Advanced Studies in Pure Mathematics 19, 1989
Integrable Systems in Quantum Field Theory and Statistical Mechanics
pp. 1–55

Eigenvalue Spectrum of the Superintegrable Chiral Potts Model

Giuseppe Albertini, Barry M. McCoy and Jacques H.H. Perk

Abstract.

We compute the eigenvalues of the 3-state superintegrable chiral Potts model and of the associated spin chain by use of a functional equation. We find that the system has four phases, two of which are massless and two of which are massive.

§1. Introduction

Recently [1–4] a new class of 2 dimensional classical statistical mechanical models has been shown to obey the integrability condition of commuting transfer matrices

$$[T(u), T(u')] = 0. \tag{1.1}$$

The model is a special case of the general N state chiral Potts model on the square lattice defined by

$$\mathcal{E} = -\sum_{j,k} \sum_{n=1}^{N-1} \{E_n^h(\sigma_{j,k}\sigma_{j,k+1}^*)^n + E_n^v(\sigma_{j,k}\sigma_{j+1,k}^*)^n\} \tag{1.2}$$

with

$$\sigma_{j,k}^N = 1. \tag{1.3}$$

We define local Boltzmann weights as

$$W_{p,q}^{v,h}(n) = \exp\beta \sum_{j=1}^{N-1} E_j^{v,h}\omega^{jn} \tag{1.4}$$

with

$$\omega = e^{2\pi i/N} \tag{1.5}$$

Received February 21, 1989.

and define the transfer matrix as (Fig. 1)

$$T_{\{\ell\},\{\ell'\}} = \prod_{j=1}^{\mathcal{N}} W^v_{p,q}(\ell_j - \ell'_j) W^h_{p,q}(\ell_j - \ell'_{j+1}) \tag{1.6}$$

where periodic boundary conditions are imposed by defining $\mathcal{N} + 1 = 1$ and the indices ℓ_i range from 1 to N.

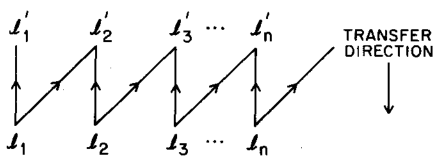

Fig. 1. The lattice used to define the transfer matrix $T_{\{\ell\},\{\ell'\}}$ for the chiral Potts model. The direction of transfer is from top to bottom. The arrows on the lines serve to define the sign of $\ell_j - \ell'_j$ (and $\ell_j - \ell'_{j+1}$).

Then the special case of (1.2) which satisfies (1.1) is defined by [4]

$$\frac{W^h_{p,q}(n)}{W^h_{p,q}(0)} = \prod_{j=1}^{n} \left(\frac{d_p b_q - a_p c_q \omega^j}{b_p d_q - c_p a_q \omega^j}\right) \tag{1.7a}$$

and

$$\frac{W^v_{p,q}(n)}{W^v_{p,q}(0)} = \prod_{j=1}^{n} \left(\frac{\omega a_p d_q - d_p a_q \omega^j}{c_p b_q - b_p c_q \omega^j}\right) \tag{1.7b}$$

where a_p, b_p, c_p, d_p and a_q, b_q, c_q, d_q lie on the generalized elliptic curve defined by

$$a^N + \lambda b^N = \lambda' d^N, \quad \lambda a^N + b^N = \lambda' c^N \tag{1.8}$$

with

$$\lambda' = (1 - \lambda^2)^{1/2}. \tag{1.9}$$

For $\lambda^2 \neq 0, 1$, and ∞ this curve has genus $N^3 - 2N^2 + 1$. The symbols q and p represent the uniformizing variable for (1.8). We regard q as the variable u in (1.1) which holds for each value of p. These results have been presented in detail by Perk in this present volume [5].

In addition to the transfer matrix $T_{p,q}$ we are interested, in fact more interested, in the eigenvalue spectrum of the associated quantum spin chain Hamiltonian. This Hamiltonian is obtained from $T_{p,q}$ by letting $q \to p$ as

(1.10a) $$a_q = a_p + \alpha u' + O(u'^2)$$

(1.10b) $$b_q = b_p + \beta u' + O(u'^2)$$

(1.10c) $$c_q = c_p + \gamma u' + O(u'^2)$$

(1.10d) $$d_q = d_p + \delta u' + O(u'^2)$$

where $\alpha, \beta, \gamma, \delta$ are constrained by (1.8). Then with the rescaling

(1.10e) $$u' = \frac{2u}{a_p\delta - \alpha d_p}\left(\frac{a_p d_p}{b_p c_p}\right)^{-N/2} a_p d_p \lambda$$

and with the normalization $W^h_{p,q}(0) = W^v_{p,q}(0) = 1$ we have $W^h_{p,p}(n) = 1, W^v_{p,p}(n) = \delta_{n,0}$,

(1.11)
$$T_{p,q} = 1\{1 + 2\mathcal{N}u\left(\frac{a_p c_p}{b_p d_p}\right)^{1-N/2} \sum_{\ell=1}^{N-1} \left(\frac{a_p c_p}{b_p d_p}\right)^{\ell-1} \frac{1}{1-\omega^{-\ell}}\} + u\mathcal{H} + O(u^2)$$

and

(1.12) $$\mathcal{H} = -\sum_{j=1}^{\mathcal{N}} \sum_{n=1}^{N-1} \{\bar{\alpha}_n (X_j)^n + \alpha_n (Z_j Z^\dagger_{j+1})^n\}.$$

Here we use

(1.13a) $$X_j = I_N \otimes \cdots \otimes X^{jth} \otimes \cdots \otimes I_N$$

(1.13b) $$Z_j = I_N \otimes \cdots \otimes Z^{jth} \otimes \cdots \otimes I_N,$$

I_N is the $N \times N$ identity matrix, the elements of the $N \times N$ matrices Z and X are

(1.14a) $$Z_{\ell,m} = \delta_{\ell,m}\omega^{\ell-1}$$

(1.14b)
$$X_{\ell,m} = \delta_{\ell,m+1} \quad (\mathrm{mod} N),$$

the parameters α_k and $\bar{\alpha}_k$ are

(1.15a)
$$\alpha_k = \exp[i(2k - N)\phi/N]/\sin(\pi k/N)$$

(1.15b)
$$\bar{\alpha}_k = \lambda \exp[i(2k - N)\bar{\phi}/N]/\sin(\pi k/N)$$

with

(1.16)
$$\cos\phi = \lambda \cos\bar{\phi}$$

and

(1.17)
$$e^{\frac{2i\phi}{N}} = \omega^{1/2}\frac{a_p c_p}{b_p d_p}, \quad e^{\frac{2i\bar{\phi}}{N}} = \omega^{1/2}\frac{a_p d_p}{b_p c_p}.$$

Our interest in this present article is to analytically study the eigenvalue spectrum of $T_{p,q}$ and of $\mathcal{H}$. Some of our results have been published in references 6 and 7.

Our study will be carried out for the case $N = 3$ and is based on the matrix equation

(1.18)
$$T_{p,q}T_{p,Rq}T_{p,R^2q} = e^{-iP}\{f^{\mathcal{N}}_{p,Rq}f^{\mathcal{N}}_{Rq,p}T_{p,q} + f^{\mathcal{N}}_{p,q}f^{\mathcal{N}}_{q,p}T_{p,R^2q} + f^{\mathcal{N}}_{p,R^5q}f^{\mathcal{N}}_{R^2q,p}T_{p,R^4q}\}.$$

Here

(1.19)
$$f_{p,q} = \{\frac{\prod_{m=1}^{N}[\sum_{k=1}^{N}\omega^{mk}W^{v}_{p,q}(k)]}{\prod_{m=1}^{N}W^{h}_{p,q}(m)}\}^{1/N}$$

is the function introduced in ref. 4 in the solution of the star triangle equation. Despite the N^{th} root which explicitly occurs in (1.19) $f_{p,q}$ can be shown to be a meromorphic function on the Riemann surface defined by (1.8). Furthermore in (1.18) R is the automorphism of (1.8) defined by

(1.20)
$$R(a_q, b_q, c_q, d_q) = (b_q, \omega a_q, d_q, c_q)$$

and P, the total momentum is obtained from $T_{p,q}$ by

(1.21a)
$$\lim_{q\to p} T_{p,Rq} = e^{-iP}.$$

The interaction (1.2) is translationally invariant so P and $T_{p,q}$ may be simultaneously diagonalized and P has the $\mathcal{N}$ eigenvalues

$$P = \frac{2\pi k}{\mathcal{N}} \tag{1.21b}$$

where $k = 0, 1, \ldots \mathcal{N} - 1 \mod \mathcal{N}$. Similarly the interaction (1.2) is invariant if all $\sigma_{j,k} \to \omega\sigma_{j,k}$. Thus the spin translation operator

$$R = e^{i2\pi Q/N} = \Pi_k X_k \tag{1.22}$$

also commutes with $T_{p,q}$ and the eigenvalues Q take on the values 0,1,2. We note that (1.18) does not involve Q.

A proof of (1.18) will not be given here. Instead, we will concentrate on its consequences. Unfortunately, as we will see in Section 2 the general solution of (1.18) seems to require the use of some deep machinery of algebraic geometry. However, as discovered in ref. 6, if we specialize to the case

$$\phi = \bar{\phi} = \pi/2 \tag{1.23}$$

first considered by Howes, Kadanoff and den Nijs [8] a remarkable simplification takes place in that *all* eigenvalues of $\mathcal{H}$ are grouped into sets which have the form

$$E = A + B\lambda + N \sum_{j=1}^{m} \pm(1 + \lambda^2 + a_j\lambda)^{1/2} \tag{1.24}$$

where A, B, m and the a_j depend on the set under consideration. This special case (1.23) we have called superintegrable. Note that when $N = 2$ the model (1.2) reduces to the Ising model and property (1.24) is that found originally by Onsager [9].

For this superintegrable case we are able to solve (1.18). The details of the solution will be presented in this paper but for orientation we conclude this introduction with a sketch of some of our major results.

We restrict our attention here to $0 \leq \lambda$ and find that the system has 4 phases (Fig. 2).

I	II	III	IV
order parameter $\neq 0$	order parameter $\neq 0$	order parameter$=0$	order parameter$=0$
disorder parameter $=0$	disorder parameter $=0$	disorder parameter $\neq 0$	disorder parameter $\neq 0$
mass gap $\neq 0$	mass gap $=0$	mass gap $=0$	mass gap$\neq 0$
no oscillation	oscillation	oscillation	no oscillation

$\lambda = 0$ $\lambda = .901292\ldots$ $\lambda = 1$ $\lambda = 1/.901292\ldots$ ∞

Fig. 2. Summary of the properties of the 4 phases of the superintegrable 3 state chiral Potts model for $0 \leq \lambda$.

In each phase we have calculated the ground state energy per site

(1.25) $$e_0(\lambda) = \lim_{\mathcal{N}\to\infty} \frac{1}{\mathcal{N}} E^0_{\mathcal{N}}(\lambda).$$

From (1.24) we see that $e_0(\lambda)$ obeys a duality relation

(1.26) $$e_0(1/\lambda) = \lambda^{-1} e_0(\lambda).$$

Phase I occurs for $0 \leq \lambda < \lambda^I = .901292\ldots$ and here the ground state energy per site is

(1.27)
$$e_0^I(\lambda) = -(1+\lambda)\{F(-\frac{1}{2},\frac{1}{3};1;\frac{4\lambda}{(1+\lambda)^2}) + F(-\frac{1}{2},\frac{2}{3};1;\frac{4\lambda}{(1+\lambda)^2})\}$$

where $F(a,b;c;z)$ is the hypergeometric function. A different form of this result has recently been derived by Baxter [10].

Phase II occurs for $\lambda^I < \lambda < 1$. Here the ground state energy per site is

(1.28a) $$e_0^{II}(\lambda) = e_0^I(\lambda) + \int_{v_L}^{v_U} dv' \rho(v') F(v')$$

where

(1.28b)

$$F(v)=2|1-\lambda|+\frac{3}{\pi}\int_1^{|\frac{1+\lambda}{1-\lambda}|^{2/3}}dy\frac{(2vy-1)v}{1-vy+v^2y^2}[\frac{4\lambda}{y^3-1}-(1-\lambda)^2]^{1/2}$$

and $\rho(v)$ satisfies the integral equation

(1.29)

$$\int_{v_L}^{v_U}dv'\rho(v')\frac{v'}{v^2+vv'+v'^2}+\frac{1}{1-v+v^2}=\frac{2\pi}{\sqrt{3}}\rho(v)\ \text{ for }\ v_L<v<v_U.$$

The limits v_L and v_U satisfy

$$(1.30)\quad F(v_L)=F(v_U)=0\quad\text{and}\quad F(v)\le 0\quad\text{for}\quad 0\le v_L\le v\le v_U.$$

The density $\rho(v)$ is positive and integrable for $v_L\le v\le v_U$.

In phase III and phase IV $e_0(\lambda)$ is obtained by use of (1.26).

We find in addition that in phase I the ground states for $Q=0,1$, and 2 are exponentially degenerate in $\mathcal{N}$. For phase IV the ground state has $Q=0$ and is different from $Q=1$ or 2 by a term $2Q|1-\lambda|$. In phases I and IV the single particle states have $Q=1$ and their energy in the $\mathcal{N}\to\infty$ limit is

(1.31)

$$\lim_{\mathcal{N}\to\infty}\{E_{\mathcal{N}}(P,\lambda)-E^0_{\mathcal{N}}(\lambda)\}=4|1-\lambda|+2(1-\lambda)$$
$$+\frac{3}{\pi}\int_1^{|\frac{1+\lambda}{1-\lambda}|^{2/3}}dy\{\frac{\omega v}{1+\omega yv}+\frac{\omega^2v}{1+\omega^2yv}\}[\frac{4\lambda}{y^3-1}-(1-\lambda)^2]^{1/2}$$

where

$$(1.32)\qquad e^{-iP}=\frac{1+v\omega^2}{1+v\omega}.$$

This spectrum always has a mass gap in phases I and IV. In phases II and III the excitation spectrum has no mass gap.

We will also argue that $\lim_{k\to\infty}|<Z_0Z_k^{\dagger}>|\neq 0$ in phases I and II but vanishes in phases III and IV and that the correlations have oscillations in phases II and III but not in phases I and IV.

In Section 2 we discuss the general method of solving (1.18) in the superintegrable case and then use the machinery developed to compute $e_0(\lambda)$ in phase I. In Section 3 we extend the procedure to calculate the eigenvalues for single particle excitations in phases I and IV. In Section

4 we show that in phase II (and Phase III) the ground state of phase I (phase IV) becomes unstable against multiparticle collapse and a new ground state occurs whose energy is given by (1.28)–(1.29). We also show why phases II and III are distinct. Finally we conclude in Section 5 with a presentation of the existing information on the order parameter and the asymptotic behavior of $< Z_0 Z_k^\dagger >$ for large k. We here also discuss the relation of this work to previous studies.

§2. Formalism and the phase I ground state energy

The discovery of relation (1.18) was inspired by the work of Bazhanov and Reshetikhin [11]. However, (1.18) as it stands is not precisely in the form of ref. 11. This is because Bazhanov and Reshetikhin follow the practice which is universally followed in the study of solvable models of factorizing the transfer matrix eigenvalues into the product of their zeroes and poles as a function of the spectral variable q. Such a factorization is possible because the eigenvalues $T_{p,q}$ are meromorphic functions of q on the Riemann surface defined by (1.8). Such a factorization is useful because the poles of $T_{p,q}$ can only come from the poles of the Boltzmann weights (1.7). Thus the universal practice is to characterize the eigenvalues by locating their zeroes.

