

EIGENVALUE ESTIMATES ON QUATERNION-KÄHLER MANIFOLDS

XIAOLONG LI AND KUI WANG

ABSTRACT. We prove lower bound for the first closed or Neumann nonzero eigenvalue of the Laplacian on a compact quaternion-Kähler manifold in terms of dimension, diameter, and scalar curvature lower bound. It is derived as large time implication of the modulus of continuity estimates for solutions of the heat equation. We also establish lower bound for the first Dirichlet eigenvalue in terms of geometric data, via a Laplace comparison theorem for the distance to the boundary function.

1. INTRODUCTION

Let (M^n, g) be a compact n -dimensional Riemannian manifold (possibly with a smooth nonempty boundary). Denote by Δ the Laplace-Beltrami operator associated to the metric g . It is well-known that the spectrum of Δ (the Neumann boundary condition is imposed if ∂M is non-empty) consists of pure points spectrum that can be arranged in the order

$$0 = \mu_0 < \mu_1 \leq \mu_2 \leq \cdots \rightarrow \infty.$$

The study of the first nonzero eigenvalue μ_1 is an important issue in both mathematics and physics. In particular, the problem of establishing lower bounds for μ_1 in terms of geometric data of the manifold received considerable attention in the past few decades and a number of results have been obtained by various authors (see for example [Cha84][SY94][BQ00][LL10]). For instance, a classical result of Lichnerowicz states that $\mu_1 \geq n\kappa$ if M is a closed n -dimensional Riemannian manifold with $\text{Ric} \geq (n-1)\kappa > 0$. This follows easily by integrating the Bochner formula. The rigidity was observed by Obata [Oba62], who showed that the equality occurs if and only if M is isometric to the round sphere of radius $1/\sqrt{\kappa}$. In the nonnegative Ricci curvature case, by refining the gradient estimates of Li [Li79] and Li and Yau [LY80], Zhong and Yang [ZY84] proved the sharp lower bound $\mu_1 \geq \frac{\pi^2}{D^2}$, where D denotes the diameter of M . Moreover, it was proved by Hang and Wang [HW07] that the equality happens if and only if M is a circle of radius D/π or the interval $[-D/2, D/2]$.

Both Lichnerowicz and Zhong-Yang's results are special cases of the following theorem, which provides sharp lower bound for μ_1 depending on dimension, Ricci curvature lower bound, and diameter.

Theorem 1.1. *Let (M^n, g) be a compact Riemannian manifold (possibly with a smooth convex boundary) with diameter D and $\text{Ric} \geq (n-1)\kappa$ for $\kappa \in \mathbb{R}$. Let μ_1 be the first nonzero eigenvalue of the Laplacian on M (with Neumann boundary condition if $\partial M \neq \emptyset$). Then*

$$\mu_1 \geq \bar{\lambda}_1(n, \kappa, D),$$

2020 *Mathematics Subject Classification.* 35P15, 53C26.

Key words and phrases. Quaternion-Kähler manifold, eigenvalue comparison, modulus of continuity, orthogonal Ricci curvature.

The research of the second author is supported by NSFC No.11601359.

where $\bar{\lambda}_1(n, \kappa, D)$ is the first nonzero Neumann eigenvalue of the one-dimensional eigenvalue problem

$$\varphi'' - (n-1)T_\kappa\varphi' = -\lambda\varphi$$

on the interval $[-D/2, D/2]$, and T_κ is defined in (1.1).

Theorem 1.1 was first proved independently by Kröger [Krö92] using gradient estimate method and by Chen and Wang [CW94] using stochastic methods. The above explicit statement appeared first in the work of Bakry and Qian [BQ00], who also extended Theorem 1.1 to the setting of smooth metric measure spaces using gradient estimates. In 2013, Andrews and Clutterbuck [AC13] gave a simple proof using modulus of continuity estimates (see also [ZW17] for an elliptic argument based on [AC13] and [Ni13]). The sharpness can be seen by constructing a sequence of Riemannian manifolds with $\text{Ric} \geq (n-1)\kappa$, which geometrically collapse to the interval $[-D/2, D/2]$ (see for example [AC13]).

However, there are very few results specifically for Kähler manifolds. Lichnerowicz [Lic58] showed that if M is closed Kähler manifold with $\text{Ric} \geq (n-1)\kappa > 0$, then $\mu_1 \geq 2(n-1)\kappa$. Notice that this is a remarkable improvement of his well-known result in the Riemannian case. Lichnerowicz's proof makes use of a complex version Bochner formula (see also [Bal06, Theorem 6.14]). A different proof using harmonic maps was given by Urakawa [Ura87]. It is only until recently that the authors of the present paper established the following analogue of Theorem 1.1 for Kähler manifold in [LW21a].

Theorem 1.2. *Let (M^m, g, J) be a compact Kähler manifold of complex dimension m and diameter D , whose holomorphic sectional curvature is bounded from below by $4\kappa_1$ and orthogonal Ricci curvature is bounded from below by $2(m-1)\kappa_2$ for some $\kappa_1, \kappa_2 \in \mathbb{R}$. Let μ_1 be the first nonzero eigenvalue of the Laplacian on M (with Neumann boundary condition if M has a strictly convex boundary). Then*

$$\mu_1 \geq \bar{\mu}_1(m, \kappa_1, \kappa_2, D),$$

where $\bar{\mu}_1(m, \kappa_1, \kappa_2, D)$ is the first Neumann eigenvalue of the one-dimensional eigenvalue problem

$$\varphi'' - (2(m-1)T_{\kappa_2} + T_{4\kappa_1})\varphi' = -\lambda\varphi$$

on $[-D/2, D/2]$, and T_κ is defined in (1.1).

Theorem 1.2 provides the first diameter-depending lower bound for μ_1 for Kähler manifolds. Its proof uses the modulus of continuity approach of Andrews and Clutterbuck [AC13]. The key idea in taking the Kählerity into consideration is that the Ricci curvature can be decomposed as the sum of holomorphic sectional curvature and orthogonal Ricci curvature. The notion of orthogonal Ricci curvature was introduced recently by Ni and Zheng [NZ18] in the study of comparison theorems on Kähler manifolds (see also [NZ19] and [NWZ18] for more results on orthogonal Ricci curvature). It turns out that holomorphic section curvature and orthogonal Ricci curvature are more suitable conditions for various comparison theorems on Kähler manifolds. On one hand, they reflect more on the Kähler structure and lead to sharper results than the Ricci curvature. On the other hand, they are weaker than the well-studied bisectional curvature lower bound, under which comparisons theorems for Kähler manifold were obtained by Li and Wang [LW05] and Tam and Yu [TY12]. See also [Liu14] for some comparison theorem for Kähler manifolds with Ricci curvature bounded from below.

Now let's turn to the quaternion-Kähler situation. When M is a closed quaternion-Kähler manifold of quaternionic dimension $m \geq 2$ (i.e., the real dimension is $4m$), it was shown by Alekseevsky and Marchiafava in [AM95] via a Bochner-type formula for 1-forms that $\mu_1 \geq 8(m+1)\kappa$, provided the scalar curvature is bound from below by $16m(m+2)\kappa > 0$. Moreover, they showed that the equality characterize the quaternionic projective space. Given the results in Theorem 1.1 and Theorem 1.2, it is natural to ask whether one can prove lower bounds for μ_1 that depends on the diameter for a quaternion-Kähler manifold.

