Springer

KSME

Journal of Mechanical Science and Technology 26 (1) (2012) 1~10
www.springerlink.com/content/1738-494x
DOI 10.1007/s12206-011-1008-5

Eigenvalue problems of rotor system with uncertain parameters †

Bao-Guo Liu*

Institute of Mechatronic Engineering, Henan University of Technology, Lianhua Street, Zhengzhou 450001, China

(Manuscript Received February 20, 2011; Revised July 27, 2011; Accepted September 18, 2011)

Abstract

A general method for investigating the eigenvalue problems of a rotor system with uncertain parameters is presented in this paper. The recurrence perturbation formulas based on the Riccati transfer matrix method are derived and used for calculating the first- and second-order perturbations of eigenvalues and their respective eigenvectors for the rotor system with uncertain parameters. In addition, these formulas can be used for investigating the independent, and repeated, as well as the complex eigenvalue problems. The general method is called the Riccati perturbation transfer matrix method (Riccati-PTMM). The formulas for calculating the mean value, variance, and co-variance of the eigenvalues and eigenvectors of the rotor system with random parameters are also given. Riccati-PTMM is used for calculating the random eigenvalues of a simply supported Timoshenko beam and a test rotor supported by two oil bearings. The results show that the method is accurate and efficient.

Keywords: Rotordynamics; Uncertain parameter; Eigenvalue problem; Perturbation; Riccati transfer matrix method

1. Introduction

Structural systems with uncertain parameters exist widely in engineering. Comprehensive reviews on the dynamic problems of these systems have been presented by Ibrahim [1], Oden et al. [2], and Sanchez [3]. The perturbation finite element method (PFEM) plays an important role in the investigation of the dynamic characteristics of these structural systems [2, 3]. The corresponding research subjects include the perturbation method of the independent eigenvalue and its eigenvector [4-8], the perturbation method of the repeated eigenvalue and its eigenvectors [9-14] and the calculation method of the random eigenvalue and its eigenvectors [15, 16], among others. However, since these methods are based on PFEM, the errors caused by truncating the high modals cannot be avoided [4-16]. On the other hand, similar to FEM, which is not the most efficient method for solving rotordynamic problems [17, 18], PFEM is also not the most efficient method for analyzing rotordynamic systems with uncertain parameters.

Rotordynamic systems with uncertain parameters also exist widely in engineering as a special kind of structure. In recent years, more and more scholars have focused on their dynamic problems [19-22]. However, the dynamic model that most researchers have studied is the Jeffcott rotor. The current study attempts to investigate a complicated rotordynamic model and present a general method to analyze its random eigenvalues and eigenvectors.

The transfer matrix methods (TMM) are very useful and efficient in undertaking the dynamic analysis of rotordynamic systems with deterministic parameters [17, 18, 23-26]. There are two kinds of TMMs: the Myklestad-Prohl transfer matrix method (MP-TMM) [17, 23] and the Riccati transfer matrix method (Riccati-TMM) [18]. MP-TMM has simple mathematical formulas and is easy to program. However, numerical difficulties can arise when higher frequencies are calculated and/or when there are too many degrees of freedom. Riccati-TMM, on the other hand, also has simple mathematical formulas and is easy to program. In addition, there are no numerical difficulties [18] involved in using this method, which is why it is more often used in engineering.

In this paper, the recurrence perturbation formulas based on the Riccati-TMM are derived to calculate the first- and second-order perturbation solutions of the eigenvalues and their eigenvectors for rotor systems with uncertain parameters. The perturbation frequency equations formulated to analyze the independent and repeated eigenvalues are also educed. Moreover, the methods used for evaluating the perturbation solutions of eigenvectors that correspond to the independent and repeated eigenvalues are also given. Thus, by not dealing with the problems of truncating modals in the process of deriving the formulas, the errors caused by truncating the high modals are avoided. Furthermore, this method can be applied to evaluate the perturbation solutions of real eigenvalues and their eigenvectors as well as evaluate the perturbation solu-

† This paper was recommended for publication in revised form by Associate Editor Ohseop Song

*Corresponding author. Tel.: +86 371 67756518, Fax.: +86 371 67756517
E-mail address: bguoliu@haut.edu.cn; bgliu1978@sina.com.cn

© KSME & Springer 2012

tions of complex eigenvalues and their eigenvectors. This perturbation method is called the Riccati perturbation transfer matrix method (Riccati-PTMM).

In the present study, based on the Riccati-PTMM, the formulas are given for calculating the mean value, variance, as well as covariance of the eigenvalues and eigenvectors for the rotor system with random parameters. The formulas are used for calculating random eigenvalues of a simply supported Timoshenko beam and a test rotor supported by two oil bearings. The results show that the method is accurate and efficient in comparison with the Monte Carlo simulated results.

2. Recurrence formulas of the Riccati perturbation transfer matrix method

To solve eigenvalue problems, the transfer and recurrence formulas of the Riccati-TMM are written as [18]:

$$\{\boldsymbol{f}\}_i = [\boldsymbol{S}]_i \{\boldsymbol{e}\}_i \tag{1}$$

$$[\boldsymbol{S}]_{i+1} = [\boldsymbol{u}_{11}\boldsymbol{S}+\boldsymbol{u}_{12}]_i [\boldsymbol{u}_{21}\boldsymbol{S}+\boldsymbol{u}_{22}]_i^{-1} = [\boldsymbol{S}_{u1}]_i [\boldsymbol{S}_{u2}]_i \tag{2}$$

$$\{\boldsymbol{e}\}_i = [\boldsymbol{u}_{21}\boldsymbol{S}+\boldsymbol{u}_{22}]_i^{-1} \{\boldsymbol{e}\}_{i+1} = [\boldsymbol{S}_{u2}]\{\boldsymbol{e}\}_{i+1} \tag{3}$$

where $\{\boldsymbol{f}\}_i$ and $\{\boldsymbol{e}\}_i$ are the state vectors with r elements at section i that satisfy the boundary conditions $\{\boldsymbol{f}\}_1 = \{\boldsymbol{0}\}$ and $\{\boldsymbol{e}\}_1 \neq \{\boldsymbol{0}\}$, respectively; $[\boldsymbol{u}_{11}]_i$, $[\boldsymbol{u}_{12}]_i$, $[\boldsymbol{u}_{21}]_i$, and $[\boldsymbol{u}_{22}]_i$ are the matrixes with $r\times r$ elements defined by the transfer relations of state vectors between sections i and i+1 on the element i; $[\boldsymbol{S}]_i$ is the so-called Riccati transfer matrix; $[\boldsymbol{S}_{u1}]_i = [\boldsymbol{u}_{11}\boldsymbol{S}+\boldsymbol{u}_{12}]_i$; and $[\boldsymbol{S}_{u2}]_i = [\boldsymbol{u}_{21}\boldsymbol{S}+\boldsymbol{u}_{22}]_i^{-1}$.

Suppose b_j (j = 1, ..., m) are the uncertain parameters of the rotor system, they are calculated using the following expression:

$$b_j = b_{j0}(1+\varepsilon_j) \quad (j=1,\cdots,m) \tag{4}$$

where b_{j0} is the initial value of the uncertain parameter b_j, and $|\varepsilon_j|<1$ is a small parameter.

Since the parameters are uncertain, the eigenvalue β of the system is also uncertain. Its perturbation expression with second-order accuracy is stated as:

$$\beta = \beta_{,0} + \sum_{j=1}^{m}\beta_{,j}\varepsilon_j + \sum_{j=1}^{m}\sum_{k=1}^{j}\beta_{,jk}\varepsilon_j\varepsilon_k \tag{5}$$

where β_0 is the initial eigenvalue of the system, while $b_j = b_{j0}$ (j = 1, ..., m); β_j and β_{jk} are the first- and second-order perturbations of β, respectively.

The perturbation expressions of all the other matrixes and vectors that are not listed here are similar to those of β.