To make this factorization we first proceed in a symbolic fashion. From (1.7) we see that $W_{p,q}^h(n)$ has 9 poles. These occur at places on the Riemann surface where

(2.1a) $$\frac{a_q}{d_q} = \omega^x \frac{b_p}{c_p}$$

(2.1b) $$\frac{b_q}{c_q} = \omega^y \frac{a_p}{d_p}$$

and

(2.1c) $$\frac{c_q}{d_q} = \omega^z \frac{d_p}{c_p}$$

where $x, y,$ and z are certain integers taking on the values 0,1 or 2. We define the symbol $[xyz]$ to represent the place (2.1). Then, defining $D_{p,q}$ to be the collection of places where (1.7a) has poles (i.e. the polar divisor of $W_{p,q}^h(n)$) we find

(2.2) $$D_{p,q} = [200, 220, 100, 201, 221, 101, 202, 222, 102].$$

Similarly we define $\bar{D}_{p,q}$ to be the polar divisor of $W^v_{p,q}(n)$ and define the symbol $[\overline{xyz}]$ to represent the place

$$\frac{a_q}{d_q} = \omega^x \frac{a_p}{d_p} \tag{2.3a}$$

$$\frac{b_q}{c_q} = \omega^y \frac{b_p}{c_p} \tag{2.3b}$$

$$\frac{d_q}{c_q} = \omega^z \frac{d_p}{c_p}. \tag{2.3c}$$

Then from (1.7b) we find

$$\bar{D}_{p,q} = [\overline{210}, \overline{110}, \overline{120}, \overline{211}, \overline{111}, \overline{121}, \overline{212}, \overline{112}, \overline{122}] \tag{2.4}$$

and we note that

$$D_{q,p} = \bar{D}_{p,Rq}. \tag{2.5}$$

We then define $T^N_{p,q}$ from

$$T_{p,q} = \frac{T^N_{p,q}}{(D_{p,q}\bar{D}_{p,q})^{\mathcal{N}}} \tag{2.6}$$

and we note the important factorization of $f_{p,q}$ (1.19)

$$f_{p,q} = \frac{D_{p,q}}{\bar{D}_{p,q}}. \tag{2.7}$$

We now put (2.6) and (2.7) into (1.18) and get

$$\begin{aligned}
&\frac{T^N_{p,q} T^N_{p,Rq} T^N_{p,R^2q}}{(D_{p,q}\bar{D}_{p,q} D_{p,Rq}\bar{D}_{p,Rq} D_{p,R^2q}\bar{D}_{p,R^2q})^{\mathcal{N}}} \\
&\quad = e^{-iP}\Big\{ \left(\frac{D_{p,Rq}}{\bar{D}_{p,Rq}} \frac{D_{Rq,p}}{\bar{D}_{Rq,p}} \right)^{\mathcal{N}} \frac{T^N_{p,q}}{(D_{p,q}\bar{D}_{p,q})^{\mathcal{N}}} \\
&\quad + \left(\frac{D_{p,q}}{\bar{D}_{p,q}} \frac{D_{q,p}}{\bar{D}_{q,p}} \right)^{\mathcal{N}} \frac{T^N_{p,R^2q}}{(D_{p,R^2q}\bar{D}_{p,R^2q})^{\mathcal{N}}} \\
&\quad + \left(\frac{D_{p,R^5q}}{\bar{D}_{p,R^5q}} \frac{D_{R^2q,p}}{\bar{D}_{R^2q,p}} \right)^{\mathcal{N}} \frac{T^N_{p,R^4q}}{(D_{p,R^4q}\bar{D}_{p,R^4q})^{\mathcal{N}}} \Big\}.
\end{aligned} \tag{2.8}$$

Then using (2.5) we may rewrite this as

$$\begin{aligned}T^N_{p,q}T^N_{p,Rq}T^N_{p,R^2q} &= e^{-iP}\{(D_{p,Rq}D_{Rq,p})^{2\mathcal{N}}T^N_{p,q} \\ &\quad + (D_{p,q}D_{q,p})^{2\mathcal{N}}T^N_{p,R^2q}\end{aligned} \tag{2.9}$$

$$+\left(\frac{D_{p,q}D_{q,p}D_{p,R^2q}D_{R^2q,p}}{D_{p,R^4q}D_{R^4q,p}}\right)^{\mathcal{N}}\left(\frac{D_{R^5q,p}D_{p,R^5q}D_{Rq,p}D_{p,Rq}}{D_{p,R^3q}D_{R^3q,p}}\right)^{\mathcal{N}}T^N_{p,R^4q}\}.$$

We now define $h^{\mathcal{N}}_{p,q}$ as the second factor of the coefficient of T_{p,R^4q} (and hence $h^{\mathcal{N}}_{p,Rq}$ as the first factor of the coefficient of T_{p,R^4q}). Then using (2.2) we find

$$\begin{aligned}h_{p,q} &= [\overline{210},\overline{000},\overline{120},\overline{211},\overline{001},\overline{121},\overline{212},\overline{002},\overline{122}]^2 \\ h_{p,Rq} &= [200,020,110,201,021,111,202,022,112]^2.\end{aligned} \tag{2.10}$$

We also use (2.2) to find

$$(D_{p,Rq}D_{Rq,p})^2 = h_{p,q}h_{p,R^2q} \tag{2.11a}$$

and

$$(D_{p,q}D_{q,p})^2 = h_{p,R^{-1}q}h_{p,Rq} \tag{2.11b}$$

Thus we obtain

$$\begin{aligned}T^N_{p,q}T^N_{p,Rq}T^N_{p,R^2q} &= e^{-iP}\{(h_{p,q}h_{p,R^2q})^{\mathcal{N}}T^N_{p,q} \\ &\quad + (h_{p,R^{-1}q}h_{p,Rq})^{\mathcal{N}}T^N_{p,R^2q} \\ &\quad + (h_{p,q}h_{p,Rq})^{\mathcal{N}}T^N_{p,R^4q}\}.\end{aligned} \tag{2.12}$$

An identical form can be obtained from (3.19) of ref. 11 if we identify the f_n of that paper with $h^{\mathcal{N}}_{p,R^nq}$.

This procedure leads to a perfectly fine result and yet it embodies a serious practical difficulty. The difficulty follows from the fact that on Riemann surfaces factorizations like (2.6) and (2.7) are not possible in terms of functions. Instead they are carried out in terms of prime forms [12]. For the case of a genus one Riemann surface the prime form is expressible in terms of the single variable Jacobi theta function and hence equations like (2.12) have been widely studied starting with Baxter's original solution of the 8-vertex model [13]. However, in our present case the genus of the Riemann surface is 10. We can still express the prime form in terms of theta functions but the theta functions now have 10 variables and involve a mapping from the original Riemann surface into the corresponding Jacobian. The most obvious way to represent the situation would be to explicitly uniformize the curve (1.8). Unfortunately we do not know how to do this in a useful fashion. Therefore we are not in a position to extract useful information from (2.12) in the general case.

However all is not quite lost because for the superintegrable case (1.23) a great miracle occurs. When $\phi = \bar{\phi} = \pi/2$ the variables a_p, b_p, c_p, and d_p are seen from (1.17) to satisfy

$$a_p = b_p \quad \text{and} \quad c_p = d_p \tag{2.13}$$

and the relation (1.8) becomes

$$\frac{a_p}{c_p} = \left(\frac{1-\lambda}{1+\lambda}\right)^{\frac{1}{2N}}. \tag{2.14}$$

When (2.13) holds, it is clear from (2.1) and (2.3) that

$$[\overline{xyz}] = [x, y, -z] \tag{2.15}$$

and the miracle occurs that the zeroes of $h_{p,q}$ given by (2.10) are identical with the zeroes of the meromorphic function

$$[\frac{a_q b_q}{c_q d_q}\eta^2 - 1] \tag{2.16}$$

where for convenience we used

$$\eta = \left(\frac{1+\lambda}{1-\lambda}\right)^{\frac{1}{2N}} \tag{2.17}$$

with $N = 3$.

Furthermore, the zeros of $D_{p,q}\bar{D}_{p,q}$ coincide with the zeros of the meromorphic function

$$\frac{(\eta\frac{a_q}{d_q})^3 - 1}{\eta\frac{a_q}{d_q} - 1}. \tag{2.18}$$

Thus if (with a slight abuse of notation) we set

$$T_{p,q} = \frac{(\eta\frac{a_q}{d_q} - 1)^{\mathcal{N}}}{[(\eta\frac{a_q}{d_q})^3 - 1]^{\mathcal{N}}} T^N_{p,q} \tag{2.19}$$

and set

$$h_{p,q} = K[\frac{a_q b_q}{c_q d_q}\eta^2 - 1] \tag{2.20}$$

where K is an appropriate normalization constant (which is irrelevant to our calculation) we find that (2.12) becomes (dropping the subscript on a_q, b_q, c_q and d_q)

$$\begin{aligned} T^N_{p,q} T^N_{p,Rq} T^N_{p,R^2 q} = K^{2\mathcal{N}} e^{-iP} \{ & (\frac{ab}{cd}\eta^2 - 1)^{\mathcal{N}} (\frac{ab}{cd}\eta^2\omega^2 - 1)^{\mathcal{N}} T^N_{p,q} \\ & + (\frac{ab}{cd}\eta^2\omega^2 - 1)^{\mathcal{N}} (\frac{ab}{cd}\eta^2\omega - 1)^{\mathcal{N}} T^N_{p,R^2 q} \\ & + (\frac{ab}{cd}\eta^2 - 1)^{\mathcal{N}} (\frac{ab}{cd}\eta^2\omega - 1)^{\mathcal{N}} T^N_{p,R^4 q} \}. \end{aligned} \tag{2.21}$$

This is an equation between meromorphic functions and all need to consider theta functions has disappeared. This is the extension to $N = 3$ of the well known fact that for the Ising case of $N = 2$ elliptic functions are not used in any step of Onsager's solution [9].

Of course, having calculated $h_{p,q}$ on the basis of these arguments about zeros and poles it would be satisfying to find an explicit algebraic calculation of (2.20) (which would also find K). We present this in appendix A.

We now turn to the solution of (2.21). As usual with any complicated equation the best way to proceed is to guess a solution and plug in. Therefore we begin our solution by studying eigenvalues of $T_{p,q}$ and $\mathcal{H}$ which have been obtained either analytically or on the computer for $\mathcal{N} = 3, \cdots, 7$. Some of these results have been published in ref. 6. From these studies we abstract the following Ansatz for the eigenvalues of $T_{p,q}$ for any N for the superintegrable case as a function of q

$$\begin{aligned} T_{p,q} = & \frac{N^{\mathcal{N}}(\eta\frac{a}{d} - 1)^{\mathcal{N}}}{[(\eta\frac{a}{d})^N - 1]^{\mathcal{N}}} (\eta\frac{a}{d})^{P_a} (\eta\frac{b}{c})^{P_b} (\frac{c^N}{d^N})^{P_c} \\ & \prod_{\ell=1}^{m_p} (\frac{1 + \omega v_\ell \eta^2 \frac{ab}{cd}}{1 + \omega v_\ell}) \prod_{\ell=1}^{m_E} (\frac{1+\lambda}{1-\lambda})^{\frac{1}{2}} \{ \frac{a^N + b^N}{2d^N} \pm \frac{w_\ell (a^N - b^N)}{(1+\lambda) d^N} \}. \end{aligned} \tag{2.22}$$

From this by (1.11) the eigenvalues of $\mathcal{H}$ are

$$\begin{aligned} E = & N(2P_c + m_E) - \mathcal{N}(N-1) \\ & + \lambda[\mathcal{N}(N-1) - N(2P_c + m_E) + 2(P_b - P_a)] \\ & + 2N \sum_{\ell=1}^{m_E} \pm w_\ell \end{aligned} \tag{2.23}$$

and from (1.21)

$$e^{-iP} = \omega^{P_b} \prod_{\ell=1}^{m_p} \left(\frac{1+\omega^2 v_\ell}{1+\omega v_\ell}\right). \tag{2.24}$$

In terms of this form we may make much more explicit the concept of "sets" of eigenvalues previously discussed by associating with each eigenvalue the quantum numbers m_E, m_p, P_a, P_b, and P_c. We have done this for all eigenvalues for $\mathcal{N} = 3, \cdots, 7$ and list the results in Table I.

There are many important properties of the eigenvalues which are reflected in Table I. For example there are the symmetry properties

1. For $\mathcal{N} \equiv 0 \bmod 3$

$$\begin{aligned} E(Q=2,P,\lambda) &= E(Q=1,-P,-\lambda) \\ E(Q=0,P,\lambda) &= E(Q=0,-P,-\lambda) \\ E(Q,P,\lambda) &= -E(Q,-P-\frac{2\pi Q}{3},-\lambda); \end{aligned}$$

2. For $\mathcal{N} \equiv 1 \bmod 3$

$$\begin{aligned} E(Q=2,P,\lambda) &= E(Q=0,-P,-\lambda) \\ E(Q=1,P,\lambda) &= E(Q=1,-P,-\lambda); \end{aligned}$$

3. For $\mathcal{N} \equiv 2 \bmod 3$

$$\begin{aligned} E(Q=1,P,\lambda) &= E(Q=0,-P,-\lambda) \\ E(Q=2,P,\lambda) &= E(Q=2,-P,-\lambda). \end{aligned} \tag{2.25}$$

The proof of these symmetries is given in Appendix B. There are also relations between m_p, P_a, P_b, and Q

1. For $\mathcal{N} \equiv 0 \bmod 3$
   a) if $m_p \equiv 0 \bmod 3$ then $P_b - P_a = Q \bmod 3$
   b) if $m_p \equiv 1 \bmod 3$ then $P_b = 1 - Q$ and $P_a = 1 + Q \bmod 3$
   c) if $m_p \equiv 2 \bmod 3$ then $P_b = 2 - Q$ and $P_a = 2 + Q \bmod 3$;
2. For $\mathcal{N} \equiv 1 \bmod 3$
   a) if $m_p \equiv 0 \bmod 3$ then $P_b = 0, P_a = 2 - Q$
   b) if $m_p \equiv 1 \bmod 3$ then $P_b - P_a = Q + 1 = 2$ so only $Q = 1$ is allowed
   c) if $m_p \equiv 2 \bmod 3$ then $P_b = Q + 1 \bmod 3$ and $P_a = 0$;
3. For $\mathcal{N} \equiv 2 \bmod 3$

$$
\begin{aligned}
&\text{a) if } m_p \equiv 0 \bmod 3 \text{ then } P_b = 0, P_a = 1 - Q \bmod 3\\
&\text{b) if } m_p \equiv 1 \bmod 3 \text{ then } P_b = Q + 2, P_a = 0 \bmod 3\\
&\text{c) if } m_p \equiv 2 \bmod 3 \text{ then } P_b - P_a = Q + 2 = 1\\
&\qquad \text{so only } Q = 2 \text{ is allowed.}
\end{aligned} \tag{2.26}
$$

These restrictions follow from the form (2.22), the equation (2.21) and a symmetry of the eigenvalues found by Baxter [10]. The proof is given in appendix C. There are inequalities necessary for $T_{p,q}$ to be a meromorphic function with poles only at $D_{p,q}\bar{D}_{p,q}$ of

$$P_a + 3P_c + 3m_E + m_p \leq 2\mathcal{N} \tag{2.27}$$

and

$$P_b + m_p \leq 3P_c$$

and there is the most important relation that eigenvalues occur in sequences such that if m_E decreases by 2 then m_p increases by 3. In other words we may group eigenvalues according to

$$3m_E + 2m_p = \text{ const.} \tag{2.28}$$

Armed with the insight of these finite chain studies, we conclude this section by solving (2.21) for the case where $m_p = 0$ and $P = 0$. Numerically it is seen that for small chains and small λ this is the ground state. This is the case recently studied by Baxter [10] for arbitrary N by use of a special case of (2.12) obtained by restricting the vector space to a set which includes only vectors for $m_p = 0$.