The purpose of this paper is to establish analogous lower bounds for the first nonzero eigenvalue of the Laplacian on a compact quaternion-Kähler manifold. The main theorem states

Theorem 1.3. *Let (M^m, g, I, J, K) be a compact quaternion-Kähler manifold (possibly with a smooth strictly convex boundary) of quaternionic dimension $m \geq 2$ and diameter D . Suppose that the scalar curvature of M is bounded from below by $16m(m+2)\kappa$ for some $\kappa \in \mathbb{R}$. Let μ_1 be the first nonzero eigenvalue of the Laplacian on M (with Neumann boundary condition if M has a smooth strictly convex boundary). Then*

$$\mu_1 \geq \bar{\mu}_1(m, \kappa, D)$$

where $\bar{\mu}_1(m, \kappa, D)$ is the first nonzero Neumann eigenvalue of the one-dimensional eigenvalue problem

$$\varphi'' - (4(m-1)T_\kappa + 3T_{4\kappa})\varphi' = -\lambda\varphi$$

on $[-D/2, D/2]$, and T_κ is defined in (1.1).

To the best of our knowledge, Theorem 1.3 provides the first diameter-dependent lower bound for μ_1 on a quaternion-Kähler manifold. It also seems to be the first lower bound for the Neumann boundary condition in the quaternion-Kähler setting. Moreover, the monotonicity of $\bar{\mu}_1(m, \kappa, D)$ in D implies that when the diameter is small, the lower bound provided in Theorem 1.3 is better than any diameter-independent lower bound.

The lower bound in Theorem 1.3 will be derived as large time implication of the modulus of continuity estimates for solutions of the heat equation. This is consistent with the Riemannian case in [AC13] and the Kähler case in [LW21a]. Recall that the modulus of continuity ω of a continuous function u defined on a metric space (X, d) is defined by

$$\omega(s) := \sup \left\{ \frac{u(y) - u(x)}{2} : d(x, y) = 2s \right\}.$$

It was first observed by Andrews and Clutterbuck [AC09a, AC09b] that the moduli of continuity for solutions to a class of quasilinear isotropic equations are subsolutions of the corresponding one-dimensional equations. This has been extended in a number of ways: to the setting of Riemannian manifolds in [AC13][Ni13], to Bakry-Emery manifolds in [AN12][LW21b], to Kähler manifolds in [LW21a], to viscosity solutions in [Li16][LW17][LTW20], to elliptic equations in [AX19], and to fully nonlinear equations in [Li20]. The most important applications of the modulus of continuity estimates are in obtaining lower bounds for the first nonzero eigenvalue (see [AC13][And15][LW19a, LW19b, LW21a]) and for proving sharp lower bound for the fundamental gap for convex domains in the Euclidean space (see [AC11]), as well as for convex domains in the sphere (see [SWW19][HWZ20][DSW18]).

It does not seem possible to express $\bar{\mu}_1(m, \kappa, D)$ explicitly in terms of elementary functions, so we provide some explicit lower bounds below, which give a sort of interpolation between the Zhong-Yang and Licherowicz lower bounds. It is similar to the one obtained by

Shi and Zhang in [SZ07] for the Riemannian case, which says for $\bar{\lambda}_1(n, \kappa, D)$ as in Theorem 1.1, it holds that for $\kappa \geq 0$,

$$\bar{\lambda}_1(n, \kappa, D) \geq \sup_{s \in (0,1)} \left(4s(1-s) \frac{\pi^2}{D^2} + s(n-1)\kappa \right).$$

A slight modification of their proof shows that in the Kähler setting, it holds that

Proposition 1.1. *Let $\bar{\mu}_1(m, \kappa_1, \kappa_2, D)$ be as in Theorem 1.2. If $\kappa_1, \kappa_2 \geq 0$, then*

$$\bar{\mu}_1(m, \kappa_1, \kappa_2, D) \geq \sup_{s \in (0,1)} \left(4s(1-s) \frac{\pi^2}{D^2} + 2s(m-1)\kappa_2 + 4s\kappa_1 \right).$$

We are grateful to Dr. Shoo Seto who pointed Proposition 1.1 to us. Similarly, in the quaternion-Kähler case, we have

Proposition 1.2. *Let $\bar{\mu}_1(m, \kappa, D)$ be as in Theorem 1.3. If $\kappa \geq 0$, then*

$$\bar{\mu}_1(m, \kappa, D) \geq \sup_{s \in (0,1)} \left(4s(1-s) \frac{\pi^2}{D^2} + 4s(m+2)\kappa \right).$$

In Section 5, we also establish lower bound for the first Dirichlet eigenvalue in terms of geometric data (see Theorem 5.4). This is done via a Laplace comparison theorem for the distance to the boundary function on quaternion-Kähler manifold. Other sections are organized as follows. In Section 2, we review some basic properties of quaternion-Kähler manifolds. In Section 3, we derive the modulus of continuity estimates for solutions of quasilinear parabolic equations on a quaternion-Kähler manifold. As an application, we prove Theorem 1.3. The explicit lower bounds in Proposition 1.1 and 1.2 are proved in Section 4.

Throughout the paper, we use the following notations. The function T_κ is defined for $\kappa \in \mathbb{R}$ by

$$(1.1) \quad T_\kappa(t) = \begin{cases} \sqrt{\kappa} \tan(\sqrt{\kappa}t), & \kappa > 0, \\ 0, & \kappa = 0, \\ -\sqrt{-\kappa} \tanh(\sqrt{-\kappa}t), & \kappa < 0. \end{cases}$$

The function c_κ is defined for $\kappa \in \mathbb{R}$ by

$$(1.2) \quad c_\kappa(t) = \begin{cases} \cos \sqrt{\kappa}t & \text{if } \kappa > 0, \\ 1 & \text{if } \kappa = 0, \\ \cosh \sqrt{-\kappa}t & \text{if } \kappa < 0. \end{cases}$$

2. QUATERNION-KÄHLER MANIFOLDS

In this section, we recall some basic properties about quaternion-Kähler manifolds that will be needed in the sequel. These are proved by Berger [Ber66] and Ishibara [Ish74] (see also [Bes08]). We shall follow the presentation of [KLZ08] here.

Definition 2.1. *A quaternion-Kähler manifold (M^m, g) of quaternionic dimension m (the real dimension is $4m$) is a Riemannian manifold with a rank three vector bundle $V \subset \text{End}(TM)$ satisfying*

- (1) *In any coordinate neighborhood U of M , there exists a local basis $\{I, J, K\}$ of V such that*

$$\begin{aligned} I^2 &= J^2 = K^2 = 1, \\ IJ &= -JI = K, \\ JK &= -KJ = I, \\ KI &= -IK = J, \end{aligned}$$

and for all $X, Y \in TM$

$$\langle X, Y \rangle = \langle IX, IY \rangle = \langle JX, JY \rangle = \langle KX, KY \rangle$$

- (2) *If $\phi \in \Gamma(V)$, then $\nabla_X \phi \in \Gamma(V)$ for all $X \in TM$.*

It is worth noting that then the tensors I, J, K may not be globally defined on M . For instance, the canonical quaternionic projective space admits no almost complex structure for topological reasons. However, the space spanned by I, J, K may always be defined globally according to the definition.

A well known fact is that a $4m$ -dimensional Riemannian manifold is quaternion-Kähler if and only if its restricted holonomy group is contained in $Sp(n)Sp(1)$. Since the 4-dimensional Riemannian manifolds with holonomy $Sp(1)Sp(1)$ are simply the oriented Riemannian manifolds, we shall only consider the case $m \geq 2$.

In this paper, we are mostly concerned about the curvature properties of quaternion-Kähler manifolds. The Riemannian curvature tensor of (M, g) is defined by

$$R(X, Y, Z, W) = \langle \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X, Y]} Z, W \rangle.$$

Definition 2.2. *Let (M, g) be a quaternion-Kähler manifold.*

- (1) *The quaternionic sectional curvature of M is defined as*

$$Q(X) := \frac{R(X, IX, X, IX) + R(X, JX, X, JX) + R(X, KX, X, KX)}{|X|^2}.$$

- (2) *The orthogonal Ricci curvature of M is defined as*

$$\text{Ric}^\perp(X, X) := \text{Ric}(X, X) - Q(X)$$

First of all, all quaternion-Kähler manifolds of quaternionic dimension $m \geq 2$ are Einstein (see for instance [KLZ08, Theorem 1.2]), namely there exists a constant κ such that

$$\text{Ric} = 4(m+2)\kappa.$$

Moreover, we have the following proposition.

