Eigenvalue techniques in design and graph theory

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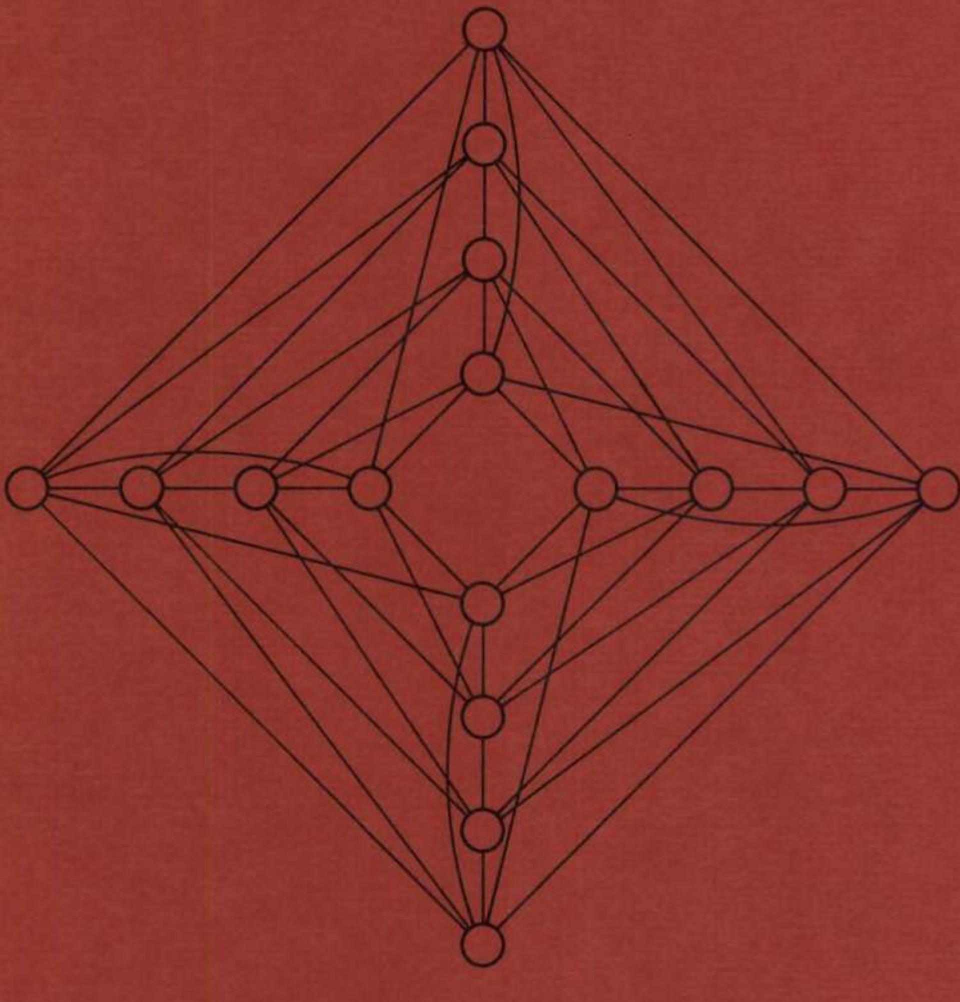
EIGENVALUE TECHNIQUES IN DESIGN AND GRAPH THEORY



WILLEM HAEMERS

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PROEFSCHRIFT

TER VERKRIJGING VAN DE GRAAD VAN DOCTOR IN DE TECHNISCHE WETENSCHAPPEN AAN DE TECHNISCHE HOGESCHOOL EINDHOVEN, OP GEZAG VAN DE RECTOR MAGNIFICUS, PROF.IR. J. ERKELENS, VOOR EEN COMMISSIE AANGEWEZEN DOOR HET COLLEGE VAN DEKANEN IN HET OPENBAAR TE VERDEDIGEN OP DINSDAG 30 OKTOBER 1979 TE 16.00 UUR

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PREFACE

The application of the theory of matrices and eigenvalues to combinatorics is certainly not new. In a certain sense the study of the eigenvalues of the adjacency matrix of a graph even became a subject of its own, see for instance [B5], [C11], [C12], [H14] and [S2]. Also in the theory of designs, matrix and eigenvalue methods have often been used successfully; see for instance [C6], [C9], [H17] and [R3]. In the present monograph the starting point is a new theorem concerning the eigenvalues of partitioned matrices. Applications of this theorem and some known matrix theorems to matrices associated to graphs or designs lead to new results, and new proofs of known results. These concern inequalities of various types, including conclusions for the case of equality. In addition we obtain guiding-principles for constructing strongly regular graphs or 2-designs. Let us give some more details.

Our theorem (Theorem 1.2.3) about eigenvalues and partitionings of matrices, which was announced in [H1], reads as follows:

<u>THEOREM</u>. *Let* A *be a complete hermitian* $n \times n$ *matrix, partitioned into* m^2 *block matrices, such that all diagonal block matrices are square. Let* B *be the* $m \times m$ *matrix whose* i,j*-th entry equals the average row sum of the* i,j*-th block matrix of* A *for* $i,j = 1,\dots,m$. *Then the eigenvalues* $\lambda_1(A) \geq \dots \geq \lambda_n(A)$ *of* A *and the eigenvalues* $\lambda_1(B) \geq \dots \geq \lambda_m(B)$ *of* B *satisfy*

$$\lambda_i(A) \geq \lambda_i(B) \geq \lambda_{n-m+i}(A) \quad \textit{for } i = 1,\dots,m\,.$$

Moreover, if for some integer k, $0 \leq k \leq m$, $\lambda_i(A) = \lambda_i(B)$ *for* $i = 1,\dots,k$ *and* $\lambda_i(B) = \lambda_{n-m+i}(A)$ *for* $i = k+1,\dots,m$, *then all the block matrices of* A *have constant row and column sums.*

The weaker inequalities $\lambda_1(A) \geq \lambda_i(B) \geq \lambda_n(A)$ were already observed by C.C. Sims (unpublished), and have been applied successfully by HESTENES & HIGMAN [H10], PAYNE [P4], [P6] and others. They are usually applied under the name Higman-Sims technique. Many proofs by means of this Higman-Sims technique can be shortened by use of our generalization. But, what is more important, our theorem leads to new results, which we shall indicate below.

Suppose G is a graph on n vertices, whose (0,1)-adjacency matrix has eigenvalues $\lambda_1(G) \geq \ldots \geq \lambda_n(G)$. DELSARTE [D1] proved that, for strongly regular G, the size of any coclique (independent set) cannot be larger than $-n\lambda_n(G) / (\lambda_1(G) - \lambda_n(G))$. A.J. Hoffman (unpublished) proved that this bound holds for any regular graph G. Using the above theorem we prove that an upperbound for the size of a coclique in any graph G is provided by

$$-n\lambda_1(G)\lambda_n(G) / (d_{min}^2 - \lambda_1(G)\lambda_n(G)) ,$$

where d_{min} denotes the smallest degree in G. This generalizes Hoffman's bound, since in case of regularity $\lambda_1(G) = d_{min}$ holds. More generally, by use of the same methods we find bounds for the size of an induced subgraph of G in terms of the average degree of the subgraph (Theorem 2.1.2). Apart from the inequalities of Delsarte and Hoffman we also find inequalities of Bumiller, De Clerck and Payne as corollaries of our result.

By applying the generalization of the Higman-Sims technique (with $m = 4$) to the adjacency matrix of the incidence graph of a design, we obtain bounds for the sizes of a subdesign in terms of the singular values of the incidence matrix (Theorem 3.1.1). For nice designs, such as 2-designs and partial geometries, this result becomes easy to apply, since then the singular values are expressible in the design parameters (for symmetric 2-designs the inequality also appeared in [H4]). Thanks to the second part of our theorem certain conclusions may be drawn easily for the case that the bounds are attained, for the graph case as well as for the design case.

We also prove results concerning the intersection numbers of designs, such as the inequalities of AGRAWAL [A1] (Theorem 3.2.1) and the results of BEKER & HAEMERS [B2] about $2-(v,k,\lambda)$ designs with an intersection number $k - \lambda(v-k)/(k-1)$.

HOFFMAN [H13] proved that the chromatic number $\gamma(G)$ of a graph G satisfies $\gamma(G) \geq 1 - \lambda_1(G) / \lambda_n(G)$. To achieve this, Hoffman first proves a generalization of the inequalities of Araonszajn concerning eigenvalues of partitioned matrices. In Section 1.3 we give a new proof of these inequalities using our generalization of the Higman-Sims technique. In Section 2.2 the application of these inequalities yields a generalization of Hoffman's bound (Theorem 2.2.1). For non-trivial strongly regular graphs this leads to $\gamma(G) \geq \max\{1 - \lambda_1(G) / \lambda_n(G) , 1 - \lambda_n(G) / \lambda_2(G)\}$. In Chapter 4 we use these bounds, and many other results from the previous chapters, in order to determine all 4-colourable strongly regular graphs. This chapter is also meant to illustrate some applications of the results and techniques obtained in the first three chapters.

Chapter 5 is in the same spirit, but rather independent from the other ones. The main result is the inequality of HAEMERS & ROOS [H3]: $t \leq s^3$ if $s \neq 1$ for a generalized hexagon of order (s,t), together with some additional regularity for the case of equality. This is proved by rather elementary eigenvalue methods. The same technique applied to generalized quadrangles of order (s,t) yields the inequality of HIGMAN [H11]: $t \leq s^2$ if $s \neq 1$, the result of BOSE [B7] for the case of equality and a theorem of CAMERON, GOETHALS & SEIDEL [C5] about pseudo-geometric graphs.

Using eigenvalue methods we obtain guiding-principles for the construction of designs and graphs. In Section 6.1 we construct a $2-(56,12,3)$ design, for which the framework is provided by Theorem 3.2.4. This design is embeddable in a symmetric $2-(71,15,3)$ design. By modifying this design we obtain eight non-isomorphic $2-(71,15,3)$ designs. All these designs seem to be new (the construction is also published in [B2]). In Section 6.2 some ideas for the construction of strongly regular graphs are described. We construct strongly regular graphs with parameter sets $(q^3+q^2+q+1,q^2+q,q-1,q+1)$ and $(2^{3n}+2^{2n+1},2^{2n}+2^n-1,2^n-2,2^n)$ for prime power q and positive integer n. Strongly regular graphs with the first parameter set are known; however, our construction yields graphs which are not isomorphic to the known ones. The second family seems to be new. Special attention is given to strongly regular graphs with parameter set $(40,12,2,4)$. Several such graphs are constructed with the help of a computer.

In the first appendix we recall some basic concepts and results from the theory of graphs and designs (including finite geometries). This appendix is meant for readers who are not familiar with the theory of designs and graphs. The second appendix exhibits explicitly some of the designs and graphs constructed earlier.

CHAPTER I

MATRICES AND EIGENVALUES

1.1. INTRODUCTION

In this chapter we shall derive some results about eigenvalues of matrices. They provide the main tools for our investigations. We shall assume familiarity with the basic concepts and results from the theory of matrices and eigenvalues. Some general references are [M3], [N1], [W5].

Let A be a square complex matrix of size n. The hermitian transpose of A will be denoted by A^*. Suppose A has n (not necessarily distinct) real eigenvalues, which for instance is the case if A is hermitian (i.e. $A = A^*$). Then we shall denote these eigenvalues by

$$\lambda_1(A) \geq \dots \geq \lambda_n(A) \quad .$$

If denoted by subscripted variables, eigenvectors will always be ordered according to the ordering of their eigenvalues. Vectors are always column vectors. The linear span of a set of vectors $u_1,\dots,u_n$ is denoted by $< u_1,\dots,u_n >$. A basic result, which is important to our purposes, is Rayley's principle (see [N1], [W5]).

1.1.1. RESULT. *Let* A *be a hermitian matrix of size* n. *For some* i, $0 \leq i \leq n$, *let* $u_1,\dots,u_i$ *be an orthonormal set of eigenvectors of* A *for* $\lambda_1(A),\dots,\lambda_i(A)$. *Then*

i. $$\lambda_i(A) \leq \frac{u^*Au}{u^*u} \quad \textit{for } u \in < u_1,\dots,u_i >,\ u \neq 0,\ i \neq 0\ ;$$

equality holds iff u *is an eigenvector of* A *for* $\lambda_i(A)$.

ii. $$\lambda_{i+1}(A) \geq \frac{u^*Au}{u^*u} \quad \textit{for } u \in < u_1,\dots,u_i >^{\perp},\ u \neq 0,\ i \neq n\ ;$$

equality holds iff u *is an eigenvector of* A *for* $\lambda_{i+1}(A)$.

For the multiplicity of an eigenvalue we shall always take the geometric one, that is, the maximal number of linearly independent eigenvectors (to be honest, this agreement is only of influence to the proof of the next lemma, because throughout the remainder of this monograph we shall only consider eigenvalues of diagonalizable matrices). Now we shall prove some easy and well known, but nevertheless useful lemmas.

<u>1.1.2. LEMMA</u>. *Let* M^* *and* N *be complex* $m_1 \times m_2$ *matrices. Put*

$$A := \begin{bmatrix} 0 & N \\ M & 0 \end{bmatrix} .$$

Then the following are equivalent.

i. $\lambda \neq 0$ *is an eigenvalue of* A *of multiplicity* f;

ii. $-\lambda \neq 0$ *is an eigenvalue of* A *of multiplicity* f;

iii. $\lambda^2 \neq 0$ *is an eigenvalue of* MN *of multiplicity* f;

iv. $\lambda^2 \neq 0$ *is an eigenvalue of* NM *of multiplicity* f.

<u>PROOF</u>.

1. (i) $\Longleftrightarrow$ (ii): let $AU = \lambda U$ for some matrix U of rank f. Write

$$U = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}, \quad \text{and define} \quad \tilde{U} := \begin{bmatrix} U_1 \\ -U_2 \end{bmatrix},$$

where U_i has m_i rows for $i = 1,2$. Then $NU_2 = \lambda U_1$ and $MU_1 = \lambda U_2$. This implies $A\tilde{U} = -\lambda\tilde{U}$. Since rank U = rank $\tilde{U}$, the first equivalence is proved.

2. (iii) $\Longleftrightarrow$ (iv): let $MNU' = \lambda U'$ for some matrix U' of rank f. Then $NM(NU') = \lambda NU'$, and rank NU' = rank U', since

$$\text{rank } \lambda U' = \text{rank } MNU' \leq \text{rank } NU' \leq \text{rank } U' ,$$

and $\lambda \neq 0$. This proves the second equivalence.

3. (i) $\Longleftrightarrow$ (iii): because

$$A^2 = \begin{bmatrix} NM & 0 \\ 0 & MN \end{bmatrix},$$

this equivalence follows immediately from the previous steps. □

The *singular values* of a complex matrix N, are the positive eigenvalues of

$$\begin{bmatrix} 0 & N \\ N^* & 0 \end{bmatrix} .$$

By the above lemma we see that they are the same as the square roots of the non-zero eigenvalues of NN^*.

1.1.3. LEMMA. *Let*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

be a complex matrix. If A_{11} *is non-singular, and* rank A_{11} = rank A, *then*

$$A_{22} = A_{21} A_{11}^{-1} A_{12} .$$

PROOF. For $i = 1,2$, let $a_{i,j}$ denote the j-th column of A_{i2}. From rank A = rank $[A_{11}^* \; A_{21}^*]^*$ it follows

$$\begin{bmatrix} a_{1,j} \\ a_{2,j} \end{bmatrix} = \begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix} u ,$$

for some vector u. But if A_{11} is non-singular, then $u = A_{11}^{-1} a_{1,j}$. Hence $a_{2,j} = A_{21} A_{11}^{-1} a_{1,j}$, which proves the lemma. □

The identity matrix of size n will be denoted by I_n or I. The matrix with all entries equal to one by J; a column vector of J is denoted by j; J_n is a square J of size n; the symbol $\otimes$ is used for the Kronecker product of matrices.

As a last result in this section we observe that, if $K := I_n \otimes J_m$, then

$$\lambda_1(K) = \ldots = \lambda_n(K) = m , \quad \lambda_{n+1}(K) = \ldots = \lambda_{mn}(K) = 0 .$$

1.2. INTERLACING OF EIGENVALUES

Suppose A and B are square complex matrices of size n and m, respectively ($m \le n$), having only real eigenvalues. If

$$\lambda_i(A) \ge \lambda_i(B) \ge \lambda_{n-m+i}(A)$$

for all $i = 1,\dots,m$, then we say that the eigenvalues of B *interlace* the eigenvalues of A. If there exists an integer k, $0 \le k \le m$, such that

$$\lambda_i(A) = \lambda_i(B) \qquad \text{for } i = 1,\dots,k$$

and

$$\lambda_{n-m+i}(A) = \lambda_i(B) \quad \text{for } i = k+1,\dots,m \ ,$$

then the interlacing will be called *tight*.

<u>1.2.1. THEOREM</u>. *Let* S *be a complex* $n \times m$ *matrix such that* $S^*S = I_m$. *Let* A *be a hermitian matrix of size* n. *Define* $B := S^*AS$, *and let* $v_1,\dots,v_m$ *be an orthonormal set of eigenvectors of* B. *Then*

i. *the eigenvalues of* B *interlace the eigenvalues of* A;

ii. *if* $\lambda_i(B) \in \{\lambda_i(A), \lambda_{n-m+i}(A)\}$ *for some* $i \in \{1,\dots,m\}$, *then there exists an eigenvector* v *of* B *for* $\lambda_i(B)$, *such that* Sv *is an eigenvector of* A *for* $\lambda_i(B)$;

iii. *if, for some* $\ell \in \{0,\dots,m\}$, $\lambda_i(A) = \lambda_i(B)$ *for all* $i = 1,\dots,\ell$, *then* Sv_i *is an eigenvector of* A *for* $\lambda_i(A)$, *for* $i = 1,\dots,\ell$;

iv. *if the interlacing is tight, then* $SB = AS$.

