Kwang-Min Won *
Youn-sik Park **
Korea Advanced Institute of Science and Technology, Taejon 305-701, Korea

Eigenvalue Sensitivity Analysis for a Combined Beam Structure with Varying Constraints

An eigenvalue sensitivity formula for a combined beam structure due to the variation of its shape or joint coordinates is proposed based on the Lagrange multiplier technique for a free vibration equation of the structure. The shape variation is decomposed into the length and the orientation variation of each beam to obtain individual effects on the total sensitivity. The sensitivity equations due to the length and the orientation variations are expressed as an energy density form and the cross product of joint forces and displacements respectively. Several numerical examples are also presented to validate the proposed formulation and to show how to implement the idea.

INTRODUCTION

Eigenvalue sensitivity analysis has attracted a great attention in the areas of structural design modifications and optimizations, since it can estimate the change of dynamic characteristics after modifying structures. In early stages, many researchers considered size of a structure as design variables such as thickness, cross sectional area, moment of inertia, and etc. Recently, shape sensitivity analysis has been popular in which design variables are structural shape itself. When considering combined substructures, composed of many substructures, the variations of both shape and orientation of each substructure must be accounted to derive eigenvalue sensitivity. This work focuses on this problem.

Son and Kwak (1993) presented the eigenvalue sensitivity formula with respect to the change of boundary positions using the material derivative concept. They utilized the tangential component of design velocities at boundaries. The derived formula was applied to finding optimal support positions of beams and plates. Chuang and Hou (1990, 1992) also proposed the eigenvalue sensitivity formula as varying the support positions of a beam or the joint positions of planar frame based on continuum approach with the material derivative method. Both the domain method and the boundary method were used in deriving the sensitivities. Twu and Choi (1992) derived a continuum based configuration design sensitivity for built-up structures using the consistent way as in driving

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*Ph. D. Student, Center for Noise and Vibration Control, Dept. of Mech. Eng.

**Professor, Center for Noise and Vibration Control, Dept. of Mech. Eng. Correspondence should be addressed to Professor Youn-sik Park.

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the shape sensitivity (Haug et al., 1986). They assumed that the change of built-up structures can be expressed with its shape and orientation changes. Wang (1993) derived the eigenvalue sensitivity formula with respect to discrete in span occurrence members using the normal mode method. He obtained an explicit form of eigenvalue sensitivity by explaining that the eigenvalue sensitivity is directly proportional to the slope of its mode shape and the reaction force at the support point. Lie et al. (1996) also derived eigenvalue sensitivity with respect to in span support location using the generalized variational principle of Rayleigh's quotient with the Lagrange multiplier method.

In this paper, a different approach is proposed to derive eigenvalue sensitivities of a combined beam structure due to its joint coordinate variations. Firstly, it formulates the free vibration equation of the combined structure using Lagrange multiplier techniques. And then, it is assumed that the variation of joint coordinates can be decomposed into the variations of free ends, joint positions, and the orientation of each subbeam. Thus, the resulting sensitivity can be obtained by summing the effects of such variations. Each effect is derived by taking a variation of the free vibration equation. A single beam structure is investigated to show the proposed idea before applying a more complex combined beam structure. Some numerical examples are also shown to compare the accuracy of proposed methods and their applicability to real problems.

EIGENVALUE SENSITIVITY ANALYSIS FOR A BEAM STRUCTURE

In the filed of eigenvalue sensitivity analysis of a structure with varying its shape or constraint positions, earlier authors considered boundary conditions or constraint positions in an implicit manner. That means that they did not include constraint equations in the free vibration equation. However, this work will incorporate constraint equations in the free vibration equation through the Lagrange multiplier technique. This approach has several advantages. Firstly, it can provide a systematic way to derive sensitivity equations when the design variables are structural shape and constraint position. Secondly, it can be applied to complexly combined structures or a structure having multiple supports or substructures. That is because the free vibration equation of those structures is easily expressed using the Lagrange multiplier technique. Lastly, the accuracy can be improved considerably, since it uses force information instead of higher derivatives of displacement at constraint positions. Those facts will be clearly shown in later sections.

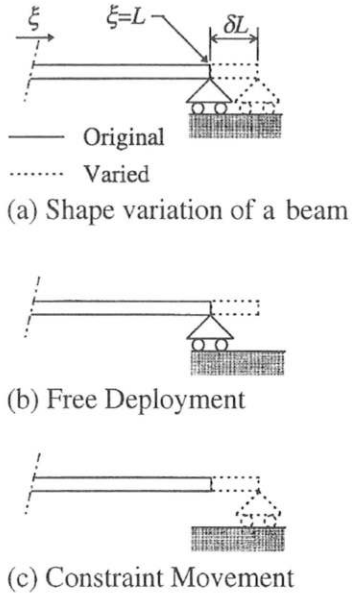

FIGURE 1 Assumptions on the shape variation of a beam.

To clarify the idea, the approach is developed for a single beam structure.

Before deriving sensitivity equations, we made an assumption regarding the shape or the length variation of the beam. Figure 1 clearly shows the assumption. As shown in that figure, the shape variation is assumed to be two consecutive steps – free end deployment and constraint movement. Then the resulting sensitivity is obtained by summing each of the effects. The effect on sensitivity due to the free deployment can be easily derived by utilizing the conventional sensitivity equation, and the one due to the constraint movement is obtained by a method proposed in this section.

The same procedure developed in this section will be extended for a combined beam structure in the following section.