By substituting the perturbation expressions into Eqs. (1)-(3) and allowing the coefficients of ε with the same power equal yields the following equations:

$$\{\boldsymbol{f}\}_{i,0} = [\boldsymbol{S}]_{i,0}\{\boldsymbol{e}\}_{i,0} \tag{6a}$$

$$\{\boldsymbol{f}\}_{i,j} = [\boldsymbol{S}]_{i,0}\{\boldsymbol{e}\}_{i,j} + [\boldsymbol{S}]_{i,j}\{\boldsymbol{e}\}_{i,0} \tag{6b}$$

$$\{\boldsymbol{f}\}_{i,jk} = [\boldsymbol{S}]_{i,0}\{\boldsymbol{e}\}_{i,jk} + [\boldsymbol{S}]_{i,j}\{\boldsymbol{e}\}_{i,k} + (1-\delta_{jk})[\boldsymbol{S}]_{i,k}\{\boldsymbol{e}\}_{i,j} + [\boldsymbol{S}]_{i,jk}\{\boldsymbol{e}\}_{i,0} \tag{6c}$$

$$[\boldsymbol{S}]_{i+1,0} = [\boldsymbol{S}_{u1}]_{i,0}[\boldsymbol{S}_{u2}]_{i,0} \tag{7a}$$

$$[\boldsymbol{S}]_{i+1,j} = [\boldsymbol{S}_{u1}]_{i,0}[\boldsymbol{S}_{u2}]_{i,j} + [\boldsymbol{S}_{u1}]_{i,j}[\boldsymbol{S}_{u2}]_{i,0} \tag{7b}$$

$$[\boldsymbol{S}]_{i+1,jk} = [\boldsymbol{S}_{u1}]_{i,0}[\boldsymbol{S}_{u2}]_{i,jk} + [\boldsymbol{S}_{u1}]_{i,j}[\boldsymbol{S}_{u2}]_{i,k} + (1-\delta_{jk})[\boldsymbol{S}_{u1}]_{i,k}[\boldsymbol{S}_{u2}]_{i,j} + [\boldsymbol{S}_{u1}]_{i,jk}[\boldsymbol{S}_{u2}]_{i,0} \tag{7c}$$

$$\{\boldsymbol{e}\}_{i,0} = [\boldsymbol{S}_{u2}]_{i,0}\{\boldsymbol{e}\}_{i+1,0} \tag{8a}$$

$$\{\boldsymbol{e}\}_{i,j} = [\boldsymbol{S}_{u2}]_{i,0}\{\boldsymbol{e}\}_{i+1,j} + [\boldsymbol{S}_{u2}]_{i,j}\{\boldsymbol{e}\}_{i+1,0} \tag{8b}$$

$$\{\boldsymbol{e}\}_{i,jk} = [\boldsymbol{S}_{u2}]_{i,0}\{\boldsymbol{e}\}_{i+1,jk} + [\boldsymbol{S}_{u2}]_{i,j}\{\boldsymbol{e}\}_{i+1,k} + (1-\delta_{jk})[\boldsymbol{S}_{u2}]_{i,k}\{\boldsymbol{e}\}_{i+1,j} + [\boldsymbol{S}_{u2}]_{i,jk}\{\boldsymbol{e}\}_{i+1,0} \tag{8c}$$

where δ_{jk} is Kronecker delta.

Eqs. (6)-(8) constitute the recurrence perturbation formulas of the Riccati-TMM.

By substituting the perturbation expressions into the matrixes $[\boldsymbol{S}_{u1}]_i$ and $[\boldsymbol{S}_{u2}]_i$ and allowing $[\boldsymbol{S}'_{u2}]_{i,0} = [\boldsymbol{S}_{u2}]_{i,0}^{-1}$, the formulas for the calculation of the matrixes $[\boldsymbol{S}_{u1}]_{i,0}$, $[\boldsymbol{S}_{u1}]_{i,j}$, and $[\boldsymbol{S}_{u1}]_{i,jk}$ as well as $[\boldsymbol{S}_{u2}]_{i,0}$, $[\boldsymbol{S}_{u2}]_{i,j}$, and $[\boldsymbol{S}_{u2}]_{i,jk}$ are respectively given as follows:

$$[\boldsymbol{S}_{u1}]_{i,0} = [\boldsymbol{u}_{11}\boldsymbol{S}+\boldsymbol{u}_{12}]_{i,0} \tag{9a}$$

$$[\boldsymbol{S}_{u1}]_{i,j} = [\boldsymbol{u}_{11}]_{i,0}[\boldsymbol{S}]_{i,j} + [\boldsymbol{u}_{11}]_{i,j}[\boldsymbol{S}]_{i,0} + [\boldsymbol{u}_{12}]_{i,j} \tag{9b}$$

$$[\boldsymbol{S}_{u1}]_{i,jk} = [\boldsymbol{u}_{11}]_{i,0}[\boldsymbol{S}]_{i,jk} + [\boldsymbol{u}_{11}]_{i,j}[\boldsymbol{S}]_{i,k} + (1-\delta_{jk})[\boldsymbol{u}_{11}]_{i,k}[\boldsymbol{S}]_{i,j} + [\boldsymbol{u}_{11}]_{i,jk}[\boldsymbol{S}]_{i,0} + [\boldsymbol{u}_{12}]_{i,jk} \tag{9c}$$

$$[\boldsymbol{S}'_{u2}]_{i,0} = [\boldsymbol{u}_{21}\boldsymbol{S}+\boldsymbol{u}_{22}]_{i,0} \tag{10a}$$

$$[\boldsymbol{S}'_{u2}]_{i,j} = [\boldsymbol{u}_{21}]_{i,0}[\boldsymbol{S}]_{i,j} + [\boldsymbol{u}_{21}]_{i,j}[\boldsymbol{S}]_{i,0} + [\boldsymbol{u}_{22}]_{i,j} \tag{10b}$$

$$[\boldsymbol{S}'_{u2}]_{i,jk} = [\boldsymbol{u}_{21}]_{i,0}[\boldsymbol{S}]_{i,jk} + [\boldsymbol{u}_{21}]_{i,j}[\boldsymbol{S}]_{i,k} + (1-\delta_{jk})[\boldsymbol{u}_{21}]_{i,k}[\boldsymbol{S}]_{i,j} + [\boldsymbol{u}_{21}]_{i,jk}[\boldsymbol{S}]_{i,0} + [\boldsymbol{u}_{22}]_{i,jk} \tag{10c}$$

$$[\boldsymbol{S}_{u2}]_{i,0} = [\boldsymbol{S}'_{u2}]_{i,0}^{-1} \tag{11a}$$

$$[\boldsymbol{S}_{u2}]_{i,j} = -[\boldsymbol{S}'_{u2}]_{i,0}^{-1}[\boldsymbol{S}'_{u2}]_{i,j}[\boldsymbol{S}'_{u2}]_{i,0}^{-1} \tag{11b}$$

$$[\boldsymbol{S}_{u2}]_{i,jk} = -[\boldsymbol{S}'_{u2}]_{i,0}^{-1}[\boldsymbol{S}'_{u2}]_{i,jk}[\boldsymbol{S}'_{u2}]_{i,0}^{-1} - [\boldsymbol{S}'_{u2}]_{i,0}^{-1}[\boldsymbol{S}'_{u2}]_{i,j}[\boldsymbol{S}_{u2}]_{i,k} - (1-\delta_{jk})[\boldsymbol{S}'_{u2}]_{i,0}^{-1}[\boldsymbol{S}'_{u2}]_{i,k}[\boldsymbol{S}_{u2}]_{i,j}. \tag{11c}$$

If the boundary conditions at both ends of the rotor system are the same (i.e., $\{\boldsymbol{f}\}_{N+1} = \{\boldsymbol{0}\}$ and $\{\boldsymbol{e}\}_{N+1} \neq \{\boldsymbol{0}\}$), then the following equation must be established:

$$\{\boldsymbol{f}\}_{N+1,0} = \{\boldsymbol{f}\}_{N+1,j} = \{\boldsymbol{f}\}_{N+1,jk} = \{\boldsymbol{0}\}. \tag{12}$$

By substituting Eq. (12) into Eq. (6a), the frequency equation for $\beta_{,0}$ is obtained as:

$$\Delta = |\boldsymbol{S}|_{N+1,0} = 0 \tag{13}$$

where $|\boldsymbol{S}|_{N+1,0}$ is the determinant of matrix $[\boldsymbol{S}]_{N+1,0}$.

Since $\beta_{,0}$ is obtained from the above equation, the matrix $[\boldsymbol{S}]_{N+1,0}$ must be singular. Conducting singular value decomposition yields:

$$[\boldsymbol{S}]_{N+1,0} = [\boldsymbol{U}][\boldsymbol{D}][\bar{\boldsymbol{V}}]^T \tag{14}$$

where the matrixes $[\boldsymbol{U}]$ and $[\boldsymbol{V}]$ are unitary matrixes, and $[\boldsymbol{D}]$ is a diagonal matrix.

3. Independent eigenvalue problem

If the eigenvalue $\beta_{,0}$ is an independent root, the last element of matrix $[\boldsymbol{D}]$ must be zero as shown in the equation below:

$$[\boldsymbol{D}] = diag[d_1, \cdots, d_{r-1}, 0]. \tag{15}$$

By substituting Eqs. (14) and (12) into Eq. (6a), the resulting expression is:

$$[\boldsymbol{U}][\boldsymbol{D}][\bar{\boldsymbol{V}}]^T \{\boldsymbol{e}\}_{N+1,0} = \{\boldsymbol{0}\}. \tag{16}$$

The nonzero solution of the vector $\{\boldsymbol{e}\}_{N+1,0}$ is:

$$\{\boldsymbol{e}\}_{N+1,0} = \{\boldsymbol{V}_r\} \tag{17}$$

where $\{\boldsymbol{V}_r\}$ is a vector combined by the elements of the last column of matrix $[\boldsymbol{V}]$.