<u>PROOF</u>. Let $u_1,\dots,u_n$ be an orthonormal set of eigenvectors of A. For any i, $1 \le i \le m$, take

$$\tilde{v}_i \in \langle v_1,\dots,v_i \rangle \cap \langle S^*u_1,\dots,S^*u_{i-1} \rangle^{\perp} \ , \quad \tilde{v}_i \neq 0 \ .$$

Then $S\tilde{v}_i \in \langle u_1,\dots,u_{i-1} \rangle^{\perp}$, hence by 1.1.1

$$\lambda_i(A) \ge \frac{\tilde{v}_i^* S^* A S \tilde{v}_i}{\tilde{v}_i^* S^* S \tilde{v}_i} = \frac{\tilde{v}_i B \tilde{v}_i}{\tilde{v}_i^* \tilde{v}_i} \ge \lambda_i(B) \ .$$

Thus also

$$-\lambda_i(B) = \lambda_{m-i+1}(-B) \le \lambda_{m-i+1}(-A) = -\lambda_{n-m+i}(A) \ .$$

This proves (i).

If $\lambda_i(B) = \lambda_i(A)$, then $\tilde{v}_i$ and $S\tilde{v}_i$ are eigenvectors of B and A respectively for the eigenvalue $\lambda_i(B) = \lambda_i(A)$. This, together with the same result applied to $-B$ and $-A$ yields (ii).

We shall prove (iii) by induction on ℓ. If $\ell = 0$, there is nothing to prove. Suppose $\ell > 0$. We have

$$\lambda_\ell(A) = \lambda_\ell(B) = v_\ell^* B v_\ell = \frac{v_\ell^* S^* A S v_\ell}{v_\ell^* S^* S v_\ell} .$$

On the other hand, $Sv_\ell \in \langle Sv_1,\dots,Sv_{\ell-1}\rangle^\perp$, and by the induction hypothesis $Sv_1,\dots,Sv_{\ell-1}$ are orthonormal eigenvectors of A for $\lambda_1(A),\dots,\lambda_{\ell-1}(A)$. Now 1.1.1.ii yields that Sv_ℓ is an eigenvector of A for $\lambda_\ell(A)$. This proves (iii).

Let the interlacing be tight. By applying (iii) to A with $\ell = k$ and to $-A$ with $\ell = m-k$, we find that $Sv_1,\dots,Sv_m$ is an orthonormal set of eigenvectors of A for $\lambda_1(B),\dots,\lambda_m(B)$. Write $V := [v_1 \ \dots \ v_m]$ and $D := \mathrm{diag}(\lambda_1(B),\dots,\lambda_m(B))$. Then $ASV = SVD$, and $BV = VD$. Hence

$$ASV = SBV .$$

Because V is non-singular, (iv) has now been proved. □

A direct consequence of the above theorem is the following theorem. This result is known under the name *Cauchy inequalities*, see [H7], [M2], [W5].

1.2.2. THEOREM. *Suppose*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{bmatrix}$$

is a hermitian matrix.

i. *The eigenvalues of* A_{11} *interlace the eigenvalues of* A.

ii. *If the interlacing is tight, then* $A_{12} = 0$.

PROOF. Let m be the size of A_{11}. Define $S := [I_m \ 0]^*$, and apply 1.2.1. □

Another consequence of 1.2.1 is the following result which was announced in [H1] (see also [H2]). This result will often be used in the forthcoming sections.

__1.2.3. THEOREM__. *Let* A *be a hermitian matrix partitioned as follows*

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1m} \\ \vdots & & \vdots \\ A_{m1} & \cdots & A_{mm} \end{bmatrix},$$

such that A_{ii} *is square for* $i = 1,\ldots,m$. *Let* b_{ij} *be the average row sum of* A_{ij}, *for* $i,j = 1,\ldots,m$. *Define the* $m \times m$ *matrix* $B := (b_{ij})$.

i. *The eigenvalues of* B *interlace the eigenvalues of* A.

ii. *If the interlacing is tight, then* A_{ij} *has constant row and column sums for* $i,j = 1,\ldots,m$.

iii. *If, for* $i,j = 1,\ldots,m$, A_{ij} *has constant row and column sums, then any eigenvalue of* B *is also an eigenvalue of* A *with not smaller a multiplicity.*

__PROOF__. Let n_i be the size of A_{ii}. Define

$$\tilde{S}^* := \begin{pmatrix} 1 \ldots 1 & 0 \ldots 0 & \ldots & 0 \ldots 0 \\ 0 \ldots 0 & 1 \ldots 1 & \ldots & 0 \ldots 0 \\ 0 \ldots 0 & 0 \ldots 0 & \ldots & 0 \ldots 0 \\ \vdots \quad \vdots & \vdots \quad \vdots & & \vdots \quad \vdots \\ \underbrace{0 \ldots 0}_{n_1} & \underbrace{0 \ldots 0}_{n_2} & \ldots & \underbrace{1 \ldots 1}_{n_m} \end{pmatrix},$$

$D := \operatorname{diag}(n_1,\ldots,n_m)$, and $S := \tilde{S} D^{-1}$. Then $S^*S = I$ and $\tilde{S}^*\tilde{S} = D^2$. We easily see that $(\tilde{S}^*A\tilde{S})_{ij}$ equals the sum of the entries of A_{ij}. Hence

$$B = \tilde{S}^*A\tilde{S}D^{-2} .$$

By 1.2.1.i we know that the eigenvalues of S^*AS interlace the eigenvalues of A. But B has the same eigenvalues as S^*AS, since

$$S^*AS = D^{-1}\tilde{S}^*A\tilde{S}D^{-1} = D^{-1}BD .$$

This proves (i).
It is easily checked that $AS = S(D^{-1}BD)$ reflects that A_{ij} has constant row sums for all $i,j = 1,\ldots,m$. Hence 1.2.1.iv implies (ii).

On the other hand, if $AS = SD^{-1}BD$ and $BU = \lambda_i(B)U$ for some matrix U and integer i, then $A(SD^{-1}U) = \lambda_i(B)SD^{-1}U$, and rank U = rank $SD^{-1}U$. This proves (iii). □

As a special case of the above theorem we have that

$$\lambda_1(A) \geq \lambda_i(B) \geq \lambda_n(A) \ ,$$

for i = 1,...,m. These inequalities are well known and usually applied under the name *Higman-Sims technique*, see [H10], [P4]. We shall also use the name Higmans-Sims technique if we apply the more general result 1.2.3. Also 1.2.3.iii is well known, see for instance [C12], [H9] (note that this result remains valid for non-hermitian A). We see that 1.2.3.ii gives a sufficient, and that 1.2.3.iii gives a necessary condition for the block matrices of A to have constant row and column sums. However, neither of these conditions is both necessary and sufficient. This is illustrated by the following partitioned matrices:

$$A := \left(\begin{array}{cc|cc} 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ \hline -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{array}\right) \ , \quad A' := \left(\begin{array}{cc|cc} 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 1 \\ \hline -1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right) .$$

For both A and A' the eigenvalues are 2,0,0,-2, and the average row sums of the block matrices are given by the entries of $B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. The block matrices of A have constant row sums, whilst the interlacing is not tight. The row sums of the block matrices of A' are not constant, whilst the eigenvalues of B are also eigenvalues of A.

1.3. MORE EIGENVALUE INEQUALITIES

In this section we shall use interlacing of eigenvalues in order to prove some known inequalities and equalities, which we shall use in later sections. The first result is due to WEYL [W2] (see also [W5]).

1.3.1. THEOREM. *Let* A_1 *and* A_2 *be hermitian matrices of size* m. *Then*

$$\lambda_{i-j}(A_1) + \lambda_{i+j}(A_2) \geq \lambda_i(A_1 + A_2) \geq \lambda_{i+j}(A_1) + \lambda_{m-j}(A_2) \ ,$$

for $i = 1,\dots,m$, $0 \leq j \leq \min\{i-1, m-i\}$.

PROOF. Define

$$A := \begin{bmatrix} A_1 - \lambda_{i-j}(A_1)I & 0 \\ 0 & A_2 - \lambda_{1+j}(A_2)I \end{bmatrix},$$

and

$$S := \tfrac{1}{2}\sqrt{2}\,[I_m \quad I_m]^* .$$

Then $\lambda_i(A) = 0$, and

$$S^*AS = \tfrac{1}{2}(A_1 + A_2 - (\lambda_{i-j}(A_1) + \lambda_{1+j}(A_2))I) .$$

With 1.2.1 we now have

$$\lambda_i(A_1 + A_2) - \lambda_{i-j}(A_1) - \lambda_{1+j}(A_2) = 2\lambda_i(S^*AS) \le 2\lambda_i(A) = 0 .$$

If we replace A_1 and A_2 by $-A_1$ and $-A_2$, we get the second inequality. □

The next theorem is due to HOFFMAN [H13].

1.3.2. THEOREM. *Let* A *be a hermitian matrix of size* n, *partitioned as follows*

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1m} \\ \vdots & & \vdots \\ A_{m1} & \cdots & A_{mm} \end{bmatrix},$$

where A_{ii} *is a square matrix of size* n_i, *for* $i = 1,\ldots,m$. *Let* $j_1,\ldots,j_m$ *be integers such that* $1 \le j_i \le n_i$ *for* $i = 1,\ldots,m$. *Then*

$$\lambda_{j_1+\ldots+j_m}(A) + \sum_{i=1}^{m-1} \lambda_i(A) \ge \sum_{i=1}^{m} \lambda_{j_i}(A_{ii}) \ge$$

$$\ge \lambda_{j_1+\ldots+j_m-m+1}(A) + \sum_{i=n-m+2}^{n} \lambda_i(A) .$$

PROOF. Let $u_1,\ldots,u_n$ be an orthonormal set of eigenvectors of A. Let $u_{i1},\ldots,u_{in_i}$ be an orthonormal set of eigenvectors of A_{ii} for $i = 1,\ldots,m$. Put $k := j_1 + \ldots + j_m$. Choose a vector $\tilde{u}_k = [\tilde{u}_{1k}^* \ \ldots \ \tilde{u}_{mk}^*]^* \ne 0$, such that

$$\tilde{u}_k \in \langle u_1,\ldots,u_{k-1} \rangle^{\perp} ,$$

and

$$\tilde{u}_{ik} \in \langle u_{i1},\ldots,u_{ij_i} \rangle \quad \text{for } i = 1,\ldots,m .$$

It follows from dimension considerations that we can always do so. Now define

$$w_i := \| \tilde{u}_{ik} \| ,$$

and

$$\hat{u}_{ik} := \begin{cases} \frac{1}{w_i} \tilde{u}_{ik} & \text{if } w_i \neq 0 , \\ u_{ij_i} & \text{if } w_i = 0 , \end{cases}$$

for $i = 1,\ldots,m$. Furthermore put

$$w := (w_1,\ldots,w_m)^* ,$$

and

$$S := \begin{bmatrix} \hat{u}_{1k} & 0 & \ldots & 0 \\ 0 & \hat{u}_{2k} & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & \hat{u}_{mk} \end{bmatrix} .$$

Then we have

$$S^*S = I , \quad Sw = \tilde{u}_k , \quad \text{and} \quad (S^*AS)_{ii} = \hat{u}_{ik}^* A_{ii} \hat{u}_{ik} \quad \text{for } i = 1,\ldots,m .$$

By 1.1.1.i and the choice of $\hat{u}_{ik}$, the last formula gives

$$(S^*AS)_{ii} \geq \lambda_{j_i}(A_{ii}) \quad \text{for } i = 1,\ldots,m .$$

Hence

$$(*) \qquad \sum_{i=1}^{m} \lambda_i(S^*AS) = \operatorname{trace} S^*AS \geq \sum_{i=1}^{m} \lambda_{j_i}(A_{ii}) .$$

On the other hand, $Sw = \tilde{u}_k$, $S^*S = I$ and 1.1.1 yield

$$(**) \qquad \lambda_m(S^*AS) \le \frac{w^*S^*ASw}{w^*w} = \frac{\tilde{u}_k^*A\tilde{u}_k}{\tilde{u}_k^*\tilde{u}_k} \le \lambda_k(A) \ .$$

Applying 1.2.1 gives

$$(***) \qquad \sum_{i=1}^{m-1} \lambda_i(S^*AS) \le \sum_{i=1}^{m-1} \lambda_i(A) \ .$$

Combining (*), (**) and (***) yields the first inequality of our theorem. Again, the second inequality follows by substituting $-A$ for A in the first one. □

If the matrix A of the above theorem is positive semi-definite and $m = 2$, then we have

$$\lambda_{j_1}(A_{11}) + \lambda_{j_2}(A_{22}) \ge \lambda_{j_1+j_2-1}(A) \ .$$

These are the inequalities of ARONSZAJN [A3] (see also [H7]).

The following consequence of 1.3.2 will turn out to be a useful tool in computations with eigenvalues.

1.3.3. THEOREM. *Suppose*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{bmatrix}$$

is hermitian of size n. *Suppose* A *has just two distinct eigenvalues, that is,*

$$\lambda_1(A) = \ldots = \lambda_f(A) > \lambda_{f+1}(A) = \ldots = \lambda_n(A)$$

for some f, $1 \le f < n$. *Let* n_1 *and* n_2 *be the sizes of* A_{11} *and* A_{22} *respectively. Then*

$$\lambda_i(A_{22}) = \lambda_1(A) + \lambda_n(A) - \lambda_{f-i+1}(A_{11})$$

$$\textit{for } \max\{1, f+1-n_1\} \le i \le \min\{f, n_2\} \ ,$$

$$\lambda_i(A_{22}) = \lambda_1(A) \quad \textit{for } 1 \le i < f+1-n_1 \ ,$$

$$\lambda_i(A_{22}) = \lambda_n(A) \quad \textit{for } f < i \le n_2 \ .$$

PROOF. By the Cauchy inequalities (1.2.2.i) we have

$$\lambda_i(A) \ge \lambda_i(A_{22}) \ge \lambda_{n_1+i} .$$

This proves the result for $1 \le i < f+1-n_1$, and for $f < i \le n_2$. For the remaining values of i, 1.3.2 gives

$$\lambda_{f+1}(A) + \lambda_1(A) \ge \lambda_{f-i+1}(A_{11}) + \lambda_i(A_{22}) \ge \lambda_f(A) + \lambda_n(A) ,$$

which proves the required result. □

It is an easy exercise to give a direct proof of the above theorem. The proof then could go analogously to the one of Theorem 5.1 of [C5], where a similar result is stated.

CHAPTER 2

INEQUALITIES FOR GRAPHS

2.1. INDUCED SUBGRAPHS

In this section we shall derive inequalities for induced subgraphs of graphs, using the results of section 1.2 on interlacing of eigenvalues.

Let G be a graph on n vertices. The *eigenvalues of* G are the eigenvalues of its (0,1)-adjacency matrix; we denote them by $\lambda_1(G) \geq \ldots \geq \lambda_n(G)$. Let G_1 be an induced subgraph of G. Then by 1.2.2 (Cauchy inequalities) the eigenvalues of G_1 interlace the eigenvalues of G. In particular, if G_1 is a coclique of size α, then $\lambda_\alpha(G) \geq \lambda_\alpha(G_1) = 0$, and $\lambda_{n-\alpha+1}(G) \leq \lambda_1(G_1) = 0$. Hence, we have the following result, which was first observed by CVETKOVIĆ [C11] (see also [C12]).

2.1.1. THEOREM. *The size of a coclique of a graph* G *cannot exceed the number of nonnegative [nonpositive] eigenvalues of* G.

Now we shall derive inequalities for induced subgraphs using the Higman-Sims technique (1.2.3). Suppose G is a graph on n vertices of average degree d. Let the vertex set of G be partitioned into two sets, and let G_1 and G_2 be the subgraphs induced by these two sets. For $i = 1,2$, let n_i be the number of vertices of G_i, let d_i be the average degree of G_i, and let $\tilde{d}_i$ be the average of the degrees in G over the vertices of G_i. Now we can state the following theorem.

2.1.2. THEOREM. *For* $i = 1,2$

i. $$\lambda_1(G)\lambda_2(G) \geq \frac{d_i dn - \tilde{d}_i^2 n_i}{n - n_i} \geq \lambda_1(G)\lambda_n(G) ,$$

ii. *if equality holds on one of the sides, then* G_1 *and* G_2 *are regular, and also the degrees in* G *are constant over the vertices of* G_1 *and* G_2.

PROOF. If G is complete we easily see that the theorem is correct. So let us assume that G is not complete. Let A_{11}, A_{22} and

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^{*} & A_{22} \end{bmatrix}$$

be the adjacency matrices of G_1, G_2, and G, respectively. Put

$$B := \begin{pmatrix} d_1 & \tilde{d}_1 - d_1 \\ \tilde{d}_2 - d_2 & d_2 \end{pmatrix} .$$

Then the entries of B are the average row sums of the block matrices of A. By 1.2.3.i we have

$$\lambda_1(A) \ge \lambda_1(B) \;,\quad \lambda_2(A) \ge \lambda_2(B) \;,\quad -\lambda_n(A) \ge -\lambda_2(B) \;.$$

Because trace A = 0, we have $\lambda_n(A) \le 0$. From 2.1.1 we know that $\lambda_2(A) \ge 0$, since otherwise G would be complete. Hence

(*) $$\lambda_1(A)\lambda_2(A) \ge \lambda_1(B)\lambda_2(B) \ge \lambda_1(A)\lambda_n(A) \;.$$

On the other hand we know

(**) $$\lambda_1(B)\lambda_2(B) = \det B = d_1\tilde{d}_2 - \tilde{d}_1(\tilde{d}_2 - d_2) \;.$$

We quickly see

$$dn = \tilde{d}_1 n_1 + \tilde{d}_2 n_2 \;,\quad n_2(\tilde{d}_2 - d_2) = n_1(\tilde{d}_1 - d_1) \;,\quad n = n_1 + n_2 \;.$$

This yields

$$d_1\tilde{d}_2 - \tilde{d}_1(\tilde{d}_2 - d_2) = (d_1 dn - \tilde{d}_1^2 n_1) / (n - n_1) \;.$$

With (*) and (**) this proves (i).
If equality holds on one of the sides, the interlacing must have been tight. Hence 1.2.3.ii gives (ii). □

Now let us look at the consequences of the above inequalities for some special cases. The size of the largest coclique and clique of G are denoted by $\alpha(G)$ and $\omega(G)$, respectively.