Previous Study

Haug et al. (1986) solved this problem using the material derivative concept. They used a variational form of the free vibration equation, and the boundary condition was treated in an implicit manner. The resulting sensitivity of a uniform beam is Eq. (1) when its end at $\xi = L$ is deployed as shown in Fig. 1(a).

$$\lambda_{,L} = -\lambda \rho A u^2(L) - EI u_{,\xi\xi\xi}(L) u_{,\xi}(L) + 2EI u_{,\xi\xi}^2(L), \tag{1}$$

where L, ρ, A, I, and E are the length, mass density, cross-sectional area, moment of inertia, and Young's modulus respectively. λ and u are the eigenvalue and the eigenfunction, respectively. A variable following a comma with a subscript variable denotes the partial derivative of the variable with respect to the subscript variable. The second and third terms in Eq. (1) can be eliminated depending on the boundary condition, e.g., both are zero for free end.

Free Vibration Equation

One of the main ideas in this work is using Lagrange multipliers to formulate the free vibration equation of a constrained structure. The free vibration equation of a beam, whose transverse displacement at $\xi = s$ is fixed, is formulated using the Lagrange multiplier method (Reddy, 1986).

$$a\big(u(\xi), \bar{u}(\xi)\big) - \lambda d\big(u(\xi), \bar{u}(\xi)\big) + l\bar{u}(\xi)\big|_{\xi=s} = 0, \tag{2a}$$

$$u(\xi)|_{\xi=s} = 0, \tag{2b}$$

where $\bar{u}$ is the kinematically admissible displacement and l is the Lagrange multiplier. The kinetic and potential energy bilinear forms for a beam, $a(\cdot,\cdot)$ and $d(\cdot,\cdot)$, are

$$a(u, \bar{u}) = \int_0^L EI u_{,\xi\xi}^{2} \bar{u}_{,\xi\xi}^{2}\, d\xi, \tag{3a}$$

$$d(u, \bar{u}) = \int_0^L \rho A u \bar{u}\, d\xi. \tag{3b}$$

The eigenvalue and eigenfunction must satisfy Eq. (2) for all kinematically admissible displacement. The eigenfunction is normalized as,

$$d(u, u) = 1. \tag{4}$$

Earlier authors did not include constraint equations in the free vibration equation, since they used kinematically admissible functions satisfying the constraints.

Eigenvalue Sensitivity on Free End Deployment

The eigenvalue sensitivity due to free deployment without the movement of constraint ($\delta s = 0$), as shown in Fig. 2, can be derived using Eq. (1). Even though Eq. (1) was derived for a beam having no kinematic constraint within its domain, it also holds for a constrained beam if the constraint does not move with the end deployment. This can be easily proven by defining a local design velocity field near the free end as in Son and Kwak (1993). Then the sensitivity becomes

$$\lambda_{,F} = -\lambda \rho A u^2(L) \tag{5}$$

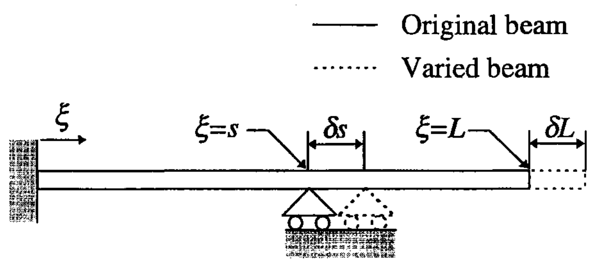

FIGURE 2 Shape variation of a constrained beam.

where $\lambda_{,F}$ denotes the sensitivity due to the deployment of the free end. Equation (5) is also satisfied when the constraints located at the end of the beam because the deployed part is also free, as shown in Fig. 1(b).

Eigenvalue Sensitivity on Constraint Movement

Many authors studied eigenvalue sensitivity analysis with varying constraint position (Hou and Chuang, 1990; Wang, 1993; Liu et al., 1996), but no one included constraint equations in the free vibration equation.

To derive the sensitivity based on Eq. (2), Eq. (2a) is differentiated with respect to constraint positions, s, after substituting u for $\bar{u}$ into that equation. In this process, the length of the beam does not change ($\delta L = 0$). Then the sensitivity becomes

$$\lambda_{,s} = 2\big[a(u, u_{,s}) - \lambda d(u, u_{,s})\big]. \tag{6}$$

A normalizing condition is used. Since the constraint equation must be satisfied before and after the variation of constraint position, the derivative of the Lagrange term becomes zero. One can also substitute $u_{,s}$ for $\bar{u}$ in Eq. (2a) to obtain Eq. (7), because the eigenfunction is also a member of admissible functions even after the variation of constraint position.

$$a(u_{,s}, u) - \lambda d(u_{,s}, u) + l u_{,s}(s) = 0. \tag{7}$$

From Eqs. (6) and (7), the sensitivity becomes

$$\lambda_{,s} = -2l u_{,s}(s). \tag{8}$$

Another important relationship is obtained by taking the derivative of the constraint equation, Eq. (2b), with respect to s. By noting that the eigenfunction is function of both the domain variable, ξ, and the constraint position, s, the derivative of the constraint equation becomes

$$\big[u(\xi, s)\big|_{\xi=s}\big]_{,s} = u_{,s}(s) + u_{,\xi}(s) = 0. \tag{9}$$

One can reach the same result using Taylor series expansion of eigenfunction at $\xi = s$. From Eqs. (8) and (9), the sensitivity becomes