When $m_p = 0$ the Ansatz (2.22) has a very simple behavior under the automorphism R^2 namely

$$T^N_{p,R^2q} = \omega^{P_a+P_b}T^N_{p,q}. \tag{2.29}$$

Thus the equation (2.21) reduces to (setting $P_b = 0$ in accordance with Table I)

$$
\begin{aligned}
\omega^{P_a}T^N_{p,q}T^N_{p,Rq} = K^{2\mathcal{N}}\{&(\frac{ab}{cd}\eta^2 - 1)^{\mathcal{N}}(\frac{ab}{cd}\eta^2\omega^2 - 1)^{\mathcal{N}}\\
&+ (\frac{ab}{cd}\eta^2\omega^2 - 1)^{\mathcal{N}}(\frac{ab}{cd}\eta^2\omega - 1)^{\mathcal{N}}\omega^{P_a}\\
&+ (\frac{ab}{cd}\eta^2 - 1)^{\mathcal{N}}(\frac{ab}{cd}\eta^2\omega - 1)^{\mathcal{N}}\omega^{2P_a}\}.
\end{aligned} \tag{2.30}
$$

Then if we note that

$$R(\frac{a^3-b^3}{a^3+b^3}) = -(\frac{a^3-b^3}{a^3+b^3}) \tag{2.31}$$

we find

$$\begin{aligned}
&\bar{K}(\eta^2\frac{ab}{cd})^{P_a}\prod_{\ell=1}^{m_E}\{\frac{(a^3+b^3)^2}{4c^3d^3} - w_\ell^2\frac{(a^3-b^3)^2}{(1+\lambda)^2c^3d^3}\}\\
=&\{(\frac{ab}{cd}\eta^2-1)^{\mathcal{N}}(\frac{ab}{cd}\eta^2\omega^2-1)^{\mathcal{N}}\omega^{-P_a}\\
&+(\frac{ab}{cd}\eta^2\omega^2-1)^{\mathcal{N}}(\frac{ab}{cd}\eta^2\omega-1)^{\mathcal{N}}\\
&+(\frac{ab}{cd}\eta^2-1)^{\mathcal{N}}(\frac{ab}{cd}\eta^2\omega-1)^{\mathcal{N}}\omega^{P_a}\}.
\end{aligned} \tag{2.32}$$

where $\bar{K}$ is an irrelevant normalizating constant.

We use this to solve for the w_ℓ. First multiply together the two equations of the curve (1.8) to get

$$\lambda(a^6+b^6)+(1+\lambda^2)a^3b^3 = (1-\lambda^2)c^3d^3. \tag{2.33}$$

Thus

$$\frac{(a^3+b^3)^2}{c^3d^3} = \lambda^{-1}\{(1-\lambda^2)-(1-\lambda)^2(\frac{ab}{cd})^3\}$$

and

$$\frac{(a^3-b^3)^2}{c^3d^3} = \lambda^{-1}\{(1-\lambda^2)-(1+\lambda)^2(\frac{ab}{cd})^3\} \tag{2.34}$$

which are functions of $\frac{ab}{cd}$ alone. Thus, defining

$$t = \eta^2\frac{ab}{cd} \tag{2.35}$$

the roots w_ℓ are determined from

$$\begin{aligned}
w_\ell^2 &= \frac{(1+\lambda)^2}{4}(\frac{a^3+b^3}{a^3-b^3})^2\\
&= \frac{(1+\lambda)^2}{4}[\frac{1-(\frac{1-\lambda}{1+\lambda})^2t_\ell^3}{1-t_\ell^3}] = \frac{1}{4}(1-\lambda)^2+\frac{\lambda}{1-t_\ell^3}
\end{aligned} \tag{2.36}$$

where t_ℓ are the roots of the polynomial equation

$$0 = P_Q(t) = t^{-P_a}\{(t-1)^{\mathcal{N}}(t\omega^2-1)^{\mathcal{N}}\omega^{-P_a} + (t\omega-1)^{\mathcal{N}}(t\omega^2-1)^{\mathcal{N}} + (t-1)^{\mathcal{N}}(t\omega-1)^{\mathcal{N}}\omega^{P_a}\}. \tag{2.37}$$

We note that

$$P_Q(\omega t) = P_Q(t) \tag{2.38}$$

and thus $P_Q(t)$ is a polynomial in t^3. Furthermore $P_Q(t)$ is real for t^3 real and thus the zeros of $P_Q(t)$ lie on the three lines where $t, \omega t$, and $\omega^2 t$ are real. Thus, since w_ℓ^2 depends on t_ℓ only through t_ℓ^3 we see that to avoid triple counting of w_ℓ we should restrict our attention to the real solutions of (2.37).

It remains to determine P_a and P_c. From Table I it is apparent that these depend on Q, and, since Q does not appear in (2.21) we clearly need some other means to find them. One such extra piece of information is the symmetry relation of Baxter [10] that if

$$a_q \to \omega a_q \quad \text{and} \quad b_q \to \omega^{-1} b_q \tag{2.39a}$$

then

$$T_{p,q} \to \omega^{-Q-\mathcal{N}} T_{p,q}. \tag{2.39b}$$

Thus we use (2.39) in (2.22) with $P_b = 0$ to find

$$P_a = -Q - \mathcal{N} \bmod 3. \tag{2.40}$$

Hence, with $0 \le P_a \le 2$ which is needed to prevent $T_{p,q}^N$ having poles at $a = 0$ or ∞, P_a is found to agree with Table I. However, for P_c we do not have such a general argument (although in ref. 10 Baxter does have a special argument which applies to the present case) and will be content to determine P_c from Table I as zero.

We now combine these considerations and obtain

$$E^0_{\mathcal{N}}(\lambda; Q) = A^Q_{\mathcal{N}} + B^Q_{\mathcal{N}}\lambda - 3\sum_{\ell=1}^{m_E^Q}[(1-\lambda)^2 + \frac{4\lambda}{1-t_\ell^3}]^{1/2} \tag{2.41}$$

where

$$m_E^Q = \text{ integral part of } [\frac{2\mathcal{N}-Q}{3}] \tag{2.42}$$

and

$$A^Q_{\mathcal{N}} + B^Q_{\mathcal{N}}\lambda = -2Q - [2\mathcal{N} - 2Q - 3m^Q_E](1+\lambda). \tag{2.43}$$

It remains to study the $\mathcal{N} \to \infty$ limit of (2.41). In this limit we see from (2.37) that all real zeroes occur for $t < -1$ where the first and third term are of equal magnitude and oscillate and the second term is real and exponentially smaller than the magnitude of terms 1 and 3. We thus use Cauchy's theorem to rewrite (2.41) as

(2.44)
$$\begin{aligned} E^0_{\mathcal{N}}(\lambda;Q) &= A^Q_{\mathcal{N}} + B^Q_{\mathcal{N}}\lambda \\ &\quad - \frac{3}{2\pi i}\int_{c_1} dt \frac{d}{dt}(\ln P_Q(t))[(1-\lambda)^2 + \frac{4\lambda}{1-t^3}]^{1/2} \\ &= A^Q_{\mathcal{N}} + B^Q_{\mathcal{N}}\lambda \\ &\quad - \frac{1}{2\pi i}\int_{c_1+c_2+c_3} dt \frac{d}{dt}(\ln P_Q(t))[(1-\lambda)^2 + \frac{4\lambda}{1-t^3}]^{1/2} \end{aligned}$$

where the contours c_1, c_2, and c_3 encircle the zeroes of $P_0(t)$ as shown in Fig. 3 and in the second line of (2.44) we have symmetrized over the three equivalent contours.

We now obtain a form useful for the $\mathcal{N} \to \infty$ limit by deforming the contour from $c_1 + c_2 + c_3$ to $c_1' + c_2' + c_3'$. However, to do this we note that the integrand of (2.44) has branch cuts at $t^3 = 1$ and $t^3 = (\frac{1+\lambda}{1-\lambda})^2$ and a pole at $t = \infty$ with residue $|1-\lambda|$ where we assume λ is real. To handle the pole we rewrite (2.41) as

$$\begin{aligned} E^0_{\mathcal{N}}(\lambda;Q) &= A^Q_{\mathcal{N}} + B^Q_{\mathcal{N}}\lambda - 3m^Q_E|1-\lambda| \\ &\quad - 3\sum_{\ell=1}^{m^Q_E}\{[(1-\lambda)^2 + \frac{4\lambda}{1-t^3_\ell}]^{1/2} - |1-\lambda|\} \end{aligned} \tag{2.45}$$

and thus (2.44) becomes

$$\begin{aligned} E^0_{\mathcal{N}}(\lambda;Q) &= A^Q_{\mathcal{N}} + B^Q_{\mathcal{N}}\lambda - 3m^Q_E|1-\lambda| \\ &\quad - \frac{1}{2\pi i}\int_{c_1'+c_2'+c_3'} dt \frac{d}{dt}(\ln P_Q(t)) \\ &\qquad \{[(1-\lambda)^2 + \frac{4\lambda}{1-t^3}]^{1/2} - |1-\lambda|\}. \end{aligned} \tag{2.46}$$

The limit $\mathcal{N} \to \infty$ may now be taken because the contours c_i' lie far away from the zeroes of $P_Q(t)$ and there is no pole at ∞. Moreover

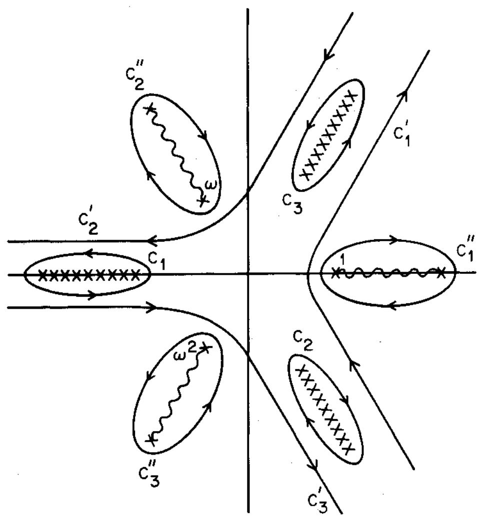

Fig. 3. The contours, c_i, c'_i, and c''_i in the complex t plane. The wiggly lines denote the branch cuts from $t^3 = 1$ to $t^3 = (\frac{1+\lambda}{1-\lambda})^2$. The crosses indicate the zeroes of the polynomial $P(t)$.

the contours c'_i all give equal contributions. Thus we may restrict our attention to c'_1 alone and deform to the contour c''_1 which circles the branch cut on the real axis. Then if we note that the second term in (2.37) is exponentially largest when t is real and positive we obtain

(2.47)
$$\begin{aligned} e_0^I(\lambda; Q) &= \lim_{\mathcal{N}\to\infty} \frac{1}{\mathcal{N}} E^0_{\mathcal{N}}(\lambda; Q) \\ &= -2|1-\lambda| \\ &\quad - \frac{3}{2\pi i}\int_{c''_1} dt\left(\frac{\omega^2}{\omega^2 t - 1} + \frac{\omega}{\omega t - 1}\right)\left[(1-\lambda)^2 + \frac{4\lambda}{1-t^3}\right]^{1/2} \end{aligned}$$

where the righthand side is independent of Q. Then, if we note by using the contours c''_2 and c''_3 of Fig. 3 and by closing on the pole at $t = \infty$

that

$$\begin{aligned}(2.48)\qquad &\frac{1}{2\pi i}\int_{c_1''} dt\{\frac{\omega}{\omega t-1}+\frac{\omega^2}{\omega^2 t-1}+\frac{1}{t-1}\}[(1-\lambda)^2+\frac{4\lambda}{1-t^3}]^{1/2}\\ &=\frac{1}{2\pi i}\int_{c_1''+c_2''+c_3''} dt\frac{1}{t-1}[(1-\lambda)^2+\frac{4\lambda}{1-t^3}]^{1/2}\\ &=-|1-\lambda|,\end{aligned}$$

we may rewrite (2.47) as

(2.49)

$$e_0^I(\lambda;Q)=$$
$$-\frac{1}{2\pi i}\int_{c_1''} dt\{\frac{\omega}{\omega t-1}+\frac{\omega^2}{\omega^2 t-1}-\frac{2}{t-1}\}[(1-\lambda)^2+\frac{4\lambda}{1-t^3}]^{1/2}.$$

The integrand in (2.49) (as opposed to (2.47) has no simple poles and is seen to be an abelian integral of the second kind over a hyperelliptic curve of genus 2.

To simplify this result further we first combine the terms in the first factor to rewrite (2.49) as

$$(2.50)\qquad e_0^I(\lambda;Q)=\frac{3}{2\pi i}\int_{c_1''} dt\frac{t+1}{t^3-1}[1+\lambda^2-2\lambda(\frac{t^3+1}{t^3-1})]^{1/2}.$$

Further reduction is obtained by deforming the contour as shown in Fig. 4 to the two rays

$$(2.51a)\qquad t=-\omega^2 z$$

and

$$(2.51b)\qquad t=-\omega z$$

where $0\le z\le\infty$.

On these contours the square root in (2.50) is real for $0\le\lambda\le 1$ and we find

$$(2.52)\qquad e_0^I(\lambda;Q)=-\frac{3\sqrt{3}}{2\pi}\int_0^\infty dz\frac{z+1}{z^3+1}[1+\lambda^2-2\lambda\frac{z^3-1}{z^3+1}]^{1/2}.$$

By sending $z\to z^{-1}$ this is manifestly seen to be an even function of λ

$$(2.53)\qquad e_0^I(\lambda;Q)=e_0^I(-\lambda;Q).$$

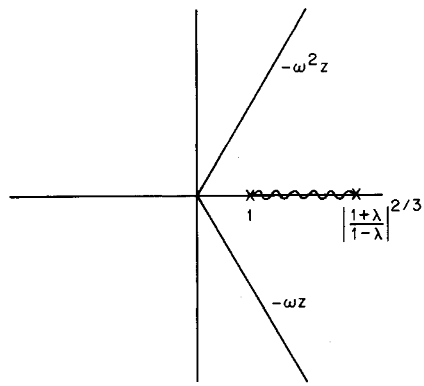

Fig. 4. The rays in the t plane $t = -\omega^2 z$ and $t = -\omega z$ for $0 \le z \le \infty$ along which the integral in (2.50) is evaluated.

Then if we let

$$z^3 = \frac{1 - \cos 2\theta}{1 + \cos 2\theta} \tag{2.54}$$

we find

$$\begin{aligned} e_0^I(\lambda; Q) = \frac{-\sqrt{3}(1+\lambda)}{\pi} \int_0^\pi & d\theta\{(\tan\theta)^{1/3} + (\tan\theta)^{-1/3}\} \\ & [1 - \frac{4\lambda}{(1+\lambda)^2}\cos^2\theta]^{1/2} \end{aligned} \tag{2.55}$$

which by using $\cos^2\theta = U$ is easily recognized as

$$\begin{aligned} e_0^I(\lambda; Q) = -(1+\lambda)\{ & F(-\frac{1}{2}, \frac{1}{3}; 1; \frac{4\lambda}{(1+\lambda)^2}) \\ & + F(-\frac{1}{2}, \frac{2}{3}; 1; \frac{4\lambda}{(1+\lambda)^2})\} \end{aligned} \tag{2.56}$$

where $F(a, b; c; z)$ is the hypergeometric function [14].