<u>2.1.3. THEOREM</u>. *If* d_{min} *and* d_{max} *are the smallest and the largest degree in the graph* G*, respectively, then*

i. $$\alpha(G)(d_{min}^2 - \lambda_1(G)\lambda_n(G)) \le -n\lambda_1(G)\lambda_n(G) ,$$

ii. $$\omega(G)(dn - d_{max}^2 + \lambda_1(G)\lambda_2(G)) \le n(d + \lambda_1(G)\lambda_2(G)) .$$

<u>PROOF</u>. (i). Substitute $\alpha(G) = n_1$, $d_1 = 0$ and $\tilde{d}_1 \ge d_{min}$ in the right hand inequality of 2.1.2.i.
(ii). Substitute $\omega(G) = n_1$, $d_1 = \omega(G) - 1$ and $\tilde{d}_1 \le d_{max}$ in the left hand inequality of 2.1.2.i. □

<u>2.1.4. THEOREM</u>. *If* G *is a regular graph on* n *vertices of degree* d*, then*

i. *any subgraph* G_1 *of* G *with* n_1 *vertices and average degree* d_1 *satisfies*

$$\lambda_2(G) \ge \frac{nd_1 - n_1 d}{n - n_1} \ge \lambda_n(G) ,$$

ii. $$\alpha(G)(d - \lambda_n(G)) \le -n\lambda_n(G) ,$$

iii. $$\omega(G)(n - d + \lambda_2(G)) \le n(1 + \lambda_2(G)) .$$

<u>PROOF</u>. If G is regular then $\lambda_1(G) = d = \tilde{d}_1 = \tilde{d}_2 = d_{min} = d_{max}$. Now 2.1.2.i, 2.1.3.i and 2.1.3.ii give the required results. □

The inequality 2.1.4.ii is an unpublished result of A.J. Hoffman (see [C12], [H2], [L2]). In fact, the inequalities (ii) and (iii) of 2.1.4 (just as the left and the right hand inequality of 2.1.4.i) are equivalent, because either one can be obtained from the other by using $\omega(\bar{G}) = \alpha(G)$, $\lambda_1(\bar{G}) = n - \lambda_1(G) - 1$ and $\lambda_i(\bar{G}) = -\lambda_{n-i+2}(G) - 1$ for $i = 2,\dots,n$ ($\bar{G}$ is the complement of G).

For the graph G with its subgraphs G_1 and G_2, we define $D(G,G_1)$ to be the incidence structure whose points and blocks are the vertices of G_1 and G_2, respectively, a point and a block being incident if the corresponding vertices are adjacent. If we have equality in any of the inequalities of 2.1.2 - 2.1.4, then 2.1.2.ii yields that $D(G,G_1)$ is a 1-design, possibly degenerate. Now let G be strongly regular. Then by use of 1.2.3.iii it is not difficult to show that equality holds in 2.1.4 iff $D(G,G_1)$ is a 1-design. If G_1 is a coclique or a clique we have a criterion for $D(G,G_1)$ to be a 2-design.

2.1.5. THEOREM. *If* G *is a strongly regular graph on* n *vertices of degree* d, *then*

i. $\alpha(G) \leq 1 + (n-d-1)/(\lambda_2(G)+1)$,

ii. $\omega(G) \leq 1 - d/\lambda_n(G)$,

iii. *if equality holds in* (i) *or* (ii), *and* G_1 *is a coclique of size* $\alpha(G)$, *or a clique of size* $\omega(G)$, *respectively, then* $D(G,G_1)$ *is a* 2-*design, possibly degenerate.*

PROOF. If G is strongly regular, we know (see [C5], [S4] or Appendix I)

$$(d-\lambda_2(G))(d-\lambda_n(G)) = n(d+\lambda_2(G)\lambda_n(G)) .$$

From this it follows in a straightforward way that (i) and (ii) are equivalent to 2.1.4.ii and 2.1.4.iii.
From the definition of a strongly regular graph we know that any two points of $D(G,G_1)$ are incident with a constant number of blocks of $D(G,G_1)$. Furthermore, equality in (i) or (ii) implies that $D(G,G_1)$ is a 1-design, so in this case $D(G,G_1)$ is then a 2-design, possibly degenerate. □

The theorems 2.1.5 and 2.1.4 for strongly regular graphs are known. They are direct consequences of the linear programming bound of DELSARTE [D1] (see also [H2]). They were also proved by BUMILLER [B9].

Applying 2.1.4.i to the point graph of a partial geometry (see Appendix I, or [B6], [T1]) gives the following result of DE CLERCK [C7] (see also [P3] for the case $\alpha = 1$, and [B9] for $t = t_1$).

2.1.6. COROLLARY. *Let* P *be a partial geometry with parameters* (s,t,α), *containing a partial subgeometry* P_1 *with parameters* (s_1,t_1,α). *Then*

$$s = s_1 \quad \text{or} \quad s \geq s_1 t_1 + \alpha - 1 .$$

PROOF. If G and G_1 are the point graphs of P and P_1, respectively, then (see Appendix I or [T1]) G_1 is an induced subgraph of G, and

$$n = (s+1)(st+\alpha)/\alpha \ , \quad n_1 = (s_1+1)(s_1t_1+\alpha)/\alpha \ ,$$

$$d = s(t+1) \ , \quad d_1 = s_1(t_1+1) \ , \quad \lambda_2(G) = s-1 \ .$$

Substitution of these values in the left hand inequality of 2.1.4.i leads to

$$(s-s_1)(st+\alpha)(s_1t_1+\alpha-s-1) \le 0 \ .$$

This proves the result. □

The next theorem gives a result in case both Hoffman's bound (2.1.4.ii) and Cvetković's bound (2.1.1) are tight.

2.1.7. THEOREM. *Let* G *be a strongly regular graph on* n *vertices. Let* $f_n(G)$ *denote the multiplicity of the eigenvalue* $\lambda_n(G)$*. Then*

i. $\alpha(G) \le f_n(G)$,

ii. $\alpha(G) \le 1 + (n-\lambda_1(G)-1)/(\lambda_2(G)+1)$,

iii. *let* G_1 *be a coclique, whose size attains both of these bounds, then* G_2*, the subgraph of* G *induced by the remaining vertices, is strongly regular with eigenvalues*

$$\lambda_1(G_2) = \lambda_1(G)\alpha(G)/(n-\alpha(G)) \ , \quad \lambda_2(G_2) = \lambda_2(G) \ ,$$

$$\lambda_{n-\alpha(G)}(G_2) = \lambda_2(G) + \lambda_n(G) \ .$$

PROOF. Theorem 2.1.1 implies (i); (ii) is the same as 2.1.5.i. Let A and A_2 be the adjacency matrices of G and G_2, respectively. Then

$$A - \frac{\lambda_1(G) - \lambda_2(G)}{n} J$$

has just two distinct eigenvalues, $\lambda_2(G)$ and $\lambda_n(G)$ of multiplicity $n-f_n(G)$ and $f_n(G) = \alpha(G)$, respectively. From 1.3.3 it follows that

$$\tilde{A}_2 := A_2 - \frac{\lambda_1(G) - \lambda_2(G)}{n} J$$

has three distinct eigenvalues, $\lambda_2(G)$, $\lambda_2(G) + \lambda_n(G)$ and $\lambda_2(G) + \lambda_n(G) + \alpha(G)(\lambda_1(G) - \lambda_2(G))/n$, where the last eigenvalue is simple (has multiplicity one). On the other hand, 2.1.5.iii gives that G_2 is regular of degree $\lambda_1(G)\alpha(G) / (n - \alpha(G))$. This shows that $\tilde{A}_2$ and A_2 have a common basis of eigenvectors, and that the simple eigenvalue of $\tilde{A}_2$ belongs to the eigenvector j. Thus A_2 has the desired eigenvalues, and therefore (see [C6] or Appendix I) G_2 is strongly regular. □

Using 1.1.3 it is not difficult to show that $D(G,G_1)$ is a quasi-symmetric 2-design (see Section 3.2), whose block graph is the complement of G_2. This situation has been studied by SHRIKHANDE [S5].

In proving 2.1.2 we applied interlacing to the product of eigenvalues. We did so in order to get reasonably nice formulas. However, for non-regular graphs the inequality for the product carries less information than the separate inequalities. For this reason, applying the Higman-Sims technique directly to the adjacency matrix of a given non-regular graph, may yield better results than 2.1.2 or 2.1.3. Also, if more is known about the structure of G or G_1, it is often possible to get better results by a more detailed application of the Higman-Sims technique. Let us illustrate this by the following result.

2.1.8. THEOREM. *Let G be a regular graph on n vertices of degree d, and let the complete bipartite graph $K_{\ell,m}$ be an induced subgraph of G. Let x_1 and x_2, $x_1 \geq x_2$, be the zeros of*

$$(n - \ell - m)x^2 + (d\ell + dm - 2\ell m)x - \ell m(n - 2d) .$$

Then

$$\lambda_2(G) \geq x_1 \quad \text{and} \quad \lambda_n(G) \leq x_2 .$$

PROOF. Without loss of generality, let G have adjacency matrix

$$A = \begin{bmatrix} O & J & A'_{12} \\ J & O & A''_{12} \\ A'_{21} & A''_{21} & A_{22} \end{bmatrix} ,$$

where the diagonal block matrices are square of sizes ℓ, m and n_2, respectively. Using the Higman-Sims technique (1.2.3) we find that the eigenvalues of

$$B := \begin{pmatrix} 0 & m & d-m \\ \ell & 0 & d-\ell \\ \ell\,\dfrac{d-m}{n_2} & m\,\dfrac{d-\ell}{n_2} & d-\dfrac{\ell d+md-2\ell m}{n_2} \end{pmatrix}$$

interlace the eigenvalues of A. Clearly $\lambda_1(B) = \lambda_1(A) = d$ and hence

$$\lambda_2(B)\lambda_3(B) = (\det B)/d = \ell m(2d-n)/n_2 ,$$

$$\lambda_2(B) + \lambda_3(B) = (\operatorname{trace} B) - d = (2\ell m - \ell d - md)/n_2 .$$

This yields $x_1 = \lambda_2(B)$ and $x_2 = \lambda_3(B)$. Now the interlacing gives the required result. □

BUMILLER [B9] showed for strongly regular G and m = 1 that

$$\ell(\mu + \lambda_2(G)) \le -d\,\lambda_n(G) ,$$

where $\mu = d + \lambda_2(G)\lambda_n(G)$. Using

$$(d-\lambda_2(G))(d-\lambda_n(G)) = n(d+\lambda_2(G)\lambda_n(G))$$

one easily checks that this follows from the second inequality of the above theorem. PAYNE [P6] proved that

$$(\ell-1)(m-1) \le s^2 ,$$

if G is the point graph of a generalized quadrangle of order (s,t) (see Chapter 5 or Appendix I). This follows after substituting

$$n = (s+1)(st+1) , \quad d = s(t+1) , \quad \lambda_2(G) = s-1$$

in the first inequality of the above theorem. It should not be surprising that for this case we obtain the same result as Payne, because he too uses the Higman-Sims technique.

2.2. CHROMATIC NUMBER

In this section we shall derive lower bounds for the chromatic number of a graph in terms of its eigenvalues. The main tool is Hoffman's generalization of Aronszajn's inequalities (1.3.2).

Let G be a non-void graph on n vertices. Then it follows immediately that $\gamma(G)$, the chromatic number of G, satisfies

$$\gamma(G)\alpha(G) \geq n \ .$$

Combining this with the upper bounds for $\alpha(G)$ found in the previous section we obtain lower bounds for $\gamma(G)$. For instance, 2.1.3 gives

$$\gamma(G) \geq 1 - d_{min}^2 / \lambda_1(G)\lambda_n(G) \ .$$

However, this is not best possible, since HOFFMAN [H13] (see also [B5], [H2], [H14], [L2]) showed that

$$\gamma(G) \geq 1 - \lambda_1(G) / \lambda_n(G) \ ,$$

which, if G is not regular, is better than the above bound. If G is regular, then the two bounds coincide. Taking into account that $\alpha(G)$ is an integer we get

$$\gamma(G) \geq n / \lfloor n\lambda_1(G)\lambda_n(G) / (\lambda_1(G)\lambda_n(G) - d_{min}^2) \rfloor \ ,$$

which is occasionally better than Hoffman's bound.

HOFFMAN [H13] proves his lower bound by use of the inequalities 1.3.2. We shall use the same technique, but in a more profound way, in order to obtain a generalization of Hoffman's inequality.

2.2.1. THEOREM. *Let* G *be a graph on* n *vertices with chromatic number* γ. *Let* k *be an integer satisfying* $0 \leq k \leq n/\gamma$. *Then*

i. $$(\gamma - 1)\lambda_{k+1}(G) \geq -\lambda_{n-k(\gamma-1)}(G) \ ,$$

ii. $$(\gamma - 1)\lambda_{n-k}(G) \leq -\lambda_{1+k(\gamma-1)}(G) \ .$$

PROOF. Let A be the adjacency matrix of G. Then without loss of generality

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1\gamma} \\ \vdots & & \vdots \\ A_{\gamma 1} & \cdots & A_{\gamma\gamma} \end{bmatrix},$$

where A_{ii} is the $n_i \times n_i$ all-zero matrix, for $i = 1,\ldots,\gamma$.
First, we assume that $n_i > k$ for $i = 1,\ldots,\gamma$. Let $u_1,\ldots,u_n$ denote an orthonormal set of eigenvectors of A. Define

$$\tilde{A} := A + (\lambda_n(A) - \lambda_1(A)) \sum_{i=1}^{k} u_i u_i^* .$$

Clearly the matrices $u_i u_i^*$ and A have a common basis of eigenvectors. This implies

$$(*) \qquad \lambda_i(\tilde{A}) = \lambda_{k+i}(A) \quad \text{for } i = 1,\ldots,n-k .$$

For $i = 1,\ldots,\gamma$, let $\tilde{A}_{ii}$ be the submatrix of $\tilde{A}$ corresponding to A_{ii}. Since $u_i u_i^*$ is positive semi-definite of rank one, $\sum_{i=1}^{k} u_i u_i^*$ is positive semi-definite of rank k. This yields that $-\tilde{A}_{ii}$ is positive semi-definite of rank at most k, hence

$$\lambda_{n_i-k}(\tilde{A}_{ii}) = 0 \quad \text{for } i = 1,\ldots,\gamma .$$

Now we apply the left hand inequality of 1.3.2 with $j_i = n_i - k$. This gives

$$\lambda_{n-\gamma k}(\tilde{A}) + \sum_{i=1}^{\gamma-1} \lambda_i(\tilde{A}) \geq 0 .$$

With (*) this yields

$$(**) \qquad \lambda_{n-k(\gamma-1)}(A) + \sum_{i=1}^{\gamma-1} \lambda_{i+k}(A) \geq 0 .$$

Hence

$$(***) \qquad (\gamma - 1)\lambda_{1+k}(A) \geq -\lambda_{n-k(\gamma-1)}(A) .$$

Now suppose that $n_i \leq k$ for some $i \in \{1,\ldots,\gamma\}$. Let $L \subset \{1,\ldots,\gamma\}$ be such that $n_i \leq k$ iff $i \in L$. Let A' be the $n' \times n'$ submatrix of A, obtained by discarding all block matrices A_{ij} with $i \in L$ or $j \in L$. Put $\ell := |L|$. From $k < n/\gamma$ it follows that $\ell < \gamma$, hence $n' > 0$. Now (***) gives

$$(\gamma - \ell - 1)\lambda_{1+k}(A') + \lambda_{n'+k\ell-k(\gamma-1)}(A') \geq 0 \ .$$

Using $n' + k\ell > n$ and the Cauchy inequalities (1.2.2) we have

$$\lambda_{n'+k\ell-k(\gamma-1)}(A') \leq \lambda_{n-k(\gamma-1)}(A') \leq \lambda_{n-k(\gamma-1)}(A) \ ,$$

$$\lambda_{1+k}(A') \leq \lambda_{1+k}(A) \ .$$

Hence

$$(\gamma - \ell - 1)\lambda_{1+k}(A) + \lambda_{n-k(\gamma-1)}(A) \geq 0 \ .$$

From $k < n/\gamma$ it follows that $\lambda_{1+k}(A) \geq \lambda_{n-k(\gamma-1)}(A)$, hence $\lambda_{1+k}(A) \geq 0$. Thus

$$(\gamma - 1)\lambda_{1+k}(A) \geq -\lambda_{n-k(\gamma-1)}(A) \ .$$

This proves (i). The proof of (ii) proceeds analogously, but also follows from the above by replacing A by $-A$. □

We see that the second inequality of the above theorem for $k = 0$ is Hoffman's bound. In Chapter 4 we shall need a sharpened version of this inequality (see [H13]):

$$-\sum_{i=0}^{\gamma-2} \lambda_{n-i}(G) \geq \lambda_1(G) \ ,$$

which is in fact just formula (**) in the above proof with $k = 0$, and A equal to minus the adjacency matrix of G.

If $k > 0$, the above inequalities are not really bounds for $\gamma(G)$, since $\gamma(G)$ also occurs in an index. However, this does not matter much if we use these inequalities for estimating the chromatic number of a given graph. It is also not difficult to derive proper bounds from these inequalities. The next results illustrate this.

2.2.2. COROLLARY. *Let* $f_n(G)$ *denote the multiplicity of the eigenvalue* $\lambda_n(G)$. *Then*

$$\gamma(G) \geq \min\{1 + f_n(G),\ 1 - \lambda_n(G)/\lambda_2(G)\} \ .$$

PROOF. Suppose $\gamma = \gamma(G) \leq f_n(G)$. Then $\lambda_n(G) = \lambda_{n-\gamma+1}(G)$. Now 2.2.1.i with $k = 1$ gives $(\gamma - 1)\lambda_2(G) \geq -\lambda_n(G)$. This proves the required result. □

For strongly regular graphs the above results lead to the following theorem.

