **106** Cambridge University Press, Cambridge, 2007

[7] Davies, E.B., Nath, J.: *Schrdinger operators with slowly decaying potentials.* J. Comput. Appl. Math. **148** (1), 1–28 (2002)

[8] Frank, Rupert L.; Laptev, Ari; Lieb, Elliott H.; Seiringer, Robert *Lieb-Thirring inequalities for Schrödinger operators with complex-valued potentials.* Lett. Math. Phys. **77** (2006), no. 3, 309–316.

[9] Ge, Jiu-Yuan; Zhang, John Z. H.: *Use of negative complex potential as absorbing potential* Journal of Chemical Physics, **108** (1998), Issue 4, 1429-1433.

[10] Fan, Ky: *Maximum properties and inequalities for the eigenvalues of completely continuous operators.* Proc. Nat. Acad. Sci., U. S. A. **37** (1951), 760–766

[11] Lieb, E. H. and Thirring, W.: *Inequalities for the moments of the eigenvalues of the Schrödinger Hamiltonian and their relation to Sobolev inequalities*, in Studies in Mathematical Physics (Essays in Honor of Valentine Bargmann), 269–303. Princeton Univ. Press, Princeton, NJ, 1976.

[12] G. Nimtz, H. Spieker and H.M. Brodowsky: *Tunneling with dissipation*, J. Phys. I France **4** (1994), 1379-1382

[13] M. Reed and B. Simon: *Methods of modern mathematical physics. IV. Analysis of operators.* Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1978.

[14] Schwinger, J.: *On the bound states of a given potential* Proc. Nat. Acad. Sci. USA **47** (1967), 122-129.

[15] Seiler, E. and Simon, B.: *Bounds in the Yukawa2 quantum field theory: upper bound on the pressure, Hamiltonian bound and linear lower bound*, Commun. Math. Phys., **45** (1975), 99-114.

[16] B. Simon: *Trace ideals and their applications.* London Mathematical Society Lecture Note Series **35**, Cambidge University Press, 1979.

[17] Von Neumann, J. and Wigner, E.: *Uber merkwurdige diskrete Eigenwerte*, Physik. Zeitschr. **30** (1929), 465

[18] Weyl, H.: *Inequalities between the two kinds of eigenvalues of a linear transformation*, Proc. Nat. Acad Sci. USA **35** (1949), 408-411.