Proposition 2.1. *Let (M^m, g) be a quaternion-Kähler manifold of quaternionic dimension $m \geq 2$ with $\text{Ric} = 4(m+2)\kappa$ for $\kappa \in \mathbb{R}$. Then*

- (1) *M has constant quaternionic sectional curvature, i.e.,*

$$Q(X) = 12\kappa|X|^2.$$

- (2) *M has constant orthogonal Ricci curvature, i.e.,*

$$\text{Ric}^\perp(X, X) = 4(m-1)\kappa|X|^2.$$

Proof. For (1), see Theorem 1.3 in [KLZ08]. (2) follows directly from the definition $\text{Ric}^\perp(X, X) = \text{Ric}(X, X) - Q(X)$. \square

We end this section with the following useful lemma.

Lemma 2.1. *Let (M^m, g) be a quaternion-Kähler manifold of quaternionic dimension $m \geq 2$ with $\text{Ric} = 4(m+2)\kappa$ for $\kappa \in \mathbb{R}$. Let $\gamma : [a, b] \rightarrow M$ be a geodesic with unit speed and $X_I(t), X_J(t), X_K(t)$ are parallel vector fields along γ such that $X_I(a) = I\gamma'(a)$, $X_J(a) = J\gamma'(a)$, $X_K(a) = K\gamma'(a)$. Then*

$$R(\gamma', X_I(t), \gamma', X_I(t)) + R(\gamma', X_J(t), \gamma', X_J(t)) + R(\gamma', X_K(t), \gamma', X_K(t)) = 12\kappa$$

for all $t \in [a, b]$.

Proof. See Lemma 1.5 in [KLZ08]. □

3. MODULUS OF CONTINUITY ESTIMATES

In this section, we prove the modulus of continuity estimates for solution of a class of quasilinear isotropic parabolic equations on a compact quaternion-Kähler manifold.

As in the Riemannian case in [AC13] or the Kähler case in [LW21a], we consider the following isotropic quasilinear equations:

$$(3.1) \quad \frac{\partial u}{\partial t} = Q[u] := \left[\alpha(|\nabla u|) \frac{\nabla_i u \nabla_j u}{|\nabla u|^2} + \beta(|\nabla u|) \left(\delta_{ij} - \frac{\nabla_i u \nabla_j u}{|\nabla u|^2} \right) \right].$$

Here α and β are smooth positive functions. Some important examples of (3.1) are the heat equation (with $\alpha = \beta = 1$), the p -Laplacian heat flows (with $\alpha = (p-1)|\nabla u|^{p-2}$ and $\beta = |\nabla u|^{p-2}$) and the graphical mean curvature flow (with $\alpha = 1/(1+|\nabla u|^2)$ and $\beta = 1$).

In the quaternion-Kähler case, the associated one-dimensional operator \mathcal{F} is given by

$$(3.2) \quad \mathcal{F}\varphi := \alpha(\varphi')\varphi'' + (4(m-1)T_\kappa + 3T_{4\kappa})\beta(\varphi')\varphi',$$

where the function T_κ is the function defined in (1.1).

The main result of this section is the following modulus of continuity estimates on a compact quaternion-Kähler manifold, in terms of initial oscillation, elapsed time, and scalar curvature lower bound.

Theorem 3.1. *Let (M^m, g, I, J, K) be a compact quaternion-Kähler manifold with diameter D whose scalar curvature is bounded from below by $16m(m+2)\kappa$ for $\kappa \in \mathbb{R}$. Let $u : M \times [0, T] \rightarrow \mathbb{R}$ be a solution of (3.1) (with Neumann boundary condition if M has a strictly convex boundary). Then the modulus of continuity $\omega : [0, D/2] \times [0, T] \rightarrow \mathbb{R}$ of u is viscosity subsolution of the one-dimensional equation*

$$(3.3) \quad \omega_t = \mathcal{F}\omega,$$

where the operator \mathcal{F} is defined in (3.2).

Proof. The proof proceeds as in the Kähler case in [LW21a] but is slightly more involved. By the definition of viscosity solutions (see [CIL92]), we need to show that for every smooth function φ that touches ω from above at $(s_0, t_0) \in (0, D/2) \times (0, T)$ in the sense that

$$\begin{cases} \varphi(s, t) \geq \omega(s, t) \text{ near } (s_0, t_0), \\ \varphi(s_0, t_0) = \omega(s_0, t_0), \end{cases}$$

it holds that

$$(3.4) \quad \varphi_t \leq \mathcal{L}\varphi,$$

at the point (s_0, t_0) . It follows from the definition of ω that for such a function φ , we have

$$(3.5) \quad u(\gamma(b), t) - u(\gamma(a), t) - 2\varphi\left(\frac{L[\gamma]}{2}, t\right) \leq 0$$

for any $t \leq t_0$ close to t_0 and any smooth path $\gamma : [a, b] \rightarrow M$ with length close to $2s_0$. Moreover, since M is compact, there exist points x_0 and y_0 in M (assume for a moment that $\partial M = \emptyset$), with $d(x_0, y_0) = 2s_0$ such that the equality in (3.5) holds for $\gamma_0 : [-s_0, s_0] \rightarrow M$, a length-minimizing unit speed geodesic connecting x_0 and y_0 . The key idea is to derive useful inequalities from the first and second tests along smooth family of variations of the curve γ_0 . For this purpose, we need to recall the first and second variation formulas of arc length. If $\gamma : (r, s) \rightarrow M$ is a smooth variation of $\gamma_0(s)$, then we have

$$\frac{d}{dr} \Big|_{r=0} L[\gamma(r, s)] = g(T, \gamma_r)|_{-s_0}^{s_0},$$

and

$$\frac{d^2}{dr^2} \Big|_{r=0} L[\gamma(r, s)] = \int_{-s_0}^{s_0} (|\nabla_s \gamma_r|^\perp|^2 - R(\gamma_s, \gamma_r, \gamma_s, \gamma_r)) ds + g(T, \nabla_r \gamma_r)|_{-s_0}^{s_0},$$

where T is the unit tangent vector to γ_0 .

It will be convenient to work in the (quaternionic) Fermi coordinates along γ_0 chosen as follows. Choose an orthonormal basis $\{e_i\}_{i=1}^{4m}$ for $T_{x_0}M$ with

$$e_1 = \gamma'_0(-s_0), e_2 = I\gamma'_0(-s_0), e_3 = J\gamma'_0(-s_0), e_4 = K\gamma'_0(-s_0),$$

and parallel transport it along γ_0 to produce an orthonormal basis $\{e_i(s)\}_{i=1}^{4m}$ for $T_{\gamma_0(s)}M$ with $e_1(s) = \gamma'_0(s)$ for each $s \in [-s_0, s_0]$. Notice that the vector fields $I\gamma'_0(t)$, $J\gamma'_0(t)$ and $K\gamma'_0(t)$ may not be parallel along γ_0 .