2.2.3. THEOREM. *Let* G *be a strongly regular graph on* n *vertices. Suppose* G *is not the pentagon or a complete* γ*-partite graph. Then*

$$\gamma(G) \geq \max\{1 - \lambda_1(G) / \lambda_n(G) , 1 - \lambda_n(G) / \lambda_2(G)\} .$$

PROOF. Due to the above results, it suffices to prove the following claim:

$$f_n(G) \geq -\lambda_n(G) / \lambda_2(G) .$$

To achieve this, we distinguish three cases.

a) $n \leq 28$. For this case it is easily checked that all feasible parameter sets for strongly regular graphs which violate our claim are those of the pentagon and the complete γ-partite graphs.

b) $\lambda_2(G) < 2$. In this case $\lambda_2(G) = 1$, or G is a conference graph (see [C9] or Appendix I). If G is a conference graph, then $\lambda_2(G) = -\frac{1}{2} + \frac{1}{2}\sqrt{n}$, and hence $n < 25$ and we are in case 1. Strongly regular graphs with $\lambda_2(G) = 1$ were determined by SEIDEL [S3]. They satisfy $n \leq 28$ or G is a ladder (disjoint union of edges), the complement of a lattice (line graph of a $K_{m,m}$) or the complement of a triangular graph (line graph of a K_m). One easily verifies that these three families of graphs satisfy our claim.

c) $\lambda_2(G) \geq 2$ and $n > 28$. If G is imprimitive (G is complete γ-partite or the disjoint union of complete graphs), the result is obvious. So assume G is primitive. Suppose the claim does not hold. Using $\lambda_2(G) \geq 2$, $\lambda_1(G) < n$ and

$$f_n(G)\lambda_n(G) + (n-1-f_n(G))\lambda_2(G) + \lambda_1(G) = 0$$

we obtain

$$f_n^2(G) < -f_n(G)\lambda_n(G) / \lambda_2(G) =$$

$$= n - 1 - f_n(G) + \lambda_1(G) / \lambda_2(G) < \tfrac{3}{2}n - f_n(G) .$$

This yields

$$f_n^2(G) + 3f_n(G) < \tfrac{3}{2}n + 2\sqrt{\tfrac{3}{2}n} .$$

For primitive G the absolute bound (see [D2], [S4]) reads

$$n \le \tfrac{1}{2}(f_n^2(G) + 3f_n(G)) \ .$$

Hence $\frac{1}{2}n < 2\sqrt{\frac{3}{2}n}$, i.e. $n < 24$. This contradicts our assumption, and therefore the theorem is proved. □

2.2.4. EXAMPLE. Let G be the Schläfli graph, which is drawn in Figure 1; two black or two white vertices are adjacent iff they are on one line, a black vertex is adjacent to a white one iff they are not on a line (see [S3], [H2]). Then G is strongly regular, n = 27 and

$$\lambda_1(G) = 16\ ,\ \lambda_2(G) = \ldots = \lambda_7(G) = 4\ ,\ \lambda_8(G) = \ldots = \lambda_{27}(G) = -2\ ,$$

$$\lambda_1(\bar{G}) = 10\ ,\ \lambda_2(\bar{G}) = \ldots = \lambda_{21}(\bar{G}) = 1,\ \lambda_{22}(\bar{G}) = \ldots = \lambda_{27}(\bar{G}) = -5\ ,$$

where $\bar{G}$ denotes the complement of G. From Figure 1 we see that

$$\alpha(G) \ge 3 \quad \text{and} \quad \alpha(\bar{G}) \ge 6\ .$$

The thin vertical lines partition G into six cliques, hence $\gamma(\bar{G}) \le 6$. The numbering gives a colouring of G with nine colours, so $\gamma(G) \le 9$. Using our bounds it follows that equality holds in all these inequalities. Indeed, by 2.1.4.ii or 2.1.5.i we have $\alpha(G) \le 3$; 2.1.1 yields $\alpha(\bar{G}) \le 6$; $\gamma(G) \ge 9$ follows from Hoffman's bound, and $\gamma(\bar{G}) \ge 6$ follows from our last theorem.

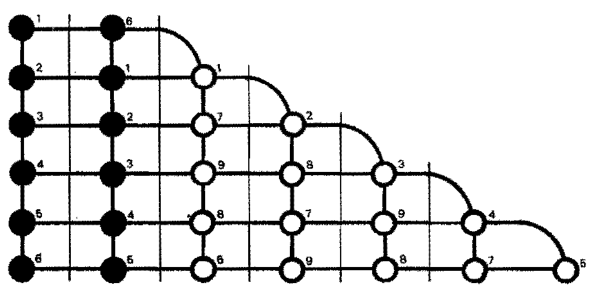


FIGURE I

The chromatic number of strongly regular graphs will be the subject of Chapter 4.

CHAPTER 3

INEQUALITIES FOR DESIGNS

3.1. SUBDESIGNS

In this section we shall derive inequalities for subdesigns of designs.

Let D be a design with incidence matrix N. It is clear that we cannot apply the Higman-Sims technique (1.2.3) to N, because N does not have to be symmetric. Instead, we apply the Higman-Sims technique to

$$A = \begin{bmatrix} 0 & N \\ N^* & 0 \end{bmatrix} .$$

By definition the positive eigenvalues of A are the singular values of N. Let $\sigma_1 \ge \sigma_2 \ge \ldots > 0$ denote these singular values. Then we can state the main result of this section.

3.1.1. THEOREM. *Let* D *be a* $1-(v,k,r)$ *design with* b *blocks. Let* D_1 *be a possibly degenerate* $1-(v_1,k_1,r_1)$ *subdesign of* D *with* b_1 *blocks. Then*

i. $$(vr_1 - b_1k)(bk_1 - v_1r) \le \sigma_2^2(v-v_1)(b-b_1) ,$$

ii. *if equality holds, then each point [block] off* D_1 *is incident with a constant number of blocks [points] of* D_1.

PROOF. Let N_1 and

$$N = \begin{bmatrix} N_1 & N_2 \\ N_3 & N_4 \end{bmatrix}$$

be the incidence matrices of D_1 and D, respectively. Put

$$A := \begin{bmatrix} 0 & 0 & N_1 & N_2 \\ 0 & 0 & N_3 & N_4 \\ N_1^* & N_3^* & 0 & 0 \\ N_2^* & N_4^* & 0 & 0 \end{bmatrix} \quad \text{and} \quad B := \begin{pmatrix} 0 & 0 & r_1 & r-r_1 \\ 0 & 0 & x & r-x \\ k & k-k_1 & 0 & 0 \\ y & k-y & 0 & 0 \end{pmatrix},$$

where

$$x := b_1(k-k_1)\,/\,(v-v_1) \quad \text{and} \quad y := v_1(r-r_1)\,/\,(b-b_1)\ .$$

Then the entries of B are the average row sums of the block matrices of A. By 1.1.2 we know

$$\lambda_i(A) = -\lambda_{b+v+1-i}(A)\ , \quad \lambda_j(B) = -\lambda_{5-j}(B)\ ,$$

for $i = 1,\ldots,b+v$, $j = 1,\ldots,4$. We easily have

$$\sigma_1 = \lambda_1(A) = \lambda_1(B) = \sqrt{rk}\ .$$

From $\det B = rk(r_1-x)(k_1-y)$ it now follows that

$$\lambda_2(B) = -\lambda_3(B) = \sqrt{(r_1-x)(k_1-y)}\ .$$

Now 1.2.3.i gives

$$(r_1-x)(k_1-y) \le \lambda_2^2(A) = \sigma_2^2\ .$$

With $b_1k_1 = v_1r_1$ this yields (i).
If equality holds, then the interlacing must be tight. Thus 1.2.3.iv gives (ii). □

From the above proof it is clear that the result also holds if D_1 is not a 1-design, but then we have to take r_1 and k_1 to be the average row and column sums of N_1.

For many 1-designs σ_2 is expressible in terms of the parameters of the design. For instance, $\sigma_2^2 = r-\lambda$ if D is a $2-(v,k,\lambda)$ design, and $\sigma_2^2 = s+t-\alpha+1$ if D is a partial geometry with parameters (s,t,α) (see Appendix I, or [C6], [T1]).

We shall make explicit two consequences of the above theorem.

3.1.2. COROLLARY. *If a symmetric* $2-(v,k,\lambda)$ *design contains a symmetric* $2-(v_1,k_1,\lambda_1)$ *subdesign, possibly degenerate, then*

$$k \ge \left(k_1 - v_1 \frac{k-k_1}{v-v_1}\right)^2 + \lambda .$$

PROOF. Substitute $b = v$, $k = r$, $b_1 = v_1$, $k_1 = r_1$ and $\sigma_2^2 = k-\lambda$ in 3.1.1.i. □

3.1.3. COROLLARY. *Let* X *and* Y *be a set of points and a set of lines, respectively, of a partial geometry with parameters* (s,t,α), *such that no point of* X *is incident with a line of* Y. *Then*

$$(\alpha|X| + (s+t+1-\alpha)(s+1))(\alpha|Y| + (s+t+1-\alpha)(t+1)) \le$$
$$\le (s+t+1-\alpha)(s+1)^2(t+1)^2 .$$

PROOF. Substitute $k_1 = r_1 = 0$, $b_1 = |Y|$, $v_1 = |X|$, $k = s+1$, $r = t+1$, $v = (s+1)(st+\alpha)/\alpha$, $b = (t+1)(st+\alpha)/\alpha$ and $\sigma_2^2 = s+t+1-\alpha$ in 3.1.1.i. □

Corollary 3.1.2 appeared in [H4]. A Baer subplane of a projective plane (see [D3]) satisfies 3.1.2 with equality. Other examples which meet this bound (hence where 3.1.1.ii applies) can be found in [H4].

The bound of 3.1.3 can also be tight. For instance, let Q be the partial geometry with parameters (2,4,1) (generalized quadrangle), whose points and lines are the vertices and the triangles of the complement of the Schläfli graph (see Example 2.2.4). There are 15 triangles which do not have a vertex in common with a double six (the black vertices in Figure 1), Thus we have an empty subgeometry (no point and line are incident) of Q having 12 points and 15 lines. This satisfies 3.1.3 with equality.

If D_1 is an empty design ($k_1 = r_1 = 0$), then one easily finds examples which meet the bound of 3.1.1.i. For instance: a projective plane with a maximal arc (see [D3]); a symmetric 2-design containing an oval without tangent blocks (see [A4]); a 2-design having a block repeated b/v times (see [L1]; here the inequality 3.1.1.i is Mann's inequality [M2]).

Although the results of this section are similar to those of Section 2.1, we did not start with a general inequality for substructures of an incidence structure like we did for subgraphs of a graph in 2.1.2. This has two reasons. Firstly, the formula for an arbitrary incidence structure is more complicated than 2.1.2. The second reason is that there does not seem to be much interest in incidence structures without any additional properties; this is certainly not true for graphs. Yet we shall give one result for an arbitrary incidence structure, namely an inequality for the sizes of an empty substructure.

3.1.4. THEOREM. *Let* D *be an incidence structure with* v *points and* b *blocks. Let every point [block] be incident with at least* r_{min} *blocks* [k_{min} *points]. Let* X *and* Y *be a set of points and a set of blocks, respectively, such that no point of* X *is incident with a block of* Y*. Then*

$$r_{min}^2\, k_{min}^2\, |X|\,|Y| \le \sigma_1^2\, \sigma_2^2 (v - |X|)(b - |Y|) \quad ,$$

where σ_1 *and* σ_2 *denote the two largest singular values of the incidence matrix of* D.

PROOF. Let the incidence matrix of D be

$$N = \begin{bmatrix} 0 & N_2 \\ N_3 & N_4 \end{bmatrix} ,$$

where 0 denotes the $|X| \times |Y|$ all-zero matrix. Let r_i and k_i be the average row sums of N_i and N_i^*, respectively, for $i = 2,3,4$. Then by 1.2.3.i the eigenvalues of

$$B := \begin{pmatrix} 0 & 0 & 0 & r_2 \\ 0 & 0 & r_3 & r_4 \\ 0 & k_3 & 0 & 0 \\ k_2 & k_4 & 0 & 0 \end{pmatrix}$$

interlace the eigenvalues of

$$A := \begin{bmatrix} 0 & N \\ N^* & 0 \end{bmatrix} .$$

Now with 1.1.2 it follows that

$$\det B = \lambda_1^2(B)\lambda_2^2(B) \le \lambda_1^2(A)\lambda_2^2(A) = \sigma_1^2\,\sigma_2^2 .$$

On the other hand we have

$$\det B = r_2r_3k_2k_3 = r_2^2\,k_3^2\,|X|\,|Y| \,/\, ((v-|X|)(b-|Y|)) .$$

Since $r_2k_3 \ge r_{min}\,k_{min}$, the theorem is proved. □

3.2. INTERSECTION NUMBERS

If two distinct blocks of a design D have exactly ρ points in common, then ρ is called an *intersection number* of D. It is obvious that an intersection number ρ of a $1-(v,k,r)$ design satisfies

$$k \ge \rho \ge \max\{0, 2k-v\} .$$

The next result, which is due to AGRAWAL [A1], gives non-trivial bounds for the intersection numbers of a 1-design. Like in the previous section, the singular values of the incidence matrix of a design D will be denoted by $\sigma_1 \ge \sigma_2 \ge \ldots > 0$.

3.2.1. THEOREM. *Let* D *be a* $1-(v,k,r)$ *design with* b *blocks. Let* B_1 *and* B_2 *be distinct blocks of* D. *Then*

i. $$|B_1 \cap B_2| \le 2\,\frac{rk - \sigma_2^2}{b} - k + \sigma_2^2 ;$$

if equality holds then $|B_1 \cap B_3| + |B_2 \cap B_3| = 2(rk-\sigma_2^2)/b$ *for any further block* B_3,

ii. $$|B_1 \cap B_2| \ge k - \sigma_2^2 ;$$

if equality holds then $|B_1 \cap B_3| = |B_2 \cap B_3|$ *for any further block* B_3.

PROOF. The result is obvious if $b \le 3$, so assume $b \ge 4$. Let N be the incidence matrix of D, such that the first two columns correspond to the blocks B_1 and B_2. Define

$$A := N^* N \ .$$

Then the off-diagonal entries of A are the intersection numbers of D, and the row and column sums of A equal kr. Put $\rho := |B_1 \cap B_2|$ and consider the following partitioning of A:

$$A = \begin{bmatrix} \begin{pmatrix} k & \rho \\ \rho & k \end{pmatrix} & A_{12} \\ A_{12}^* & A_{22} \end{bmatrix} .$$

Define

$$x := 2(kr - k - \rho) \ / \ (b - 2) \quad \text{and} \quad B := \begin{pmatrix} k+\rho & kr-k-\rho \\ x & kr-x \end{pmatrix} .$$

Then the entries of B are the average row sums of the block matrices of A. Clearly

$$\lambda_1(B) = \lambda_1(A) = rk \quad \text{and} \quad \lambda_2(B) = k + \rho - x \ .$$

By 1.2.3.i we have

$$\lambda_2(B) \le \lambda_2(A) = \sigma_2^2 \ .$$

Hence

$$\sigma_2^2(b-2) \le (k+\rho)(b-2) - 2(kr - k - \rho) \ .$$

This yields

$$\rho \le \sigma_2^2 - k + 2(rk - \sigma_2^2) \ / b \ .$$

If equality holds, the interlacing is tight and 1.2.3.i gives that every column sum of A_{12} equals x. This proves (i).
To prove (ii) we apply 1.2.1 to A with

$$S := \begin{pmatrix} -1 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 1 \end{pmatrix}^* \cdot \begin{pmatrix} 2 & 0 \\ 0 & n-2 \end{pmatrix}^{-\frac{1}{2}} .$$

Then

$$B := S^* A S = \begin{pmatrix} k-\rho & 0 \\ 0 & kr-x \end{pmatrix} .$$

It is easily seen that $kr - x \geq k - \rho$ if $b \geq 4$. So $\lambda_2(B) = k - \rho$. Hence, by 1.2.1.i

$$k - \rho \leq \lambda_2(A) = \sigma_2^2 .$$

Here equality does not have to imply that the interlacing is tight. Therefore we shall use 1.2.1.ii. If $\sigma_2^2 = k - \rho = kr - x$, then $r = 1$, $\rho = 0$ or $r = 2$, $\rho = 0$, $b = 4$, and the result is easily checked to be true. If $\sigma_2^2 = k - \rho < kr - x$, then 1.2.1.ii implies that $S(1,0)^* = (-\sqrt{2},\sqrt{2},0,\ldots,0)^*$ is an eigenvector of A for the eigenvalue $k - \rho$. Thus $A_{12}^*(-1,1)^* = 0$. This proves (ii). □

It is straightforward to verify that equality in (i) or (ii) for a pair of blocks of D implies also equality for the corresponding blocks of the complement of D.

Although Agrawal's proof of the above theorem is different from ours, it also uses eigenvalue techniques (in essence the Cauchy inequalities 1.2.2). MAJUMDAR [M1] gives a proof of this theorem for the case that D is a 2-design, using counting arguments. See also BUSH [B10] and CONNOR [C8] for similar results.

It is clear that our method also leads to inequalities if we consider the intersection pattern of more than two blocks.

3.2.2. THEOREM. *Let* D *be a* $1-(v,k,r)$ *design with* b *blocks. Let* Y *be a set of blocks which mutually have* ρ *points in common. Then*

i. $$|Y|\ (b\rho - rk) \geq b(\rho - k) ,$$

ii. $$|Y|\ (b\rho - rk + \sigma_2^2) \leq b(\sigma_2^2 - k + \rho) .$$

PROOF. Let N be the incidence matrix of D. We apply the Higman-Sims technique to $A := N^*N$, partitioned according to Y and the other blocks of D. Put

$$x := |Y|\ \frac{k(r-1) - \rho(|Y|-1)}{b - |Y|} .$$

Then

$$B := \begin{pmatrix} k - \rho + \rho|Y| & rk - k + \rho - \rho|Y| \\ x & rk - x \end{pmatrix}$$

carries the average row sums of the block matrices of A. Clearly

$$\lambda_2(B) = k - \rho + \rho|Y| - x = (b(k-\rho) + (b\rho - kr)|Y|) \,/\, (b - |Y|) \ .$$

From 1.2.3.i we have

$$0 \le \lambda_b(A) \le \lambda_2(B) \le \lambda_2(A) = \sigma_2^2 \ .$$

This lower and upper bound for $\lambda_2(B)$ yields (i) and (ii), respectively. □

We define two blocks B_1 and B_2 of a $1-(v,k,r)$ design to be *equivalent* if

$$|B_1 \cap B_2| \in \{k\,,\,k-\sigma_2^2\} \ .$$

Then from 3.2.1.ii it is clear that this indeed defines an equivalence relation, and that the number of common points of two blocks only depends on the equivalence classes of these blocks. By the use of 3.2.2 we find bounds for the size of the equivalence classes.