3.1 The first-order perturbation of independent eigenvalue and its eigenvector

Substituting Eqs. (14) and (12) into Eq. (6b) yields:

$$[\boldsymbol{U}][\boldsymbol{D}][\bar{\boldsymbol{V}}]^T \{\boldsymbol{e}\}_{N+1,j} + [\boldsymbol{S}]_{N+1,j} \{\boldsymbol{e}\}_{N+1,0} = \{\boldsymbol{0}\} \tag{18}$$

where

$$\{\tilde{e}\}_{N+1,j} = [\bar{\boldsymbol{V}}]^T \{\boldsymbol{e}\}_{N+1,j}. \tag{19}$$

Substituting Eq. (19) into Eq. (18), which is then premultiplied by the matrix $[\bar{\boldsymbol{U}}]^T$, obtains the following statement:

$$[\boldsymbol{D}]\{\tilde{e}\}_{N+1,j} + [\bar{\boldsymbol{U}}]^T [\boldsymbol{S}]_{N+1,j} \{\boldsymbol{e}\}_{N+1,0} = \{\boldsymbol{0}\}. \tag{20}$$

Since the last element of the matrix $[\boldsymbol{D}]$ is zero, the condition that makes Eq. (20) equal is:

$$\hat{e}^r_{N+1,j} = \left\{[\bar{\boldsymbol{U}}]^T [\boldsymbol{S}]_{N+1,j} \{\boldsymbol{e}\}_{N+1,0}\right\}_r = 0. \tag{21}$$

Eq. (21) is the frequency equation for the first-order perturbation β_j (j = 1, ..., m) of the eigenvalue β, where the symbol $\{\bullet\}_r$ denotes the *rth* element of the vector $\{\bullet\}$.

After β_j is obtained, substituting the results into Eq. (20) yields:

$$\tilde{e}^p_{N+1,j} = -\frac{1}{d_p}\hat{e}^p_{N+1,j} \quad (p = 1, \cdots, r-1) \tag{22a}$$

$$\tilde{e}^r_{N+1,j} = 0 \tag{22b}$$

where $\tilde{e}^p_{N+1,j}$ is the *pth* element of the vector $\{\tilde{e}\}_{N+1,j}$. Accordingly, Eq. (22) is substituted into Eq. (19) and premultiplied by the matrix $[\boldsymbol{V}]$. The resulting equation becomes:

$$\{\boldsymbol{e}\}_{N+1,j} = [\boldsymbol{V}]\{\tilde{e}\}_{N+1,j}. \tag{23}$$

3.2 The second-order perturbation of independent eigenvalue and its eigenvector

Similarly, by substituting Eqs. (14) and (12) into Eq. (6c), the frequency equation is derived for the second-order perturbation β_{jk} (j = 1, ..., m; k= 1, ..., j) of the eigenvalue β, which is expressed as:

$$\hat{e}^r_{N+1,jk} = \left\{[\bar{\boldsymbol{U}}]^T \left([\boldsymbol{S}]_{N+1,j}\{\boldsymbol{e}\}_{N+1,k} + (1-\delta_{jk})[\boldsymbol{S}]_{N+1,k}\{\boldsymbol{e}\}_{N+1,j} + [\boldsymbol{S}]_{N+1,jk}\{\boldsymbol{e}\}_{N+1,0}\right)\right\}_r = 0. \tag{24}$$

The solution of the vector $\{\boldsymbol{e}\}_{N+1,jk}$ is as follows:

$$\{\boldsymbol{e}\}_{N+1,jk} = [\boldsymbol{V}]\{\tilde{e}\}_{N+1,jk}. \tag{25}$$

The elements of the vector $\{\tilde{e}\}_{N+1,jk}$ are:

$$\tilde{e}^p_{N+1,jk} = -\frac{1}{d_p}\hat{e}^p_{N+1,jk} \quad (p = 1, \cdots, r-1) \tag{26a}$$

$$\tilde{e}^r_{N+1,jk} = 0 \tag{26b}$$

where $\tilde{e}^{t,p}_{N+1,jk}$ is the *pth* element of the vector $\{\boldsymbol{e}\}^t_{N+1,jk}$; and $\hat{e}^{t,p}_{N+1,jk}$ is defined by Eq. (24).

Eqs. (15)-(26) are the given formulas to solve independent eigenvalue problems. Using the frequency Eqs (13), (21), and (24), the initial eigenvale $\beta_{,0}$ and its first-order perturbations β_j (j = 1, ..., m) as well as its second-order perturbations β_{jk} (j = 1, ..., m; k = 1, ..., j) can be searched, respectively. By substituting Eqs. (17), (23), and (25) into the recurrence formulas (8) and (6), the corresponding eigenvector and its first- and second-order perturbations can be calculated.

4. Repeated eigenvalue problem

If the eigenvalue $\beta_{,0}$ is a repeated root with s orders, the last s diagonal elements of the matrix [$\boldsymbol{D}$] in Eq. (14) must be zeroes. There must be s nonzero solutions of vector $\{\boldsymbol{e}\}_{N+1,0}$ in Eq. (16). These s nonzero vectors are derived by:

$$\{\boldsymbol{e}\}^t_{N+1,0} = \{\boldsymbol{V}_{r-t+1}\} \quad (t = 1,\cdots,s) \tag{27}$$

where $\{\boldsymbol{V}_{r-t+1}\}$ is a vector combined by the elements of the (r-t+1)th column of the matrix; and $\{\boldsymbol{e}\}^t_{N+1,0}$ is the tth solution of the vector $\{\boldsymbol{e}\}_{N+1,0}$.

4.1 The first-order perturbations of repeated eigenvalue and its eigenvectors

By substituting Eqs. (14) and (12) as well as the s solutions of the vector $\{\boldsymbol{e}\}_{N+1,0}$ into Eq. (6b), the s equations below are obtained:

$$\left[\boldsymbol{U}\right]\left[\boldsymbol{D}\right]\left[\bar{\boldsymbol{V}}\right]^T\{\boldsymbol{e}\}^t_{N+1,j} + \left[\boldsymbol{S}\right]_{N+1,j}\{\boldsymbol{e}\}^t_{N+1,0} = \{\boldsymbol{0}\} \quad (t = 1,\cdots,s)\,. \tag{28}$$

Multiplied by s nonzero constants, the above s equations are rewritten in the following matrix form:

$$\left[\boldsymbol{U}\right]\left[\boldsymbol{D}\right]\left[\bar{\boldsymbol{V}}\right]^T\left[\bar{\boldsymbol{e}}\right]_{N+1,j}\{\boldsymbol{u}\} + \left[\boldsymbol{S}\right]_{N+1,j}\left[\bar{\boldsymbol{e}}\right]_{N+1,0}\{\boldsymbol{u}\} = \{\boldsymbol{0}\} \tag{29}$$

where $\{\boldsymbol{e}\}^t_{N+1,j}$ is the tth solution of the vector $\{\boldsymbol{e}\}_{N+1,j}$; $\left[\bar{\boldsymbol{e}}\right]_{N+1,0} = \left[\{\boldsymbol{e}\}^1_{N+1,0},\cdots,\{\boldsymbol{e}\}^s_{N+1,0}\right]$; $\left[\bar{\boldsymbol{e}}\right]_{N+1,j} = \left[\{\boldsymbol{e}\}^1_{N+1,j},\cdots,\{\boldsymbol{e}\}^s_{N+1,j}\right]$; and $\{\boldsymbol{u}\}$ is a column vector combined by the s nonzero constants.

Define

$$\left[\tilde{\boldsymbol{e}}\right]_{N+1,j} = \left[\bar{\boldsymbol{V}}\right]^T\left[\bar{\boldsymbol{e}}\right]_{N+1,j}\,. \tag{30}$$

Substituting Eq. (30) into Eq. (29) and pre-multiplied by the matrix $\left[\bar{U}\right]^T$ yields the following:

$$\left[\boldsymbol{D}\right]\left[\tilde{\boldsymbol{e}}\right]_{N+1,j}\{\boldsymbol{u}\} + \left[\bar{\boldsymbol{U}}\right]^T\left[\boldsymbol{S}\right]_{N+1,j}\left[\bar{\boldsymbol{e}}\right]_{N+1,0}\{\boldsymbol{u}\} = \{\boldsymbol{0}\}\,. \tag{31}$$

Since the last s diagonal elements of the matrix [$\boldsymbol{D}$] are zeroes, the condition that makes Eq. (31) equal is:

$$\left[\boldsymbol{W}\right]_1\{\boldsymbol{u}\} = \{\boldsymbol{0}\} \tag{32}$$

where $[\boldsymbol{W}]_1$ is a matrix made up by the last s element rows of the matrix $\left[\bar{\boldsymbol{U}}\right]^T\left[\boldsymbol{S}\right]_{N+1,j}\left[\bar{\boldsymbol{e}}\right]_{N+1,0}$. Obviously, if the determinant of matrix $[\boldsymbol{W}]_1$ is equal to zero, Eq. (32) has a nonzero solution. Therefore, the frequency equation for the first-order perturbation β_j (j = 1, ..., m) of the eigenvalue β is:

$$\left|\boldsymbol{W}_1\right| = 0. \tag{33}$$

Eq. (33) has s roots which correspond to the s first-order perturbations of the eigenvalue β. By substituting the s roots into Eq. (28), the s solutions of the vector $\{\boldsymbol{e}\}_{N+1,j}$ are obtained as follows:

$$\{\boldsymbol{e}\}^t_{N+1,j} = \left[\boldsymbol{V}\right]\{\tilde{\boldsymbol{e}}\}^t_{N+1,j} \quad (t = 1,\cdots,s). \tag{34}$$

The elements of the vector $\{\tilde{\boldsymbol{e}}\}_{N+1,j}$ are:

$$\tilde{e}^{t,p}_{N+1,j} = -\frac{1}{d_p}\hat{e}^{t,p}_{N+1,j} \quad (t = 1,\cdots,s;\, p = 1,\cdots,r-s) \tag{35a}$$

$$\tilde{e}^{t,p}_{N+1,j} = 0 \quad \left(t = 1,\cdots,s;\, p = r-s+1,\cdots,r\right) \tag{35b}$$

where $\tilde{e}^{t,p}_{N+1,j}$ is the pth element of the vector $\{\boldsymbol{e}\}^t_{N+1,j}$; and $\hat{e}^{t,p}_{N+1,j}$ is the pth element of the vector $\left[\bar{\boldsymbol{U}}\right]^T\left[\boldsymbol{S}\right]_{N+1,j}\{\boldsymbol{e}\}^t_{N+1,0}$.