We have thus obtained an analytic expression for the function whose expansion to order λ^{88} was derived in ref. 6.

Near $\lambda = 1$ we use [14]

(2.57)

$$F(a,b;c;\frac{4\lambda}{(1+\lambda)^2}) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \times F(a,b;a+b-c+1;(\frac{1-\lambda}{1+\lambda})^2) + \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}(\frac{1-\lambda}{1+\lambda})^{2(c-a-b)} \times F(c-a,c-b;c-a-b+1;(\frac{1-\lambda}{1+\lambda})^2)$$

to find that

$$e_0^I(1;Q) = -4\pi^{-1/2}\{\frac{\Gamma(\frac{7}{6})}{\Gamma(\frac{2}{3})} + \frac{\Gamma(\frac{5}{6})}{\Gamma(\frac{1}{3})}\} \tag{2.58}$$

which agrees with the numerical result of ref. 6. Moreover it is also clear from (2.56) and (2.57) that near $\lambda = 1$ we have

$$e_0^I(\lambda;Q) = \sum_{n=0}^{\infty}(\frac{1-\lambda}{1+\lambda})^{2n}a_n + (\frac{1-\lambda}{1+\lambda})^{2-\frac{1}{3}}\sum_{n=0}^{\infty}(\frac{1-\lambda}{1+\lambda})^{2n}b_n + (\frac{1-\lambda}{1+\lambda})^{2+\frac{1}{3}}\sum_{n=0}^{\infty}(\frac{1-\lambda}{1+\lambda})^{2n}c_n \tag{2.59}$$

which also confirms [6].

We also note that Baxter [10] has studied this problem for $N > 3$ and that his result for general N can be put in the form (2.56); namely

$$e_0^I(\lambda;Q,N) = -(1+\lambda)\sum_{\ell=1}^{N-1} F(-\frac{1}{2},\frac{\ell}{N};1;\frac{4\lambda}{(1+\lambda)^2}) \tag{2.60}$$

and at $\lambda = 1$

$$e_0^I(1;Q,N) = -4\pi^{-1/2}\sum_{\ell=1}^{N-1}\frac{\Gamma(\frac{3}{2}-\frac{\ell}{N})}{\Gamma(\frac{N-\ell}{N})}. \tag{2.61}$$

For $N = 4$ and 5 these agree with the results of ref. 6.

For the purpose of studying the ground state energy per site this completes our study in phase I. However, for the study of the excited states we need slightly more information than the order $\mathcal{N}$ term in $E_{\mathcal{N}}^0(\lambda;Q)$, we need the order 1 term in $E_{\mathcal{N}}^0(\lambda;Q) - E_{\mathcal{N}}^0(\lambda;Q=0)$. From

Table I or from (2.42) and (2.43) we see that the details of the calculation will depend on whether $\mathcal{N}$ is congruent to 0,1, or 2 mod 3. However, the final result is expected to be independent of this fact. We illustrate the calculation for $\mathcal{N} \equiv 0$ mod 3.

From (2.42), (2.43) and (2.46) we find for $\mathcal{N} \equiv 0$

(2.62)
$$\begin{aligned} &E^0_{\mathcal{N}}(\lambda; Q=1) - E^0_{\mathcal{N}}(\lambda; Q=0) \\ &= -(3+\lambda) - 3(m^1_E - m^0_E)|1-\lambda| \\ &- \frac{3}{2\pi i}\int_{c''_1} dt \frac{d}{dt}\ln\frac{P_1(t)}{P_0(t)}\{[(1-\lambda)^2 + \frac{4}{1-t^3}]^{1/2} - |1-\lambda|\} \end{aligned}$$

and $m^1_E - m^0_E = -1$. We may now use (2.37) and (2.40) to show that on c'_1

$$\lim_{\mathcal{N}\to\infty} \frac{d}{dt}\ln\frac{P_1(t)}{P_0(t)} = -2t^{-1} \tag{2.63}$$

and hence from (2.62)

$$\begin{aligned} &\lim_{\mathcal{N}\to\infty}\{E^0_{\mathcal{N}}(\lambda; Q=1) - E^0_{\mathcal{N}}(\lambda, Q=0)\} \\ &= -(3+\lambda) + 3|1-\lambda| \\ &+ \frac{3}{\pi i}\int_{c''_1} \frac{dt}{t}\{[(1-\lambda)^2 + \frac{4\lambda}{1-t^3}]^{1/2} - |1-\lambda|\}. \end{aligned} \tag{2.64}$$

This contour integral may be explicitly evaluated if we first set

$$z = t^3 \tag{2.65}$$

to write

$$\begin{aligned} &\lim_{\mathcal{N}\to\infty}\{E^0_{\mathcal{N}}(\lambda; Q=1) - E^0_{\mathcal{N}}(\lambda; Q=0)\} \\ &= -(3+\lambda) + 3|1-\lambda| \\ &+ \frac{1}{\pi i}\int_{\Gamma} \frac{dz}{z}\{[(1-\lambda)^2 + \frac{4\lambda}{1-z}]^{1/2} - |1-\lambda|\}. \end{aligned} \tag{2.66}$$

when Γ encircles the branch cuts of the square root in the integrand. The only other singularity in the integrand is at $z = 0$ so the integral is readily evaluated and we find for $\mathcal{N} \equiv 0$ mod 3

(2.67)
$$\lim_{\mathcal{N}\to\infty}\{E^0_{\mathcal{N}}(\lambda;Q=1)-E^0_{\mathcal{N}}(\lambda;Q=0)\}$$
$$=-(3+\lambda)+3|1-\lambda|+2\{1+\lambda-|1-\lambda|\}$$
$$=|1-\lambda|-(1-\lambda).$$

For $\mathcal{N}\equiv 1$ and $\mathcal{N}\equiv 2 \bmod 3$ the calculation is identical except that

(2.68) $$A^1_{\mathcal{N}}+B^1_{\mathcal{N}}\lambda-(A^0_{\mathcal{N}}+B^0_{\mathcal{N}}\lambda)=2\lambda,\quad m^1_E-m^0_E=0$$

and

(2.69) $$P_a(Q=0)-P_a(Q=1)=1.$$

A completely equivalent calculation can be done for $Q=2$. Thus we conclude that for $0\le\lambda<1$

$$\lim_{\mathcal{N}\to\infty}\{E^0_{\mathcal{N}}(\lambda;Q)-E^0_{\mathcal{N}}(\lambda;0)\}=0$$

but for $1<\lambda$

(2.70) $$\lim_{\mathcal{N}\to\infty}\{E^0_{\mathcal{N}}(\lambda;Q)-E^0_{\mathcal{N}}(\lambda;0)\}=2Q(\lambda-1).$$

Indeed, for $0\le\lambda<1$ a more accurate calculation shows that $E^0_{\mathcal{N}}(\lambda,Q)$ is not only independent of Q to order 1 as $\mathcal{N}\to\infty$ but that the dependence on Q vanishes exponentially as $\mathcal{N}\to\infty$. Thus we say that these eigenvalues are asymptotically 3 fold degenerate. This is the analogue of the 2 fold asymptotic degeneracy of the Ising model in the low temperature phase [9].

§3. Single particle excitation in phases I and IV

Even though the eigenvalue $E^0_{\mathcal{N}}(\lambda;Q)$ calculated in Section 2 is the lowest eigenvalue for small λ and even though $e_0(\lambda;Q)$ fails to be analytic only at $\lambda^2=1$, this is not sufficient to guarantee that no phase transition occurs for $0<\lambda<1$ because, clearly, if for some value of λ the system is unstable against single particle excitation a phase transition will occur. In this section we continue our investigations by studying these single particle excitations. More precisely, we study the states in the sector $Q=1$ with $m_p=1$ and $P_b=0$. From Table I we see that such states exist for all $\mathcal{N}$ and all $P\neq 0$, or $-2\pi/3$.

We commence our study by using the form (2.22) with $m_p = 1$ and $P_b = 0$ in (2.21) and note again that in the resulting expression $a, b, c,$ and d can be eliminated in favor of the single variable t to give

$$
\begin{aligned}
&c' t^{\bar{P}_a}(1+t^3v^3)\prod_{\ell=1}^{\bar{m}_E}[1-(\frac{1-\lambda}{1+\lambda})^2 t^3 - w_\ell^2\frac{4}{(1+\lambda)^2}(1-t)^3)] \\
&= e^{-iP}\{(t-1)^{\mathcal{N}}(t\omega^2-1)^{\mathcal{N}}(1+\omega v t)\omega^{-\bar{P}_a} \\
&+ (t\omega-1)^{\mathcal{N}}(t\omega^2-1)^{\mathcal{N}}(1+vt) \\
&+ (t-1)^{\mathcal{N}}(t\omega-1)^{\mathcal{N}}(1+\omega^2 vt)\omega^{\bar{P}_a}\}.
\end{aligned}
\tag{3.1}
$$

Here we use the symbols $\bar{P}_a$ and $\bar{m}_E$ to distinguish from P_a and m_E of Section 2. This equation must be used now to determine v as well as w_ℓ.

We first obtain v by letting $t = -v^{-1}$ (and $-\omega v^{-1}$ and $-\omega^2 v^{-1}$ as well). The left hand side of (3.1) vanishes as well as the second term the right. Thus we find that v satisfies the equation

$$
\begin{aligned}
0 &= (-v^{-1}-1)^{\mathcal{N}}(-\omega^2 v^{-1}-1)^{\mathcal{N}}(1-\omega)\omega^{-\bar{P}_a} \\
&+ (-v^{-1}-1)^{\mathcal{N}}(-\omega v^{-1}-1)^{\mathcal{N}}(1-\omega^2)\omega^{\bar{P}_a}
\end{aligned}
\tag{3.2}
$$

which we rewrite as

$$
\omega^{\mathcal{N}}(\frac{1+\omega v}{1+\omega^2 v})^{\mathcal{N}} = \omega^{2\bar{P}_a-1}. \tag{3.3}
$$

Then if we use the relation (2.24) between P and v we find

$$
\omega^{\mathcal{N}} e^{i\mathcal{N}P} = \omega^{2\bar{P}_a-1} \tag{3.4}
$$

which, because of the quantization of P (1.21b) gives the identity

$$
2\bar{P}_a - 1 - \mathcal{N} \equiv 0 \mod 3 \tag{3.5}
$$

which, with $0 \le \bar{P}_a \le 2$ fixes $\bar{P}_a$ and agrees with Table I.

We now can determine the w_ℓ as before in terms of the roots of the polynomial

$$
\begin{aligned}
\bar{P}_1(t) = t^{-\bar{P}_a}\{&(\omega^2 t-1)^{\mathcal{N}}(\omega t-1)^{\mathcal{N}}\frac{1+tv}{1+t^3v^3}\omega^{\bar{P}_a} \\
&+ (t-1)^{\mathcal{N}}(\omega^2 t-1)^{\mathcal{N}}\frac{1+\omega tv}{1+t^3v^3} \\
&+ (t-1)^{\mathcal{N}}(\omega t-1)^{\mathcal{N}}\frac{1+\omega^2 tv}{1+t^3v^3}\omega^{-\bar{P}_a}\}.
\end{aligned}
\tag{3.6}
$$

Thus, using the linear terms as found in Table I we follow the procedure of the previous secion and find

$$E_{\mathcal{N}}(P,\lambda) = \bar{A}_{\mathcal{N}} + \bar{B}_{\bar{N}}\lambda - 3\bar{m}_E|1-\lambda| - \frac{1}{2\pi i}\int_{c_1'+c_2'+c_3'} dt \frac{d}{dt}\ln \bar{P}_1(t)[(1-\lambda)^2 + \frac{4\lambda}{1-t^3}]^{1/2} \tag{3.7}$$

where

$$\bar{A}_{\mathcal{N}} = 0, \bar{B}_{\mathcal{N}} = -4 \quad \text{if} \quad \mathcal{N} \equiv 0 \bmod 3$$
$$\bar{A}_{\mathcal{N}} = 1, \bar{B}_{\mathcal{N}} = -3 \quad \text{if} \quad \mathcal{N} \equiv 1 \bmod 3$$

and

$$\bar{A}_{\mathcal{N}} = 2, \bar{B}_{\mathcal{N}} = -2 \quad \text{if} \quad \mathcal{N} \equiv 2 \bmod 3. \tag{3.8}$$

We finally must compute the difference $E_{\mathcal{N}}(P,\lambda) - E^0_{\mathcal{N}}(\lambda; Q=0)$ and let $\mathcal{N} \to \infty$. Again separate calculations must be done for $\mathcal{N} \equiv 0, 1,$ or 2 mod 3. However, the calculation is slightly simplified if we note that for all $\mathcal{N}$

$$\bar{P}_a = P_a(Q=1). \tag{3.9}$$

Thus from (3.8) and (2.43) we have for all $\mathcal{N}$

$$\bar{A}_{\mathcal{N}} + \bar{B}_{\mathcal{N}}\lambda - (A^1_{\mathcal{N}} + B^1_{\mathcal{N}}\lambda) = 3(1-\lambda) \tag{3.10}$$

and

$$\bar{m}_E - m^1_E = -1. \tag{3.11}$$

Here, subtracting (2.45) from (3.7) and using the result that for $1 \le t$

$$\lim_{\mathcal{N}\to\infty} \frac{d}{dt}\ln\frac{\bar{P}_1(t)}{P_1(t)} = \frac{d}{dt}\ln\frac{1+tv}{1+t^3v^3} = -(\frac{\omega v}{1+\omega tv} + \frac{\omega^2 v}{1+\omega^2 tv}) \tag{3.12}$$

we find

$$\lim_{\mathcal{N}\to\infty}\{E_{\mathcal{N}}(P;\lambda) - E^0_{\mathcal{N}}(\lambda;1)\} = 3(1-\lambda) + 3|1-\lambda| + \frac{3}{\pi}\int_1^{|\frac{1+\lambda}{1-\lambda}|^{2/3}} dt\{\frac{\omega v}{1+\omega tv} + \frac{\omega^2 v}{1+\omega^2 tv}\} \times [\frac{4\lambda}{t^3-1} - (1-\lambda)^2]^{1/2} \tag{3.13}$$

and, thus, using (2.67)

$$\lim_{\mathcal{N}\to\infty}\{E_{\mathcal{N}}(P,\lambda)-E^0_{\mathcal{N}}(\lambda;0)\}=4|1-\lambda|+2(1-\lambda)$$
$$+\frac{3}{\pi}\int_1^{|\frac{1+\lambda}{1-\lambda}|^{2/3}}dt\{\frac{wv}{1+\omega tv}+\frac{\omega^2 v}{1+\omega^2 tv}\}\times[\frac{4\lambda}{t^3-1}-(1-\lambda)^2]^{1/2}. \quad (3.14)$$

We now can investigate the question of stability posed at the beginning of the section. When $\lambda=0$ (3.14) reduces to

$$E_{\mathcal{N}}(P,0)-E^0_{\mathcal{N}}(0;0)=6. \quad (3.15)$$

However when $\lambda=1$ (3.14) becomes

$$\lim_{\mathcal{N}\to\infty}\{E_{\mathcal{N}}(P,1)-E^0_{\mathcal{N}}(1;0)\}=\frac{6}{\pi}v\int_1^{\infty}dt\{\frac{\omega}{1+\omega tv}+\frac{\omega^2}{1+\omega^2 tv}\}(t^3-1)^{-1/2}. \quad (3.16)$$

This is clearly an analytic function of v which vanishes linearly in v at $v=0$. Therefore there is a range of P for which $E_{\mathcal{N}}(P,1)$ lies *below* $E^0_{\mathcal{N}}(1,0)$ if $\mathcal{N}$ is sufficiently large. Thus there is a range of λ about $\lambda=1$ for which the state whose eigenvalue is $E^0_{\mathcal{N}}(\lambda;0)$ is *not* the ground state.