$$\lambda_{,s} = 2lu_{,\xi}(s). \quad (10)$$

Examining Eq. (2a), it is clear that the Lagrange multiplier is the constraint force with a minus sign. Thus, the final form of sensitivity due to constraint movement becomes

$$\lambda_{,s} = -2f_V u_{,\xi}(s) \quad (11)$$

where f_V is the vertical reaction force at the constraint position. If the constraint is in rotational displacement instead of in transverse displacement, following the same procedure, the sensitivity becomes

$$\lambda_{,s} = -2f_M \tilde{u}_{,\xi\xi}(s), \quad (12)$$

where f_M is the moment at the constraint. Since $u_{,\xi\xi}$ is not continuous across the constraint position, the averaged value, $\tilde{u}_{,\xi\xi}$, is used. Let the position of the displacement constraint and the rotational displacement constraint coincide and move to the end of the beam, then the sensitivity becomes

$$\lambda_{,s} = -2f_V u_{,\xi}(L) - f_M u_{,\xi\xi}(L). \quad (13)$$

The averaged value of the second derivative of the eigenfunction becomes $u_{,\xi\xi}/2$ as the constraint point moves to the free end.

The resulting form of the shape eigenvalue sensitivity is obtained by combining Eqs. (5) and (13),

$$\lambda_{,L} = -\lambda\rho A u^2(L) - 2f_V u_{,\xi}(L) - f_M u_{,\xi\xi}(L). \quad (14)$$

It is clear that Eq. (14) is the other form of Eq. (1) by considering the classical beam theory which associates internal forces and displacements. Thus, the proposed procedure can be an alternative to deriving sensitivity equations. The usefulness of this procedure is verified in later sections.

EIGENVALUE SENSITIVITY ANALYSIS FOR A COMBINED BEAM STRUCTURE

A combined beam structure is a structure composed of multiple beams connected by constraints or joints. The shape eigenvalue sensitivity of the structure is the variation of the eigenvalue due to the variation of its joint coordinates. Some authors have derived the sensitivity formula using the material derivative concept (Chuang and Hou, 1992; Twu and Choi, 1992).

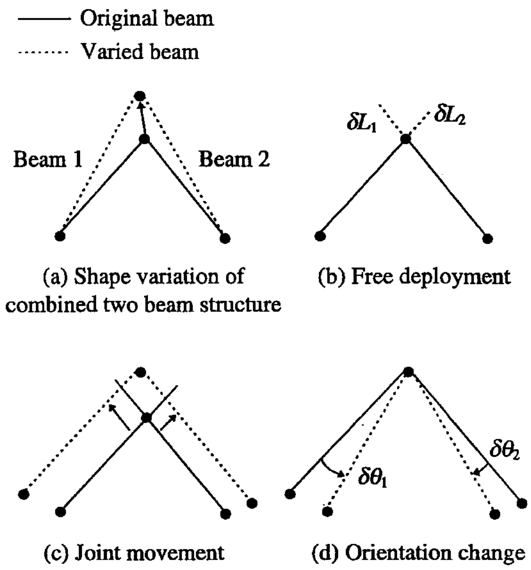

FIGURE 3 Shape variation of combined two beams.

The shape eigenvalue sensitivity for a combined beam structure is derived in this section by using the same procedure developed in the previous section. A similar assumption, as made in the previous section, is applied to describe the variation of shape or configuration of a combined beam structure. In order to achieve the final configuration, it is assumed that the original structure is varied in three consecutive steps – end deployments, joint movements, and orientation changes of sub-beams as shown in Fig. 3. Thus, the shape eigenvalue sensitivity can be considered as combined effects of free end deployment, joint movement, and orientation change of each sub-beam.

Only a planar beam is considered in this work, but the proposed approach can be extended to spatial beam structures without further difficulties.

Free Vibration of Combined Beam Structure

Figure 4 shows two beams, which are connected by joint j at position $\mathbf{s}_j$ of a structure composed of n beams and m rigid joints. The free vibration equation of the structure is formulated using the Lagrange multiplier method as follows.

$$\sum_{i=1}^{n}\left(a(\mathbf{u}^{(i)}, \bar{\mathbf{u}}^{(i)}) - \lambda d(\mathbf{u}^{(i)}, \bar{\mathbf{u}}^{(i)})\right) + \sum_{j=1}^{m} \mathbf{l}_j^T\left(\bar{\mathbf{u}}^{(j1)}(\mathbf{s}_j) - \bar{\mathbf{u}}^{(j2)}(\mathbf{s}_j)\right) + \sum_{j=1}^{m} l_{Mj}\left(\bar{u}_{,\xi}^{(j1)}(\mathbf{s}_j) - \bar{u}_{,\xi}^{(j2)}(\mathbf{s}_j)\right) = 0, \quad (15a)$$

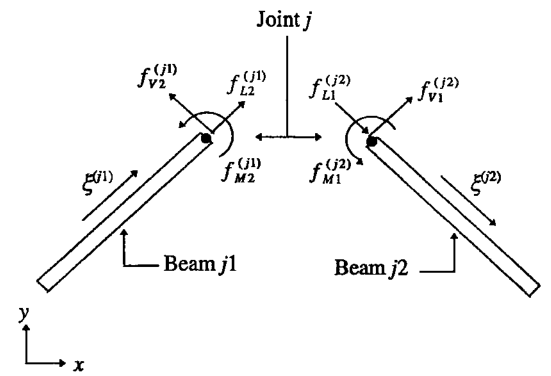

FIGURE 4 Two beams connected by joint j.