4.2 The second-order perturbations of repeated eigenvalue and its eigenvectors

By substituting Eqs. (14) and (12), as well as the solutions of the vectors $\{\boldsymbol{e}\}_{N+1,0}$ and $\{\boldsymbol{e}\}_{N+1,j}$ into Eq. (6c), the frequency equation is derived for the second-order perturbation β_{jk} (j = 1, ..., m; k= 1, ..., j) of the eigenvalue β as shown below:

$$\left|\boldsymbol{W}_2\right| = 0. \tag{36}$$

The matrix $[\boldsymbol{W}]_2$ is made up by the last s element rows of the following matrix $[\boldsymbol{W}]$:

$$[W] = \left[\bar{U}\right]^T\left(\left[\boldsymbol{S}\right]_{N+1,j}\left[\bar{\boldsymbol{e}}\right]_{N+1,k} + \left(1-\delta_{jk}\right)\left[\boldsymbol{S}\right]_{N+1,k}\left[\bar{\boldsymbol{e}}\right]_{N+1,j} + \left[\boldsymbol{S}\right]_{N+1,jk}\left[\bar{\boldsymbol{e}}\right]_{N+1,0}\right) \tag{37}$$

where $\left[\bar{\boldsymbol{e}}\right]_{N+1,jk} = \left[\{\boldsymbol{e}\}^1_{N+1,jk},\cdots,\{\boldsymbol{e}\}^s_{N+1,jk}\right]$.

Eq. (36) has s roots that correspond to the s second-order perturbations of the eigenvalue β. The s solutions of the vector $\{\boldsymbol{e}\}_{N+1,jk}$ are derived as follows:

$$\{\boldsymbol{e}\}^t_{N+1,jk} = \left[\boldsymbol{V}\right]\{\tilde{\boldsymbol{e}}\}^t_{N+1,jk} \quad (t = 1,\cdots,s). \tag{38}$$

The elements of the vector $\{\tilde{\boldsymbol{e}}\}_{N+1,jk}$ are:

$$\tilde{e}^{t,p}_{N+1,jk} = -\frac{1}{d_p}\hat{e}^{t,p}_{N+1,jk} \quad (t = 1,\cdots,s;\, p = 1,\cdots,r-s) \tag{39a}$$

$$\tilde{e}_{N+1,jk}^{t,p} = 0 \quad \left(t = 1,\cdots,s; p = r-s+1,\cdots,r\right) \tag{39b}$$

where $\tilde{e}_{N+1,jk}^{t,p}$ is the *pth* element of the vector $\{\tilde{e}\}_{N+1,jk}^{t}$; and $\hat{e}_{N+1,jk}^{t,p}$ is the element of matrix $[\boldsymbol{W}]$ at the *pth* row and *tth* column.

Eqs. (27)-(39) are the derived formulas for the repeated eigenvalue problems. Using the frequency Eqs. (13), (33) and (36), the initial repeated eigenvalue β_0 with s orders along with their first-order perturbations β_j (j = 1, ..., m) and second-order perturbations β_{jk} (j = 1, ..., m; k= 1, ..., j) can be searched, respectively. Thus, by substituting Eqs. (27), (34), and (38) into the recurrence formulas (8) and (6), the corresponding eigenvectors and their first- and second-order perturbations can be calculated.

5. Random eigenvalue problem

If β_p is the *pth* eigenvalue of a rotor system, Eq. (5) is rewritten as:

$$\beta_p = \beta_{p,0} + \sum_{j=1}^{m} \beta_{p,j}\varepsilon_j + \sum_{j=1}^{m}\sum_{k=1}^{m} \hat{\beta}_{p,jk}\varepsilon_j\varepsilon_k \tag{40}$$

where

$$\hat{\beta}_{p,jk} = \hat{\beta}_{p,kj} = \frac{1}{2}\beta_{p,jk} \quad (j \neq k). \tag{41}$$

The corresponding eigenvector is:

$$\{\boldsymbol{\psi}\}_p = \{\boldsymbol{\psi}\}_{p,0} + \sum_{j=1}^{m} \{\boldsymbol{\psi}\}_{p,j}\varepsilon_j + \sum_{j=1}^{m}\sum_{k=1}^{m} \{\hat{\boldsymbol{\psi}}\}_{p,jk}\varepsilon_j\varepsilon_k. \tag{42}$$

If ε_j (j = 1, …, m) are random parameters in Eq. (4), their mean values are zeroes, and their standard deviations are σ_j (j = 1, …, m). Thus, the relation between the variation coefficient v_j of the parameter b_j and the standard deviations σ_j of the parameter ε_j is obtained as follows:

$$v_j = E_{bj} / \sigma_{bj} = \sigma_j \quad (j = 1,\cdots,m) \tag{43}$$

where E_{bj} and σ_{bj} are the mean values and the standard deviation of parameter b_j, respectively.

From Eq. (40), the expressions of the mean value and variance of the eigenvalue β_p are derived as:

$$E\left(\beta_p\right) = \beta_{p,0} + \sum_{j=1}^{m}\sum_{k=1}^{m} \hat{\beta}_{p,jk} E\left(\varepsilon_j\varepsilon_k\right) \tag{44}$$

$$\begin{aligned} D\left(\beta_p\right) &= E\left[\beta_p - E\left(\beta_p\right)\right]^2 \\ &= \sum_{j=1}^{m}\sum_{k=1}^{m} \beta_{p,j}\beta_{p,k} E\left(\varepsilon_j\varepsilon_k\right) + 2\sum_{j=1}^{m}\sum_{k=1}^{m}\sum_{l=1}^{m} \beta_{p,j}\hat{\beta}_{p,kl} E\left(\varepsilon_j\varepsilon_k\varepsilon_l\right) \\ &+ \sum_{j=1}^{m}\sum_{k=1}^{m}\sum_{l=1}^{m}\sum_{s=1}^{m} \beta_{p,jk}\hat{\beta}_{ls}\left[E\left(\varepsilon_j\varepsilon_k\varepsilon_l\varepsilon_s\right) - E\left(\varepsilon_j\varepsilon_k\right)E\left(\varepsilon_l\varepsilon_s\right)\right]. \end{aligned} \tag{45}$$

In addition, from Eq. (42), the respective expressions of the mean value and covariance of the eigenvector $\{\boldsymbol{\psi}\}_p$ are derived as follows:

$$E\left(\{\boldsymbol{\psi}\}_p\right) = \{\boldsymbol{\psi}\}_{p,0} + \sum_{j=1}^{m}\sum_{k=1}^{m} \{\hat{\boldsymbol{\psi}}\}_{p,jk} E\left(\varepsilon_j\varepsilon_k\right) \tag{46}$$

$$\begin{aligned} Cov\left(\{\boldsymbol{\psi}\}_p, \{\boldsymbol{\psi}\}_p^T\right) &= E\left[\left(\{\boldsymbol{\psi}\}_p - E\left(\{\boldsymbol{\psi}\}_p\right)\right)\left(\{\boldsymbol{\psi}\}_p^T - E\left(\{\boldsymbol{\psi}\}_p^T\right)\right)\right] \\ &= \sum_{j=1}^{m}\sum_{k=1}^{m} \{\boldsymbol{\psi}\}_{p,j}\{\boldsymbol{\psi}\}_{p,k}^T E\left(\varepsilon_j\varepsilon_k\right) \\ &+ \sum_{j=1}^{m}\sum_{k=1}^{m}\sum_{l=1}^{m} \left[\{\boldsymbol{\psi}\}_{p,j}\{\hat{\boldsymbol{\psi}}\}_{p,kl}^T + \{\hat{\boldsymbol{\psi}}\}_{p,kl}\{\boldsymbol{\psi}\}_{p,j}^T\right] E\left(\varepsilon_j\varepsilon_k\varepsilon_l\right) \\ &+ \sum_{j=1}^{m}\sum_{k=1}^{m}\sum_{l=1}^{m}\sum_{s=1}^{m} \{\hat{\boldsymbol{\psi}}\}_{p,jk}\{\hat{\boldsymbol{\psi}}\}_{p,ls}^T\left[E\left(\varepsilon_j\varepsilon_k\varepsilon_l\varepsilon_s\right) - E\left(\varepsilon_j\varepsilon_k\right)E\left(\varepsilon_l\varepsilon_s\right)\right]. \end{aligned} \tag{47}$$