To be more quantitative about this instability we have plotted $\Delta e(P,\lambda)=\lim_{\mathcal{N}\to\infty}\{E_{\mathcal{N}}(P,\lambda)-E^0_{\mathcal{N}}(\lambda;0)\}$ for several values of λ between 0 and 1 in Fig. 5a.

We note that as we increase λ $\Delta e(P,\lambda)$ first vanishes at $\lambda=.9735\ldots$ and $P=.1612\ldots$. When $\lambda=1$ $\Delta e(P,\lambda)$ is negative for $0<P<.3518\ldots$. Thus we conclude that phase I surely does not extend to

$$.9735\ldots<\lambda \quad (3.17)$$

because of instability against one particle excitation.

However, this is not the entire story. From the duality relation (1.26) the value λ^I at which phase I ceases to be stable and the value λ^{IV} at which phase IV ceases to be stable should satisfy

$$\lambda^I=1/\lambda^{IV}. \quad (3.18)$$

Due to the presence of the term $4|1-\lambda|+2(1-\lambda)$ in (3.14) we find (see Fig. 5b) that for $\lambda>1$ the region of P where $\Delta e(P,\lambda)$ is negative ocurs

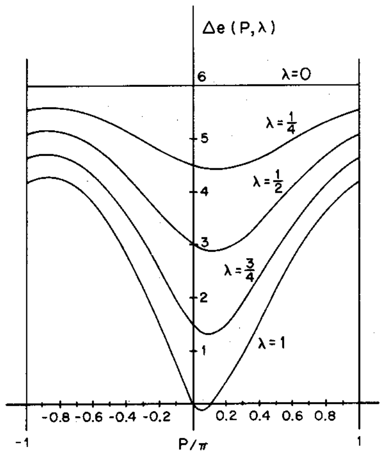

Fig. 5a The excitation energy
$\Delta e(P,\lambda) = \lim_{\mathcal{N}\to\infty}\{E_{\mathcal{N}}(P,\lambda) - E^0_{\mathcal{N}}(\lambda;0)\}$
for several values of λ as a function of P for $0 < \lambda < 1$. This excitation curve is tangent to the P axis at $P = .1612\ldots$ for $\lambda = .9735\ldots$.

for

$$.9013\ldots = 1/\lambda^{IV} < 1/\lambda \tag{3.19}$$

This is clearly a smaller value than that of (3.17) and indicates that phase I will be unstable against multiparticle excitation even though it is stable against single particle excitation. This is the first piece of evidence that the full phase diagram of Fig. 2 is correct and that phase II and Phase III are in fact distinct.

§4. Multiparticle excitations for $Q = 0$

Once it has been shown that "single particle excitations" lie below the "ground state" then it is to be expected that multiparticle excitations will be lower still and thus the true ground state energy per site will lie below the $e_0(\lambda)$ calculated in Section 2.

The basic equation for eigenvalues is still valid as is the form (2.22) but, as before, some guidance as to the choice of allowed quantum numbers must be gained from the finite chain studies summarized in Table

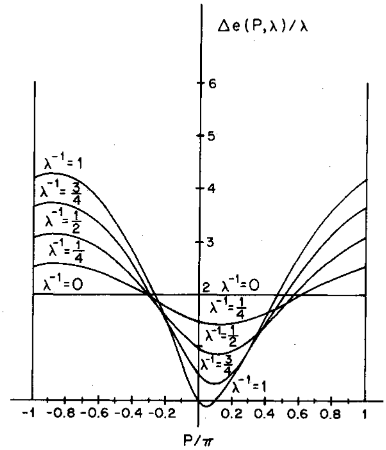

Fig. 5b The excitation energy $\Delta e(P, \lambda)/\lambda$ for several values of λ as a function of P for $1 < \lambda < \infty$. This excitation curve is tangent to the real P axis at $P = .225\ldots$ for $1/\lambda = .9013\ldots$.

I. Here we confine our attention to the sector $Q = 0$ and make the observation that multiparticle excitations occurs in triplets. For example, consider $\mathcal{N} = 7$. The $P = 0, m_p = 0$ state has the linear term $-2(1+\lambda)$. Accompanying the eigenvalue is a set with $m_p = 3$ but not $m_p = 1$ or 2. Similarly for $P \neq 0$ the same linear term of $-2(1+\lambda)$ has states of $m_p = 3$ and $m_p = 6$. This occurance of triplets is caused by the fact that each m_p single particle excitation has a Z_3 charge of 1. Thus triplets of excitations do not change the Z_3 charge.

In this section we will compute the ground state energy per site for a finite density of these excitations when

$$\lim_{\mathcal{N}\to\infty} m_p/\mathcal{N} > 0. \tag{4.1}$$

We first must determine the equation for the v_ℓ. This is easily obtained by first putting the form (2.22) in (2.21) to obtain the generalization of (3.1) of

$$c'\omega^{2P_b+P_a}t^{P_a+P_b}\prod_{\ell=1}^{m_p}(1+t^3v_\ell^3)\prod_{\ell=1}^{m_E}[1-(\frac{1-\lambda}{1+\lambda})^2t^3 - w_\ell^2\frac{4}{(1+\lambda)^2}(1-t^3)]$$

$$
\begin{aligned}
&= e^{-iP}\{(t-1)^{\mathcal{N}}(t\omega^2-1)^{\mathcal{N}}\prod_{\ell=1}^{m_p}(1+\omega v_\ell t)\\
(4.2)\qquad &+(t\omega-1)^{\mathcal{N}}(t\omega^2-1)^{\mathcal{N}}\omega^{P_a+P_b}\prod_{\ell=1}^{m_p}(1+v_\ell t)\\
&+(t-1)^{\mathcal{N}}(t\omega-1)^{\mathcal{N}}\omega^{-(P_a+P_b)}\prod_{\ell=1}^{m_p}(1+\omega^2 v_\ell t)\}.
\end{aligned}
$$

We obtain m_p equations for the m_p unknowns v_ℓ by setting $t=-v_k$. Both the left hand side and the second term on the left hand side of (4.2) vanish here and we have

$$
\begin{aligned}
&0=(-v_k^{-1}-1)^{\mathcal{N}}(-v_k^{-1}\omega^2-1)^{\mathcal{N}}\prod_{\ell=1}^{m_p}(1-\omega v_\ell/v_k)\\
(4.3)\qquad &+(-v_k^{-1}-1)^{\mathcal{N}}(-v_k^{-1}\omega-1)^{\mathcal{N}}\omega^{-(P_a+P_b)}\prod_{\ell=1}^{m_p}(1-\omega^2 v_\ell/v_k)
\end{aligned}
$$

for $k=1,\cdots m_p$ which we rewrite as

$$
(4.4)\qquad (\frac{\omega^2+v_k}{\omega+v_k})^{\mathcal{N}}=-\omega^{-(P_a+P_b)}\prod_{\ell=1}^{m_p}(\frac{v_k-\omega^2 v_\ell}{v_k-\omega v_\ell}).
$$

Once the v_k are determined from (4.4) the w_ℓ are obtained from (2.36) where now the t_ℓ are the real roots of the polynomial

$$
\begin{aligned}
&P(t)=t^{-(P_a+P_b)}\{(\omega^2t-1)^{\mathcal{N}}(\omega t-1)^{\mathcal{N}}\omega^{P_a+P_b}\prod_{\ell=1}^{m_p}(\frac{1+tv_\ell}{1+t^3v_\ell^3})\\
(4.5)\qquad &+(t-1)^{\mathcal{N}}(\omega^2t-1)^{\mathcal{N}}\prod_{\ell=1}^{m_p}(\frac{1+\omega tv_\ell}{1+t^3v_\ell^3})\\
&+(t-1)^{\mathcal{N}}(\omega t-1)^{\mathcal{N}}\omega^{-(P_a+P_b)}\prod_{\ell=1}^{m_p}(\frac{1+\omega^2tv_\ell}{1+t^3v_\ell^3})\}.
\end{aligned}
$$

We now restrict our attention to the case $m_p\equiv 0 \mod 3$

$$
(4.6)\qquad Q=0, P_b=0,\quad 2P_a-\mathcal{N}\equiv 0,\quad \mathrm{mod}\ 3
$$

where

$$\text{linear term} = \begin{cases} 0 & \text{for } \mathcal{N} \equiv 0 \bmod 3 \\ -2(1+\lambda) & \text{for } \mathcal{N} \equiv 1 \bmod 3 \\ -(1+\lambda) & \text{for } \mathcal{N} \equiv 2 \bmod 3. \end{cases} \tag{4.7}$$

Then since

$$2m_p + 3m_E = \text{ const } = 3m_E^0 \tag{4.8}$$

and $m_p = 3s$ where $s = 0, 1, \cdots$ we have

(4.9)

$$\begin{aligned} E_{\mathcal{N}}(P,\lambda,m_p) - E_{\mathcal{N}}^0(\lambda;0) &= -3(m_E - m_E^0)|1-\lambda| \\ &\quad - \frac{3}{2\pi i}\int_{c_1''} dt \frac{d}{dt}\ln\frac{P(t)}{P_0(t)}[(1-\lambda)^2 + \frac{4}{1-t^3}]^{1/2} \\ &= 2m_p|1-\lambda| \\ &\quad + \frac{3}{\pi}\int_1^{|\frac{1+\lambda}{1-\lambda}|^{2/3}} dt\frac{d}{dt}\ln(\frac{P(t)}{P_0(t)})[\frac{4\lambda}{t^3-1} - (1-\lambda)^2]^{1/2} \\ &= 2m_p|1-\lambda| \\ &\quad + \frac{3}{\pi}\int_1^{|\frac{1+\lambda}{1-\lambda}|^{2/3}} dt\sum_{\ell=1}^{m_p}\{\frac{\omega v_\ell}{1+\omega t v_\ell} + \frac{\omega^2 v_\ell}{1+\omega^2 t v_\ell}\} \\ &\qquad \times [\frac{4\lambda}{t^3-1} - (1-\lambda)^2]^{1/2}. \end{aligned}$$

where in the last line the $\mathcal{N} \to \infty$ limit is taken.

The solution of our problem has now been reduced to the solution of (4.4). This, however, is a very familiar equation as can be seen by letting

$$v_k = \frac{1-\omega z_k}{z_k - \omega} \tag{4.10}$$

to obtain

$$z_k^{\mathcal{N}} = \omega^{2P_a + P_b + \mathcal{N} - m_p}(-1)^{m_p+1}\prod_{\ell=1}^{m_p}\frac{1+z_\ell+z_k z_\ell}{1+z_k+z_k z_\ell}. \tag{4.11}$$

where from (2.24)

$$e^{+iP} = \omega^{m_p - P_b}\prod_{\ell=1}^{m_p} z_\ell. \tag{4.12}$$

Up to the phase factor in front, this is the well known Bethe's Ansatz equation for the XXZ spin chain with $\Delta = -1/2$ [15] where m_p is the number of down spins in the chain. We thus would like to study the $\mathcal{N} \to \infty$ limit by means of integral equations as is done for the Bethe's Ansatz problems. However, in order to do this we need to answer two questions.

1. Do solutions occur for which $v_k = v_j$? and
2. Do solutions occur where some v_k are complex?

To answer the first question we first note that there are indeed solutions of (4.4) when 2 or more v_ℓ are equal. However whenever these solutions have been compared with the finite chain studies they always have led to eigenvalues which do not exist. This is, of course, not a surprising happening because we do not as yet have any information on the eigenvectors and hence the degeneracy of our eigenvalues is not really known. This is a problem which affects any method of solution based on equations for eigenvalues alone [9,16]. For the purposes of this paper we will assume that solutions of (4.4) with equal roots do not contribute to the eigenvalues because of the information obtained from the finite chain studies. A better argument should surely be found because the physics of the result depends on it.

To answer the second question we again look at finite chain calculations. In contrast to the first question we do indeed find that there are eigenvalues corresponding to complex v_ℓ. Thus we must pursue this question further.

If v_k is real then z_k is on the unit circle and each side of (4.11) has magnitude 1 and (4.12) is satisfied with P real. However if v_k is complex then z_k will not be on the unit circle and if there were only one such z_k (4.12) could never be satisfied. Thus we study the possibility of a pair of z_k, one inside and one outside the unit circle. Moreover because of (4.6) the equation (4.11) reduces to

$$z_k^{\mathcal{N}} = \omega^{2\mathcal{N}}(-1)^{3m_p+1} \prod_{\ell=1}^{m_p} \frac{1 + z_\ell + z_k z_\ell}{1 + z_k + z_k z_\ell} \tag{4.13}$$

from which we conclude that the pair of roots z_1, z_2 off the unit circle satisfy

$$z_1^* = z_2^{-1} \tag{4.14}$$

which we parametrize in the form

$$z_1 = \zeta e^{i\phi}, \quad z_2 = \zeta^{-1} e^{i\phi} \tag{4.15}$$

with $0 \leq \zeta < 1$ where ζ and ϕ are real. We now recall the standard discussion of pair solutions of Bethe's Ansatz equations. If $k = 1$ then as $\mathcal{N} \to \infty$ the left hand side of (4.13) goes to zero. Thus on the right hand side there must be some factor which vanishes. The factor is

$$1 + z_2 + z_1 z_2. \tag{4.16}$$

Equating this to zero and using (4.15) we find for $\mathcal{N} \to \infty$ the relation between ϕ and ζ of

$$\zeta^{-1} = -2\cos\phi, \tag{4.17}$$

where, to satisfy the requirement that $0 \leq \zeta < 1$ we need

$$\frac{2\pi}{3} < \phi < \frac{4\pi}{3}. \tag{4.18}$$

We now use this in (4.10) and find that the v_k which correspond to these pair solutions are

$$v_1 = -\omega\left(\frac{\omega e^{i\phi} - \omega^{-1}e^{-i\phi}}{\omega^{-1}e^{i\phi} - \omega e^{-i\phi}}\right) = -\omega\frac{\sin(\phi + \frac{2\pi}{3})}{\sin(\phi - \frac{2\pi}{3})}\} \tag{4.19a}$$

and

$$v_2 = v_1^*. \tag{4.19b}$$

Then from (4.9) we see that the contribution of these pair states make to the energy is

$$\frac{3}{\pi}\int_1^{|\frac{1+\lambda}{1-\lambda}|^{2/3}} dt\{\frac{\omega v_r}{1 + \omega t v_r} + \frac{\omega^2 v_r}{1 + \omega^2 t v_r} + 2\frac{v_r}{1 + v_r t}\} \times [\frac{4\lambda}{t^3 - 1} - (1 - \lambda)^2]^{1/2} \tag{4.20}$$

with

$$v_r = -\frac{\sin(\phi + \frac{2\pi}{3})}{\sin(\phi - \frac{2\pi}{3})} \geq 0 \tag{4.21}$$

where the positivity statement is a consequence of (4.18). Thus, rewriting (4.20) as

$$\frac{3}{\pi}\int_1^{|\frac{1+\lambda}{1-\lambda}|^{2/3}} dt\frac{v_r[1 - v_r t + 4(v_r t)^2]}{1 + t^3 v_r^3}\left[\frac{4\lambda}{t^3 - 1} - (1 - \lambda)^2\right]^{1/2} \tag{4.22}$$

and noting that for the allowed range of v_r (4.22) this is positive we conclude that complex pairs of v_ℓ will only serve to raise the energy $E_{\mathcal{N}}(P, \lambda, m_p) - E^0_{\mathcal{N}}(\lambda; 0)$. Thus these complex pairs will not contribute to the lowest energy state and for our purposes the answer to question 2 is negative.