<u>3.2.3. THEOREM</u>. *Let* D *be a* $1-(v,k,r)$ *design with* b *blocks. Let* Y *be an equivalence class of blocks. Then*

i. k *and* $k-\sigma_2^2$ *cannot both be an intersection number of* D,

ii. *if* $k-\sigma_2^2$ *is an intersection number of* D, *then*

$$|Y| \le \frac{b\sigma_2^2}{b\sigma_2^2 - bk + rk} ,$$

iii. *if* k *is an intersection number of* D, *then*

$$|Y| \le \frac{b\sigma_2^2}{bk - rk + \sigma_2^2} .$$

<u>PROOF</u>. Assume $2r \le b$; we may do so because of the remark right after Theorem 3.2.1. Suppose $k-\sigma_2^2$ is an intersection number. Then $k-\sigma_2^2 \ge 0$, hence $\sigma_2^2 - k + 2(rk - \sigma_2^2)\,/\,b \le 2k(r-1)\,/\,b < k$. So 3.2.1.i yields that k cannot be an intersection number.

Formulas (ii) and (iii) follow immediately from (i) and (ii) of 3.2.2 by substitution of $\rho = k - \sigma_2^2$ and $\rho = k$, respectively. □

Suppose D is a $2-(v,k,\lambda)$ design, so $\sigma_2^2 = r - \lambda = (bk - rk)/(v-1)$. From 3.2.3.iii it follows that D has at most b/v repeated blocks; this is the inequality of MANN [M2] (see also [L1]). If D has an intersection number $k - \sigma_2^2 = k - r + \lambda$, then 3.2.3.ii implies that the size of any equivalence class is at most $b/(b-v+1)$; this bound appeared in [B2]. This paper also contains the next result (see also [B1]).

A 2-design with just two distinct intersection numbers is called *quasi-symmetric*. Consider the graph G, whose vertices are the blocks of a quasi-symmetric 2-design D, two vertices being adjacent if the number of points which the corresponding blocks have in common equals the larger intersection number. We call G the *block graph* of D. GOETHALS & SEIDEL [G2] (see also [C6]) proved that the block graph of a quasi-symmetric 2-design is strongly regular.
Now suppose D is a $2-(v,k,\lambda)$ design with just three distinct intersection numbers $k-r+\lambda$, ρ_1 and ρ_2 $(\rho_1 > \rho_2)$. We have already observed that the number of points which two blocks have in common only depends on the equivalence classes of these blocks. For this reason the following definition is legitimate. The *class graph* of D is the graph whose vertices are the equivalence classes, two vertices being adjacent if two blocks representing the corresponding classes have ρ_1 points in common.

3.2.4. THEOREM. *Let* D *be a* $2-(v,k,\lambda)$ *design with just three intersection numbers,* $k-r+\lambda$, ρ_1 *and* ρ_2. *Then the class graph of* D *is a strongly regular graph on*

$$n := \frac{b(k-r+\lambda-\rho_1)(k-r+\lambda-\rho_2)}{\lambda k^2 - k(r-\lambda) + (r-\lambda)^2 + b\rho_1\rho_2 - \lambda v(\rho_1+\rho_2)}$$

vertices, with eigenvalues

$$\lambda_1(G) = (\lambda vn - b(k-r+\lambda-\rho_2+\rho_2 n))/b(\rho_1-\rho_2) ,$$

$$\lambda_2(G) = (\rho_2-k+r-\lambda)/(\rho_1-\rho_2) ,$$

$$\lambda_n(G) = (b(\rho_1-k+r-\lambda) - n(r-\lambda))/b(\rho_1-\rho_2) .$$

PROOF. Let N be the incidence matrix of D. Define $A := N^*N$. Then

$$(*) \qquad A^2 = N^*(NN^*)N = N^*(\lambda J + (r-\lambda)I)N = \lambda k^2 J + (r-\lambda)A \ .$$

Put $\rho_0 := k-r+\lambda$. Let x_j $(j = 0,1,2)$ denote the number of times that ρ_j occurs in the i-th row of A, for some $i \in \{1,\dots,b\}$. Then

$$(AJ)_{i1} = k + x_0(k-r+\lambda) + x_1\rho_1 + x_2\rho_2 = rk \ ,$$

$$(A^2)_{ii} = k^2 + x_0(k-r+\lambda)^2 + x_1\rho_1^2 + x_2\rho_2^2 = \lambda k^2 + k(r-\lambda) \ ,$$

on using (*). Substitute $x_2 = b-1-x_0-x_1$ and subtract the first equation multiplied by $(\rho_1+\rho_2)$ from the second one. This yields

$$x_0(k-r+\lambda-\rho_1)(k-r+\lambda-\rho_2) =$$

$$= (b-1)\rho_1\rho_2 - k(r-1)(\rho_1+\rho_2) + k(r-\lambda) + k^2(\lambda-1) \ .$$

Hence x_0 does not depend on i and therefore all equivalence classes have size x_0+1. Now $n = b/(x_0+1)$ yields the given formula for n.
Now we partition A according to the equivalence classes. Let $\tilde{A}$ denote the adjacency matrix of G. Then the definition of G yields that the entries of

$$(**) \qquad B := (b/n)((k-r+\lambda-\rho_2)I + (\rho_1-\rho_2)\tilde{A} + \rho_2 J)$$

are the row sums (which are constant) of the block matrices of A. Since A has three distinct eigenvalues, rk, $r-\lambda$ and 0, it follows from 1.2.3.iii that each eigenvalue of B is equal to rk, $r-\lambda$ or 0. We easily check that rk is a simple eigenvalue of B, belonging to the all-one vector j. Now from (**) the eigenvalues of $\tilde{A}$ follow. Hence $\tilde{A}$ has an eigenvector j and just two distinct eigenvalues not belonging to j. This implies (see Appendix I or [C6]) that G is strongly regular. □

Examples of designs which satisfy the hypothesis of the above theorem can be found in [B1], [B3] or [M8]. For all these examples the class graph is a complete multipartite graph. In Section 6.1 we shall give an example for the above theorem where the class graph is primitive (not complete multipartite or the complement). For other results on 2-designs with an intersection number $k-r+\lambda$, see [B2].

CHAPTER 4

4-COLOURABLE STRONGLY REGULAR GRAPHS

4.1. INTRODUCTION

In this chapter we shall illustrate the use of the results and techniques obtained in the previous chapters. The result will be the determination of all 4-colourable strongly regular graphs.

It is obvious that a regular complete γ-partite graph, and a disjoint union of complete graphs on γ vertices are strongly regular graphs with chromatic number γ. Strongly regular graphs, not belonging to one of these two families, are called *primitive*.

Let G be a strongly regular graph with parameters (n,d,p^1_{11},p^2_{11}). Then (see [C5], [C9] or Appendix I)

$$d = \lambda_1(G) \ , \quad p^2_{11} - d = \lambda_2(G)\lambda_n(G) \ , \quad p^1_{11} - p^2_{11} = \lambda_2(G) + \lambda_n(G) \ ,$$

$$np^2_{11} = (d - \lambda_2(G))(d - \lambda_n(G)) \ .$$

Moreover, G has at most three distinct eigenvalues:

$$\lambda_1(G) \geq \lambda_2(G) = \ldots = \lambda_{f+1}(G) \geq 0 \ , \quad -1 \geq \lambda_{f+2}(G) = \ldots = \lambda_n(G) \ ,$$

where $f = f_2(G)$, the multiplicity of $\lambda_2(G)$, satisfies

$$f = (-\lambda_n(G)(n-1) - d) \ / \ (\lambda_2(G) - \lambda_n(G)) \ .$$

If G is primitive, then $p^2_{11} > 0$, $\lambda_2(G) > 0$, and $\lambda_n(G) < -1$.

4.1.1. LEMMA. *If* G *is a primitive strongly regular graph, not the pentagon, then*

i. $\quad d \leq -\lambda_n(G)(\gamma(G) - 1)$,

ii. $\quad -\lambda_n(G) \leq \lambda_2(G)(\gamma(G) - 1)$,

iii. $\quad \lambda_2(G) < \gamma(G) - 1$.

PROOF. (i) and (ii) are quoted from 2.2.3. Since G is primitive, $0 < p_{11}^2 = d - \lambda_2(G)\lambda_n(G)$. Hence (i) gives $\gamma(G) - 1 \geq -d/\lambda_n(G) > \lambda_2(G)$. □

As a direct consequence of this lemma we have the following theorem.

4.1.2. THEOREM. *Given $\gamma \in \mathbb{N}$, the number of primitive strongly regular graphs with chromatic number γ is finite.*

PROOF. If the graph G is primitive, then $p_{11}^2 \geq 1$ and hence by use of the formulas above

$$n \leq np_{11}^2 = (d - \lambda_2(G))(d - \lambda_n(G)) \leq d(d - \lambda_n(G)) .$$

By Lemma 4.1.1 we have

$$d(d - \lambda_n(G)) < \gamma(G)(\gamma(G) - 1)^5 .$$

This completes the proof. □

Now let us examine the case $\gamma(G) \leq 4$.

4.1.3. LEMMA. *Let G be a 4-colourable strongly regular graph. Suppose G has a non-integral eigenvalue. Then G is the pentagon.*

PROOF. Since G has a non-integral eigenvalue, we have (see [C9] or Appendix I)

$$\lambda_1(G) = \tfrac{1}{2}(n-1) \ , \quad \lambda_2(G) = -\tfrac{1}{2} + \tfrac{1}{2}\sqrt{n}\,, \quad \lambda_n(G) = -\tfrac{1}{2} - \tfrac{1}{2}\sqrt{n}\,,$$

$$n \equiv 1 \pmod 4 \ , \quad \sqrt{n} \notin \mathbb{N} .$$

By 2.1.5 we have $\alpha(G) \leq \sqrt{n}$, hence

$$4 \geq \gamma(G) \geq n/\alpha(G) \geq n/\lfloor \sqrt{n} \rfloor ,$$

therefore $n = 16$ or $n \leq 12$. Combining the restrictions for n we have $n = 5$, hence G is the pentagon. □

4.1.4. LEMMA. *A 4-colourable primitive strongly regular graph has one of the following parameter sets:*

i.	(5,2,0,1) ,	vii.	(16,9,4,6) ,
ii.	(9,4,1,2) ,	viii.	(40,12,2,4),
iii.	(10,3,0,1),	ix	(50,7,0,1) ,
iv.	(15,6,1,3),	x.	(56,10,0,2),
v.	(16,5,0,2),	xi.	(64,18,2,6),
vi	(16,6,2,2),	xii.	(77,16,0,4).

PROOF. Let G be such a graph. Suppose G is not the pentagon (which has parameter set (i)). Then by 4.1.3, the eigenvalues of G are integers. The primitivity of G yields $\lambda_2(G) > 0$, $\lambda_n(G) < -1$. Now 4.1.1.iii gives

$$\lambda_2(G) \in \{1,2\} .$$

Suppose $\lambda_2(G) = 1$. Then by 4.1.1

$$\lambda_n(G) \in \{-2,-3\} , \quad d = \lambda_1(G) \leq 9 .$$

Straightforward computations give that the only feasible parameter sets satisfying these conditions are (ii) - (v), (vii) and (10,6,3,4). However, a graph G with this last parameter set satisfies $\alpha(G) \leq 2$, therefore $\gamma(G) \geq 5$. Suppose $\lambda_2 = 2$. Then 4.1.1 implies

$$\lambda_n(G) \in \{-2,-3,-4,-5,-6\} , \quad d \leq 18 .$$

With a little more work than for the previous case, this leads to the feasible parameter sets (vi), (viii) - (xii) and (57,14,1,4). However, WILBRINK & BROUWER [W4] proved the nonexistence of a strongly regular graph with this last parameter set. □

For graphs with parameters (i) - (v) existence and uniqueness is known (see [S3]). Cases (i), (ii) and (iii) are the pentagon, the line graph of $K_{3,3}$ (also called the lattice graph $L_2(3)$), and the Petersen graph, respectively. It is easily seen that these three graphs have chromatic number three. From 4.1.1 it is clear that none of the other graphs is 3-colourable. Case (iv) is the complement of the line graph of K_6 (also called the complement of the triangular graph $T(6)$), which is easily seen to be 4-colourable. Case (v) is the Clebsch graph (see [S3]). This graph is given in Figure 2, where two black or two white vertices are adjacent iff they are not on one line, whilst a black vertex is adjacent to a white one iff they are on one line. We almost immediately see that this graph is

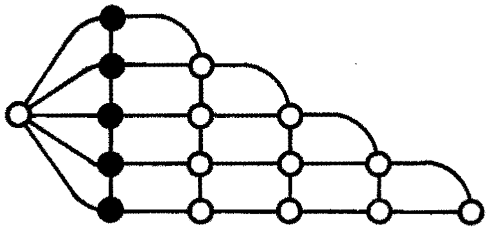


FIGURE 2

4-colourable. There are precisely two nonisomorphic strongly regular graphs with parameter set (vi) (see [S6]): the line graph of $K_{4,4}$ ($L_2(4)$) and the Shrikhande graph (see cover), both graphs are easily seen to be 4-colourable. Case (vii) is the complementary parameter set of (vi). We quickly see that the complement of the line graph of $K_{4,4}$ is 4-colourable, however, the complement of the Shrikhande graph is not 4-colourable. Indeed, the size of the largest coclique equals three. The remaining cases are more difficult. They will be treated in the next section.

4.2. STRONGLY REGULAR GRAPHS ON 40, 50, 56, 64 AND 77 VERTICES

In this section we shall study the feasible parameter sets for 4-colourable strongly regular graphs, which remain from the previous section.

The first case is the parameter set (40,12,2,4). Although several strongly regular graphs with these parameters are known (see Section 6.2), it will turn out that no such graph has chromatic number four. To prove this we use the following lemma.

4.2.1. LEMMA. *There is no regular bipartite graph on* 20 *vertices with eigenvalues* 4, 2, 0, -2, -4 *of multiplicity* 1, 6, 6, 6, 1, *respectively*.

PROOF. Suppose G were such a bipartite graph. Let

$$A = \begin{bmatrix} 0 & N \\ N^* & 0 \end{bmatrix}$$

be the adjacency matrix of G. Then N is the incidence matrix of a 1 - (10,4,4) design, D say, with singular values $\sigma_1 = 4$, $\sigma_2 = \ldots = \sigma_7 = 2$. Let B_1 and B_2 be two distinct blocks of D. Then 3.2.1.i yields

$$|B_1 \cap B_2| \leq 12/5 \ .$$

Suppose B_1 and B_2 are disjoint. Let x and y be the two points of D which are not incident with B_1 and B_2. Let B_3 be a block through x. Using 3.2.1.ii it follows that

$$|B_1 \cap B_3| = |B_2 \cap B_3| = 1 \ ,$$

so B_3 is incident with y. Hence, any block incident with x is also incident with y. However, this is not possible, since two points of D have at most two blocks in common, as follows from 3.2.1.i applied to the dual of D. So we have

$$|B_1 \cap B_2| \in \{1,2\} \ .$$

This implies that $B := N^*N - J - 3I$ is the adjacency matrix of a (strongly) regular graph with eigenvalues -3, 1 and 3 of multiplicity 3, 6 and 1, respectively. This is impossible, since

$$30 = \text{trace } B^2 \neq 3.(-3)^2 + 6.1^2 + 1.3^2 \ . \qquad \square$$

4.2.2. THEOREM. *There exists no 4-colourable strongly regular graph with parameters* (40,12,2,4).

PROOF. Let G be a strongly regular graph with parameter set (40,12,2,4). The eigenvalues of A, the adjacency matrix of G, are 12, 2 and -4 of multiplicity 1, 24 and 15, respectively. Suppose G is 4-colourable. Then without loss of generality

$$A = \begin{bmatrix} 0 & A_{12} & A_{13} & A_{14} \\ A_{21} & 0 & A_{23} & A_{24} \\ A_{31} & A_{32} & 0 & A_{34} \\ A_{41} & A_{42} & A_{43} & 0 \end{bmatrix}.$$

By 2.1.5.i all block matrices are square of size 10, and by 2.1.5.iii all row and column sums of A_{ij} are equal to 4, for $i,j = 1,2,3,4$, $i \neq j$. Define

$$A_1 := \begin{bmatrix} 0 & A_{12} \\ A_{21} & 0 \end{bmatrix}, \quad A_2 := \begin{bmatrix} 0 & A_{34} \\ A_{43} & 0 \end{bmatrix}, \quad \tilde{A} := A - \tfrac{1}{4}J, \quad \tilde{A}_i := A_i - \tfrac{1}{4}J,$$

for $i = 1,2$. Let $i \in \{1,2\}$. Let G_i be the graph with adjacency matrix A_i. Now $\tilde{A}$ has just two distinct eigenvalues 2 and -4 of multiplicity 25 and 15, respectively. Furthermore, A_i and $\tilde{A}_i$ have the same eigenvalues, except for the one belonging to the eigenvector j, which equals 4 for A_i and -1 for $\tilde{A}_i$. Since G_i is bipartite, A_i also has an eigenvalue -4. Now by the Cauchy inequalities (1.2.2) it follows that $\tilde{A}_i$, and hence also A_i, has at least five times the eigenvalue 2. But G_i is bipartite, therefore A_i, and hence also $\tilde{A}_i$, has at least five times the eigenvalue -2. Now from 1.3.3 it follows that $\tilde{A}_{3-i}$, and hence also $\tilde{A}_i$ and A_i, has at least five times the eigenvalue 0. Since G_i is bipartite on an even number of vertices, the multiplicity of the eigenvalue 0 is even, so at least six. Going backwards through the above reasoning we conclude that the multiplicities of the eigenvalues 2 and -2 of A_i are also at least six. Thus A_i has eigenvalues 4, 2, 0, -2, -4 of multiplicity 1, 6, 6, 6, 1, respectively. Now Lemma 4.2.1 finishes the proof. □

HOFFMAN & SINGLETON [H15] showed the existence and uniqueness of a strongly regular graph with parameters (50,7,0,1). So we only have to determine whether this graph is 4-colourable or not. To do so, we shall use a description of the Hoffman-Singleton graph (this description seems to be folklore, since it is well known; however, I could not find a reference) based on the following result, see [B11] or [C10].