If ε_j (j = 1, …, m) are subject to the normal distribution, and independent, Eqs. (44)-(47) are simplified as follows:

$$E\left(\beta_p\right) = \beta_{p,0} + \sum_{j=1}^{m} \hat{\beta}_{p,jj}\sigma_j^2 \tag{48}$$

$$D\left(\beta_p\right) = \sum_{j=1}^{m}\left(\beta_{p,j}^2\sigma_j^2 + 2\hat{\beta}_{p,jj}^2\sigma_j^4\right) \tag{49}$$

$$E\left(\{\boldsymbol{\psi}\}_p\right) = \{\boldsymbol{\psi}\}_{p,0} + \sum_{j=1}^{m} \{\hat{\boldsymbol{\psi}}\}_{p,jj}\sigma_j^2 \tag{50}$$

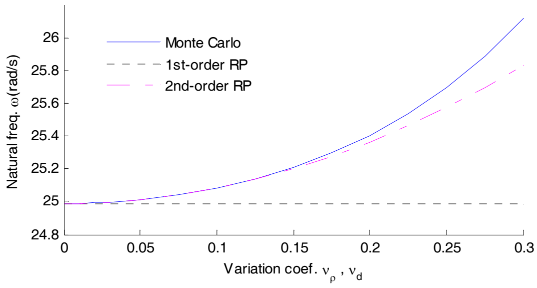
$$Cov\left(\{\boldsymbol{\psi}\}_p, \{\boldsymbol{\psi}\}_p^T\right) = \sum_{j=1}^{m}\left[\{\boldsymbol{\psi}\}_{p,j}\{\boldsymbol{\psi}\}_{p,j}^T\sigma_j^2 + 2\{\hat{\boldsymbol{\psi}}\}_{p,jj}\{\hat{\boldsymbol{\psi}}\}_{p,jj}^T\sigma_j^4\right]. \tag{51}$$

6. Numerical examples

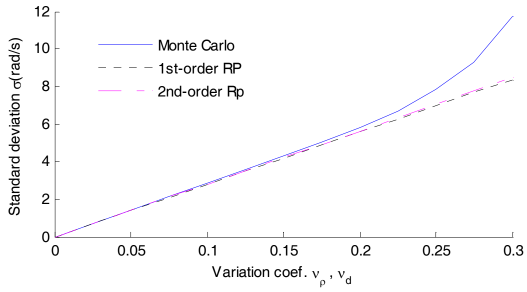
6.1 The simply supported Timoshenko beam

The accuracy of the method presented in this paper was first tested in calculating the random eigenvalues of a simply supported Timoshenko beam. Suppose that the mass density ρ and the section diameter d of the beam are independent random variables to follow normal distribution, the mean value of the mass density is E_ρ = 7800 kg/m^3, and the mean value of the diameter is E_d = 0.2 m. The other parameters are certain, such as the length L = 10 m and the Young's modulus $E = 2.0\times10^{11}$ N/m^2.

When the variation coefficient of the mass density v_ρ is the same as that of the section diameter v_d, the mean value curves of the first natural frequency versus the variation coefficients are shown in Fig. 1(a). Based on the analytical formula of natural frequency of the simply supported Timoshenko beam, the solid curve is 1 million times the Monte Carlo random simulated results. The dotted curve is the first-order random

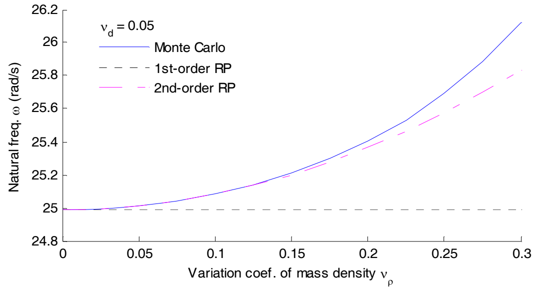


(a) Mean values vs. variation coefficient v_ρ and v_d

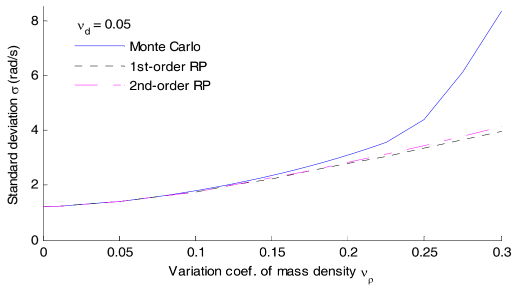


(b) Standard deviations vs. variation coefficient v_ρ and v_d.

Fig. 1. Random perturbation and Monte Carlo simulation results of the first natural frequency of the Timoshenko beam as $v_\rho = v_d$.



(a) Mean values vs. variation coefficient v_ρ



(b) Standard deviations vs. variation coefficient v_ρ

Fig. 2. Random perturbation and Monte Carlo simulation results of the first natural frequency of the Timoshenko beam as $v_d = 0.05$.

perturbation results. Meanwhile, the dash-dot curve represents the second-order random perturbation results.

Fig. 1(a) shows that the mean values of the second-order random perturbation are in agreement with those of the Monte Carlo simulation results, and the errors are less than 0.2% as the variation coefficients $v_\rho = v_d <= 0.20$. However, the errors of the first-order random perturbation are less than 1.7% in the same case. It is apparent that the second-order mean values are more accurate than those of the first-order. The curve of the first-order mean values is a horizontal line, and its value is the natural frequency with variation coefficients $v_\rho = v_d = 0$ from Eq. (44).

Fig. 1(b) plots the standard deviation curves of the first natural frequency. The figure shows that the accuracies of the first- and second-order standard deviations are close at less than 4.4% and 3.9%, respectively, as the variation coefficients $v_\rho = v_d <= 0.20$.

With the variation coefficient of the section diameter v_d = 0.05, Fig. 2 plots the mean value and the standard deviation curves of the first natural frequency versus the variation coefficient of the mass density v_ρ. The errors of the first-order mean values and standard deviations are less than 1.7% and 9.7%, respectively, as $v_d = 0.05$ and $v_\rho <= 0.20$; and the errors of the second-order mean values and standard deviations are less than 0.2% and 8.1%, respectively, in the same case.

Figs. 1 and 2 show that the second-order random perturbation is more accurate than the first-order random perturbation. The results obtained by the random perturbation method presented in this paper are in agreement with the results of the Monte Carlo simulation method within a larger range. However, the errors tend to increase as the variation coefficients increase.

The random analyzed results of the second, third, and fourth natural frequencies have the same accuracies as well. However, the related curves and data are not listed here.

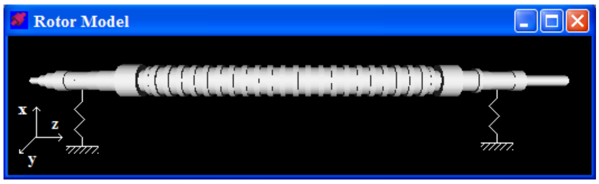


Fig. 3. A test rotor with two oil bearings: the distance between the two bearings is 4.205 *m*, the total length is 5.35 *m*, and the total mass is 2680.6 *kg*.

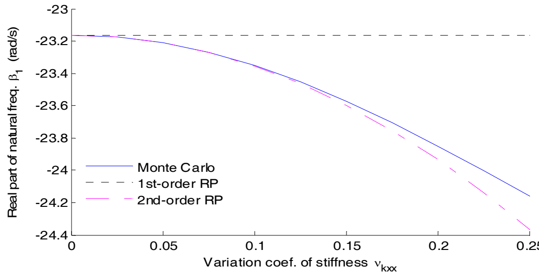
Since the theoretical vibration modes of the simply supported Timoshenko beam are not related with the mass density ρ and the section diameter d, the random variations of these two parameters do not influence vibration modes. The random perturbation results are similar to those that have been presented previously.