Having thus disposed of complex pair solutions and argued that the v_k obey an exclusion principle we may now follow the procedure used for the Heisenberg Ising chain and consider the v_k to be distributed along the real v axis in an interval

$$v_L < v < v_U. \tag{4.23}$$

We then take the logarithm of (4.4) and choose the logarithm of 1 to be $2\pi i I_k$ where I_k are a set of distinct integers with

$$I_{k+1} - I_k = 1 \tag{4.24}$$

to find

$$\mathcal{N} \ln\left(\frac{\omega^2 + v_k}{\omega + v_k}\right) = \frac{4\pi i}{3} P_a + 2\pi i (I_k + \frac{1}{2}) + \sum_{\ell=1}^{m_p} \ln\left(\frac{v_k - \omega^2 v_\ell}{v_k - \omega v_\ell}\right). \tag{4.25}$$

We now may obtain an integral equation by considering the interval $\Delta v_k = v_{k+1} - v_k$ and define the density function $\rho(v; \lambda)$ as

$$\mathcal{N} \rho(v_k; \lambda) \Delta v_k = 1 \tag{4.26}$$

and use

$$\frac{1}{\mathcal{N}} \sum_\ell = \sum_\ell \rho(v_i; \lambda) \Delta v_\ell \to \int dv \rho(v; \lambda) \tag{4.27}$$

where

$$\rho(v; \lambda) \geq 0 \quad \text{for} \quad v_L < v < v_U \tag{4.28}$$

and

$$\int_{v_L}^{v_U} dv \rho(v; \lambda) = \lim_{\mathcal{N} \to \infty} m_p / \mathcal{N}. \tag{4.29}$$

Thus subtracting (4.25) from the same equation with $k \to k+1$ and using (4.24), (4.26) and (4.27) we obtain

$$\int_{v_L}^{v_U} dv' \rho(v'; \lambda) \frac{v'}{v^2 + vv' + v'^2} + \frac{1}{1 - v + v^2} = \frac{2\pi}{\sqrt{3}} \rho(v; \lambda) \tag{4.30}$$

and hence the ground state energy per site is obtained from (4.9) as

$$(4.31)\quad e_0^{II}(\lambda) = \lim_{\mathcal{N}\to\infty} \frac{1}{\mathcal{N}} E_{\mathcal{N}}(P,\lambda,m_p) = e_0^I(\lambda) + \int_{v_L}^{v_U} dv\rho(v;\lambda)F(v,\lambda)$$

where

(4.32)

$$F(v,\lambda) = 2|1-\lambda| + \frac{3}{\pi}\int_1^{|\frac{1+\lambda}{1-\lambda}|^{2/3}} dy \frac{(2vy-1)v}{1-vy+v^2y^2}\left[\frac{4\lambda}{y^3-1} - (1-\lambda)^2\right]^{1/2}.$$

We note that $F(v,\lambda)$ is identical with the $Q=1$ single particle excitation spectrum $\lambda\Delta e(P,\lambda^{-1})$ for $1<\lambda$ found in Section 3. Thus we see that if

$$(4.33)\qquad \lambda^I = .901292\cdots < \lambda < 1$$

then $F(v)$ has two zeroes and is negative for v between them. There zeroes determine the v_L and v_U as $F(v_L) = F(v_U) = 0$ and can be used in (4.29) to compute $\frac{m_p}{\mathcal{N}}$.

We thus have found in (4.30), (4.31) and (4.32) a set of equations to determine $e_0^{II}(\lambda)$, the ground state energy per site in phase II. However, just as with the Heisenberg Ising chain in the presence of a nonzero magnetic field, this integral equation (4.30) has not been solved explicitly. Thus we will content ourselves here with a few qualitative features of the solution.

We have already noted that at λ^I the energy per site $e_0^I(\lambda)$ (2.56) is analytic. However the same is not true for e_0^{II}. When $\lambda \to \lambda^I$ from above, v_L and v_U approach

$$(4.34)\qquad v_c \simeq .12248\ldots,$$

the integral in (4.30) can be neglected as being of order $v_U - v_L$ and we find that $\rho(v)$ can be approximated as

$$(4.35)\qquad \rho_c = \rho(v_c;\lambda^I) = \frac{\sqrt{3}}{2\pi}\frac{1}{1-v_c+v_c^2}$$

and $F(v,\lambda)$ is approximated by

$$(4.36)\qquad F(v,\lambda) \sim a_1(v-v_c)^2 - a_0(\lambda-\lambda^I)$$

where

$$(4.37)\qquad a_0 = -\frac{\partial}{\partial\lambda} f(v_c,\lambda)|_{\lambda=\lambda^I}$$

and

$$a_1 = \frac{1}{2}\frac{\partial^2}{\partial v^2}F(v,\lambda^I)|_{v=v_c}. \tag{4.38}$$

Thus

$$v_{U,L} = v_c \pm [\frac{a_0}{a_1}(\lambda - \lambda^I)]^{1/2} \tag{4.39}$$

and we find that

$$\begin{aligned} e_0^{II}(\lambda) - e_0^I(\lambda) &\sim \int_{v_L}^{v_U} dv \rho_c[a_1(v-v_c)^2 - a_0(\lambda - \lambda^I)] \\ &= \rho_c[\frac{a_1}{3}(v-v_c)^3 - a_0(\lambda-\lambda^I)(v-v_c)]_{v_L}^{v_U} \\ &= -\frac{4}{3}\rho_c\frac{a_0^{3/2}}{a_1^{1/2}}(\lambda-\lambda^I)^{3/2}. \end{aligned} \tag{4.40}$$

Hence $e_0^{II}(\lambda)$ has a singularity at $\lambda = \lambda^I$ even though $e_0^I(\lambda)$ does not. Thus it is surely true that λ^I marks a phase boundary.

We close this section with a discussion of the behavior of $e_0^{II}(\lambda)$ as $\lambda \to 1$. First we study the value of e_0^{II} at $\lambda = 1$ by setting $v_L = 0$ in (4.30) and $\lambda = 1$ in (4.32). Thus $\rho(v;1)$ satisfies

$$\int_0^{v_U} dv'\rho(v';1)\frac{v'}{v^2+vv'+v'^2} + \frac{1}{1-v+v^2} = \frac{2\pi}{\sqrt{3}}\rho(v;1) \tag{4.41}$$

and

$$e_0^{II}(1) = e_0^I(1) + \int_0^{v_U} dv\rho(v;1)\frac{6}{\pi}\int_1^\infty dy\frac{(2vy-1)v}{1-vy+v^2y}(y^3-1)^{-1/2}. \tag{4.42}$$

Secondly, we wish to study the singularity in $e_0^{II}(\lambda)$ as $\lambda \to 1$. For this we must note from (4.41) that $\rho(v;1)$ is not bounded as $v \to 0$ because the kernal of the integral equation is v'^{-1} if $v = 0$. Thus $\rho(v;1)$ is of the form

$$\rho(v;1) \sim Av^{-\alpha} \quad \text{as} \quad v \to 0 \tag{4.43}$$

and, using this form in (4.41) we find that α satisfies

$$\int_0^\infty dx x^{-\alpha}\frac{x}{1+x+x^2} = \frac{2\pi}{\sqrt{3}} \tag{4.44}$$

and the integral on the left is evaluated as

$$\frac{2\pi}{\sqrt{3}} \frac{\sin \frac{\pi}{3}(\alpha+2)}{\sin \pi\alpha}. \tag{4.45}$$

Thus the equation for α is

$$\frac{\sin \frac{\pi}{3}(\alpha+2)}{\sin \pi\alpha} = 1 \tag{4.46}$$

from which we readily find

$$\alpha = \frac{1}{4}. \tag{4.47}$$

We may now compute the form of the singularity in $e_0^{II}(\lambda)$ at $\lambda = 1$ by noting that the region in the y and v integrals which contributes to the leading singularity is

$$vy = 0(1) \tag{4.48}$$

where separately

$$y = x\left(\frac{1-\lambda}{1+\lambda}\right)^{-2/3} \quad \text{with} \quad x = 0(1) \tag{4.49}$$

and

$$v = \left(\frac{1-\lambda}{1+\lambda}\right)^{2/3} z \quad \text{with} \quad z = 0(1). \tag{4.50}$$

Then if we note that $v_L = c(1-\lambda)$ which is much smaller than (4.50) we see that $\rho(v;\lambda)$ may be approximated by (4.43) and (4.47). Thus from (4.42) the leading singular behavior is

$$\begin{aligned} &A \int_0^\infty dz \left(\frac{1-\lambda}{1+\lambda}\right)^{2/3} \left(\frac{1-\lambda}{1+\lambda}\right)^{-\frac{2}{3}\cdot\frac{1}{4}} z^{-1/4} \left(\frac{1-\lambda}{1+\lambda}\right) \\ &\times \int^1 dx \frac{z(2zx-1)}{1-zx+z^2x^2} \left(\frac{1-x^3}{x^3}\right)^{1/2}. \end{aligned} \tag{4.51}$$

The x integral must be cut off in an appropriate fashion to show the constant term (4.42). This subtraction, however, can only affect the amplitude and not the form of the singularity. Thus we find that as $\lambda \to 1$

$$e_0^{II}(\lambda) \sim e_0^{II}(1) + \text{ const } (1-\lambda)^{\frac{3}{2}} + \cdots \tag{4.52}$$

This singularity is clearly larger than the singularity $(1-\lambda^2)^{\frac{5}{3}}$ present in $e_0^I(\lambda)$. We thus conclude that phase II is separated from phase III by a singularity in $e_0^{II}(\lambda)$ at $\lambda = 1$ (which, by duality also occurs in phase III).

§5. Discussion

Our calculations are complete and we now turn to the physical interpretation of our results and a comparison with previous work.

The most detailed previous study of this system is that of Howes, Kadanoff, and den Nijs [8]. We strongly advise the reader at this point to consult ref. 8 because the situation is most subtle.

In Fig. 6 we reproduce the phase diagram of ref. 8.

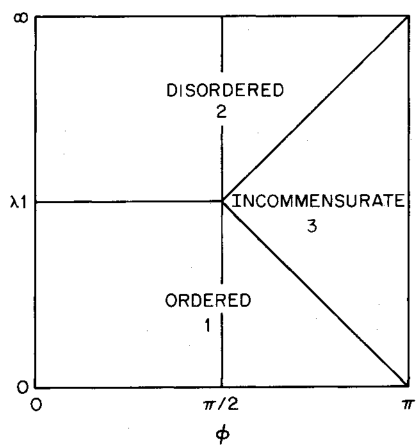

Fig. 6 The proposed phase diagram of ref. 8.

This diagram is in the $\lambda, \phi = \bar{\phi}$ plane and contains 3 phases:

1. An ordered phase where for $k \to \infty$

$$< Z_0 Z_k^\dagger >= m^2 + O(e^{-k/\xi}) \tag{5.1}$$

and $m = (1-\lambda^2)^{1/9}$ for $\phi = \bar{\phi} = \pi/2$ [8].

2. A disordered phase, dual to the ordered phase where

$$< Z_0 Z_k^\dagger >= O(e^{-k/\xi}) \tag{5.2}$$

3. A single incommensurate phase where

$$< Z_0 Z_k^\dagger > \simeq e^{iQ_\theta k}/k^\eta \tag{5.3}$$

for $0 < Q_\theta < 2\pi/3$.

As the boundary between 1 and 3 is approached from phase 1 there is no singularity in the ground state energy. As the boundary between phase 1 and 3 is approached from phase 3 the ground state energy has a singularity of the form $(\lambda - \lambda_c)^{3/2}$. This phase diagram is arrived at through series expansions in λ and a fermion analysis in phase 3 about the point $\lambda = 0, \phi = \pi$.

It is quite clear that there are major differences between Fig. 6 and the phase diagram on the line $\phi = \bar{\phi} = \pi/2$ of Fig. 2. The most striking of the differences is that not only does the phase boundary cross the $\phi = \bar{\phi} = \pi/2$ line for $\lambda_I < 1$ but that there are 4 phases instead of 3. Moreover, neither our phase II nor phase III have all the characteristics of the incommensurate phase 3. We conclude this because from Section 3 and Section 4 we see that in phase II the energy of three $Q = 1$ excitations is *not* the sum of three separate $Q = 1$ energies. This is inferred from the coefficient of $1 - \lambda$ being 6 in (3.14) if $\lambda < 1$ while it is 2 if $\lambda > 1$. This lack of additive energies in phase II indicates that if the total number of $Q = 1$ domain walls is not a multiple of 3 there is a mismatched energy seam in the system which can only be caused by a long range spin order. However, it is clear that if we extend the Bethe's Ansatz analysis of the eigenvalues in phase II we will see that there is no mass gap in this phase. Hence we conclude that in contrast to the incommensurate phase 3 we have for phase II

$$< Z_0 Z_k^\dagger > = \bar{m}_{II}^2 e^{i\theta_{II} k} + K/k^\eta \tag{5.4}$$

where K could contain a term $e^{i\theta_{II} k}$ as well. In phase III there will be no long range order but by duality there will be long range disorder. Such long range disorder is also absent in the phase 3 of ref. 8.

How can these two situations be resolved?

We first note that the analysis of ref. 8 makes no claim to be valid near $\lambda = 1$ and that we have no results near $\phi = \pi$ and $\lambda = 0$. Thus, unless some subtlety causes the perturbative results of ref. 8 to break down if the expansion is carried out to all orders there has to be a new phase boundary which separates the incommensurate phase 3 of ref. 8 from the phases II and III of the paper.

We also note, however, that phases II and III do have much in common with phase 3. The argument that the incommensurate oscillation in the incommensurate phase 3 is given by the density of domain walls $m_p/\mathcal{N}$ surely applies in our case as well. Indeed the singularity of $(\lambda - \lambda_c)^{3/2}$ in phase 3 at the phase 1 boundary is exactly the same as our $(\lambda - \lambda^I)^{3/2}$ singularity in phase II at the phase I boundary. The only difference is that we have interacting fermions and ref. 8 has free fermions. All in all the physical situation is most delicate.

Our final remark is of a more general nature. As was emphasized in ref. 8 the reason that there are any transitions into phase 3 or phases II and III at all is because of level crossing in the ground state. However, if we were dealing with a two dimensional statistical mechanical system with real interactions all elements of the transfer matrix would be positive for all real λ and the ground state would always be non-degenerate by the Perron-Frobenius theorem [17]. Thus level crossing in the ground state would not occur.

There is a submanifold in the space of Boltzmann weights (1.7) where this positivity occurs [4] namely

$$\begin{aligned} a_p^* c_p &= \omega^{1/2} b_p^* d_p, \\ |a_p| &= |d_p|, \\ |b_p| &= |c_p|. \end{aligned} \tag{5.5}$$

On this submanifold there will be no ground state level crossing and the ground state energy found by Baxter [18] will be correct for all real λ.

On the other hand, for the superintegrable case considered here $\phi = \bar{\phi} = \pi/2$, the Boltzmann weights (1.7) are in general complex, the Perron-Frobenius theorem does not apply, and hence there can be level crossing in the ground state of $\mathcal{H}$. The existence of phases II and III is due to this crossing.