<u>4.2.3. RESULT</u>. *The thirtyfive lines of* PG(3,2) *can be represented by the thirtyfive triples of a set with seven elements, such that two lines*

intersect iff the corresponding triples have one element in common.

REMARK. This result is directly related to the isomorphism of the groups PSL(4,2) and the alternating group on eight symbols (see for instance [B4] or [C10]).

Now we construct the Hoffman-Singleton graph as follows. The vertices are the fifteen points and the thirtyfive lines of PG(3,2). Points are mutually non-adjacent; a point is adjacent to a line iff the point is on that line; two lines are adjacent iff the triples, which correspond with these lines according to the above result, are disjoint. It is an easy exercise to check that this construction indeed gives the desired strongly regular graph.

4.2.4. THEOREM. *The Hoffman-Singleton graph has chromatic number four.*

PROOF. Colour the fifteen points red. Fix two elements x and y of the 7-set of Result 4.2.3. Colour lines blue, if they correspond to a triple containing x. Of the remaining lines, colour those yellow, whose corresponding triple contains y, and colour the other ones green. From our definition it is obvious that this is a correct colouring of the Hoffman-Singleton graph. □

GEWIRTZ [G1] showed existence and uniqueness of a strongly regular graph with parameters (56,10,0,2). Before giving a description of the Gewirtz graph we first prove the following.

4.2.5. PROPOSITION. *If the Gewirtz graph has two disjoint cocliques of size 16, then its chromatic number equals four.*

PROOF. Assume that the Gewirtz graph has adjacency matrix

$$A = \begin{bmatrix} 0 & A_{12} & A_{13} \\ A_{21} & 0 & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix},$$

where $A_{12} = A_{21}^*$ is square of size 16. We know that A has three distinct eigenvalues, 10, 2 and -4. From 2.1.5 it follows that A_{12}, A_{21}, A_{31} and A_{32}

have constant row sums equal to 4. Therefore the graph G_3, whose adjacency matrix is A_{33}, is a disjoint union of cycles. Suppose one of these cycles has c vertices. Partition A_{33} according to the vertices of this cycle, and the remaining ones. This induces a partition of A into sixteen block matrices, and the entries of the matrix

$$B := \begin{bmatrix} 0 & 4 & \frac{1}{4}c & 6-\frac{1}{4}c \\ 4 & 0 & \frac{1}{4}c & 6-\frac{1}{4}c \\ 4 & 4 & 2 & 0 \\ 4 & 4 & 0 & 2 \end{bmatrix}$$

are the average row sums of these block matrices. We immediately see that the eigenvalues of B are

$$\lambda_1(B) = 10\ ,\quad \lambda_2(B) = 2\ ,\quad \lambda_3(B) = \lambda_4(B) = -4\ .$$

However, we know that

$$\lambda_1(A) = 10\ ,\quad \lambda_2(A) = 2\ ,\quad \lambda_{55}(A) = \lambda_{56}(A) = -4\ .$$

Hence the eigenvalues of B interlace the eigenvalues of A tightly, thus by 1.2.3.ii all block matrices have constant row sums. Therefore, $\frac{1}{4}c$ is an integer. This proves that each component of G_3 is a cycle of even length. Thus G_3 is bipartite, and therefore the whole graph is 4-colourable. □

We use the description of the Gewirtz graph given in [G2], where this graph is obtained as the complement of the block graph of a quasi-symmetric $2-(21,6,4)$ design with intersection numbers 0 and 2 (see Section 3.2).

<u>4.2.6. THEOREM</u>. *The Gewirtz graph has chromatic number four.*

<u>PROOF</u>. Let D be the quasi-symmetric $2-(21,6,4)$ design. It is clear that all blocks through a fixed point of D yield a coclique in our graph of size 16. To see that there is another coclique of the same size, disjoint from this one, we proceed as follows. D can be obtained from a $3-(22,6,1)$ design $\tilde{D}$ (the extension of PG(2,4), see for instance [C6]) by deleting one point and all blocks through that point (i.e. D is a residual design of $\tilde{D}$). Take a block B of $\tilde{D}$, which is not a block of D. An elementary counting argument (see [C2]) shows that there are 16 blocks of D which are disjoint

from B, and which mutually have 2 points in common. Hence these 16 blocks provide a coclique of size 16 in the Gewirtz graph, which, if B has been chosen appropiately, is disjoint from our previous coclique. Application of 4.2.5 completes the proof. □

Three 2-(16,6,2) designs on a common point set are called *linked* if any two blocks from distinct designs have 3 or 1 points in common (see [C1] or [M7]). Let N_1, N_2 and N_3 be incidence matrices of three linked 2-(16,6,2) designs. Then we know that

$$N_i^*J = N_iJ = 6J \ , \quad N_i^*N_i = N_iN_i^* = 2J + 4I \ ,$$

for $i = 1,2,3$. Moreover, for $i,j = 1,2,3$, $i \neq j$, the matrix

$$N_{ij} := \tfrac{1}{2}(3J - N_i^*N_j)$$

is a (0,1) matrix by definition. In fact, N_{ij} is the incidence matrix of a 2-(16,6,2) design, since

$$N_{ij}N_{ij}^* = \tfrac{1}{4}(3J - N_i^*N_j)(3J - N_j^*N_i) = 2J + 4I \ ,$$

by use of the above formulas. Define

$$N_{ii} := \tfrac{1}{2}J - 2I \ , \quad N_{0j} := N_j \ , \quad N_{j0} := N_j^* \ ,$$

for $i = 0,1,2,3$, $j = 1,2,3$. Then

$$N_{ij} = N_{ji}^* \ , \quad N_{ij}N_{jk} = 3J - 2N_{ik} \ ,$$

for $i,j,k = 0,1,2,3$, as follows readily from the formulas above. This implies that

$$A := \begin{bmatrix} 0 & N_{01} & N_{02} & N_{03} \\ N_{10} & 0 & N_{12} & N_{13} \\ N_{20} & N_{21} & 0 & N_{23} \\ N_{30} & N_{31} & N_{32} & 0 \end{bmatrix}$$

is a symmetric matrix, which satisfies

$$A^2 = 18I + 2A + 6(J - I - A) \ .$$

Hence A is the adjacency matrix of a strongly regular graph with parameters (64,18,2,6), which is 4-colourable. We call this graph the *incidence graph* of the three linked designs.

<u>4.2.7. THEOREM</u>. *Let* G *be a strongly regular graph with parameters* (64,18,2,6) *and chromatic number four. Then* G *is the incidence graph of three linked* 2 - (16,6,2) *designs.*

<u>PROOF</u>. Suppose G has adjacency matrix

$$A = \begin{bmatrix} 0 & A_{01} & A_{02} & A_{03} \\ A_{10} & 0 & A_{12} & A_{13} \\ A_{20} & A_{21} & 0 & A_{23} \\ A_{30} & A_{31} & A_{32} & 0 \end{bmatrix}.$$

A has eigenvalues 18, 2, and -6 of multiplicity 1, 45 and 18, respectively. By 2.1.5, A_{ij} is square of size 16, and all row and column sums of A_{ij} are equal to 6, for $i,j = 0,1,2,3$, $i \neq j$. This implies that A and

$$K := I_4 \otimes J_{16}$$

have a common basis of eigenvectors. Using this, we obtain that

$$\tilde{A} := A - 2I + \tfrac{1}{2}K$$

has eigenvalues 24, 0 and -8 of multiplicity 1, 48 and 15, respectively. Thus

$$\operatorname{rank} \tilde{A} = 16 \ .$$

For $i = 0,1,2,3$, put $A_{ii} := \tfrac{1}{2}J_{16} - 2I_{16}$. Then

$$\operatorname{rank} A_{ii} = 16 \quad \text{and} \quad \tilde{A} = \begin{bmatrix} A_{00} & \cdots & A_{03} \\ \vdots & & \vdots \\ A_{30} & \cdots & A_{33} \end{bmatrix}.$$

From $\operatorname{rank} A_{00} = \operatorname{rank} \tilde{A}$, it follows that $A_{i0} A_{00}^{-1} A_{0j} = A_{ij}$, for $i,j = 1,2,3$, on applying 1.1.3. By use of

$$A_{00}^{-1} = (\tfrac{1}{2}J - 2I)^{-1} = (1/24)J - \tfrac{1}{2}I \quad \text{and} \quad A_{ij}J = 6J$$

this implies that for i,j = 1,2,3

$$(*)\qquad \frac{3}{2} J - \frac{1}{2} A_{0i}^{*} A_{0j} = A_{ij} .$$

This completes the proof. Indeed, if i = j, then (*) implies $A_{0i}A_{0i}^{*} = 2J + 4I$, showing that A_{0i} is the incidence matrix of a 2-(16,6,2) design, and if $i \neq j$, then (*) reflects that the designs represented by A_{01}, A_{02} and A_{03} are linked, and that A_{ij} is of the desired form for i,j = 1,2,3. □

MATHON [M7] proved that there are exactly twelve non-isomorphic triples of linked 2-(16,6,2) designs, which lead to eleven non-isomorphic incidence graphs. Hence there are precisely eleven non-isomorphic 4-colourable strongly regular graphs with parameters (64,18,2,6). It is fairly easy to show that one of these graphs is the point graph of the known generalized quadrangle of order (3,5) (see Chapter 5; a construction is described in 6.2.3). For completeness we list in Appendix II the systems of three linked 2-(16,6,2) designs, which provide the ten remaining graphs; these systems are taken from Mathon's paper. It is not known whether there are any further strongly regular graphs with these parameters, which are not 4-colourable.

Finally, the next theorem deals with the last set of parameters of Lemma 4.1.4.

4.2.8. THEOREM. *There exists no 4-colourable strongly regular graph with parameters* (77,16,0,4).

PROOF. Let G be a strongly regular graph with parameters (77,16,0,4). Then G has eigenvalues 16, 2 and -6 of multiplicity 1, 55 and 21, respectively. Suppose G is 4-colourable, and let c be the size of the largest colour class. Then $c \geq \lceil 77/4 \rceil = 20$. Let

$$A = \begin{bmatrix} 0 & A_{12} \\ A_{21} & A_{22} \end{bmatrix} ,$$

where 0 is square of size c, be the adjacency matrix of G. Then G_2, the graph with adjacency matrix A_{22}, is 3-colourable. From 2.1.7 it follows that $c \leq \alpha(G) \leq 21$, and that $c = \alpha(G) = 21$ implies that G_2 is the Gewirtz graph.

Since the Gewirtz graph has chromatic number four, G cannot be coloured with four colours if $c = 21$ (this shows that G is 5-colourable if $\alpha(G) = 21$, which is the case for the known strongly regular graph with these parameters; see [G2]). Suppose $c = 20$. Now we apply 1.3.3 to $A - (2/11)J$, so as to obtain that

$$\lambda_{57}(A_{22} - \frac{2}{11} J) = -6 , \quad \lambda_{56}(A_{22} - \frac{2}{11} J) = -4 .$$

Now 1.3.1 gives

$$\lambda_{57}(A_{22}) \geq -6 , \quad \lambda_{56}(A_{22}) \geq -4 .$$

On the other hand, the average row sum of A_{22} equals 592/57. Hence by 1.2.3.i (take $m = 1$) we have

$$\lambda_1(A_{22}) \geq 592/57 .$$

Now the sharpened version of Hoffman's inequality for the chromatic number (see Section 2.2), applied to the 3-colourable graph G_2 gives

$$-\lambda_{57}(A_{22}) - \lambda_{56}(A_{22}) \geq \lambda_1(A_{22}) .$$

This is a contradiction, proving the theorem. □

4.3. RECAPITULATION.

All cases of Lemma 4.1.4 have been treated now. The only thing left is to state the main theorem.

4.3.1. THEOREM. *If* G *is a* 4-*colourable strongly regular graph, then one of the following holds:*

i. $\gamma(G) = 2$, *and* G *is a regular complete bipartite graph, or a disjoint union of edges;*

ii. $\gamma(G) = 3$, *and* G *is a regular complete* 3-*partite graph, a disjoint union of triangles, the pentagon, the line graph of* $K_{3,3}$, *or the Petersen graph;*

iii. $\gamma(G) = 4$, *and* G *is a regular complete* 4-*partite graph, a disjoint union of* K_4'*s, the complement of the line graph of* K_6, *the line graph of* $K_{4,4}$, *or its complement, the Shrikhande graph, the*

Clebsch graph, the Hoffman-Singleton graph, the Gewirtz graph, or one of the eleven incidence graphs of three linked 2 - (16,6,2) *designs.*

CHAPTER 5

GENERALIZED POLYGONS

5.1. INTRODUCTION

A *generalized* n-*gon of order* (s,t), s > 0, t > 0, is a 1 - (v,s+1,t+1) design whose incidence graph has girth 2n and diameter n (see [T4], [D3], [F1], [H12] or Appendix I). A *generalized polygon* is a generalized n-gon for some n.

Generalized n-gons were introduced by TITS [T4]. An important result is the theorem of FEIT & HIGMAN [F1] (see also [H12], [K2]), which states that a generalized n-gon of order (s,t) is an ordinary n-gon (s = t = 1) or $n \in \{2,3,4,6,8,12\}$.

For a generalized polygon we speak of lines rather than blocks. We shall often omit the adjective "generalized". If D is a polygon of order (s,t), then we immediately see that the dual of D (points and lines interchanged) is a polygon of order (t,s).

Suppose N is the incidence matrix of the incidence graph of an n-gon of order (s,s). Then N is the incidence matrix of a 2n-gon of order (1,s). Conversely, it can be proved easily that all generalized n-gons of order (1,s), s > 1, are of this form. FEIT & HIGMAN [F1] also proved that s = 1 or t = 1 for a 12-gon of order (s,t), thus in a sense generalized 12-gons are the same as generalized hexagons of order (s,s). Generalized n-gons of order (s,t) with s > 1, t > 1, are called *thick*.

A generalized 2-gon is degenerate (every point is incident with every line). It is not difficult to verify that a generalized triangle of order (s,t) is a $2-(s^2+s+1,s+1,1)$ design, which is the same as a projective plane of order s (thus s = t). So 3-gons of order (s,s) exist for every prime power s. For projective planes see DEMBOWSKI [D3] or HUGHES & PIPER [H16].

Thick generalized quadrangles of order (s,t) are known to exist for (s,t), (t,s) = (q,q), (q,q^2), (q^2,q^3), $(q-1,q+1)$, for every prime power q ($q \neq 2$ for the last case). Constructions are due to AHRENS & SZEKERES [A2], HALL [H6], KANTOR [K1], PAYNE [P1], [P2] and TITS [T4], see also [D3], [T3]. HIGMAN [H11] showed that

$$t \le s^2 \le t^4$$

for thick quadrangles of order (s,t). Several other proofs of this inequality have been found, see [C3], [C5], [H12], [P6] and Section 3 of this chapter; some of these proofs also lead to consequences for the case of equality. There is an extended literature on generalized quadrangles. We mention the survey papers [P4], [T1] and [T3].

Thick generalized hexagons are known to exist for the orders (s^3,s), (s,s) and (s,s^3) for prime power s, see [T4]. A necessary condition for existence of a hexagon of order (s,t) is that st be a square, see [F1]. HAEMERS & ROOS [H3] showed that

$$s \le t^3 \le s^9$$

for thick hexagons of order (s,t). This inequality, together with a result for the case of equality, will be the subject of the next section. For more information about generalized hexagons we refer to [M4], [R1], [S1], [T4], [Y1].

Thick generalized octagons of order (s,t) are only known to exist for (t,s), $(s,t) = (2^m,2^{2m})$, for odd m. The construction is due to J. Tits, see [D3]. A necessary condition for existence of an octagon of order (s,t) is that 2st be a square, see [F1]. HIGMAN [H12] showed that

$$s \le t^2 \le s^4$$

for thick octagons. There is hardly any literature about octagons.

Let G be a connected graph of diameter m. For vertices x and y of G, let $\rho(x,y)$ denote the distance between x and y. For $i,j = 0,\dots,m$, define

$$p_{ij}(x,y) := \left|\{z \mid \rho(x,z) = i \ \&\ \rho(y,z) = j\}\right| .$$

If $p_{ij}(x,y)$ depends on i, j and $\rho(x,y)$ only, then G is called *distance regular* (see [B5]), and we write $p_{ij}^k := p_{ij}(x,y)$ where $k := \rho(x,y)$, and $d_i := p_{ii}^0$, for $i,j = 0,\dots,m$. The numbers p_{ij}^k are called the *intersection numbers* of G. Clearly, a distance regular graph is regular of degree d_1, and a distance regular graph of diameter 2 is the same as a connected strongly regular graph (in general, a distance regular graph of diameter m is equivalent to a metric association scheme with m classes, see [D1]). For a distance regular graph G of diameter m, we define the matrices $A_0,\dots,A_m$, indexed by the vertices of G, by

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } \rho(x,y) = i\ , \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $A_0 = I$, $\sum_{i=0}^{m} A_i = J$ and A_1 is the adjacency matrix of G. Moreover,

$$(A_i\ A_j)_{xy} = p_{ij}^{\rho(x,y)}$$

implies

$$A_i\ A_j = \sum_{k=0}^{m} p_{ij}^{k}\ A_k\ , \quad A_i\, J = d_i\, J\ , \quad \text{for } i,j = 0,\dots,m\ .$$

These equations show that $A_0,\dots,A_m$ generate an $m+1$ dimensional algebra. This type of algebra turns out to be useful in the study of distance regular graphs and similar configurations, see [B5], [B8], [D1], [H12], [W1].