6.2 The rotor supported by two oil bearings

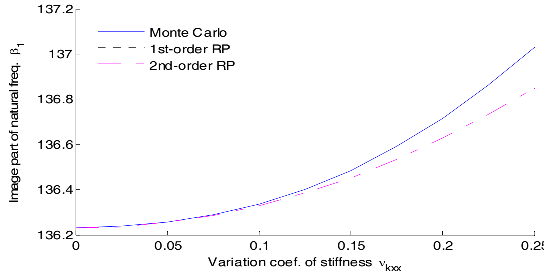
Fig. 3 shows a test rotor supported by two oil bearings. The working speed of the rotor was 1000 *rpm*. The stiffness and damping matrixes of the bearings are respectively given as:

$$[\boldsymbol{K}] = \begin{bmatrix} k_{xx} & k_{xy} \\ k_{yx} & k_{yy} \end{bmatrix} = \begin{bmatrix} 2.385 & 1.761 \\ 0.0616 & 0.7533 \end{bmatrix} \times 10^8 \; N/m$$

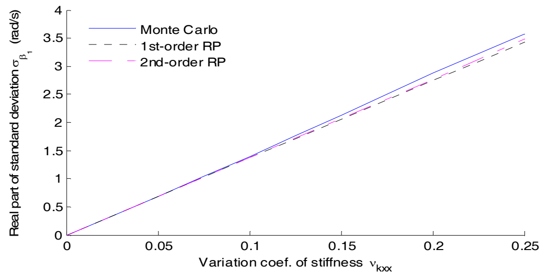
$$[\boldsymbol{C}] = \begin{bmatrix} c_{xx} & c_{xy} \\ c_{yx} & c_{yy} \end{bmatrix} = \begin{bmatrix} 3.173 & 0.8171 \\ 0.8093 & 0.5946 \end{bmatrix} \times 10^6 \; N \cdot s/m \, .$$



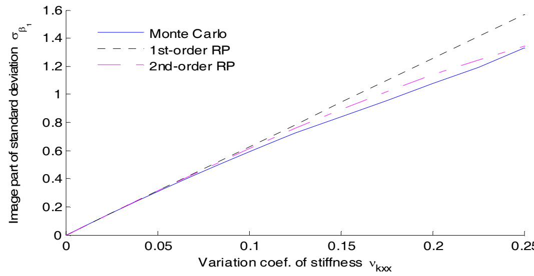
(a) Real parts of mean values of the first natural frequency



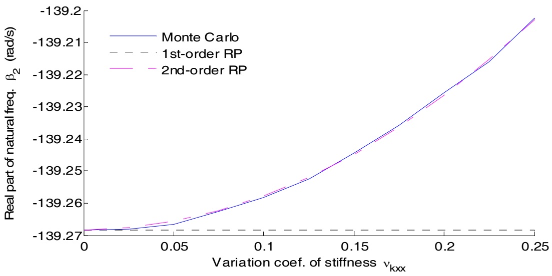
(b) Image parts of mean values of the first natural frequency



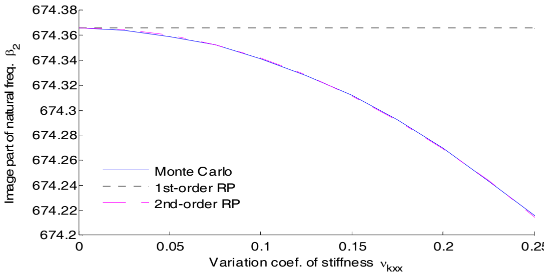
(c) Real parts of standard deviations of the first natural frequency



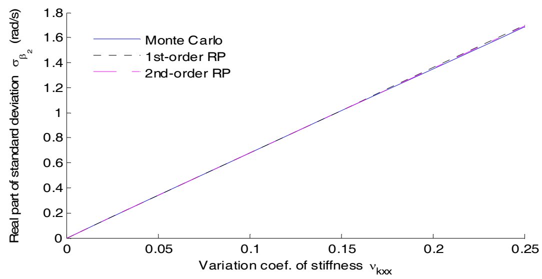
(d) Image parts of standard deviations of the first natural frequency



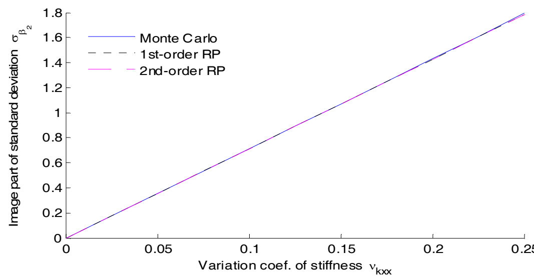
(e) Real parts of mean values of the second natural frequency



(f) Image parts of mean values of the second natural frequency



(g) Real parts of standard deviations of the second natural frequency



(h) Image parts of standard deviations of the second natural frequency

Fig. 4. Random perturbation and Monte Carlo simulation curves of the natural frequencies of the rotor with two oil bearings versus the variation coefficient v_{kxx} .

The rotor was segmented to 72 sections according to the diameter, length, and mass of the sections. Polar moments and transverse moments of inertia were lumped at both ends of the sections.

Suppose primarily that the stiffness k_{xx} of the two oil bearings are independent random variables to follow normal distribution, their variation coefficient values may vary synchronously to facilitate the drawing.

This is a complex eigenvalue problem. Fig. 4 plots the real- and image-part curves of the mean value and the standard deviation of the first and second natural frequencies versus the variation coefficient of the stiffness k_{xx}. The solid curves are 100,000 times the Monte Carlo random simulated results based on the Riccati-TMM.

Fig. 4 shows that the results of the random perturbations and the Monte Carlo simulations are in agreement with each other within a larger range. However, to complete a calculation on the dot of the curve, the first- and second-order random perturbations take only 0.19 and 0.62 second, respectively. Meanwhile, the Monte Carlo simulation method needs 8,668 seconds on the same computer.

With the variation coefficient of the stiffness v_{kxx} = 0.15, the Monte Carlo simulation and the random perturbation results of the first vibration mode in X-direction are presented in the appendix. The first- and second-order errors of the norm of mean values of the vibration mode are 1.73% and 0.01%, respectively; the corresponding errors of standard deviations are 14.65% and 12.74%. The other vibration modes are not listed in the current paper due to limited length.

7. Conclusions

The Riccati-TMM has been used widely in the field of engineering because it is highly useful and efficient in conducting a dynamic analysis of the rotor system. From this method, the Riccati-PTMM is developed for the perturbation analysis of the eigenvalue problems of the rotor system with uncertain or random parameters. The recurrence perturbation formulas presented in this paper can be used for the first- and second-order perturbation analyses of the independent, repeated, as well as complex eigenvalues and their eigenvectors. Consequently, the formulas for calculating the mean value, variance, and covariance of the eigenvalues and eigenvectors of the rotor system with random parameters are given.

The Riccati-PTMM is used for the random lateral eigenvalue problem analyses of a simply supported Timoshenko beam and a test rotor supported by two oil bearings. The first numerical example reveals that the method is highly accurate within a larger range. The second example, on the other hand, demonstrates that the method is efficient. Both examples show that the second-order random perturbation results are more accurate than those of the first-order. However, these precision improvements are not obvious. The second-order random perturbation analysis is much more complex. Hence, the first-order random perturbation analysis of Riccati-PTMM is a useful choice for engineers. The calculation time of the first-order perturbation analysis is also shorter than that of the second-order.

For concrete problems, as long as the element transfer matrix and the element perturbation expressions are given, the method can be used for longitudinal, lateral, and/or torsional vibration eigenvalue problems of the rotor system with uncertain or random parameters. Therefore, the method presented in this paper can be used generally in other applications.

Acknowledgment

The author would like to thank the Committee of National Natural Science Foundation of China (No. 10872064 and No. 11172092) and the Zhengzhou Bureau of Science and Technology (No. 10LJRC187) for the financial support.

References

[1] R. A. Ibrahim, Structural dynamics with parameter uncertainties, *Applied Mechanics Review*, 40 (3) (1987) 309-328.

[2] J. T. Oden, T. Belytschko, I. Babuska and T. J. R. Hughes, Research directions in computational mechanics, *Computer Methods in Applied Mechanics and Engineering*, 192 (7-8) (2003) 913-22.

[3] E. N. Sanchez, A view to the new perturbation technique valid for large parameters, *Journal of Sound and Vibration*, 282 (3-5) (2005) 1309-1316.

[4] L. C. Rogers, Derivatives of eigenvalues and eigenvectors, *American Institute of Aeronautics and Astronautics Journal*, 15 (5) (1977) 943-944.

[5] J. C. Chen and B. K. Wada, Matrix perturbation for structural dynamics, *American Institute of Aeronautics and Astronautics Journal*, 17 (6) (1979) 1095-1110.

[6] R. B. Nelson, Simplified calculation of eigenvector derivatives, *American Institute of Aeronautics and Astronautics Journal*, 14 (9) (1976) 1201-1205.

[7] B. B. William, An improved computational technique for perturbations of the generalized symmetric linear algebraic eigenvalue problem, *International Journal of Numerical Methods in Engineering*, 24 (3) (1987) 529-541.

[8] B. P. Wang, Improved approximate method for computing eigenvalue derivatives in structural dynamics, *American Institute of Aeronautics and Astronautics Journal*, 29 (6) (1991) 1018-1029.

[9] E. J. Haug and B. Rousselet, Design sensitivity analysis in structural dynamics II, Eigenvalue variations, *Journal of Structural Mechanics*, 8 (2) (1980) 161-169.

[10] K. B. Lim, J. N. Juang and P. Ghaemmaghami, Eigenvector derivatives of repeated eigenvalues using singular value decomposition, *Journal of Guidance - Control and Dynamics*, 12 (2) (1989) 282-283.

[11] W. C. Mills-Curran, Calculation of eigenvector derivatives for structures with repeated eigenvalues, *American Institute of Aeronautics and Astronautics Journal*, 26 (7) (1988) 867-871.