$$\begin{aligned} &\mathbf{u}^{(j1)}(\mathbf{s}_j) - \mathbf{u}^{(j2)}(\mathbf{s}_j) = 0, \\ &u,_{\xi}^{(j1)}(\mathbf{s}_j) - u,_{\xi}^{(j2)}(\mathbf{s}_j) = 0, \\ &\quad \text{for } j = 1, \ldots, m, \end{aligned} \tag{15b}$$

where $\mathbf{u}$ is displacement vector composed of transverse displacement, u, and longitudinal displacement, v. $\mathbf{l}$ is the vector of Lagrange multipliers for the displacement constraint and l_M is the Lagrange multiplier for the rotational displacement constraint. The superscript T denotes transpose of a vector or a matrix and (i) on a variable denotes that the variable belongs to beam i. The eigenvalue and eigenfunction must satisfy Eq. (15) for all kinematically admissible displacements, $\bar{v}$ and $\bar{u}$. The kinetic and potential energy bilinear form of each beam is

$$a(\mathbf{u}^{(i)}, \bar{\mathbf{u}}^{(i)}) = \int_0^{L^{(i)}} \big(E^{(i)}A^{(i)}v,_{\xi}^{(i)}\, \bar{v},_{\xi}^{(i)} + E^{(i)}I^{(i)}u,_{\xi\xi}^{(i)}\, \bar{u},_{\xi\xi}^{(i)}\big)\, d\xi^{(i)}, \tag{16a}$$

$$d(\mathbf{u}^{(i)}, \bar{\mathbf{u}}^{(i)}) = \int_0^{L^{(i)}} \rho^{(i)}A^{(i)}\big(v^{(i)}\bar{v}^{(i)} + u^{(i)}\bar{u}^{(i)}\big)\, d\xi^{(i)}. \tag{16b}$$

Normalization condition is also used.

$$\sum_{i=1}^{n} d\big(\mathbf{u}^{(i)}, \bar{\mathbf{u}}^{(i)}\big) = 1. \tag{17}$$

Eigenvalue Sensitivity on Free End Deployment

As shown in Fig. 3(b), since the internal force and the moment are zero in the deployed section of the beam, only the kinetic energy terms have effect on the sensitivity due to the free deployment as in the case of the single beam of previous section. Thus the eigenvalue variation due to free deployment is

$$\delta\lambda_F = \sum_{i=1}^{n} \lambda\rho^{(i)}A^{(i)}\big(\|\mathbf{u}_1^{(i)}\|^2 \delta L_1^{(i)} - \|\mathbf{u}_2^{(i)}\|^2 \delta L_2^{(i)}\big) \tag{18}$$

where subscripts 1 or 2 denote that the value is evaluated at $\xi^{(l)} = 0$ or $\xi^{(l)} = L^{(i)}$ respectively. δL_1 and δL_2 are the length variation at the ends of a beam. $\|\cdot\|$ denotes a norm operator.

Eigenvalue Sensitivity on Joint Movement

Following the same procedure for the case of a constrained single beam, the sensitivity is obtained when joints move to the varied ends as shown in Fig. 3(c). One can obtain the equation caused by the joint movements as follows.

$$\begin{aligned} \delta\lambda_C = &- \sum_i \big(f_{L1}^{(i)} v_1,_{\xi}^{(i)} + 2 f_{V1}^{(i)} u_1,_{\xi}^{(i)} + f_{M1}^{(i)} u_1,_{\xi\xi}^{(i)}\big)\delta L_1^{(i)} \\ &- \sum_i \big(f_{L2}^{(i)} v_2,_{\xi}^{(i)} + 2 f_{V2}^{(i)} u_2,_{\xi}^{(i)} + f_{M2}^{(i)} u_2,_{\xi\xi}^{(i)}\big)\delta L_2^{(i)} \end{aligned} \tag{19}$$

where f_{Li}, f_{Vi}, and f_{Mi} are the constraint forces as in Fig. 4. The detailed derivation will not be shown because of its complexity.

Energy Density Form

Even though the sensitivity due to shape or length variation of each sub-beam is the sum of Eqs. (18) and (19), there is a general form which governs it. The form can be derived by applying those equations to a single beam base, which is considered as two combined beams as shown in Fig. 5. Then, the sensitivity is the sum of the effects of free deployment and joint movement of beam 1.

$$\begin{aligned} \delta\lambda_L = &-\big(\lambda\rho A\|\mathbf{u}^{(1)}(L_1)\|^2 + f_{L2}^{(1)} v,_{\xi}^{(1)}(L_1) \\ &+ 2 f_{V2}^{(1)} u,_{\xi}^{(1)}(L_1) + f_{M2}^{(1)} u,_{\xi\xi}^{(1)}(L_1)\big)\delta L_1. \end{aligned} \tag{20}$$

Since the constraint forces in Eq. (20) are also the internal forces at $\xi = s$ of the original beam, Eq. (20) can be rewritten as

$$\begin{aligned} \delta\lambda_L = &-\big(\lambda\rho A\|\mathbf{u}(s)\|^2 + g_L(s) v,_{\xi}(s) \\ &+ 2 g_V u,_{\xi}(s) + g_M(s) u,_{\xi\xi}(s)\big)\delta L, \end{aligned} \tag{21}$$