The first variation consideration gives

$$(3.6) \quad Q[u](y_0, t_0) - Q[u](x_0, t_0) - 2\varphi_t \geq 0.$$

and

$$(3.7) \quad \nabla u(x_0, t_0) = -\varphi' e_1(-s_0), \text{ and } \nabla u(y_0, t_0) = \varphi' e_1(s_0).$$

The variation $\gamma(r, s) = \gamma_0\left(s + r\frac{2s-1}{L}\right)$ yields

$$(3.8) \quad u_{11}(y_0, t_0) - u_{11}(x_0, t_0) - 2\varphi'' \leq 0.$$

For $i = 2, 3, 4$, the variations $\gamma(r, s) = \exp_{\gamma_0(s)}(r\eta(s)e_i(s))$ produce

$$(3.9) \quad u_{ii}(y_0, t_0) - u_{ii}(x_0, t_0) - \varphi' \int_{-s_0}^{s_0} ((\eta')^2 - \eta^2 R(e_1, e_i, e_1, e_i)) ds \leq 0,$$

and for $5 \leq j \leq 4m$, the variations $\gamma(r, s) = \exp_{\gamma_0(s)}(r\zeta(s)e_j(s))$ yield

$$(3.10) \quad u_{jj}(y_0, t_0) - u_{jj}(x_0, t_0) - \varphi' \int_{-s_0}^{s_0} ((\zeta')^2 - \zeta^2 R(e_1, e_j, e_1, e_j)) ds \leq 0.$$

We derive, by choosing $\eta(s) = \frac{c_{4\kappa}(s)}{c_{4\kappa}(s_0)}$ and summing (3.9) for $i = 2, 3, 4$, that

$$\begin{aligned}
& \sum_{i=2}^4 (u_{ii}(y) - u_{ii}(x)) \\
& \leq \varphi' \int_{-s_0}^{s_0} \left(3(\eta')^2 - \eta^2 \left(\sum_{i=2}^4 R(e_1, e_i, e_1, e_i) \right) \right) ds \\
& \leq 3\varphi' \eta'|_{-s_0}^{s_0} - \int_{-s_0}^{s_0} \eta^2 \left(\sum_{i=2}^4 R(e_1, e_i, e_1, e_i) - 12\kappa \right) ds \\
(3.11) \quad & = -6T_{4\kappa}\varphi'
\end{aligned}$$

where we have used $\sum_{i=2}^4 R(e_1, e_i, e_1, e_i) \geq 12\kappa$ by in Proposition 2.1.

Note that

$$\sum_{i=5}^{4m} R(e_1, e_i, e_1, e_i) = \text{Ric}(e_1, e_1) - \sum_{i=2}^4 R(e_1, e_i, e_1, e_i) \geq 4(m-1)\kappa.$$

Choosing $\zeta(s) = \frac{c_\kappa(s)}{c_\kappa(s_0)}$ and summing (3.10) for $5 \leq j \leq 4m$ yield

$$\begin{aligned}
& \sum_{j=5}^{4m} (u_{jj}(y) - u_{jj}(x)) \\
& \leq \varphi' \int_{-s_0}^{s_0} \left(4(m-1)(\eta')^2 - \zeta^2 \left(\sum_{j=5}^{4m} R(e_1, e_j, e_1, e_j) \right) \right) ds \\
& \leq 4(m-1)\varphi' \zeta\zeta'|_{-s_0}^{s_0} - \int_{-s_0}^{s_0} \zeta^2 \left(\sum_{j=5}^{4m} R(e_1, e_j, e_1, e_j) - 4(m-1)\kappa \right) ds \\
& = -8T_{4\kappa}\varphi'.
\end{aligned}$$

Combining (3.11) and (3.12) together, we have

$$\begin{aligned}
& Q[u](y_0, t_0) - Q[u](x_0, t_0) \\
& = \alpha(\varphi') (u_{11}(y_0, t_0) - u_{11}(x_0, t_0)) + \beta(\varphi') \sum_{i=2}^{4m} (u_{ii}(y_0, t_0) - u_{ii}(x_0, t_0)) \\
& \leq 2\alpha(\varphi')\varphi'' - 2\beta(\varphi')\varphi' (4(m-1)T_\kappa + 3T_{4\kappa}) = 2\mathcal{F}\varphi.
\end{aligned}$$

(3.12)

It follows from (3.6) and (3.12) that

$$\varphi_t \leq \mathcal{F}\varphi.$$

This completes the proof.

In case M has a strictly convex boundary, the same argument as in [LW21a] rule out the possibility that either $x_0 \in \partial M$ or $y_0 \in \partial M$. So the above argument remains valid. \square

The following corollary is immediate.

Corollary 3.2. *Let M and u be the same as in Theorem 3.1. Suppose $\varphi : [0, D/2] \times [0, T] \rightarrow \mathbb{R}$ satisfies*

- (1) $\varphi_t \geq \mathcal{F}\varphi$;
- (2) $\varphi' \geq 0$ on $[0, D/2] \times [0, T]$;
- (3) $|u(y, 0) - u(x, 0)| \leq 2\varphi\left(\frac{d(x, y)}{2}, 0\right)$

Then

$$|u(y, t) - u(x, t)| \leq 2\varphi\left(\frac{d(x, y)}{2}, t\right)$$

for all $x, y \in M$ and $t \in [0, T]$.

As in the Riemannian or Kähler case, the modulus of continuity estimate implies lower bounds for the first nonzero eigenvalue of the Laplacian on a quaternion-Kähler manifold. We restate Theorem 1.3 here.

Theorem 3.3. *Let (M^m, g, I, J, K) be a compact quaternion-Kähler manifold (possibly with a smooth strictly convex boundary) of quaternionic dimension m and diameter D . Suppose that the scalar curvature of M is bounded from below by $16m(m+2)\kappa$ for some $\kappa \in \mathbb{R}$. Let μ_1 be the first nonzero eigenvalue of the Laplacian on M (with Neumann boundary condition if $\partial M \neq \emptyset$). Then*

$$\mu_1 \geq \bar{\mu}_1(m, \kappa, D)$$

where $\bar{\mu}_1(m, \kappa, D)$ is the first nonzero Neumann eigenvalue of the one-dimensional eigenvalue problem

$$\varphi'' - (4(m-1)T_\kappa + 3T_{4\kappa})\varphi' = -\lambda\varphi$$

on $[-D/2, D/2]$, and T_κ is defined in (1.1).

Proof of Theorem 3.3. The proof is a slight modification of the proof in the Kähler case in [LW21a], so we leave the details to interested reader. \square

4. EXPLICIT LOWER BOUNDS

Let a, b be positive integers and κ_1, κ_2 be nonnegative real numbers. Consider the one-dimensional eigenvalue problem

$$(4.1) \quad \varphi'' - [2a(m-1)\sqrt{\kappa_2} \tan(\sqrt{\kappa_2}t) + 2b\sqrt{\kappa_1} \tan(2\sqrt{\kappa_1})] \varphi' + \lambda\varphi = 0$$

with Neumann boundary condition $\varphi'(-D/2) = \varphi'(D/2) = 0$.

Proposition 4.1. *Let λ_1 be the first nonzero Neumann eigenvalue of (4.1). Then*

$$\lambda_1 \geq \sup_{s \in (0,1)} \left(4s(1-s) \frac{\pi^2}{D^2} + s(2a(m-1)\kappa_2 + 4b\kappa_1) \right).$$

Proof. Let φ be the eigenfunction of (4.1) associated to the first nonzero Neumann eigenvalue λ_1 . It's easy to observe that the function $y = \varphi'$ satisfies the ODE

$$\begin{aligned} & y'' - [2a(m-1)\sqrt{\kappa_2} \tan(\sqrt{\kappa_2}t) + 2b\sqrt{\kappa_1} \tan(2\sqrt{\kappa_1})] y' \\ & - [2a(m-1)\kappa_1 \sec^2(\sqrt{\kappa_2}t) + 4b\kappa_1 \sec^2(2\sqrt{\kappa_1})] y + \lambda y = 0. \end{aligned}$$

with Dirichlet boundary condition $y(-D/2) = y(D/2) = 0$. Multiplying by $y^{\gamma-1}$ for some $\gamma > 1$ and integrating both sides, we have

$$\begin{aligned}
(4.2) \quad & \int_{-\frac{D}{2}}^{\frac{D}{2}} y'' y^{\gamma-1} dt + \lambda \int_{-\frac{D}{2}}^{\frac{D}{2}} y^\gamma dt \\
&= \int_{-\frac{D}{2}}^{\frac{D}{2}} \left(2a(m-1)\sqrt{\kappa_2} \tan(\sqrt{\kappa_2}t) + 2b\sqrt{\kappa_1} \tan(2\sqrt{\kappa_1}t) \right) y' y^{\gamma-1} dt \\
&\quad + \int_{-\frac{D}{2}}^{\frac{D}{2}} \left(2a(m-1)\kappa_2 \sec^2(\sqrt{\kappa_2}t) + 4b\kappa_1 \sec^2(2\sqrt{\kappa_1}t) \right) y^\gamma dt
\end{aligned}$$