[12] R. L. Dailey, Eigenvector derivatives with repeated eigenvalues, *American Institute of Aeronautics and Astronautics Journal*, 27 (4) (1989) 486-491.

[13] J. Shaw and S. Jayasuriya, Modal sensitivities for repeated eigenvalues and eigenvectors derivatives, *American Institute of Aeronautics and Astronautics Journal*, 30 (3) (1992) 850-852.

[14] G. J. W. Hou and S. P. Kenny, Eigenvalue and eigenvector approximate analysis for repeated eigenvalues problems, *American Institute of Aeronautics and Astronautics Journal*, 30 (9) (1992) 2317-2324.

[15] W. Gao, Natural frequency and mode shape analysis of structures with uncertainty, *Mechanical Systems and Signal Processing*, 21 (1) (2007) 24-39.

[16] J. Dai, W. Gao, N. Zhang and N. G. Liu, Seismic random vibration analysis of shear beams with random structural parameters, Journal of Mechanical Science and Technology, 24 (2) (2010) 497-504.

[17] M. A. Prohl, A general method for calculating critical speeds of flexible rotors, *Journal of Applied Mechanics - Transactions of the ASME*, 12 (3) (1945) 142-148.

[18] G. C. Horner and W. D. Pillkey, The Riccati transfer matrix method, *Journal of Mechanical Design-Transactions of the ASME*, 100 (4) (1978) 297-302.

[19] Y. M. Zhang, B. C. Wen and Q. L. Liu, Uncertain responses of rotor-stator systems with rubbing, *Mechanical Systems, Machine Elements and Manufacturing - International Journal of the JSME*, 46 (1) (2003) 150-154.

[20] M. F. Dimentberg, Vibration of a rotating shaft with randomly varying internal damping. *Journal of Sound and Vibration*, 285 (3) (2005) 759-765.

[21] M. F. Dimentberg, Transverse vibrations of rotating shafts: probability density and first-passage time of whirl radius, *International Journal of Non-Linear Mechanics*, 40 (10) (2005) 1263-1267.

[22] M. F. Dimentberg, D. V. Iourtchenko and A. Naess, Coherence function of transverse random vibrations of a rotating shaft, *Journal of Sound and Vibration*, 292 (3-5) (2006) 983-986.

[23] N. O. Myklestad, A new method for calculating natural modes of uncoupled bending vibration of airplane wings and other types of beams, *Journal of the Aeronautical Sciences*, 11 (2) (1944) 153-162.

[24] S. C. Hsieh, J. H. Chen and A. C. Lee, A modified transfer matrix method for the coupling lateral and torsional vibrations of symmetric rotor-bearing systems, *Journal of Sound and Vibration*, 289 (1-2) (2006) 294-333.

[25] Y. Kang, A. C. Lee and Y. P. Shih, A modified transfer matrix method for asymmetric rotor-bearing systems, *Journal of Vibration and Acoustics*, 116 (3) (1994) 309-317.

[26] S. J. Oh, Influence coefficients on rotor having thick shaft elements and resilient bearings, *Journal of Sound and Vibration,* 272 (3-5) (2004) 657-673.

Appendix: Data of the first vibration mode of the rotor with two bearings

Mean values in X-direction			Standard deviations in X-direction		
Monte Carlo simulation	First-order random perturbation	Second-order random perturbation	Monte Carlo simulation	First-order random perturbation	Second-order random perturbation
0.17499 + 0.05929i	0.17387 + 0.06079i	0.17558 + 0.06068i	0.04009 + 0.01189i	0.04061 + 0.00210i	0.04030 + 0.00170i
0.16769 + 0.07555i	0.16674 + 0.07733i	0.16818 + 0.07690i	0.03973 + 0.01150i	0.04021 + 0.00240i	0.03991 + 0.00200i
0.16039 + 0.09182i	0.15960 + 0.09388i	0.16079 + 0.09312i	0.03939 + 0.01115i	0.03985 + 0.00270i	0.03956 + 0.00231i
0.14862 + 0.11805i	0.14811 + 0.12057i	0.14887 + 0.11929i	0.03889 + 0.01067i	0.03936 + 0.00318i	0.03910 + 0.00280i
0.13559 + 0.14714i	0.13537 + 0.15017i	0.13565 + 0.14831i	0.03840 + 0.01028i	0.03896 + 0.00371i	0.03871 + 0.00335i
0.11955 + 0.18295i	0.11969 + 0.18661i	0.11940 + 0.18403i	0.03790 + 0.01001i	0.03868 + 0.00436i	0.03844 + 0.00403i
0.10426 + 0.21716i	0.10475 + 0.22141i	0.10390 + 0.21815i	0.03755 + 0.00999i	0.03862 + 0.00495i	0.03839 + 0.00465i
0.08907 + 0.25121i	0.08991 + 0.25606i	0.08851 + 0.25211i	0.03730 + 0.01021i	0.03878 + 0.00551i	0.03854 + 0.00525i
0.07367 + 0.28578i	0.07486 + 0.29123i	0.07290 + 0.28659i	0.03713 + 0.01066i	0.03912 + 0.00603i	0.03886 + 0.00581i
0.05492 + 0.32777i	0.05654 + 0.33396i	0.05389 + 0.32848i	0.03701 + 0.01145i	0.03974 + 0.00656i	0.03946 + 0.00641i
0.04648 + 0.34662i	0.04829 + 0.35314i	0.04534 + 0.34728i	0.03698 + 0.01188i	0.04009 + 0.00676i	0.03979 + 0.00663i
0.02237 + 0.40032i	0.02475 + 0.40777i	0.02091 + 0.40083i	0.03703 + 0.01334i	0.04138 + 0.00718i	0.04102 + 0.00715i
0.01472 + 0.41733i	0.01727 + 0.42508i	0.01315 + 0.41779i	0.03709 + 0.01387i	0.04187 + 0.00727i	0.04149 + 0.00727i
0.01023 + 0.42730i	0.01288 + 0.43522i	0.00860 + 0.42773i	0.03713 + 0.01418i	0.04218 + 0.00730i	0.04177 + 0.00733i
0.00634 + 0.43590i	0.00909 + 0.44397i	0.00466 + 0.43631i	0.03716 + 0.01446i	0.04245 + 0.00733i	0.04203 + 0.00737i
0.00119 + 0.44730i	0.00406 + 0.45556i	-0.00056 + 0.44767i	0.03722 + 0.01483i	0.04283 + 0.00734i	0.04240 + 0.00741i
-0.00618 + 0.46361i	-0.00314 + 0.47216i	-0.00803 + 0.46393i	0.03732 + 0.01539i	0.04341 + 0.00735i	0.04294 + 0.00745i
-0.01099 + 0.47423i	-0.00783 + 0.48297i	-0.01291 + 0.47452i	0.03739 + 0.01576i	0.04381 + 0.00734i	0.04332 + 0.00746i
-0.01784 + 0.48933i	-0.01452 + 0.49833i	-0.01985 + 0.48957i	0.03751 + 0.01629i	0.04441 + 0.00730i	0.04389 + 0.00745i
-0.02227 + 0.49909i	-0.01884 + 0.50826i	-0.02434 + 0.49930i	0.03760 + 0.01665i	0.04481 + 0.00726i	0.04427 + 0.00743i
-0.02852 + 0.51284i	-0.02494 + 0.52225i	-0.03068 + 0.51301i	0.03773 + 0.01716i	0.04541 + 0.00717i	0.04484 + 0.00737i
-0.03254 + 0.52166i	-0.02886 + 0.53122i	-0.03474 + 0.52180i	0.03782 + 0.01749i	0.04581 + 0.00710i	0.04522 + 0.00732i
-0.03814 + 0.53396i	-0.03433 + 0.54374i	-0.04043 + 0.53406i	0.03796 + 0.01796i	0.04640 + 0.00698i	0.04577 + 0.00723i
-0.04170 + 0.54176i	-0.03780 + 0.55167i	-0.04404 + 0.54183i	0.03805 + 0.01826i	0.04679 + 0.00688i	0.04614 + 0.00715i
-0.04662 + 0.55251i	-0.04260 + 0.56261i	-0.04902 + 0.55254i	0.03818 + 0.01868i	0.04736 + 0.00672i	0.04667 + 0.00702i
-0.04970 + 0.55923i	-0.04559 + 0.56945i	-0.05214 + 0.55923i	0.03827 + 0.01895i	0.04772 + 0.00660i	0.04702 + 0.00691i
-0.05388 + 0.56835i	-0.04967 + 0.57872i	-0.05638 + 0.56831i	0.03839 + 0.01932i	0.04825 + 0.00641i	0.04751 + 0.00674i
-0.05644 + 0.57393i	-0.05216 + 0.58440i	-0.05898 + 0.57387i	0.03846 + 0.01955i	0.04859 + 0.00627i	0.04783 + 0.00661i
-0.05985 + 0.58134i	-0.05548 + 0.59193i	-0.06244 + 0.58123i	0.03856 + 0.01987i	0.04907 + 0.00604i	0.04828 + 0.00641i
-0.06189 + 0.58574i	-0.05746 + 0.59641i	-0.06450 + 0.58562i	0.03862 + 0.02006i	0.04937 + 0.00588i	0.04856 + 0.00626i