The use of positivity properties of quantum mechanical hamiltonians to prevent level crossing is well-known for hamiltonians of the form

$$\mathcal{H} = \text{KINETIC ENERGY} \quad + \quad \text{POTENTIAL ENERGY} \tag{5.6}$$

where in the coordinate representation

$$\text{POTENTIAL ENERGY} = \sum_{i<j} V(q_i - q_j) \tag{5.7}$$

is a diagonal operator which may be made positive by adding a suitable constant and

$$\text{KINETIC ENERGY} = \frac{1}{2m} \sum_j P_i^2 = -\frac{1}{2m} \sum_j \frac{\partial^2}{\partial q_j^2} \tag{5.8}$$

is an off diagonal operator with non-negative elements. For this hamiltonian the (operator analogue of) Perron-Frobenius theorem tells us that the ground state is unique and has no nodes. For this type of hamiltonian the level crossing transition of this paper is forbidden.

But there are many familiar quantum hamiltonians which do not enjoy positivity properties. The most ubiquitous are those obtained from the minimal coupling with an electromagnetic field

$$P_\mu \to P_\mu - A_\mu \tag{5.9}$$

where A_μ is a vector potential. The simplest example is the phenomenon of Landau diamagnetism [19] where A_μ is chosen to give a uniform magnetic field and $V = 0$. In this case the ground state eigenvalue has a macroscopic degeneracy. If instead of Schroedinger's equation we consider the free Dirac equation in an external magnetic field we also have a system which has no positivity properties and has level crossing [20]. Clearly nonpositive hamiltonians, degenerate ground states, and level crossing are common in quantum mechanics.

In quantum field theory there are also simple field theories such as $\lambda\phi^4$ which have positivity properties for which we can prove uniqueness of the vacuum. But when gauge interactions are introduced via (5.9) these positivity properties can be destroyed. Once the positivity properties are gone, level crossing phenomena can occur that change the vacuum. One such example is the Adler, Bell, Jackiw [21] anomaly which can be shown to be most similar to the behavior of electrons in the presence of a magnetic field [22].

Finally, we remark that many of the models proposed to explain high temperature superconductivity do not have positivity properties and level crossing in the ground state must be considered.

On the other hand, even though there are many examples of quantum systems where level crossing occurs, the physics of these systems is far from understood if, for no other reason, than that perturbation theory is an inadequate tool to investigate these phenomena. One of the main uses of the integrable chiral Potts model is to provide physical insight into strong coupling level crossing transitions. In this way the $N \geq 3$ integrable chiral Potts model plays the role with respect to level crossing transitions that the $N = 2$ Ising model has played for 45 years with respect to second order phase transitions.

Acknowledgement

B.M.M. and J.H.H.P. wish to take this opportunity to thank Prof. M. Jimbo and Prof. T. Miwa and the Taniguchi Foundation for the opportunity of participating in this Taniguchi conference on integrable systems and conformal field theory. This research was supported by NSF grant DMR-8803678.

Appendix A

We give here an algebraic derivation of the normalized recursion relation (2.21). As a first step, we want to find an explicit expression for $f_{p,q}f_{q,p}$ and $f_{p,Rq}f_{q,p}$. This has already been done in [18], but an alternative calculation will be given here for completeness. Consider the star-triangle relations [4,18]

$$\begin{aligned}\text{(A.1)}\qquad &\sum_{i=0}^{N-1} W^v_{p,q}(k-i)W^h_{r,q}(j-i)W^v_{r,p}(i-\ell)\\ &= \frac{f_{r,p}f_{p,q}}{f_{r,q}} W^h_{r,p}(j-k)W^v_{r,q}(k-\ell)W^h_{p,q}(j-\ell).\end{aligned}$$

Note that, from the definition (1.7)

$$\text{(A.2)}\qquad W^{(h)}_{q,q}(n)=1,\quad W^{(v)}_{q,q}(n)=\delta_{n,0} \mod N.$$

Therefore, if we replace $q\to Rq, r\to q$ in (A.1), we find

$$\text{(A.3)}\qquad f_{p,Rq}f_{q,p}=f_{q,Rq}.$$

Actually, by the definition (1.19), the RHS is in the form $\frac{0}{0}$, so it must be understood as $\lim_{q'\to q} f_{q,Rq'}$. This limit can be computed and it gives

$$\text{(A.4)}\qquad f_{p,Rq}f_{q,p}=\lim_{q'\to q} f_{q,Rq'}=\frac{N}{\lambda^{(N-1)/N}}.$$

(A.4) allows us to rewrite the recursion relation (1.18) as

$$\begin{aligned}\text{(A.5)}\qquad T_{p,q}T_{p,Rq}T_{p,R^2q} &= e^{-iP}\{(f_{p,q}f_{q,p})^{\mathcal{N}}T_{p,R^2q}+(f_{p,Rq}f_{Rq,p})^{\mathcal{N}}T_{p,q}\\ &+\left(\frac{3^3}{\lambda^2 f_{p,R^3q}f_{R^3q,p}f_{p,R^4q}f_{R^4q,p}}\right)^{\mathcal{N}} T_{p,R^4q}\}\end{aligned}$$

where only $f_{p,q}f_{q,p}$ need to be computed, the other coefficients being obtained applying the automorphism R to the q variables. Define then

$$\text{(A.6)}\qquad S_{p,q}(k,\ell)\stackrel{\text{def}}{=}\sum_{q=0}^{N-1} W^v_{p,q}(k-i)W^v_{q,p}(i-\ell)$$

and specialize (A.1) to the case $r = q$. Properties (A.2) of the Boltzmann weights must be supplemented by

$$W^{(h)}_{p,q}(n)W^{(h)}_{q,p}(n) = 1, \qquad f_{q,q} = 1 \tag{A.7}$$

which are also easy consequences of the definitions (1.7), (1.18). So we find that

$$S_{p,q}(k,\ell) = f_{p,q}f_{q,p}\delta_{k,\ell} \tag{A.8}$$

and $f_{p,q}f_{q,p}$ is nothing else than $S(0,0)$. We then proceed to compute $S(0,0)$. Since $W^{v}_{p,q}(N+n) = W^{v}_{p,q}(n)$, the definition of the Boltzmann weights (1.7) yields

$$\begin{aligned} S_{p,q}(0,0) &= \sum_{i=0}^{N-1}\prod_{j=1}^{N-i}\frac{\omega a_p d_q - d_p a_q \omega^j}{c_p b_q - b_p c_q \omega^j}\prod_{j=1}^{i}\frac{\omega d_p q_q - a_p d_q \omega^j}{b_p c_q - c_p b_q \omega^j} \\ &= A\sum_{i=0}^{N-1}\frac{\omega^i}{(b_p c_q - c_p b_q \omega^i)(d_p a_q - a_p d_q \omega^i)} \end{aligned} \tag{A.9}$$

where we have set

$$A = \frac{(d_p a_q)^N - (a_p d_q)^N}{(b_p c_q)^N - (c_p b_q)^N}(b_p c_q - c_p b_q)(d_p a_q - a_p d_q). \tag{A.10}$$

By means of the geometric sum representation

$$\frac{1}{1-x} = \frac{1}{1-x^N}\sum_{k=0}^{N-1} x^k. \tag{A.11}$$

(A.7) is written as

$$A'\sum_{i=0}^{N-1}\sum_{k_1,k_2=0}^{N-1}\omega^i\left(\frac{c_p b_q}{b_p c_q}\omega^i\right)^{k_1}\left(\frac{a_p d_q}{d_p a_q}\omega^i\right)^{k_2} \tag{A.12}$$

with

$$A' = \frac{A}{a_q b_p c_q d_p[1-(\frac{c_p b_q}{b_p c_q})^N][1-(\frac{a_p d_q}{d_p a_q})^N]}. \tag{A.13}$$

Interchange the sums in (A.10), to get

$$A'\sum_{k_1,k_2=0}^{N-1}\left(\frac{c_p b_q}{b_p c_q}\right)^{k_2}\left(\frac{a_p d_q}{d_p a_q}\right)^{k_2}\delta_{1+k_1+k_2,0}(\mathrm{mod}\, N). \tag{A.14}$$

Since $k_1, k_2 \epsilon \{0, 1 \ldots N-1\}$, one has that $k_1 + k_2 + 1 \epsilon \{1, 2 \ldots 2N-1\}$ and $\delta_{k_1+k_2+1,0} (\mathrm{mod}\ N)$ can be replaced by $\delta_{k_1+k_2+1,N}$. Eliminating the δ one remains with a geometric sum and it is easy to conclude that

$$\text{(A.15)}\qquad \begin{aligned} S_{p,q}(0,0) = & \frac{N(d_p a_q - a_p d_q)(b_p c_q - c_p b_q)}{a_p b_p c_q d_q - a_q b_q c_p d_p} \\ & \times \frac{(a_p b_p c_q d_q)^N - (a_q b_q c_p d_p)^N}{[(b_p c_q)^N - (c_p b_q)^N][(d_p a_q)^N - (a_p d_q)^N]} . \end{aligned}$$

This awkward expression can be slightly reduced, by means of (1.8)

$$\text{(A.16)}\qquad S_{p,q}(0,0) = \frac{N(d_p a_q - a_p d_q)(b_p c_q - c_p b_q)}{a_p b_p c_q d_q - a_q b_q c_p d_p} \frac{a_p^N a_q^N - b_p^N b_q^N}{a_q^N b_p^N - a_p^N b_q^N} .$$

In the superintegrable case, when $a_p = b_p = 1$ and $c_p = d_p = \eta$, this expression further reduces to

$$\text{(A.17)}\qquad f_{p,q} f_{q,p} = \frac{N(\eta \frac{a_q}{d_q} - 1)(\eta \frac{b_q}{c_q} - 1)}{\lambda(\eta^2 \frac{a_q b_q}{c_q d_q} - 1)} .$$

The notation is somewhat simplified if, following [10], we set $x = \eta \frac{a_q}{d_q}$ $y = \frac{c_q}{\eta b_q}$ with $(x^N - 1)(y^N - 1) = \lambda(y^N - x^N)$, owing to (1.8). Therefore, when $N = 3$

$$\text{(A.18)}\qquad f_{p,q} f_{q,p} = \frac{3(x-1)(y-1)}{\lambda(y-x)}$$

and $f_{p,R^k q} f_{R^k q,p}$ is found noting that the action of the automorphism R on the pair (x, y) is $R(x, y) = (1/y, 1/\omega x)$. Finally, defining the normalized transfer matrix $T_{p,q}^N$ from

$$\text{(A.19)}\qquad T_{p,q} = \frac{(x-1)^{\mathcal{N}}}{(x^3 - 1)^{\mathcal{N}}} T_{p,q}^N$$

and inserting (A.16), (A.17) into (A.2), it is straightforward to obtain (2.12) with

$$\text{(A.20)}\qquad h_{p,q} = 3^{1/2}(x/y - 1) = 3^{1/2} \left(\eta^2 \frac{a_q b_q}{c_q d_q} - 1 \right) .$$

Appendix B

The Hamiltonian (1.12) enjoys many symmetries, but we will prove here only those listed in (2.25). Further information can be found in [8,23]. The definition (1.14) of X and Z implies that

$$ZX = \omega XZ \quad X^N = Z^N = 1 \quad X^\dagger = X^{-1} \quad Z^\dagger = Z^{-1} \tag{B.1}$$

where X and Z are taken to be on the same site. Operators on different sites commute.

Here we prefer to work in the basis when X is diagonal. In this basis Z acts as the cyclic step-down operator, (see (B.1))

$$X|n> = \omega^n|n> \quad Z|n> = |n-1> \quad (\mathrm{mod}\, N) \tag{B.2}$$

Define an operator V that, on each site of the chain, acts as follows

$$V|n> = |N-1-n> \tag{B.3}$$

Obviously $V = V^{-1}$ and

$$\begin{aligned} VZ_kV^{-1} &= Z_k^\dagger \\ VX_kV^{-1}X_k^\dagger &= \omega^{-1}X_k^\dagger. \end{aligned} \tag{B.4}$$

Furthermore, we will need the space inversion operator I. It slightly simplifies the notation to consider the Hamiltonian (1.12) to be centered about the origin, even though the conclusions are by no means affected by this choice. Thus

$$\begin{aligned} IX_kI^{-1} &= X_{-k} \\ IZ_kI^{-1} &= Z_{-k}. \end{aligned} \tag{B.5}$$

Set $U = IV$. We have that

$$UH(\bar{\alpha}_n, \alpha_n)U^{-1} = H(\omega^n\bar{\alpha}_{-n}, \alpha_n) \tag{B.6}$$

This is true for any value of $\lambda, \phi, \bar{\phi}$. In the superintegrable case, $\phi = \bar{\phi} = \pi/2$ entails that

$$\omega^n\bar{\alpha}_{-n} = -\bar{\alpha}_n \tag{B.7}$$

which amounts to change the sign of λ. Therefore, we conclude that in the superintegrable case, for any length of the chain, the whole spectrum is invariant under $\lambda \to -\lambda$

$$\{E(\lambda)\} = \{E(-\lambda)\}. \tag{B.8}$$

To prove how the various sectors are mapped into each other, we observe that with P the momentum and e^{iP} the (right) shift operator,

$$Ue^{iP}U^{-1} = e^{-iP} \tag{B.9}$$

since U includes the space-inversion operator. As to the Z_N charge operator $R = \prod_k X_k$, (B.4) implies that

$$URU^{-1} = \omega^{\mathcal{N}} R^{-1} \tag{B.10}$$

Therefore, calling Q the z_N charge, defined mod N by $R = e^{\frac{2\pi i}{N}Q}$, the complete transformation properties of the spectrum are

$$E(Q, P, \lambda) = E(-Q - \mathcal{N}, -P, -\lambda). \tag{B.11}$$

When $\mathcal{N} = 0$ mod N, a further symmetry of the spectrum is present. To show it, we introduce $\tilde{U}$ defined by

$$\tilde{U} = I(\prod_k X_k^{-k}). \tag{B.12}$$

The transformation property of the basic operators X, Z is

$$\begin{aligned} \tilde{U} X_k \tilde{U}^{-1} &= X_{-k} \\ \tilde{U} Z_k \tilde{U}^{-1} &= \omega^k Z_{-k} \end{aligned} \tag{B.13}$$

When applying $\tilde{U}$ to the Hamiltonian, it is easily seen that the term $(Z_k Z_{k+1}^\dagger)^j$ goes into $\omega^{-j}(Z_{-k} Z_{-k-1}^\dagger)^j$ but the boundary term, where periodic boundary conditions are imposed, acquires an extra factor $\omega^{\mathcal{N}j}$. This factor is identically 1 if we choose $\mathcal{N} = 0$ mod N. Under this restriction

$$\tilde{U} H(\bar{\alpha}_n, \alpha_n) \tilde{U}^{-1} = H(\bar{\alpha}_n, \alpha_{-n}\omega^n) \tag{B.14}$$

In the superintegrable case $\alpha_{-n}\omega^n = -\alpha_n$, and therefore the whole spectrum has the symmetry

$$\{E(\lambda)\} = \{-E(-\lambda)\}. \tag{B.15}$$

Incidentally, (B.15) along with (B.8), assures that for $\mathcal{N} = 0$ mod N, the whole spectrum is reflection symmetric about 0 [23].