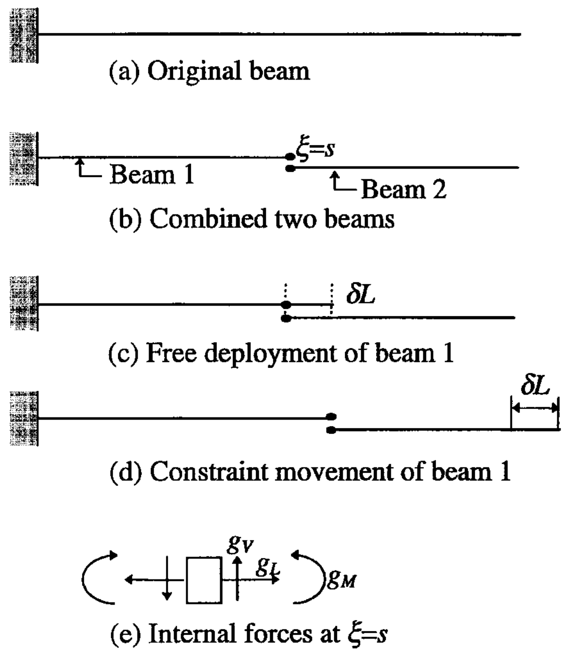

FIGURE 5 Shape variation of a single beam with a concept of two beam structure.

where g_L, g_V, and g_M are the internal forces of the original beam at s as in Fig. 5(e). However, the sensitivity is independent of the constraint position, s, Eq. (21) holds for every point, ξ. Thus, we can define a constant variable for a uniform beam which governs the shape eigenvalue sensitivity of a beam

$$E(\xi) = \text{constant}, \tag{22}$$

where,

$$E(\xi) = E_K(\xi) + E_P(\xi), \tag{23a}$$

$$E_K(\xi) \equiv \lambda\rho A\|\mathbf{u}(\xi)\|^2, \tag{23b}$$

$$E_P(\xi) \equiv g_L v_{,\xi}(\xi) + 2g_V u_{,\xi}(\xi) + g_M u_{,\xi\xi}(\xi), \tag{23c}$$

E, E_K, and E_P will be called energy density form and kinetic and potential energy density form respectively. It is noticeable that the shape sensitivity of a beam can be obtained from the properties at any point of the beam domain including the boundary. That is the extension of the conventional sensitivity, Eq. (1), which is evaluated only at the boundary.

Eigenvalue Sensitivity on Orientation Change

Figure 3(d) shows the orientation variation after the deployment of the free end and the movement of the joint. The eigenvalue sensitivity is also derived by applying the proposed procedure. Then the sensitivity becomes

$$\delta\lambda_\theta = -2\sum_{j=1}^{m}\mathbf{l}_j^{\mathrm{T}}\left(\frac{\partial}{\partial\theta^{(j1)}}\mathbf{u}^{(j1)}\left(\mathbf{s}_j^{(j1)}\right)\delta\theta^{(j1)} - \frac{\partial}{\partial\theta^{(j2)}}\mathbf{u}^{(j2)}\left(\mathbf{s}_j^{(s2)}\right)\delta\theta^{(j2)}\right). \tag{24}$$

Equation (24) is also rewritten as follows:

$$\delta\lambda_\theta = -2\sum_{i=1}^{n}\left(\mathbf{f}_1^{(i)\mathrm{T}}\frac{\partial}{\partial\theta^{(i)}}\mathbf{u}_1^{(i)} + \mathbf{f}_2^{(i)\mathrm{T}}\frac{\partial}{\partial\theta^{(i)}}\mathbf{u}_2^{(i)}\right)\delta\theta^{(i)}, \tag{25}$$

where $\mathbf{f}$ denotes the joint force vector. The term in Eq. (25) can be also written as

$$\mathbf{f}^{\mathrm{T}}\frac{\partial}{\partial\theta}\mathbf{u} = \mathbf{f}^{\mathrm{T}}\frac{\partial}{\partial\theta}\left(\mathbf{A}(\theta)\hat{\mathbf{u}}\right) = \mathbf{f}^{\mathrm{T}}\frac{\partial}{\partial\theta}\left(\mathbf{A}(\theta)\right)\mathbf{A}^{\mathrm{T}}(\theta)\mathbf{u}, \tag{26}$$

where $\hat{\mathbf{u}}$ is the displacement vector expressed in the local coordinate of a beam and $A(\theta)$ is the transformation matrix from the local coordinate system of a beam to the global coordinate system. Using the cross product of vectors, Eq. (26) becomes

$$\mathbf{f}^{\mathrm{T}}\frac{\partial}{\partial\theta}\mathbf{u} = -\mathbf{f}\times\mathbf{u}\cdot\mathbf{n}_z, \tag{27}$$

where $\mathbf{n}_z$ is the unit normal vector to the plane where the structure locate. Thus the eigenvalue variation can be expressed as

$$\delta\lambda_\theta = 2\sum_{i=1}^{n}\left(\mathbf{f}_1^{(i)}\times\mathbf{u}_1^{(i)} + \mathbf{f}_2^{(i)}\times\mathbf{u}_2^{(i)}\right)\cdot\mathbf{n}_z\delta\theta^{(i)}. \tag{28}$$

Finally, the shape eigenvalue sensitivity of a combined beam structure is obtained using the energy density form and Eqs. (21), (23), and (28)

$$\delta\lambda = -\sum_{i=1}^{n}E^{(i)}\delta L^{(i)} + 2\sum_{i=1}^{n}\left(\mathbf{f}_1^{(i)}\times\mathbf{u}_1^{(i)} + \mathbf{f}_2^{(i)}\times\mathbf{u}_2^{(i)}\right)\cdot\mathbf{n}_z\delta\theta^{(i)}. \tag{29}$$