The *point graph* of a generalized n-gon D is the graph whose vertices are the points of D, two points being adjacent whenever they are on one line of D. It is well known (see [D3], [Y2]) that the point graph G of an n-gon of order (s,t) is distance regular of diameter $\lfloor \frac{1}{2}n \rfloor$, and that the intersection numbers of G can be expressed in terms of s and t (in the forthcoming sections of this chapter we shall exhibit this result for $n = 6$ and $n = 4$).

A graph G is called *geometric for an n-gon* if G is the point graph of an n-gon. A graph G is called *pseudo-geometric for an n-gon* if G is distance regular of diameter $\lfloor \frac{1}{2}n \rfloor$ and its intersection numbers are such that G could be geometric, that is, there exist integers s and t, such that the intersection numbers of G depend on s and t as for geometric graphs.

Let D be a generalized polygon. An *element* of D is a point or a line of D. A sequence of $\ell+1$ elements $e_0,\dots,e_\ell$, is called a *path of length* ℓ *between* e_0 *and* e_ℓ, if e_i is incident with e_{i-1} for $i = 1,\dots,\ell$ (thus in $e_0,\dots,e_\ell$, points and lines alternate). The *distance* between elements e_0 and e_ℓ of D, denoted by $\lambda(e_0,e_\ell)$, is the length of the shortest path between e_0 and e_ℓ. Thus, if e_0 and e_ℓ are both points, then $\lambda(e_0,e_\ell)$ is twice the distance between e_0 and e_ℓ in the point graph of D.

In the next two sections we shall describe a method which for quadrangles and hexagons leads to the inequalities mentioned above, and to the results in case of equality (unfortunately, this method does not work for octagons). The same method also yields a new proof of a theorem of CAMERON, GOETHALS & SEIDEL [C5], which states that a pseudo-geometric graph for a quadrangle of order (s,s^2) is geometric.

5.2. AN INEQUALITY FOR GENERALIZED HEXAGONS

For $n = 6$ the definition of a generalized n-gon is equivalent to the following one:

5.2.1. DEFINITION. *A generalized hexagon of order* (s,t) *is an incidence structure with points and lines, such that*

i. *each line has* $s+1$ *points,*

ii. *each point is on* $t+1$ *lines,*

iii. *two distinct lines meet in at most one point,*

iv. *for any non-incident point-line pair* x,L *there is a unique path of length* < 6 *between* x *and* L.

Throughout this section H will denote a generalized hexagon of order (s,t). By use of the above definition it is straightforward to count the intersection numbers p_{ij}^k of the point graph of H. They are exhibited in Table 1. The amount of work in computing these numbers can be reduced by use of the equalities

$$\sum_{i=0}^{3} p_{ij}^k = d_j \ , \quad \sum_{j=0}^{3} d_j = v = (s+1)(s^2t^2+st+1) \ .$$

This counting also shows that the point graph of a generalized hexagon is distance regular.

TABLE 1

k	p_{11}^k	p_{12}^k	p_{22}^k	p_{13}^k	p_{23}^k	p_{33}^k
0	$s(t+1)$	0	$s^2t(t+1)$	0	0	s^3t^2
1	$s-1$	st	$st(s-1)$	0	s^2t^2	$s^2t^2(s-1)$
2	1	$s-1$	$s(t^2+t-1)$	st	$st(s-1)(t+1)$	$st(s^2t-st-s+t)$
3	0	$t+1$	$(s-1)(t+1)^2$	$(s-1)(t+1)$	$(t+1)(s^2t-st-s+t)$	$t(t-1)(s^2t-s+t)$

Let $A_0,\ldots,A_3$ be the matrices of the point graph of H. Define

$$E := A_2 - (s-1)A_1 + (s^2-s+1)I - \frac{1}{s+1} J .$$

Then we have the following lemma.

5.2.2. LEMMA. *The eigenvalues of* E *are*

$$0 \quad \textit{and} \quad s^2 + st + t^2$$

of multiplicity

$$1 + st(s+1)(t+1) \frac{s^2t+st^2-st+s+t}{s^2+st+t^2} \quad \textit{and} \quad s^3 \frac{s^2t^2+st+1}{s^2+st+t^2} ,$$

respectively.

PROOF. It is clear that $EJ = 0$, and that

$$\left(E - (s^2-s+1)I + \frac{1}{s+1} J\right)^2 = (s-1)^2 A_1^2 - (s-1)(A_1A_2+A_2A_1) + A_2^2 .$$

From $A_iA_j = \sum_{k=0}^{3} p_{ij}^k A_k$ and Table 1 we have

$$A_2^2 = s^2t(t+1)I + st(s-1)A_1 + (st^2+st-s)A_2 + (s-1)(t+1)^2 A_3 ,$$

$$A_1A_2 = A_2A_1 = st\, A_1 + (s-1)A_2 + (t+1)A_3 ,$$

$$A_1^2 = s(t+1)I + (s-1)A_1 + A_2 .$$

By use of $A_1 + A_2 + A_3 = J - I$ and $EJ = 0$, this leads to

$$E(E - (s^2+st+t^2)I) = 0 .$$

Hence 0 and s^2+st+t^2 are the eigenvalues of E. Finally,

$$\text{trace } E = v(s^2-s+1-(s+1)^{-1}) = s^3(s^2t^2+st+1)$$

yields the multiplicities. □

REMARK. In the terminology of DELSARTE [D1], the matrix $(s^2+st+t^2)^{-1} E$ is a minimal idempotent in the Bose-Mesner algebra of the association scheme on the points of H. The underlying theory provides a more elegant way to prove the previous lemma.

Let L_0 be a line of H, let L_i $(i = 1,2)$ denote the set of points at distance $2i+1$ from L_0. Partition E according to L_0, L_1 and L_2:

$$E = \begin{bmatrix} E_{00} & E_{01} & E_{02} \\ E_{10} & E_{11} & E_{12} \\ E_{20} & E_{21} & E_{22} \end{bmatrix} ,$$

so $E_{ij} = E^*_{ji}$ for $i,j = 0,1,2$.

5.2.3. LEMMA. *The eigenvalues of* E_{11} *are* 0, s^2 *and* st *of multiplicity* $st-s+t$, $t(s^2-1)$ *and* s, *respectively.*

PROOF. Let $A_{1,11}$ and $A_{2,11}$ be the submatrices of A_1 and A_2, respectively, corresponding to L_1. We easily see that without loss of generality

$$I + A_{1,11} = I_{st+t} \otimes J_s \quad \text{and} \quad I + A_{1,11} + A_{2,11} = I_{s+1} \otimes J_{st}$$

holds, hence

$$E_{11} = I_{s+1} \otimes J_{st} - s(I_{st+t} \otimes J_s) + s^2 I - \frac{1}{s+1} J .$$

From the eigenvalues of $I_{s+1} \otimes J_{st}$, $I_{st+t} \otimes J_s$, I and J, and the fact that these four matrices have a common basis of eigenvectors, the eigenvalues of E_{11} and their multiplicities follow. □

5.2.4. THEOREM. *A generalized hexagon with* $s+1$ *points on a line and* $t+1$ *lines through a point satisfies*

i. $\quad t \le s^3$ *or* $s = 1$,

ii. $\quad s \le t^3$ *or* $t = 1$.

PROOF. From 5.2.2 and 5.2.3 it follows that

$$\text{rank } E = s^3(s^2t^2+st+1)/(s^2+st+t^2) \;, \quad \text{rank } E_{11} = s^2t+s-t .$$

Since rank $E_{11} \le$ rank E, we have

$$(s^2t+s-t)(s^2+st+t^2) \le s^3(s^2t^2+st+1) .$$

This yields

$$t^2(s^2-1)(t-s^3) \le 0 .$$

Thus $s = 1$ or $t \le s^3$. Applying this result to the dual of H yields (ii). □

For another proof of this inequality see [H3]. Next we shall derive some additional regularity for hexagons meeting the above bound. To achieve this we need some properties of the matrices E_{ij}. First we observe that

$$(E_{02})_{xy} = \begin{cases} 1-1/(s+1) & \text{if } \lambda(x,y) = 4 , \\ -1/(s+1) & \text{otherwise,} \end{cases}$$

as follows directly from the definition of E and E_{02}. This implies

$$(E_{02}^* E_{02})_{xy} = \begin{cases} 1-1/(s+1) & \text{if } \lambda(x,z) = \lambda(y,z) \text{ for some } z \in L_0 , \\ -1/(s+1) & \text{otherwise.} \end{cases}$$

Hence, without loss of generality

$$E_{02} = j^* \otimes I_{s+1} - \frac{1}{s+1} J_{r(s+1)} ,$$

$$E_{02}^* E_{02} = J_r \otimes I_{s+1} - \frac{1}{s+1} J_{r(s+1)} ,$$

where $r := s^2t^2$. Now the positions of the points of L_2 relative to the points of L_0 give rise to a partition of E_{22} into $(s+1)^2$ square block matrices F_{ij} of size s^2t^2:

$$E_{22} = \begin{bmatrix} F_{00} & \cdots & F_{0s} \\ \vdots & & \vdots \\ F_{s0} & \cdots & F_{ss} \end{bmatrix} .$$

It is a matter of straightforward counting to see that F_{ij} has constant row sums equal to $t^2(s-1+\delta_{ij}) - s^2t^2/(s+1)$ for $i,j = 0,\ldots,s$. This, and the structure of E_{02} imply that

$$E_{22} E_{02}^* = t^2 E_{02}^* .$$

The following identities are now quickly seen to be true:

$$E_{00} = s^2(I - \frac{1}{s+1} J) , \quad E_{ij}J = 0 \quad \text{for } i,j = 0,1,2,$$

$$E_{00}^2 = s^2 E_{00} , \quad E_{00} E_{02} = s^2 E_{02} , \quad E_{02} E_{02}^* = t^2 E_{00} .$$

5.2.5. LEMMA. *The matrix*

$$\tilde{E}_{22} := E_{22} - s^{-2} E_{02}^* E_{02}$$

has eigenvalues

$$0 , \quad st + t^2 \quad \textit{and} \quad s^2 + st + t^2 ;$$

the multiplicity of $s^2 + st + t^2$ *equals*

$$t^2(s^2 - 1)(s^3 - t) / (s^2 + st + t^2) .$$

PROOF. Define

$$U := \begin{bmatrix} E_{00} \\ E_{20} \end{bmatrix} , \quad E' := \begin{bmatrix} E_{00} & E_{02} \\ E_{20} & E_{22} \end{bmatrix} .$$

Using the above formulas we obtain

$$(*) \qquad UU^*U = s^2(s^2 + t^2)U , \quad E'U = (s^2 + t^2)U .$$

Since rank $U = s$, the last formula reflects that the columns of U span an s-dimensional eigenspace of E' corresponding to the eigenvalue $s^2 + t^2$. Thanks to 5.2.2 and 5.2.3 the eigenvalues of E and E_{11} are known. By use of 1.3.3 we obtain that the non-zero eigenvalues of E' are

$$s^2 + t^2 , \quad st + t^2 \quad \text{and} \quad s^2 + st + t^2 ,$$

of multiplicity

$$s , \quad t(s^2 - 1) \quad \text{and} \quad t^2(s^2 - 1)(s^3 - t) / (s^2 + st + t^2) ,$$

respectively. Now from (*) it follows that

$$E' - s^{-2} U^* U$$

has just two distinct non-zero eigenvalues $st + t^2$ and $s^2 + st + t^2$, with the same multiplicities as before. On the other hand one easily verifies that

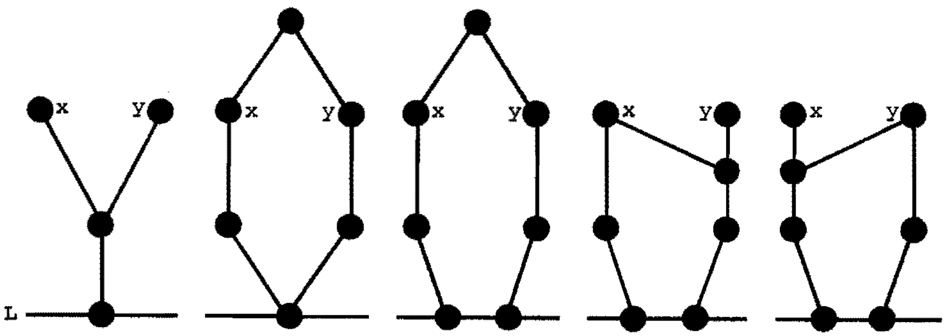
$$E' - s^{-2}U^*U = \begin{bmatrix} 0 & 0 \\ 0 & \tilde{E}_{22} \end{bmatrix},$$

which proves the lemma. □

The important thing in the last lemma is that the eigenvalue s^2+st+t^2 disappears if $t = s^3$. In order to give a combinatorial interpretation of this phenomenon, we need two definitions. For a line L and points x and y of a generalized hexagon we define:

$$p_{ijk}(L,x,y) := |\{z \mid \lambda(z,L) = 2i+1,\ \lambda(z,x) = 2j,\ \lambda(z,y) = 2k\}|\ ,$$

for $i = 0,1,2$, $j,k = 0,1,2,3$; the configuration *induced* by L, x and y is the configuration formed by the points and the lines, which are on a shortest path between L and x, L and y, or x and y. For example, Figure 3 gives all possible configurations induced by L, x and y if $\lambda(L,x) = \lambda(L,y) = 5$ and $\lambda(x,y) = 4$.



$s^7+s^4-s^3-2s$ $\quad$ $s^7+s^4-s^3-s^2-1$ $\quad$ $s^7+s^4-s^3-1$ $\quad$ $s^7+s^4-s^3-s$ $\quad$ $s^7+s^4-s^3-s$

FIGURE 3

5.2.6. THEOREM. *If a generalized hexagon has order* (s,s^3), *then* $p_{ijk}(L,x,y)$ *only depends on* i, j, k *and the configuration induced by* L, x *and* y.

PROOF. First observe that

$$\sum_{i=0}^{2} p_{ijk}(L,x,y)\ ,\quad \sum_{j=0}^{3} p_{ijk}(L,x,y) \quad\text{and}\quad \sum_{k=0}^{3} p_{ijk}(L,x,y)$$

only depend on i, j, k and the configuration induced by L, x and y (in fact, $\sum_{i=0}^{2} p_{ijk}(L,x,y) = p_{jk}^{\rho(x,y)}$). Subsequently, we verify (this is an easy but tiresome job), that the theorem is true if $i = 0$, or $j \le 1$, or $k \le 1$, and also if $\lambda(L,x) < 5$ or $\lambda(L,y) < 5$. Thus it suffices to prove the theorem for $i = j = k = 2$, $\lambda(L,x) = \lambda(L,y) = 5$. From the definitions of E, E_{22} and $p_{ijk}(L,x,y)$ it follows that

$$\begin{aligned}((E_{22} - \tfrac{1}{s+1} J)^2)_{xy} &= (s^2-s+1)^2\, p_{200}(L_0,x,y) + \\ &- 2(s-1)(s^2-s+1)p_{201}(L_0,x,y) + 2(s^2-s+1)p_{202}(L_0,x,y) + \\ &+ (s-1)^2\, p_{211}(L_0,x,y) - (s-1)(p_{212}(L_0,x,y) + p_{221}(L_0,x,y)) + \\ &+ p_{222}(L_0,x,y)\ .\end{aligned}$$

Now take $t = s^3$. Lemma 5.2.5 implies that $\tilde{E}_{22}$ has just two distinct eigenvalues 0 and $s^4(1+s^2)$. Hence

$$\tilde{E}_{22}^2 = s^4(1+s^2)\tilde{E}_{22}\ .$$

Using the formulas for the matrices E_{ij} this yields

$$(E_{22} - \tfrac{1}{s+1} J)^2 = s^4(1+s^2)E_{22} - s^2\, E_{02}^{*}\, E_{02} + s^8(s+1)^{-1} J\ .$$

This implies that $((E_{22} - (s+1)^{-1} J)^2)_{xy}$ only depends on x, y and the configuration induced by L_0, x and y. Combination with the previous steps yields that for $x,y \in L_2$, $p_{222}(L_0,x,y)$ only depends on the configuration induced by L_0, x and y. This completes the proof. □

With the available formulas the values of $p_{ijk}(L,x,y)$ are readily computed. For example, in Figure 3 we give $p_{222}(L,x,y)$ for the given configurations.

RONAN [R1], [R2] and THAS [T2], give sufficient conditions for a generalized hexagon to be one of the known ones. One hopes of course that a result like the one above will imply such a sufficient condition. Unfortunately, the gap between the condition we have and the condition we need, still seems to be too large to close up. It is worthwhile to remark that the known hexagons of order (s,t) with $t \neq s^3$ do not satisfy the condition of the above theorem.

Finally we remark that similar techniques yield the inequality (see MATHON [M6])

$$t \leq s^3 + t_2(s^2 - s + 1)$$

for a regular near hexagon with parameters (s,t,t_2), as introduced by SHULT & YANUSHKA [S7] (see also [Y2]). If $t_2 = 0$, then a near hexagon is the same as a generalized hexagon.

5.3. GEOMETRIC AND PSEUDO-GEOMETRIC GRAPHS FOR GENERALIZED POLYGONS

In this section we deal with the question whether a pseudo-geometric graph is geometric for a generalized n-gon. It is clear that for $n \in \{2,3\}$ the point graph of an n-gon is the complete graph. Assume G is the point graph of an n-gon D with $n > 3$. Then three points of D which form a triangle in G, must lie on one line of D. This implies that the graph  cannot be an induced subgraph of G. The next result states that the converse is also true.

5.3.1. LEMMA. *For a generalized* n*-gon with* $n > 3$*, a pseudo-geometric graph* G *is geometric iff*  *is not an induced subgraph of* G.

PROOF. Only the "if" part remains to be proved. Take n even (the case n odd is not difficult, but superfluous because of the Feit-Higman theorem). Let D be the incidence structure whose points are the vertices of G, and whose lines are the cliques (= complete subgraphs) of G of size $p^1_{11} + 2$. For two adjacent vertices of G, there are p^1_{11} vertices adjacent to both, but all these vertices are mutually adjacent since otherwise  occurs. This means that every edge of G determines a unique line of D. This proves that D is a $1-(v,s+1,t+1)$ design, where $s = p^1_{11} + 1$ and $t = p^n_{1,n-1} - 1$.