By integration by parts, we get

$$\begin{aligned}
(4.3) \quad & \sqrt{\kappa_2} \int_{-\frac{D}{2}}^{\frac{D}{2}} \tan(\sqrt{\kappa_2}t) y^{\gamma-1} y' dt \\
&= \frac{\sqrt{\kappa_2}}{\gamma} \int_{-\frac{D}{2}}^{\frac{D}{2}} \tan(\sqrt{\kappa_2}t) (y^\gamma)' dt \\
&= -\frac{\kappa_2}{\gamma} \int_{-\frac{D}{2}}^{\frac{D}{2}} \sec^2(\sqrt{\kappa_2}t) y^\gamma dt.
\end{aligned}$$

Similarly, we have that

$$(4.4) \quad 2\sqrt{\kappa_1} \int_{-\frac{D}{2}}^{\frac{D}{2}} \tan(2\sqrt{\kappa_1}t) y' y^{\gamma-1} dt = -\frac{4\kappa_1}{\gamma} \int_{-\frac{D}{2}}^{\frac{D}{2}} \sec^2(2\sqrt{\kappa_1}t) y^\gamma dt$$

On the other hand, integration by parts implies

$$\int_{-\frac{D}{2}}^{\frac{D}{2}} y^{\gamma-1} y'' dt = -(\gamma-1) \int_{-\frac{D}{2}}^{\frac{D}{2}} y^{\gamma-2} y' y' dt = -\frac{4(\gamma-1)}{\gamma^2} \int_{-\frac{D}{2}}^{\frac{D}{2}} \left((y^{\frac{\gamma}{2}})' \right)^2 dt.$$

By Wirtinger's inequality, we get that

$$(4.5) \quad \int_{-\frac{D}{2}}^{\frac{D}{2}} y^{\gamma-1} y'' dt \leq -\frac{4(\gamma-1)}{\gamma^2} \frac{\pi^2}{D^2} \int_{-\frac{D}{2}}^{\frac{D}{2}} y^\gamma dt$$

Plugging (4.3), (4.4), and (4.5) into (4.2), we obtain

$$\begin{aligned}
& -\frac{4(\gamma-1)}{\gamma^2} \frac{\pi^2}{D^2} \int_{-\frac{D}{2}}^{\frac{D}{2}} y^\gamma dt + \lambda \int_{-\frac{D}{2}}^{\frac{D}{2}} y^\gamma dt \\
&\geq \left(1 - \frac{1}{\gamma} \right) \int_{-\frac{D}{2}}^{\frac{D}{2}} \left(2a(m-1)\kappa_2 \sec^2(\sqrt{\kappa_2}t) + 4b\kappa_1 \sec^2(2\sqrt{\kappa_1}t) \right) y^\gamma dt \\
&\geq \left(1 - \frac{1}{\gamma} \right) (2a(m-1)\kappa_2 + 4b\kappa_1) \int_{-\frac{D}{2}}^{\frac{D}{2}} y^\gamma dt
\end{aligned}$$

where we have used $\sec^2(x) \geq 1$ for all x in the last line. It then follows that we must have

$$\lambda \geq \frac{4(\gamma-1)}{\gamma^2} \frac{\pi^2}{D^2} + \left(1 - \frac{1}{\gamma} \right) (2a(m-1)\kappa_2 + 4b\kappa_1)$$

Letting $s = 1 - \frac{1}{\gamma}$, we have

$$\lambda \geq 4s(1-s) \frac{\pi^2}{D^2} + s(2a(m-1)\kappa_2 + 4b\kappa_1).$$

Since $\gamma > 1$ is arbitrary, we get

$$\lambda \geq \sup_{s \in (0,1)} \left(4s(1-s) \frac{\pi^2}{D^2} + s(2a(m-1)\kappa_2 + 4b\kappa_1) \right).$$

□

Proof of Proposition 1.1 and 1.2. Proposition 1.1 is a special case of Proposition 4.1 with $a = b = 1$ and Proposition 1.2 is a special case of Proposition 4.1 with $a = 2$ and $b = 3$. □

5. FIRST DIRICHLET EIGENVALUE

Throughout this section, $d(x, \partial M)$ denotes the distance function to ∂M given by

$$d(x, \partial M) = \inf\{d(x, y) : y \in \partial M\}$$

and R denotes the inradius of M given by

$$R = \sup\{d(x, \partial M) : x \in M\}.$$

For convenience, denote by $C_{\kappa, \Lambda}(t)$ the unique solution of the initial value problem

$$(5.1) \quad \begin{cases} \phi'' + \kappa\phi = 0, \\ \phi(0) = 1, \phi'(0) = -\Lambda, \end{cases}$$

and define $T_{\kappa, \Lambda}$ for $\kappa, \Lambda \in \mathbb{R}$ by

$$(5.2) \quad T_{\kappa, \Lambda}(t) := -\frac{C'_{\kappa, \Lambda}(t)}{C_{\kappa, \Lambda}(t)}.$$

In the Riemannian setting, Li and Yau [LY80] and Kause [Kas84] proved the following well-known result.

Theorem 5.1. *Let (M^n, g) be a compact Riemannian manifold with smooth boundary $\partial M \neq \emptyset$. Suppose that the Ricci curvature of M is bounded from below by $(n-1)\kappa$ and the mean curvature of ∂M is bounded from below by $(n-1)\Lambda$ for some $\kappa, \Lambda \in \mathbb{R}$. Let λ_1 be the first Dirichlet eigenvalue of the Laplacian on M . Then*

$$\lambda_1 \geq \bar{\lambda}_1(n, \kappa, \Lambda, R),$$

where $\bar{\lambda}_1(n, \kappa, \Lambda, R)$ is the first eigenvalue of the one-dimensional eigenvalue problem

$$(5.3) \quad \begin{cases} \varphi'' - (n-1)T_{\kappa, \Lambda}\varphi' = -\lambda\varphi, \\ \varphi(0) = 0, \varphi'(R) = 0. \end{cases}$$

In [LW21a], the authors of the present paper obtained an analogous theorem in the Kähler setting.

Theorem 5.2. *Let (M^m, g, J) be a compact Kähler manifold with smooth nonempty boundary ∂M . Suppose that the holomorphic sectional curvature is bounded from below by $4\kappa_1$ and the orthogonal Ricci curvature is bounded from below by $2(m-1)\kappa_2$ for some $\kappa_1, \kappa_2 \in \mathbb{R}$, and the second fundamental form on ∂M is bounded from below by $\Lambda \in \mathbb{R}$. Let λ_1 be the first Dirichlet eigenvalue of the Laplacian on M . Then*

$$\lambda_1 \geq \bar{\lambda}_1(m, \kappa_1, \kappa_2, \Lambda, R)$$

where $\bar{\lambda}_1(m, \kappa_1, \kappa_2, \Lambda, R)$ is the first eigenvalue of the one-dimensional eigenvalue problem

$$(5.4) \quad \begin{cases} \varphi'' - (2(m-1)T_{\kappa_2, \Lambda} + T_{4\kappa_1, \Lambda})\varphi' = -\lambda\varphi, \\ \varphi(0) = 0, \varphi'(R) = 0. \end{cases}$$

Remark 5.3. *Diameter-independent lower bounds for λ_1 were obtained by Guedj, Kolev and Yeganefar [GKY13]. It has been generalized via a p -Reilly formula to the first Dirichlet eigenvalue of the p -Laplacian by Blacker and Seto [BS19] when $p \geq 2$.*

Here we prove the following quaternion-Kähler version.