Calculation of Eigenvalue Sensitivity

From the classical beam theory which associates internal forces and deflections, the energy density form, Eq. (23), can be expressed in three different ways:

$$E(\xi) = \lambda\rho A(v^2 + u^2) + g_L v,_\xi + 2g_V u,_\xi + g_M u,_{\xi\xi}, \tag{30a}$$

$$E(\xi) = \lambda\rho A(v^2 + u^2) + EAv,_\xi^2 - 2EIu,_{\xi\xi\xi}\, u,_\xi + EIu,_{\xi\xi}^2, \tag{30b}$$

$$E(\xi) = \lambda\rho A(v^2 + u^2) + \frac{g_L^2}{EA} + 2g_V u,_\xi + \frac{g_M^2}{EI}. \tag{30c}$$

If $E(\xi)$ is to be evaluated at the joint position ($\xi = 0$ or L), then the internal forces become joint forces. The joint forces are obtained by using the discrete version of free vibration equation, Eqs. (15)

$$[\mathbf{K} - \lambda\mathbf{M}]\{\mathbf{u}\} + \mathbf{C}\{\mathbf{l}\} = \{\mathbf{0}\}, \tag{31a}$$

$$\mathbf{C}^{\mathrm{T}}\{\mathbf{l}\} = \{\mathbf{0}\}, \tag{31b}$$

where $\{\mathbf{u}\}$ and $\{\mathbf{l}\}$ are the eigenvector and the Lagrange multiplier vector respectively. The mass and stiffness matrices, **M** and **K**, are made by simply augmenting the mass and stiffness matrices of sub-beams, thus those are block diagonal matrices. **C** is a constraint matrix to connect sub-beams at joints, and that is usually Boolean matrix. The external force vector is

$$\{\mathbf{f}\} = -\mathbf{C}\{\mathbf{l}\} = [\mathbf{K} - \lambda\mathbf{M}]\{\mathbf{u}\}. \tag{32}$$

Since the joint forces are only the external forces, the joint forces can be obtained from $\{\mathbf{f}\}$. The procedure to obtain energy density form can be summarized as follows:

(i) Solve eigenvalue problem of the corresponding structure.
(ii) Obtain $v,_\xi$, $u,_{\xi\xi}$, and $u,_{\xi\xi\xi}$ from known element shape functions and eigenvector.
(iii) Form **M** and **K** using mass and stiffness matrices of sub-beams.
(iv) Obtain joint forces from the external force vector.
(v) Obtain E using Eqs. (30).

If the internal forces need to be evaluated at a certain point of a beam, the same approach can be applied by treating the beam as two beams which are rigidly connected at that point.

From the first of Eqs. (30), we will call the mixed, displacement, and force equation of method. The displacement method uses only displacement information, thus it is readily applicable after solving the eigenvalue problem of considering structure. The mixed method is the direct result of the proposed approach. The force method utilizes any possible force information. The accuracy of those methods will be compared in the following section by investigating several examples.

EXAMPLES

Analytic Beam

Since the exact eigenvalue of a cantilevered beam, which one end is fixed at $\xi = 0$ and the other end is free at $\xi = L$, is known as Eq. (33a) in Blevins (1979), the analytical form of its shape eigenvalue sensitivity is Eq. (33b)

$$\lambda = \frac{c^4}{L^4}\frac{EI}{\rho A}, \tag{33a}$$

$$\lambda,_L = -4\frac{c^4}{L^5}\frac{EI}{\rho A}, \tag{33b}$$

where c is a constant and that is 1.8751 for the first mode. The same sensitivity equation can be obtained using the conventional sensitivity equation, Eq. (1), and the exact eigenfunction, Eq. (34), at $\xi = L$

$$u = \frac{1}{\sqrt{\rho AL}}\left[\cosh\frac{c}{L}x - \cos\frac{c}{L}x - \frac{\sinh c - \sin c}{\cosh c + \cos c}\left(\sinh\frac{c}{L}x - \sin\frac{c}{L}x\right)\right]. \tag{34}$$

But it is not difficult to show that the sensitivity can be also obtained using the analytic eigenfunction irrespective of ξ from the proposed equations, Eqs. (29) and (30b). Figure 6 shows the kinetic and potential energy densities for the first mode in case of $L = 1$ and $EI/\rho A = 1$. SI units are used in all examples.

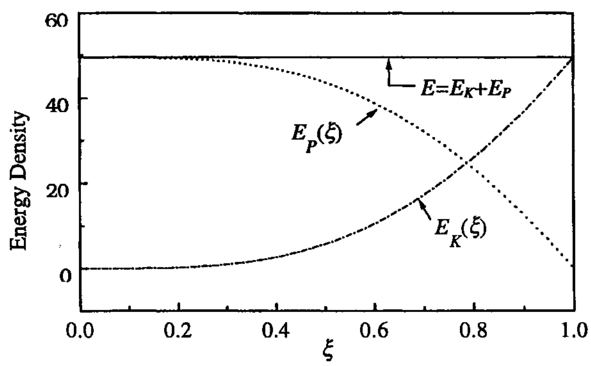

FIGURE 6 Energy densities of a cantilevered beam.