-0.06449 + 0.59138i	-0.05999 + 0.60214i	-0.06715 + 0.59122i	0.03870 + 0.02031i	0.04980 + 0.00563i	0.04896 + 0.00602i
-0.06598 + 0.59457i	-0.06143 + 0.60539i	-0.06865 + 0.59439i	0.03874 + 0.02045i	0.05006 + 0.00545i	0.04920 + 0.00585i
-0.06776 + 0.59839i	-0.06315 + 0.60926i	-0.07046 + 0.59818i	0.03878 + 0.02064i	0.05042 + 0.00517i	0.04953 + 0.00558i
-0.06868 + 0.60035i	-0.06404 + 0.61125i	-0.07140 + 0.60012i	0.03880 + 0.02074i	0.05064 + 0.00497i	0.04973 + 0.00539i
-0.06963 + 0.60232i	-0.06494 + 0.61326i	-0.07236 + 0.60207i	0.03882 + 0.02086i	0.05093 + 0.00467i	0.05000 + 0.00510i
-0.06998 + 0.60303i	-0.06527 + 0.61397i	-0.07272 + 0.60276i	0.03881 + 0.02091i	0.05110 + 0.00446i	0.05015 + 0.00489i
-0.07008 + 0.60314i	-0.06535 + 0.61408i	-0.07282 + 0.60285i	0.03879 + 0.02096i	0.05131 + 0.00413i	0.05034 + 0.00458i
-0.06986 + 0.60259i	-0.06512 + 0.61351i	-0.07260 + 0.60228i	0.03876 + 0.02097i	0.05143 + 0.00391i	0.05045 + 0.00435i
-0.06911 + 0.60084i	-0.06436 + 0.61173i	-0.07185 + 0.60051i	0.03870 + 0.02095i	0.05156 + 0.00356i	0.05056 + 0.00402i
-0.06832 + 0.59903i	-0.06357 + 0.60989i	-0.07105 + 0.59869i	0.03864 + 0.02091i	0.05162 + 0.00332i	0.05061 + 0.00378i
-0.06673 + 0.59544i	-0.06200 + 0.60622i	-0.06945 + 0.59509i	0.03854 + 0.02082i	0.05168 + 0.00297i	0.05065 + 0.00342i
-0.06537 + 0.59239i	-0.06066 + 0.60312i	-0.06808 + 0.59203i	0.03846 + 0.02073i	0.05168 + 0.00272i	0.05065 + 0.00318i
-0.06296 + 0.58698i	-0.05827 + 0.59761i	-0.06564 + 0.58661i	0.03832 + 0.02058i	0.05165 + 0.00234i	0.05061 + 0.00280i
-0.06105 + 0.58273i	-0.05639 + 0.59327i	-0.06371 + 0.58234i	0.03821 + 0.02045i	0.05161 + 0.00209i	0.05055 + 0.00254i
-0.05784 + 0.57554i	-0.05322 + 0.58595i	-0.06046 + 0.57515i	0.03803 + 0.02022i	0.05150 + 0.00170i	0.05043 + 0.00216i
-0.05540 + 0.57011i	-0.05082 + 0.58042i	-0.05799 + 0.56971i	0.03790 + 0.02005i	0.05139 + 0.00144i	0.05032 + 0.00189i
-0.05141 + 0.56121i	-0.04689 + 0.57135i	-0.05395 + 0.56081i	0.03769 + 0.01976i	0.05120 + 0.00105i	0.05012 + 0.00149i
-0.04846 + 0.55464i	-0.04398 + 0.56466i	-0.05097 + 0.55423i	0.03753 + 0.01954i	0.05104 + 0.00078i	0.04996 + 0.00122i
-0.04373 + 0.54410i	-0.03933 + 0.55393i	-0.04618 + 0.54369i	0.03728 + 0.01919i	0.05077 + 0.00038i	0.04969 + 0.00081i
-0.04029 + 0.53644i	-0.03595 + 0.54613i	-0.04270 + 0.53603i	0.03710 + 0.01893i	0.05057 + 0.00011i	0.04948 + 0.00054i
-0.03486 + 0.52435i	-0.03062 + 0.53381i	-0.03721 + 0.52394i	0.03682 + 0.01852i	0.05022 + 0.00029i	0.04914 + 0.00013i
-0.03097 + 0.51567i	-0.02679 + 0.52498i	-0.03327 + 0.51526i	0.03663 + 0.01822i	0.04997 + 0.00056i	0.04888 + 0.00015i
-0.02489 + 0.50212i	-0.02082 + 0.51118i	-0.02712 + 0.50171i	0.03633 + 0.01776i	0.04955 + 0.00096i	0.04847 + 0.00056i
-0.02058 + 0.49250i	-0.01658 + 0.50138i	-0.02275 + 0.49209i	0.03611 + 0.01743i	0.04925 + 0.00123i	0.04817 + 0.00084i
-0.01391 + 0.47760i	-0.01003 + 0.48622i	-0.01600 + 0.47720i	0.03579 + 0.01691i	0.04877 + 0.00162i	0.04770 + 0.00124i
-0.00921 + 0.46711i	-0.00542 + 0.47553i	-0.01125 + 0.46671i	0.03557 + 0.01655i	0.04843 + 0.00189i	0.04736 + 0.00152i
-0.00200 + 0.45100i	0.00166 + 0.45913i	-0.00395 + 0.45061i	0.03524 + 0.01599i	0.04790 + 0.00227i	0.04684 + 0.00191i
0.00304 + 0.43973i	0.00660 + 0.44765i	0.00115 + 0.43934i	0.03501 + 0.01560i	0.04753 + 0.00253i	0.04647 + 0.00218i
0.00684 + 0.43122i	0.01033 + 0.43899i	0.00500 + 0.43084i	0.03485 + 0.01530i	0.04724 + 0.00272i	0.04619 + 0.00237i
0.01124 + 0.42136i	0.01466 + 0.42895i	0.00946 + 0.42098i	0.03465 + 0.01496i	0.04691 + 0.00293i	0.04587 + 0.00259i
0.01882 + 0.40436i	0.02209 + 0.41164i	0.01713 + 0.40399i	0.03432 + 0.01436i	0.04631 + 0.00326i	0.04528 + 0.00294i
0.04291 + 0.35019i	0.04573 + 0.35650i	0.04152 + 0.34987i	0.03331 + 0.01246i	0.04436 + 0.00421i	0.04338 + 0.00393i
0.05140 + 0.33107i	0.05406 + 0.33703i	0.05012 + 0.33076i	0.03299 + 0.01178i	0.04367 + 0.00449i	0.04270 + 0.00423i
0.06557 + 0.29911i	0.06795 + 0.30450i	0.06446 + 0.29882i	0.03245 + 0.01065i	0.04248 + 0.00492i	0.04155 + 0.00469i
0.07076 + 0.28736i	0.07305 + 0.29254i	0.06972 + 0.28709i	0.03226 + 0.01024i	0.04204 + 0.00506i	0.04112 + 0.00484i
0.08656 + 0.25166i	0.08854 + 0.25620i	0.08571 + 0.25142i	0.03172 + 0.00898i	0.04069 + 0.00544i	0.03980 + 0.00524i
0.10186 + 0.21711i	0.10354 + 0.22102i	0.10120 + 0.21690i	0.03125 + 0.00775i	0.03939 + 0.00573i	0.03854 + 0.00556i
0.11727 + 0.18238i	0.11865 + 0.18568i	0.11681 + 0.18221i	0.03087 + 0.00652i	0.03811 + 0.00593i	0.03730 + 0.00579i
0.13343 + 0.14603i	0.13450 + 0.14867i	0.13318 + 0.14589i	0.03056 + 0.00522i	0.03681 + 0.00606i	0.03604 + 0.00594i
0.14977 + 0.10947i	0.15052 + 0.11145i	0.14972 + 0.10937i	0.03037 + 0.00392i	0.03556 + 0.00606i	0.03482 + 0.00597i
0.16619 + 0.07296i	0.16663 + 0.07428i	0.16634 + 0.07289i	0.03031 + 0.00261i	0.03436 + 0.00595i	0.03366 + 0.00588i
0.18265 + 0.03648i	0.18277 + 0.03714i	0.18300 + 0.03644i	0.03035 + 0.00131i	0.03321 + 0.00569i	0.03254 + 0.00564i
0.19912 + 0.00000i	0.19892 + 0.00000i	0.19968 + 0.00000i	0.03050 + 0.00000i	0.03212 + 0.00528i	0.03148 + 0.00526i

Bao-Guo Liu was born in Henan, China in 1962. He received his B.E. at the Henan University of Technology, China in 1982, M.E. at the Dalian University of Technology, China in 1986, and his Ph.D at the Chongqing University, China, in 2002. He worked for Zheng-zhou Research Institute of Mechanical Engineering (1986-2002), and works currently as a professor in Henan University of Technology. His research focuses on rotordynamics, random parameter vibration and fault diagnosis of rotational machinery.