Theorem 5.4. *Let (M^m, g, I, J, K) be a compact quaternion-Kähler manifold with smooth nonempty boundary ∂M . Suppose that the scalar curvature is bounded from below by $16m(m+2)\kappa$ for some $\kappa \in \mathbb{R}$, and the second fundamental form on ∂M is bounded from below by $\Lambda \in \mathbb{R}$. Let λ_1 be the first Dirichlet eigenvalue of the Laplacian on M . Then*

$$\lambda_1 \geq \bar{\lambda}_1(m, \kappa, \Lambda, R)$$

where $\bar{\lambda}_1(m, \kappa, \Lambda, R)$ is the first eigenvalue of the one-dimensional eigenvalue problem

$$(5.5) \quad \begin{cases} \varphi'' - (4(m-1)T_{\kappa, \Lambda} + 3T_{4\kappa, \Lambda})\varphi' = -\lambda\varphi, \\ \varphi(0) = 0, \varphi'(R) = 0. \end{cases}$$

Remark 5.5. *When the boundary is convex, namely $\Lambda = 0$, it is easily seen that we have*

$$\bar{\lambda}_1(m, \kappa, 0, R) = \bar{\mu}_1(m, \kappa, R).$$

Thus we can obtain the same explicit lower bounds as in Proposition 1.2 for $\bar{\lambda}_1(m, \kappa, 0, R)$.

Remark 5.6. *With the help of a generalized Barta's inequality for the p -Laplacian (see [LW20, Theorem 3.1]), the same argument here indeed yields such lower bounds for the first Dirichlet eigenvalue of the p -Laplacian for all $1 < p < \infty$.*

The proof of Theorem 5.4 relies on a comparison theorem for the second derivatives of $d(x, \partial M)$.

Theorem 5.7. *Let (M^m, g, I, J, K) be a compact quaternion-Kähler manifold with smooth nonempty boundary ∂M . Suppose that the scalar curvature of M is bounded from below by $16m(m+2)\kappa$ for some $\kappa \in \mathbb{R}$, and the second fundamental form on ∂M is bounded from below by $\Lambda \in \mathbb{R}$. Let $\varphi : [0, R] \rightarrow \mathbb{R}_+$ be a smooth function with $\varphi' \geq 0$. Then the function $v(x) = \varphi(d(x, \partial M))$ is a viscosity supersolution of*

$$Q[v] = [\alpha(\varphi')\varphi'' - \beta(\varphi')\varphi'(4(m-1)T_{\kappa, \Lambda} + 3T_{4\kappa, \Lambda})]_{|_{d(x, \partial M)}},$$

on M , where Q is the operator defined in (3.1).

Proof of Theorem 5.7. By approximation, it suffices to consider the case $\varphi' > 0$ on $[0, R]$. By definition of viscosity solutions (see [CIL92]), it suffices to prove that for any smooth function ψ touching v from below at $x_0 \in M$, i.e.,

$$\psi(x) \leq v(x) \text{ on } M, \quad \psi(x_0) = v(x_0),$$

it holds that

$$Q[\psi](x_0) \leq [\alpha(\varphi')\varphi'' - \beta(\varphi')\varphi' (4(m-1)T_{\kappa,\Lambda} + 3T_{4\kappa,\Lambda})]_{d(x_0, \partial M)}.$$

Since the function $d(x, \partial M)$ may not be smooth at x_0 , so we need to replace it by a smooth function $\bar{d}(x)$ defined in a neighborhood $U(x_0)$ of x_0 satisfying $\bar{d}(x) \geq d(x, \partial M)$ for $x \in U(x_0)$ and $\bar{d}(x_0) = d(x_0, \partial M)$. The construction is standard (see e.g. [Wu79, pp. 73-74] or [AX19, pp. 1187]), which we state below for reader's convenience.

Since M is compact, there exists $y_0 \in \partial M$ such that

$$d(x_0, y_0) = d(x_0, \partial M) := s_0.$$

Let $\gamma : [0, s_0] \rightarrow M$ be the unit speed length-minimizing geodesic with $\gamma(0) = x_0$ and $\gamma(s_0) = y_0$. Choose an orthonormal basis $\{e_i\}_{i=1}^{4m}$ for $T_{x_0}M$ with

$$e_1 = \gamma'_0(0), e_2 = I\gamma'_0(0), e_3 = J\gamma'_0(0), e_4 = K\gamma'_0(0),$$

and parallel transport it along γ_0 to produce an orthonormal basis $\{e_i(s)\}_{i=1}^{4m}$ for $T_{\gamma_0(s)}M$ with $e_1(s) = \gamma'_0(s)$ for each $s \in [0, s_0]$.

For any vector $X \in \exp_{x_0}^{-1}U(x_0)$, let $X(s), s \in [0, s_0]$ be the vector field obtained by parallel translating X along γ , and decompose it as

$$X(s) = aX_1(s) + b\gamma'(s) + cX_2(s),$$

where a, b and c are constants along γ with $a^2 + b^2 + c^2 = |X|^2$, and $X_1(s)$ and $X_2(s)$ are unit components in $\text{span}\{e_5(s), \dots, e_{4m}(s)\}$ and $\text{span}\{e_2(s), e_3(s), e_4(s)\}$. Define

$$W(s) = a\eta(s)X_1(s) + b\left(1 - \frac{s}{s_0}\right)\gamma'(s) + c\zeta(s)X_2(s),$$

where $\eta, \zeta : [0, s_0] \rightarrow \mathbb{R}_+$ are two C^2 functions to be chosen later. Next we define the $4m$ -parameter family of curves $\gamma_X : [0, s_0] \rightarrow M$ such that

- (1) $\gamma_0 = \gamma$;
- (2) $\gamma_X(0) = \exp_{x_0}(W(0))$ and $\gamma_X(s_0) \in \partial M$;
- (3) $W(s)$ is induced by the one-parameter family of curves $\varepsilon \rightarrow \gamma_{\varepsilon X}(s)$ for $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ and $s \in [0, s_0]$;
- (4) γ_X depends smoothly on X .

Finally let $\bar{d}(x)$ be the length of the curve γ_X where $x = \exp_{x_0}(X) \in U(x_0)$. Then we have $\bar{d}(x) \geq d(x, \partial M)$ on $U(x_0)$, $\bar{d}(x_0) = d(x_0, \partial M)$.

Recall the first and second variation formulas:

$$(5.6) \quad \nabla \bar{d} = -e_1(0),$$

and

$$\begin{aligned} & \nabla^2 \bar{d}(X, X) \\ &= -A\left(a\eta(s_0)X_1(s_0) + c\zeta(s_0)X_2(s_0), a\eta(s_0)X_1(s_0) + c\zeta(s_0)X_2(s_0)\right) \\ & \quad + \int_0^{s_0} (a^2(\eta')^2 + c^2(\zeta')^2 - R(a\eta X_1 + c\zeta X_2, \gamma', a\eta X_1 + c\zeta X_2, \gamma')) ds \end{aligned}$$

where A denotes the second fundamental form of ∂M at y_0 . Here and below the derivatives of \bar{d} are all evaluated at x_0 . Then we have

$$(5.7) \quad \nabla^2 \bar{d}(e_1(0), e_1(0)) = 0.$$

For $2 \leq i \leq 4$, we obtain by choosing $\zeta(s) = C_{4\kappa, \Lambda}(s_0 - s)/C_{4\kappa, \Lambda}(s_0)$ that

$$\begin{aligned} (5.8) \quad & \nabla^2 \bar{d}(e_i(0), e_i(0)) \\ &= -\zeta^2(s_0)A(e_i(s_0), e_i(s_0)) + \int_0^{s_0} (\zeta')^2 - \zeta^2 R(e_i, \gamma', e_i, \gamma') ds \\ &\leq -\frac{\Lambda}{C_{4\kappa, \Lambda}^2(s_0)} + \int_0^{s_0} (\zeta')^2 - \zeta^2 R(e_i, \gamma', e_i, \gamma') ds \end{aligned}$$