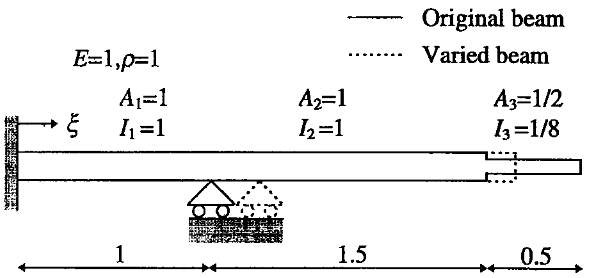

FIGURE 7 A stepped beam with a support.

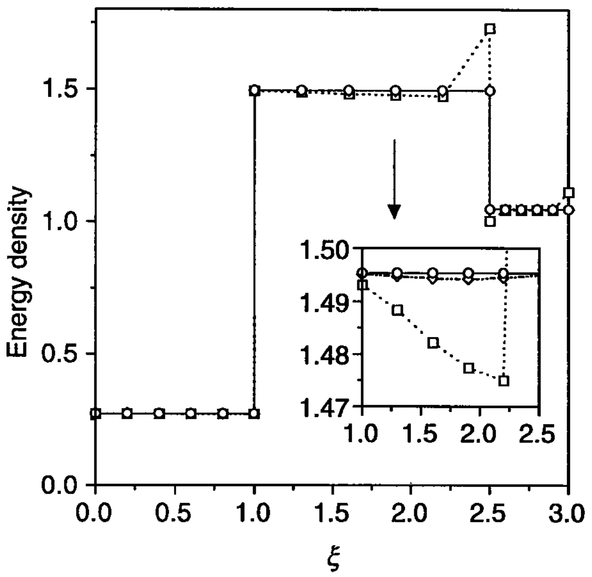

FIGURE 8 Energy densities of a stepped beam with a support (□: displacement method, ◇: mixed method, and o: force method).

A Stepped Beam with a Support

The proposed method can be easily applied to obtain eigenvalue sensitivities of a beam when its in-span discrete components move. The components could be a support, step, spring, substructure, and etc. Figure 7 shows an example of that beam. It has a support at $\xi = 1$ and a step at $\xi = 2.5$. To apply the proposed method, the beam is treated as a three uniform beam structure joined at the support and the step position.

The energy densities are numerically obtained using the proposed three methods to verify the conservative property of the densities in the uniform beam domain and to study the error characteristics of the methods. Five elements per beam are used for finite element model. Figure 8 shows the conservative property clearly. The force method provides the most accurate results but the displacement method shows some errors especially near joints. This aspect is identical with the characteristics of the finite element displacement method which usually provides inaccurate results for the derivative of displacements or stresses at discontinuous points.

The sensitivity formula due to the support or the step movement can be expressed explicitly by applying the proposed method, since the sensitivity is simply the difference of energy density forms of two neighboring beams. For the support movement, because only the internal shear forces are not continuous at the support position, the sensitivity becomes

$$\begin{aligned} \lambda' &= E^{(2)}(0) - E^{(1)}(1) \\ &= 2\left(f_V^{(2)}(0) - f_V^{(1)}(1)\right)u_{,\xi}(1). \end{aligned} \tag{35}$$

The difference of internal shear forces in Eq. (35) is the support reaction force.

The sensitivity with respect to the variation of the step position becomes

$$\begin{aligned} \lambda' &= E^{(3)}(0) - E^{(2)}(1.5) \\ &= \lambda\rho(A_2 - A_1)u^2(2.5) \\ &\quad + f_m^2(2.5)\frac{1}{E}\left(\frac{1}{I_2} - \frac{1}{I_1}\right). \end{aligned} \tag{36}$$

Equation (36) shows that the discontinuities of area and inertia moment affect the eigenvalue sensitivity.

The eigenvalue sensitivities with respect to step position variation are numerically obtained and the results are compared with those of the finite difference method. The forward and backward finite difference methods are used.

$$\lambda'_{\mathrm{FDM}} = \frac{\lambda(s+\delta s) - \lambda(s-\delta s)}{2\delta s}. \tag{37}$$

Two, five, and ten elements are used in meshing each beam, and the results are summarized in Table 1. The first eigenvalue decreases as the step moves to right but the second and third eigenvalues increase. The results of the mixed and force methods agree well with those of the finite difference method. The displacement method gives almost useless results for this example. The reason is that it uses the derivation of v and the second and third derivation of u at the stepped point, and those values from the finite element technique are usually inaccurate especially at such a discontinuous point. The eigenvalues and sensitivities are converge as the number of elements increase.

Two Beam Structure

The proposed methods are applied to two beam structures as shown in Fig. 9. If the joint position is moved in the ξ direction, the length of the lower beam and the length and orientation of the upper beam are changed.

Table 1. Eigenvalue Sensitivities for a Stepped Beam with a Support

			Disp. M.	Mixed M.	Force M.	
Mode	Mesh/Beam	λ	$\frac{\lambda'_D}{\lambda'_{FDM}}$ %	$\frac{\lambda'_M}{\lambda'_{FDM}}$ %	$\frac{\lambda'_F}{\lambda'_{FDM}}$ %	λ'_{FDM}
	2	0.760	240.11	99.96	100.12	−0.340
1	5	0.760	162.10	99.96	100.00	−0.340
	10	0.760	132.12	99.99	100.00	−0.340
	2	23.569	137.82	94.39	99.39	828.62
2	5	23.442	109.01	99.61	99.98	809.57
	10	23.438	102.53	99.95	100.00	808.96
	2	116.838	53.80	93.69	94.21	2.620E4
3	5	114.397	76.59	98.71	99.81	2.335E4
	10	114.313	88.21	99.64	99.99	2.327E4