Obviously $[\tilde{U}, R] = 0$ and a sector of specified Q is mapped into itself under $\tilde{U}$, but an eigenspace of momentum is not, because $\tilde{U}$ is not translation invariant. Instead, remembering that $e^{iP}I = Ie^{-iP}$ we find that for an arbitrary vector of momentum P and Z_N charge Q

(B.16)
$$e^{iP}\tilde{U}|P,Q> = e^{iP}I(\prod_k X_k^{-k})|P,Q> = Ie^{-iP}(\prod_k X_k^{-k})e^{iP}e^{-iP}|P,Q>$$
$$= \tilde{U}R^{-1}e^{-iP}|P,Q> = \tilde{U}e^{-i(P+\frac{2\pi Q}{N})}|P,Q>$$

Hence $\tilde{U}|P,Q>$ has momentum $-(P+\frac{2\pi Q}{N})$. Now (B.14) allows us to conclude that, given an eigenvector of $H(\lambda)$, with momentum P, Z_N charge Q and energy $E(\lambda)$

(B.17) $$H(-\lambda)\tilde{U}|E(\lambda),P,Q> = -E(\lambda,P,Q)\tilde{U}|E(\lambda),P,Q>$$

so that $|E(\lambda),P,Q>$ is also an eigenvector of $H(-\lambda)$ and the relation holds

(B.18) $$\{E(\lambda,P,Q)\} = \{-E(-\lambda,-P-\frac{2\pi Q}{N},Q)\}$$

which completes the demonstration of (2.25).

Appendix C

The set of equations (2.26) comes from a requirement of consistency between (4.12) and (4.13) and (2.39). In a $\mathcal{N}$-site chain, $e^{i\mathcal{N}P}=1$, so, (4.13) yields

(C.1) $$\prod_{\ell=1}^{m_p} z_\ell^{\mathcal{N}} = \omega^{\mathcal{N}(P_b-m_p)}$$

If we multiply the m_p equations in (4.12) we have

(C.2) $$\prod_{\ell=1}^{m_p} z_\ell^{\mathcal{N}} = \omega^{m_p(\mathcal{N}-P_a-P_b-m_p)}$$

which is consistent with (C.1) provided that

(C.3) $$\mathcal{N}(P_b-m_p) = m_p(\mathcal{N}-P_a-P_b-m_p) \pmod 3$$

This restriction on P_a, P_b has to be supplemented by

(C.4) $$P_b - P_a = Q + \mathcal{N} \pmod 3$$

which follows from the symmetry (2.39). By inspecting all the possible values of $\mathcal{N}$ and m_p (mod 3) in (C.3) and C.4) we solve for P_a and P_b and find (2.26). Note that for some choices of $\mathcal{N}$ and m_p, a solution can be found only for particular values of Q.

Table I The quantum numbers for the eigenvalues of the transfer matrix established from analytical and numerical studies on finite size chains of length $\mathcal{N} = 3, 4, 5, 6,$ and 7. Here Q is the Z_3 charge, P is the total momentum, and the quantum numbers P_a, P_b, P_c, m_E, and m_p are defined by (2.22).

$\mathcal{N} = 3$

Q	P	Linear Term	P_a	P_b	P_c	m_E	m_P	Number of Sets	Number of Eigenvalues
0	0	0	0	0	0	2	0	1	4
		0	0	0	1	0	3	1	1
	$\pm\frac{2\pi}{3}$	0	0	0	1	0	3	2	2
1	0	$-(3+\lambda)$	2	0	0	1	0	1	2
		2λ	0	1	1	0	2	1	1
	$+\frac{2\pi}{3}$	-4λ	2	0	1	0	1	1	1
		2λ	0	1	1	0	2	2	2
	$\frac{-2\pi}{3}$	$3-\lambda$	0	1	1	1	0	1	2
		2λ	0	1	1	0	2	1	1
2	0	$-3+\lambda$	1	0	0	1	0	1	2
		-2λ	1	0	1	0	2	1	1
	$\frac{+2\pi}{3}$	$3+\lambda$	0	2	1	1	0	1	2
		-2λ	1	0	1	0	2	1	1
	$\frac{-2\pi}{3}$	4λ	0	2	1	0	1	1	1
		-2λ	1	0	1	0	2	2	2

$\mathcal{N} = 4$

Q	P	Linear Term	P_a	P_b	P_c	m_E	m_P	Number of Sets	Number of Eigenvalues
0	0	$-2(1+\lambda)$	2	0	0	2	0	1	4
		$1+\lambda$	0	1	1	1	2	2	4
	$\frac{\pm\pi}{2}$	$1+\lambda$	0	1	1	1	2	2	4
		$-2(1+\lambda)$	2	0	1	0	3	2	2
	π	$1+\lambda$	0	1	1	1	2	3	6
		$-2(1+\lambda)$	2	0	1	0	3	1	1
1	0	-2	1	0	0	2	0	1	4
		-2	1	0	1	0	3	2	2
		4	0	2	2	0	3	2	2
	$\frac{\pm\pi}{2}$	$1-3\lambda$	1	0	1	1	1	1	2
		$1+3\lambda$	0	2	1	1	1	1	2
		-2	1	0	1	0	3	2	2
	π	$1-3\lambda$	1	0	1	1	1	1	2
		$1+3\lambda$	0	2	1	1	1	1	2
		-2	1	0	1	0	3	2	2
		4	0	2	2	0		1	1
2	0	$-2+2\lambda$	0	0	0	2	0	1	4
		$1-\lambda$	0	0	1	1	2	2	4
	$\frac{\pm\pi}{2}$	$1-\lambda$	0	0	1	1	2	2	4
		$-2+2\lambda$	0	0	1	0	3	2	2
	π	$1-\lambda$	0	0	1	1	2	3	6
		$-2+2\lambda$	0	0	1	0	3	1	1

$\mathcal{N} = 5$

Q	P	Linear Term	P_a	P_b	P_c	m_E	m_P	Number of Sets	Number of Eigenvalues
0	0	$-(1+\lambda)$	1	0	0	3	0	1	8
			1	0	1	1	3	3	6
		$2(1+\lambda)$	0	2	2	0	3	3	3
	All $P \neq 0$	$-(1+\lambda)$	1	0	1	1	3	5	10
		$2(1+\lambda)$	0	2	1	2	1	1	4
			0	2	2	0	4	2	2
1	0	$-1+\lambda$	0	0	0	3	0	1	8
			0	0	1	1	3	3	6
		$2-2\lambda$	0	0	2	0	3	3	3
	All P$\neq$ 0	$-1+\lambda$	0	0	1	1	3	5	10
		$2-2\lambda$	0	0	1	2	1	1	4
			0	0	2	0	4	2	2
2	0	-4	2	0	0	2	0	1	4
		2	0	1	1	2	1	1	4
		$-1-3\lambda$	2	0	1	1	2	1	2
		$-1+3\lambda$	0	1	1	1	2	1	2
		-4	2	0	1	0		1	1
		2	0	1	2	0		4	4
	All P$\neq$ 0	2	0	1	1	2	1	1	4
		$-1-3\lambda$	2	0	1	1	2	2	4
		$-1+3\lambda$	0	1	1	1	2	2	4
		2	0	1	2	0	4	3	3
		-4	2	0	1	0		1	1

$\mathcal{N} = 6$

Q	P	Linear Term	P_a	P_b	P_c	m_E	m_P	Number of Sets	Number of Eigenvalues
0	0	0	0	0	0	4	0	1	16
			0	0	1	2	3	5	20
			0	0	2	0	6	10	10
	$\frac{\pm 2\pi}{6}$	0	0	0	1	2	3	8	32
			0	0	2	0	6	6	6
	$\frac{\pm 4\pi}{6}$	0	0	0	1	2	3	8	32
			0	0	2	0	6	8	8
	π	0	0	0	2	2	3	9	36
			0	0	2	0	6	5	5
1	0	$-3-\lambda$	2	0	0	3	0	1	8
			2	0	1	1	3	2	4
		$3-\lambda$	0	1	2	1	3	5	10
		2λ	0	1	1	2	2	3	12
			0	1	2	0	5	5	5
		-4λ	2	0	2	0		3	3
	$\frac{2\pi}{6}$, $\frac{4\pi}{6}$, $\frac{6\pi}{6}$, $\frac{-2\pi}{6}$	-4λ	2	0	1	2	1	1,1,1,1	4,4,4,4
			2	0	2	0	4	0,4,0,1	0,4,0,1
		$-3-\lambda$	2	0	1	1	3	5,3,4,4	10,6,8,8
		$3-\lambda$	0	1	2	1	3	4,3,5,4	8,6,10,8
		2λ	0	1	1	2	2	3,4,3,3	12,16,12,12
			0	1	2	0	5	5,6,5,6	5,6,5,6
	$\frac{-4\pi}{6}$	$3-\lambda$	0	1	1	3	0	1	8
			0	1	2	1	3	2	4
		$-3-\lambda$	2	0	1	1	3	5	10
		2λ	0	1	1	2	2	3	12
			0	1	2	0	5	5	5
		-4λ	2	0	2	0		3	3

Q	P	Linear Term	P_a	P_b	P_c	m_E	m_P	Number of Sets	Number of Eigenvalues
2	0	$-3+\lambda$	1	0	0	3	0	1	8
			1	0	0	1	3	2	4
		$3+\lambda$	0	2	2	1	3	5	10
		-2λ	1	0	1	2	2	3	12
			1	0	2	0	5	5	5
		4λ	0	2	2	0		3	3
	$\frac{2\pi}{6}$, $\frac{6\pi}{6}$, $\frac{-4\pi}{6}$, $\frac{-2\pi}{6}$	4λ	0	2	1	2	1	1,1,1,1	4,4,4,4
			0	2	2	0	4	1,0,4,0	1,0,4,0
		$-3+\lambda$	1	0	1	1	3	4,4,3,5	8,8,6,10
		$3+\lambda$	0	2	2	1	3	4,5,3,4	8,10,6,8
		-2λ	1	0	1	2	2	3,3,4,3	12,12,16,12
			1	0	2	0	5	6,5,6,5	6,5,6,5
	$\frac{4\pi}{6}$	$3+\lambda$	0	2	1	3	0	1	8
			0	2	2	1	3	2	4
		$-3+\lambda$	1	0	1	1	3	5	10
		-2λ	1	0	1	2	2	3	12
			1	0	2	0	5	5	5
		4λ	0	2	2	0		3	3

$\mathcal{N} = 7$

Q	P	Linear Term	P_a	P_b	P_c	m_E	m_P	Number of Sets	Number of Eigenvalues
0	0	$-2(1+\lambda)$	2	0	0	4	0	1	16
			2	0	1	2	3	4	16
			2	0	2	0	6	7	7
		$1+\lambda$	0	1	1	3	2	3	24
			0	1	2	1	5	21	42
	$P \neq 0$	$-2(1+\lambda)$	2	0	1	2	3	7	28
			2	0	2	0	6	6	6
		$1+\lambda$	0	1	1	3	2	4	32
			0	1	0	1	5	19	38
1	0	-2	1	0	0	4	0	1	16
			1	0	1	2	3	7	28
			1	0	2	0	6	10	10
		4	0	2	2	2	2	3	12
			0	2	3	0	5	3	3
		$1+3\lambda$	0	2	2	1	4	9	18
		$1-3\lambda$	1	0	2	1	4	9	18
	P$\neq$ 0	$1-3\lambda$	1	0	1	3	1	1	8
			1	0	2	1	4	8	16
		-2	1	0	1	2	3	9	36
			1	0	2	0	6	9	9
		$1+3\lambda$	0	2	1	3	1	1	8
			0	2	2	1	4	8	16
		4	0	2	2	2	2	2	8
			0	2	3	0	5	3	3
2	0	$-2(1-\lambda)$	0	0	0	4	0	1	16
			0	0	1	2	3	4	16
			0	0	2	0	6	7	7
		$1-\lambda$	0	0	1	3	2	3	24
			0	0	2	1	5	21	42
	P$\neq$0	$-2(1-\lambda)$	0	0	1	2	3	7	28
			0	0	2	0	6	6	6
		$1-\lambda$	0	0	1	3	2	4	32
			0	0	2	1	5	19	38

References

[1] H. Au-Yang, B. M. McCoy, J. H. H. Perk, S. Tang, and M. L. Yan, Phys. Letts., **A123** (1987), 219.

[2] B. M. McCoy, J. H. H. Perk, S. Tang, and C. H. Sah, Phys. Letts., **A125** (1987), 9.

[3] H. Au-Yang, B. M. McCoy, J. H. H. Perk, and S. Tang, "Algebraic Analysis", ed. M. Kashiwara and T. Kawai, Academic Press, 1988, pp. 29.

[4] R. J. Baxter, J. H. H. Perk, and H. Au-Yang, Phys. Letts., **A128** (1988), 138.

[5] J. H. H. Perk, Taniguchi lecture, this volume.

[6] G. Albertini, B. M. McCoy, J. H. H. Perk, and S. Tang, Nucl. Phys., **314** (1989), 741.

[7] G. Albertini, B. M. McCoy, and J. H. H. Perk, Phys. Letts., **A135** (1989), 159.

[8] S. Howes, L. P. Kadanoff and M. den Nijs, Nucl. Phys., **B215** [**FS7**] (1983), 169.

[9] L. Onsager, Phys. Rev., **65** (1944), 117; "Critical Phenomena in Alloys, Magnets, and Superconductors", edited by R. E. Mills, E. Ascher, and R. I. Jaffee,, McGraw-Hill, 1971, p. 3.

[10] R. J. Baxter, see also this volume, Phys. Letts., **A133** (1988), 185.

[11] V. V. Bazhanov and N. Yu Reshetikhin, Int. J. of Mod. Phys. A, **4** (1989), 115.

[12] D. Mumford, "Tata Lectures on Theta", Birkhäuser, Boston, 1983.

[13] R. J. Baxter, Phys. Rev. Lett., **26** (1971), 832.

[14] "Higher Transcendental Functions", ed. A. Erdelyi, McGraw Hill, New York, 1953.

[15] C. N. Yang and C. P. Yang, Phys. Rev., **150** (1966), 321.

[16] R. J. Baxter, "Exactly Solved Models in Statistical Mechanics", Academic Press, London, 1982.

[17] O. Perron, Math. Ann., **64** (1907), 248.
G. Frobenius, S. B. Deutsch. Akad. Wiss. Berlin. Math-Nat. Kl (1912), 456.
see also F. R. Gantmacher "Matrix Theory" Chelsea Publishing Co. 1959, vol. 2..

[18] R. J. Baxter, J. Stat. Phys., **52** (1988), 639.

[19] L. Landau, Z.f. Phys., **64** (1930), 629.

[20] V. B. Berestetskii, E. M. Lifschitz, L. P. Pitaevskii, "Relativistic Quantum Theory", Pergamon, 1971, p. 101.

[21] S. Adler, Phys. Rev., **177** (1969), 2426.
J. S. Bell and R. Jackiw, Nuovo Cimento, **Series 10, 60A** (1969), 47.

[22] H. B. Nielsen and M. Ninomiya, Proc. 12th Int. Conf. on High Energy Physics, Paris (1982).
J. Ambjorn, J. Greensite and C. Peterson, Nucl. Phys., **B221** (1983), 381.

[23] G. von Gehlen and V. Rittenberg, Nucl. Phys., **B257** [**FS14**] (1985), 351.

Giuseppe Albertini
A. Della Riccia, fellow
Institute for Theoretical Physics
State University of New York at Stony Brook
Stony Brook, NY 11794-3840

Barry M. McCoy
Institute for Theoretical Physics
State University of New York at Stony Brook
Stony Brook, NY 11794-3840

Jacques H. H. Perk
Physics Department
Oklahoma State University
Stillwater, Oklahoma 74078-0444