Summing over $2 \leq i \leq 4$ gives

$$\begin{aligned} (5.9) \quad & \sum_{i=2}^4 \nabla^2 \bar{d}(e_i(0), e_i(0)) \\ &\leq -\frac{3\Lambda}{C_{4\kappa, \Lambda}^2(s_0)} + \int_0^{s_0} 3(\zeta')^2 - \zeta^2 \sum_{i=2}^4 R(e_i, \gamma', e_i, \gamma') ds \\ &\leq -\frac{3\Lambda}{C_{4\kappa, \Lambda}^2(s_0)} + \int_0^{s_0} 3(\zeta')^2 - 12\kappa\zeta^2 ds \\ &= -3T_{4\kappa, \Lambda}(s_0). \end{aligned}$$

For $5 \leq i \leq 4m$, we have

$$\nabla^2 \bar{d}(e_i(0), e_i(0)) = -\eta^2(s_0)A(e_i(s_0), e_i(s_0)) + \int_0^{s_0} (\eta')^2 - \eta^2 R(e_i, \gamma', e_i, \gamma') ds.$$

Summing over $5 \leq i \leq 4m$ and choosing $\eta(s) = C_{\kappa, \Lambda}(s_0 - s)/C_{\kappa, \Lambda}(s_0)$ gives

$$\begin{aligned} (5.10) \quad & \sum_{i=5}^{4m} \nabla^2 \bar{d}(e_i(0), e_i(0)) \\ &= -\frac{\sum_{i=5}^{4m} A(e_i(s_0), e_i(s_0))}{C_{\kappa, \Lambda}^2(s_0)} + \int_0^{s_0} 4(m-1)(\eta')^2 - \eta^2 \sum_{i=5}^{4m} R(e_i, \gamma', e_i, \gamma') ds \\ &\leq -\frac{4(m-1)\Lambda}{C_{\kappa, \Lambda}^2(s_0)} + 4(m-1) \int_0^{s_0} (\eta')^2 - \kappa\eta^2 ds \\ &= -4(m-1)T_{\kappa, \Lambda}(s_0). \end{aligned}$$

Since the function $\psi(x) - \varphi(d(x, \partial M))$ attains its maximum at x_0 and $\varphi' > 0$, it follows that the function $\psi(x) - \varphi(\bar{d}(x))$ attains a local maximum at x_0 . The first and second derivative tests yield

$$\nabla \psi(x_0) = -\varphi' e_1(0), \quad \psi_{11}(x_0) \leq \varphi'',$$

and

$$\psi_{ii}(x_0) \leq \varphi' \nabla^2 \bar{d}(e_i(0), e_i(0))$$

for $2 \leq i \leq 4m$, where we used (5.6) and (5.7). Here and below the derivatives of φ are all evaluated at $s_0 = d(x_0, \partial M)$. Thus we have

$$\begin{aligned} (5.11) \quad Q[\psi](x_0) &= \alpha(\varphi')\psi_{11} + \beta(\varphi') \sum_{i=2}^{4m} \psi_{ii} \\ &\leq \alpha(\varphi')\varphi'' + \beta(\varphi')\varphi' \left(\sum_{i=2}^{4m} \nabla^2 \bar{d}(e_i(0), e_i(0)) \right) \\ &\leq \alpha(\varphi')\varphi'' - \beta(\varphi')\varphi' (4(m-1)T_{\kappa, \Lambda} + 3T_{4\kappa, \Lambda}). \end{aligned}$$

The proof is complete. \square

By choose $\varphi(s) = s$, we have

Corollary 5.8. *Let (M^m, g, I, J, K) be a compact quaternion-Kähler manifold with smooth nonempty boundary ∂M . Suppose that the scalar curvature of M is bounded from below by $16m(m+2)\kappa$ for some $\kappa \in \mathbb{R}$, and the second fundamental form on ∂M is bounded from below by $\Lambda \in \mathbb{R}$. Then*

$$\Delta d(x, \partial M) \leq -4(m-1)T_{\kappa, \Lambda}(d(x, \partial M)) - 3T_{4\kappa, \Lambda}(d(x, \partial M))$$

in the viscosity sense.

Proof of Theorem 5.4. The proof is a slight modification of the Kähler case presented in [LW21a, Section 7], so we omit the details. \square

REFERENCES

- [AC09a] Ben Andrews and Julie Clutterbuck. Lipschitz bounds for solutions of quasilinear parabolic equations in one space variable. *J. Differential Equations*, 246(11):4268–4283, 2009.
- [AC09b] Ben Andrews and Julie Clutterbuck. Time-interior gradient estimates for quasilinear parabolic equations. *Indiana Univ. Math. J.*, 58(1):351–380, 2009.
- [AC11] Ben Andrews and Julie Clutterbuck. Proof of the fundamental gap conjecture. *J. Amer. Math. Soc.*, 24(3):899–916, 2011.
- [AC13] Ben Andrews and Julie Clutterbuck. Sharp modulus of continuity for parabolic equations on manifolds and lower bounds for the first eigenvalue. *Anal. PDE*, 6(5):1013–1024, 2013.
- [AM95] Dmitry V. Alekseevsky and Stefano Marchiafava. Transformations of a quaternionic Kähler manifold. *C. R. Acad. Sci. Paris Sér. I Math.*, 320(6):703–708, 1995.
- [AN12] Ben Andrews and Lei Ni. Eigenvalue comparison on Bakry-Emery manifolds. *Comm. Partial Differential Equations*, 37(11):2081–2092, 2012.
- [And15] Ben Andrews. Moduli of continuity, isoperimetric profiles, and multi-point estimates in geometric heat equations. In *Surveys in differential geometry 2014. Regularity and evolution of nonlinear equations*, volume 19 of *Surv. Differ. Geom.*, pages 1–47. Int. Press, Somerville, MA, 2015.
- [AX19] Ben Andrews and Changwei Xiong. Gradient estimates via two-point functions for elliptic equations on manifolds. *Adv. Math.*, 349:1151–1197, 2019.
- [Bal06] Werner Ballmann. *Lectures on Kähler manifolds*. ESI Lectures in Mathematics and Physics. European Mathematical Society (EMS), Zürich, 2006.
- [Ber66] Marcel Berger. Remarques sur les groupes d’holonomie des variétés riemanniennes. *C. R. Acad. Sci. Paris Sér. A-B*, 262:A1316–A1318, 1966.
- [Bes08] Arthur L. Besse. *Einstein manifolds*. Classics in Mathematics. Springer-Verlag, Berlin, 2008. Reprint of the 1987 edition.
- [BQ00] Dominique Bakry and Zhongmin Qian. Some new results on eigenvectors via dimension, diameter, and Ricci curvature. *Adv. Math.*, 155(1):98–153, 2000.