Table 2. Eigenvalue Sensitivities for Two Beam Structure

		Displacement variation (x)				Angle variation (θ)		
Mode	$\frac{\lambda\rho A}{EI}$	$\frac{\lambda'_D}{\lambda'_{FDM}}$ %	$\frac{\lambda'_M}{\lambda'_{FDM}}$ %	$\frac{\lambda'_F}{\lambda'_{FDM}}$ %	$\lambda'_{FDM}\frac{\rho A}{EI}$	$\frac{\lambda'_D}{\lambda'_{FDM}}$ %	$\frac{\lambda'_F}{\lambda'_{FDM}}$ %	$\lambda'_{FDM}\frac{\rho A}{EI}$
1	6.45	96.12	100.05	100.03	−11.329	99.85	100.00	6.910
2	22.93	106.04	99.74	100.13	−37.832	100.12	100.00	4.430
3	36.32	127.16	99.95	100.21	−67.660	102.02	100.00	14.996
4	107.2	100.86	100.07	101.20	−97.368	106.83	100.60	−7.515

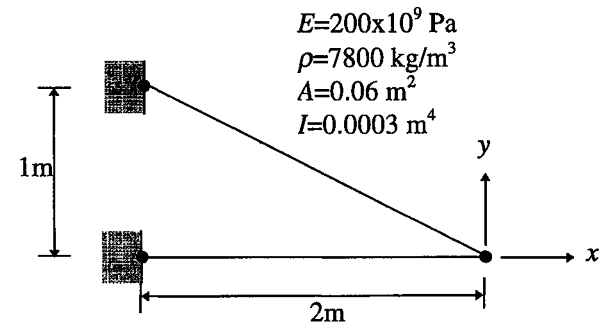

FIGURE 9 Example of two beam structure.

Thus, the eigenvalue variation is the sum of the shape sensitivities of each beam and the orientation sensitivity of the upper beam multiplied by corresponding variations. The results are compared with those of the finite difference method and summarized in Table 2. Five elements per beam are used for the finite element model. The energy densities are taken at the joint. The results show that if the joint position moves in x direction, the eigenvalues decrease. The mixed and force methods provide almost equal results with the finite difference method. But the displacement method gives inaccurate results. Table 2 also shows the sensitivities when the orientation between two beams is varied. There are almost no differences between the results from the force method and the finite difference method. The displacement method shows little discrepancies.

A Truss Beam Structure

The proposed methods are applied to a truss structure composed of twelve beams to study the applicability of the method to a complex beam structure. The original and the perturbed configuration are shown in Fig. 10. After the configuration variation, the length of nine beams and the orientation of six beams are varied. The sensitivities by force, mixed, and displacement methods are also obtained and the results are compared with those from the finite difference method. Five elements per beam are also used in meshing the structure. As displayed in Table 3, the first four eigenvalues are raised for the varied configuration. All of the three methods provide satisfactory results and the pattern of errors is also the same as that of the previous examples. From this example, it is believed that

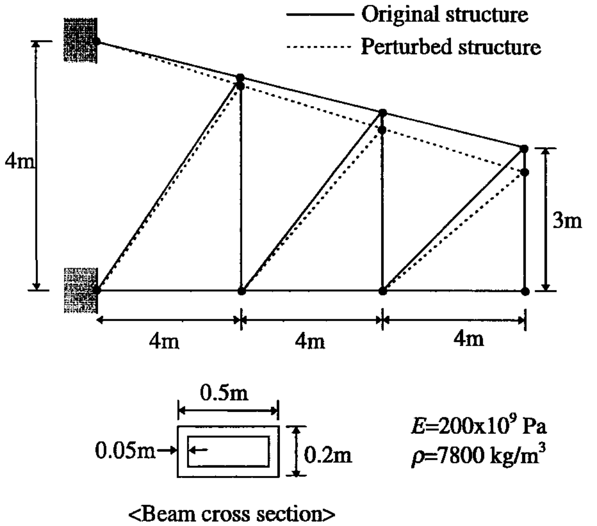

FIGURE 10 Example of a truss beam structure.

Table 3. Eigenvalue Sensitivities for Truss Structure

Mode	$\frac{\lambda \rho A}{EI}$	$\frac{\lambda'_D}{\lambda'_{FDM}}\%$	$\frac{\lambda'_M}{\lambda'_{FDM}}\%$	$\frac{\lambda'_F}{\lambda'_{FDM}}\%$	$\lambda'_{FDM}\frac{\rho A}{EI}$
1	0.0718	99.12	99.80	100.02	0.0127
2	0.1443	97.02	99.57	100.08	0.0213
3	0.1779	96.20	99.57	100.07	0.0460
4	0.2704	94.77	99.83	100.26	0.1088

the proposed method can be applied to complex beam structure. The accuracy of the force method and the mixed methods is almost the same as the finite difference method.

CONCLUSIONS

A procedure is proposed to derive the eigenvalue sensitivity of a combined beam structure due to the variation of its joint coordinates based on the Lagrange multiplier technique for the free vibration equation. The sensitivity is obtained by taking the variation of the equation. The configuration variation of the structure is decomposed into the variations of lengths and orientations of sub-beams. It is found that the sensitivity due to length variations of sub-beams is determined by an energy density form, and this form is conservative on the uniform beam section. The sensitivity due to orientation variation is represented by the cross product of joint forces and displacements. The displacement, mixed, and force methods are suggested to calculate the sensitivities, and it is shown that the force method provides the most accurate results.

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