- [BS19] Casey Blacker and Shoo Seto. First eigenvalue of the p -Laplacian on Kähler manifolds. *Proc. Amer. Math. Soc.*, 147(5):2197–2206, 2019.
- [Cha84] Isaac Chavel. *Eigenvalues in Riemannian geometry*, volume 115 of *Pure and Applied Mathematics*. Academic Press, Inc., Orlando, FL, 1984. Including a chapter by Burton Randol, With an appendix by Jozef Dodziuk.
- [CIL92] Michael G. Crandall, Hitoshi Ishii, and Pierre-Louis Lions. User’s guide to viscosity solutions of second order partial differential equations. *Bull. Amer. Math. Soc. (N.S.)*, 27(1):1–67, 1992.
- [CW94] Mu Fa Chen and Feng Yu Wang. Application of coupling method to the first eigenvalue on manifold. *Sci. China Ser. A*, 37(1):1–14, 1994.
- [DSW18] Xianzhe Dai, Shoo Seto, and Guofang Wei. Fundamental gap estimate for convex domains on sphere – the case $n = 2$. *Comm. Anal. Geom.*, to appear, *arXiv:1803.01115*, 2018.
- [GKY13] Vincent Guedj, Boris Kolev, and Nader Yeganefar. A Lichnerowicz estimate for the first eigenvalue of convex domains in Kähler manifolds. *Anal. PDE*, 6(5):1001–1012, 2013.
- [HW07] Fengbo Hang and Xiaodong Wang. A remark on Zhong-Yang’s eigenvalue estimate. *Int. Math. Res. Not. IMRN*, (18):Art. ID rnm064, 9, 2007.
- [HWZ20] Chenxu He, Guofang Wei, and Qi S. Zhang. Fundamental gap of convex domains in the spheres. *Amer. J. Math.*, 142(4):1161–1192, 2020.
- [Ish74] Shigeru Ishihara. Quaternion Kählerian manifolds. *J. Differential Geometry*, 9:483–500, 1974.
- [Kas84] Atsushi Kasue. On a lower bound for the first eigenvalue of the Laplace operator on a Riemannian manifold. *Ann. Sci. École Norm. Sup. (4)*, 17(1):31–44, 1984.
- [KLZ08] Shengli Kong, Peter Li, and Detang Zhou. Spectrum of the Laplacian on quaternionic Kähler manifolds. *J. Differential Geom.*, 78(2):295–332, 2008.
- [Krö92] Pawel Kröger. On the spectral gap for compact manifolds. *J. Differential Geom.*, 36(2):315–330, 1992.
- [Li79] Peter Li. A lower bound for the first eigenvalue of the Laplacian on a compact manifold. *Indiana Univ. Math. J.*, 28(6):1013–1019, 1979.
- [Li16] Xiaolong Li. Moduli of continuity for viscosity solutions. *Proc. Amer. Math. Soc.*, 144(4):1717–1724, 2016.
- [Li20] Xiaolong Li. Modulus of continuity estimates for fully nonlinear parabolic equations. *arXiv:2006.16631*, 2020. .
- [Lic58] André Lichnerowicz. *Géométrie des groupes de transformations*. Travaux et Recherches Mathématiques, III. Dunod, Paris, 1958.
- [Liu14] Gang Liu. Kähler manifolds with Ricci curvature lower bound. *Asian J. Math.*, 18(1):69–99, 2014.
- [LL10] Jun Ling and Zhiqin Lu. Bounds of eigenvalues on Riemannian manifolds. In *Trends in partial differential equations*, volume 10 of *Adv. Lect. Math. (ALM)*, pages 241–264. Int. Press, Somerville, MA, 2010.
- [LTW20] Xiaolong Li, Yucheng Tu, and Kui Wang. On a class of quasi-linear operators on smooth metric measure spaces. *arXiv:2009.10418*, 2020.
- [LW05] Peter Li and Jiaping Wang. Comparison theorem for Kähler manifolds and positivity of spectrum. *J. Differential Geom.*, 69(1):43–74, 2005.
- [LW17] Xiaolong Li and Kui Wang. Moduli of continuity for viscosity solutions on manifolds. *J. Geom. Anal.*, 27(1):557–576, 2017.
- [LW19a] Xiaolong Li and Kui Wang. Sharp lower bound for the first eigenvalue of the weighted p -Laplacian I. *J. Geom. Anal.*, to appear *arxiv:1910.02295*, 2019.
- [LW19b] Xiaolong Li and Kui Wang. Sharp lower bound for the first eigenvalue of the weighted p -Laplacian II. *Math. Res. Lett.*, to appear, *arXiv:1911.04596*, 2019.
- [LW20] Xiaolong Li and Kui Wang. First Robin eigenvalue of the p -Laplacian on Riemannian manifolds. *Math. Z.*, to appear, *arXiv:2002.06472*, 2020.
- [LW21a] Xiaolong Li and Kui Wang. Lower bounds for the first eigenvalue of the Laplacian on Kähler manifolds. *Trans. Amer. Math. Soc.*, to appear, *arXiv:2010.12792*, 2021.
- [LW21b] Xiaolong Li and Kui Wang. Sharp lower bound for the first eigenvalue of the weighted p -Laplacian I. *J. Geom. Anal.*, to appear, *arxiv:1910.02295*, 2021.
- [LY80] Peter Li and Shing Tung Yau. Estimates of eigenvalues of a compact Riemannian manifold. In *Geometry of the Laplace operator (Proc. Sympos. Pure Math., Univ. Hawaii, Honolulu, Hawaii, 1979)*, Proc. Sympos. Pure Math., XXXVI, pages 205–239. Amer. Math. Soc., Providence, R.I., 1980.

- [Ni13] Lei Ni. Estimates on the modulus of expansion for vector fields solving nonlinear equations. *J. Math. Pures Appl. (9)*, 99(1):1–16, 2013.
- [NWZ18] Lei Ni, Qingsong Wang, and Fangyang Zheng. Manifolds with positive orthogonal ricci curvature. *Amer. J. Math.*, to appear, *arXiv:1806.10233v2*, 2018.
- [NZ18] Lei Ni and Fangyang Zheng. Comparison and vanishing theorems for Kähler manifolds. *Calc. Var. Partial Differential Equations*, 57(6):Art. 151, 31, 2018.
- [NZ19] Lei Ni and Fangyang Zheng. On orthogonal Ricci curvature. In *Advances in complex geometry*, volume 735 of *Contemp. Math.*, pages 203–215. Amer. Math. Soc., Providence, RI, 2019.
- [Oba62] Morio Obata. Certain conditions for a Riemannian manifold to be isometric with a sphere. *J. Math. Soc. Japan*, 14:333–340, 1962.
- [SWW19] Shoo Seto, Lili Wang, and Guofang Wei. Sharp fundamental gap estimate on convex domains of sphere. *J. Differential Geom.*, 112(2):347–389, 2019.
- [SY94] R. Schoen and S.-T. Yau. *Lectures on differential geometry*. Conference Proceedings and Lecture Notes in Geometry and Topology, I. International Press, Cambridge, MA, 1994.
- [SZ07] Yu Min Shi and Hui Chun Zhang. Lower bounds for the first eigenvalue on compact manifolds. *Chinese Ann. Math. Ser. A*, 28(6):863–866, 2007.
- [TY12] Luen-Fai Tam and Chengjie Yu. Some comparison theorems for Kähler manifolds. *Manuscripta Math.*, 137(3-4):483–495, 2012.
- [Ura87] Hajime Urakawa. Stability of harmonic maps and eigenvalues of the Laplacian. *Trans. Amer. Math. Soc.*, 301(2):557–589, 1987.
- [Wu79] H. Wu. An elementary method in the study of nonnegative curvature. *Acta Math.*, 142(1-2):57–78, 1979.
- [ZW17] Yuntao Zhang and Kui Wang. An alternative proof of lower bounds for the first eigenvalue on manifolds. *Math. Nachr.*, 290(16):2708–2713, 2017.
- [ZY84] Jia Qing Zhong and Hong Cang Yang. On the estimate of the first eigenvalue of a compact Riemannian manifold. *Sci. Sinica Ser. A*, 27(12):1265–1273, 1984.

DEPARTMENT OF MATHEMATICS AND STATISTICS, MCMASTER UNIVERSITY, HAMILTON, ONTARIO, L8S 4K1, CANADA

Email address: `li1304@mcmaster.ca`

SCHOOL OF MATHEMATICAL SCIENCES, SOOCHOW UNIVERSITY, SUZHOU, 215006, CHINA

Email address: `kuiwang@suda.edu.cn`