Let G' denote the incidence graph of D. Then, because $p^k_{i,k-i} = 1$ for $k = 0,\ldots,\frac{1}{2}n-1$, $i = 0,\ldots,k$, the girth of G' is at least 2n. Now, since each point of D is incident with $p^n_{1,n-1}$ lines, it follows that the distance between a point and a line of D (regarded as vertices of G') is at most

$n-1$. Hence G' has diameter n and girth $2n$. This proves that D is an n-gon, whose point graph is G. □

A direct consequence of this lemma (which was pointed out to me by D.E. Taylor) is the following.

5.3.2. COROLLARY. *For a generalized* n*-gon with* $n > 4$*, a pseudo-geometric graph is geometric.*

PROOF. If $n > 4$, then $p^2_{11} = 1$. Hence  does not occur in a pseudo-geometric graph. Now 5.3.1 gives the result. □

What remains to be studied are generalized quadrangles. A very easy counting argument shows that the point graph G of a quadrangle of order (s,t) is strongly regular with intersection numbers $p^1_{11} = s-1$, $p^2_{11} = t+1$, $p^0_{11} = d_1 = s(t+1)$ (this proves that the point graph of an n-gon is distance regular, in the case $n = 4$). This implies (see [C5], [T1] or Appendix I) that the eigenvalues of G, and hence the eigenvalues of any pseudo-geometric graph for a quadrangle of order (s,t), are

$$s(t+1)\ , \quad s-1 \quad \text{and} \quad -t-1$$

of multiplicity

$$1\ , \quad s^2(st+1)\,/\,(s+t) \quad \text{and} \quad st(s+1)(t+1)\,/\,(s+t)\ ,$$

respectively.

There exist many pseudo-geometric graphs for quadrangles, which are not geometric. The Shrikhande graph (see cover) is one of them. More examples (including an infinite family) are given in Section 6.2. The following theorem, which is due to CAMERON, GOETHALS & SEIDEL [C5], gives a sufficient condition for a pseudo-geometric graph to be geometric, as well as the extension of Higman's inequality to pseudo-geometric graphs for generalized quadrangles.

5.3.3. THEOREM. *Let* G *be a pseudo-geometric graph for a thick generalized quadrangle of order* (s,t)*. Then*

i. $t \le s^2$,

ii. *if equality holds, then* G *is geometric,*

iii. *equality implies that all subconstituents of* G *are strongly regular.*

PROOF. Let A be the adjacency matrix of G. Define

$$E := -(s+1)A + (s^2-1)I + J \ .$$

Then from the eigenvalues of A it follows that E has just one non-zero eigenvalue $(s+1)(s+t)$ of multiplicity $s^2(st+1)/(s+t)$. Hence

$$(*) \qquad \operatorname{rank} E = s^2(st+1)/(s+t) \ .$$

Lat x be a vertex of G. Partition A and E according to x, the vertices adjacent to x, and the vertices not adjacent to x:

$$A = \begin{bmatrix} 0 & j^* & 0 \\ j & A_{11} & A_{12} \\ 0 & A_{21} & A_{22} \end{bmatrix}, \quad E = \begin{bmatrix} s^2 & -sj^* & j^* \\ -sj & E_{11} & E_{12} \\ j & E_{21} & E_{22} \end{bmatrix},$$

where $A_{12} = A_{21}^*$ and $E_{12} = E_{21}^*$. For $i = 1,2$, let G_i be the graph with adjacency matrix A_{ii} (so G_i is a subconstituent of G). Then G_1 has $s(t+1)$ vertices, and is regular of degree $p_{11}^1 = s-1$. Hence $s-1$ is an eigenvalue of A_{11} of multiplicity c, say. It is known (see [B5], [C12] or Appendix I) that c equals the number of components of G_1. Clearly each component has at least s vertices. Hence

$$(**) \qquad cs \le s(t+1) \ .$$

The matrices A_{11}, I and J have a common basis of eigenvectors. Using this it follows that E_{11} has an eigenvalue 0 of multiplicity $c-1$ (one of the eigenvalues $s-1$ of A_{11} corresponding to the eigenvector j leads to the eigenvalue $s(t+1)$ of E_{11}). Hence

$$\operatorname{rank} E_{11} = s(t+1) - (c-1) \ .$$

Now using (*) and (**) we have

$$s(t+1) - t \le \operatorname{rank} E_{11} \le \operatorname{rank} E = s^2(st+1)/(s+t) \ .$$

This yields

$$t(s-1)(t-s^2) \le 0 \ ,$$

proving (i). Suppose equality holds. Then we must have equality in (**), which means that G_1 is a disjoint union of complete graphs on s vertices. Since x is arbitrary, this yields that  does not occur in G. Now (ii) follows on applying 5.3.1.

Since the disjoint union of complete graphs of the same size is strongly regular, it only remains to be proved that G_2 is strongly regular. This we shall prove analogously to the proof of 5.2.5. We know that the eigenvalues of E_{11} are 0, $s(s+1)$ and $s(s^2+1)$ of multiplicity s^2, $(s-1)(s^2+1)$ and 1, respectively. Now using 1.3.3 and the eigenvalues of E, we obtain that the matrix

$$E' := \begin{bmatrix} s^2 & j^* \\ j & E_{22} \end{bmatrix}$$

has eigenvalues 0, $s^2(s+1)$ and $2s^2$, where $2s^2$ is a simple eigenvalue with eigenvector $[s^2 \ \ j^*]^*$. Hence E_{22}, and also A_{22}, has just two distinct eigenvalues not belonging to the eigenvector j. This proves that G_2 is strongly regular. □

From (iii) of the above theorem it follows that the number of points adjacent to three mutually non-adjacent points of a quadrangle of order (s,s^2) is constant. This result was first proved by BOSE [B7].

A quadrangle of order (s,t) is the same as a partial geometry with parameters (s,t,1) (see [H11], [T1] or Appendix I). For pseudo-geometric graphs for a partial geometry with parameters (s,t,α), where $\alpha > 1$, a result like Lemma 5.3.1 does not hold anymore. Therefore the question in the beginning of this section is much more difficult to answer for these geometries.

CHAPTER 6

CONSTRUCTIONS

6.1. SOME 2 - (71,15,3) DESIGNS

In this section we shall construct eight non-isomorphic 2 - (71,15,3) designs. First we shall construct a 2 - (56,12,3) design D, which satisfies the hypothesis of Theorem 3.2.4. Next we show that D is embeddable in a 2 - (71,15,3) design. A less extensive treatment of this construction appeared in [B2]. Designs with these parameters seem to be new (see [C4] p.104, or [H5] p.297).

The most important ingredient for our construction is $\mathbb{F}_8$, the field with eight elements. Let G be the full automorphism group of $\mathbb{F}_8$, that is, the group of order 168 defined by $x \mapsto ax^{2^i} + b$, $a,b \in \mathbb{F}_8, a \neq 0$, $i \in \mathbb{Z}$. We shall identify $\mathbb{F}_8$ with AG(3,2), the 3-dimensional affine space over $\mathbb{F}_2$. Although G is the full automorphism group of $\mathbb{F}_8$, G is not the full automorphism group of AG(3,2). Yet G acts transitively on the elements (we reserve the word points for points of a design), the lines (i.e. unordered pairs of elements), the planes (sets of four linearly dependent elements), and the sets of four linearly independent elements. Moreover, the stabilizer of a line L has four orbits on lines: L itself, the lines intersecting L, the lines parallel to L, and the lines skew to L.

Now we shall define the incidence structure D. The points of D are the fiftysix ordered pairs of distinct elements of $\mathbb{F}_8$. The blocks of D are the seventy 4-subsets of $\mathbb{F}_8$. Let α, satisfying $\alpha^3 = \alpha + 1$, be a primitive element of $\mathbb{F}_8$. The point (0,1) is defined to be incident with the following blocks:

$$\{0,1,\alpha,\alpha^3\},\ \{0,1,\alpha,\alpha^2\},\ \{0,1,\alpha^2,\alpha^3\},\ \{\alpha,\alpha^2,\alpha^3,\alpha^4\},\ \{\alpha^2,\alpha^3,\alpha^4,\alpha^5\},$$

$$(*) \qquad \{0,1,\alpha^2,\alpha^6\},\ \{0,1,\alpha^2,\alpha^4\},\ \{0,1,\alpha^4,\alpha^6\},\ \{\alpha,\alpha^2,\alpha^4,\alpha^6\},\ \{\alpha,\alpha^3,\alpha^4,\alpha^6\},$$

$$\{0,1,\alpha^4,\alpha^5\},\ \{0,1,\alpha,\alpha^4\},\ \{0,1,\alpha,\alpha^5\},\ \{\alpha,\alpha^2,\alpha^4,\alpha^5\},\ \{\alpha,\alpha^2,\alpha^5,\alpha^6\}.$$

Now we let G act on D. This defines D, because G acts transitively on the points of D, and because the map $x \mapsto x^2$, which fixes the point (0,1), also fixes the set of blocks incident with (0,1). A point (x,y) is called *equivalent* to a point (w,z) if $\{x,y\} = \{w,z\}$. Two blocks $b \subset \mathbb{F}_8$ and $b' \subset \mathbb{F}_8$ are called *equivalent* if $b = b'$, or $b \cap b' = \emptyset$ (i.e. $b \cup b' = \mathbb{F}_8$). We know that G has two orbits on the blocks of D. The first orbit contains fiftysix blocks, they are the linearly independent 4-subsets of $\mathbb{F}_8$; these blocks will be called blocks of *type* I. The second orbit consists of the fourteen linearly dependent 4-subsets of $\mathbb{F}_8$ (i.e. planes), that is, they have the form $\{x,y,z,x+y+z\}$; these blocks are of *type* II. It is clear that equivalent blocks are of the same type.

<u>6.1.1. LEMMA</u>. *Let* (x,y) *be a point of* D. *Let* $b \subset \mathbb{F}_8$ *and* $b' \subset \mathbb{F}_8$ *be distinct equivalent blocks of* D.

i. *If* (x,y) *is incident with* b, *then* $\{x,y\} \subset b$ *or* $\{x,y\} \subset b'$.

ii. *If* $\{x,y\} \subset b$, b *of type* I, *then exactly one of the following two statements is true.*

   1. (x,y) *is incident with* b *and not with* b', *and* (y,x) *is incident with* b' *and not with* b;
   2. (x,y) *is incident with* b' *and not with* b *and* (y,z) *is incident with* b *and not with* b'.

iii. *If* b *is of type* II, *then* (x,y) *is incident with* b *iff* $\{x,y\} \subset b$.

iv. b *and* b' *have no points of* D *in common.*

<u>PROOF</u>. Without loss of generality take $(x,y) = (0,1)$. We may do so, because G is transitive on the points of D. Blocks incident with $(0,1)$ are given in (*). On applying the map $x \mapsto x+1$ to (*) we find that the blocks incident with $(1,0)$ are the following ones:

$$\begin{array}{ll} & \{0,1,\alpha,\alpha^3\},\ \{0,1,\alpha^3,\alpha^6\},\ \{0,1,\alpha,\alpha^6\},\ \{\alpha,\alpha^3,\alpha^5,\alpha^6\},\ \{\alpha,\alpha^4,\alpha^5,\alpha^6\}, \\ (**) & \{0,1,\alpha^2,\alpha^6\},\ \{0,1,\alpha^5,\alpha^6\},\ \{0,1,\alpha^2,\alpha^5\},\ \{\alpha^2,\alpha^3,\alpha^5,\alpha^6\},\ \{\alpha,\alpha^2,\alpha^3,\alpha^5\}, \\ & \{0,1,\alpha^4,\alpha^5\},\ \{0,1,\alpha^3,\alpha^5\},\ \{0,1,\alpha^3,\alpha^4\},\ \{\alpha^3,\alpha^4,\alpha^5,\alpha^6\},\ \{\alpha^2,\alpha^3,\alpha^4,\alpha^6\}. \end{array}$$

The first column of (*) and (**) consists of blocks of type II, all other blocks are of type I. Now (i), (ii) and (iii) are just a matter of verification. (iv) follows immediately from (ii) for blocks of type I, and from (iii) for blocks of type II. ∎

From (ii) and (iii) of this lemma we conclude the following. If each point of D is replaced by its equivalent partner and each block of D of type I by its equivalent partner, then incidence is not changed. Hence the permutation of the points of D, which interchanges the equivalent pairs of points, is an automorphism of D. This automorphism is different from, and commutes with, the automorphisms provided by the group G.

6.1.2. THEOREM. D *is a* 2-(56,12,3) *design*.

PROOF. First we prove that D is a 1-design. Since every point is incident with fifteen blocks, the average number of points incident with a block equals twelve. By 6.1.1.iii a block of type II is incident with exactly twelve points. Now, since G acts transitively on blocks of the same type, also the blocks of type I are incident with exactly twelve points. We have seen that G has three orbits on the (unordered) pairs of lines of $\mathbb{F}_8$ (intersecting, parallel, skew), for which the following pairs are representatives:

$$\{\{0,1\},\{0,\alpha\}\}\ ,\quad \{\{0,1\},\{\alpha,\alpha^3\}\}\ ,\quad \{\{0,1\},\{\alpha^2,\alpha^4\}\}\ .$$

From this it follows that the group $2\times G$, which is an automorphism group of D, has seven orbits on the (unordered) pairs of points of D, for which the following ones are representatives:

$$\{(0,1),(0,\alpha)\}\ ,\quad \{(0,1),(\alpha,\alpha^3)\}\ ,\quad \{(0,1),(\alpha^2,\alpha^4)\}\ ,$$

$$\{(1,0),(0,\alpha)\}\ ,\quad \{(1,0),(\alpha,\alpha^3)\}\ ,\quad \{(1,0),(\alpha^2,\alpha^4)\}\ ,$$

$$\{(1,0),(0,1)\}\ .$$

Blocks incident with (0,1) and (1,0) are given in (*) and (**), respectively. Using the maps $x\mapsto \alpha x$, $x\mapsto x+\alpha$ and $x\mapsto \alpha(x+\alpha)$, we obtain the blocks incident with $(0,\alpha)$, (α,α^3) and (α^2,α^4). The blocks incident with $(0,\alpha)$ are

$$\{0,\alpha,\alpha^2,\alpha^4\},\ \{0,\alpha,\alpha^2,\alpha^3\},\ \{0,\alpha,\alpha^3,\alpha^4\},\ \langle\alpha^2,\alpha^3,\alpha^4,\alpha^5\},\ \{\alpha^3,\alpha^4,\alpha^5,\alpha^6\rangle,$$

$$\langle 0,1,\alpha,\alpha^3\rangle,\ \{0,\alpha,\alpha^3,\alpha^5\},\ \langle 0,1,\alpha,\alpha^5\},\ \{1,\alpha^2,\alpha^3,\alpha^5\},\ \{1,\alpha^2,\alpha^4,\alpha^5\}\ ,$$

$$\{0,\alpha,\alpha^5,\alpha^6\},\ \{0,\alpha,\alpha^2,\alpha^5\},\ \{0,\alpha,\alpha^2,\alpha^6\},\ \{\alpha^2,\alpha^3,\alpha^5,\alpha^6\rangle,\ \{1,\alpha^2,\alpha^3,\alpha^6\}.$$

The blocks incident with (α,α^3) are

$$\blacktriangleleft\{0,1,\alpha,\alpha^3\}\blacktriangleright,\ \{0,\alpha,\alpha^3,\alpha^4\},\ \{1,\alpha,\alpha^3,\alpha^4\},\ \blacktriangleleft\{0,1,\alpha^2,\alpha^4\},\ \{1,\alpha^2,\alpha^4,\alpha^6\},$$
$$\{\alpha,\alpha^3,\alpha^4,\alpha^5\},\ \blacktriangleleft\{\alpha,\alpha^2,\alpha^3,\alpha^4\},\ \{\alpha,\alpha^2,\alpha^3,\alpha^5\}\blacktriangleright,\ \{0,\alpha^2,\alpha^4,\alpha^5\},\ \{0,1,\alpha^2,\alpha^5\}\blacktriangleright,$$
$$\{\alpha,\alpha^2,\alpha^3,\alpha^6\},\ \{0,\alpha,\alpha^2,\alpha^3\},\ \{0,\alpha,\alpha^3,\alpha^6\},\ \{0,\alpha^2,\alpha^4,\alpha^6\},\ \{0,\alpha^4,\alpha^5,\alpha^6\}.$$

The blocks incident with (α^2,α^4) are

$$\{0,\alpha,\alpha^2,\alpha^4\},\ \{0,\alpha^2,\alpha^4,\alpha^5\},\ \blacktriangleleft\{\alpha,\alpha^2,\alpha^4,\alpha^5\},\ \{0,\alpha,\alpha^3,\alpha^5\},\ \{1,\alpha,\alpha^3,\alpha^5\},$$
$$\{\alpha^2,\alpha^4,\alpha^5,\alpha^6\},\ \blacktriangleleft\{\alpha^2,\alpha^3,\alpha^4,\alpha^5\},\ \{\alpha^2,\alpha^3,\alpha^4,\alpha^6\}\blacktriangleright,\ \{0,\alpha^3,\alpha^5,\alpha^6\},\ \{0,\alpha,\alpha^3,\alpha^6\},$$
$$\{1,\alpha^2,\alpha^3,\alpha^4\},\ \{0,\alpha^2,\alpha^3,\alpha^4\},\ \blacktriangleleft\{0,1,\alpha^2,\alpha^4\},\ \{0,1,\alpha^3,\alpha^5\}\blacktriangleright,\ \{0,1,\alpha^5,\alpha^6\}\blacktriangleright.$$

Blocks incident with $(0,1)$ are marked by $\blacktriangleleft\{$. Blocks incident with $(1,0)$ are marked by $\}\blacktriangleright$. We see that for each of the seven pairs of points there are exactly three blocks incident with both points. Hence D is a $2-(56,12,3)$ design. □

Next we shall see that D satisfies the hypothesis of 3.2.4. The *line graph* of a geometry is the graph whose vertices are the lines, two vertices being adjacent iff the lines intersect.

<u>6.1.3. THEOREM</u>. D *has just three intersection numbers* 3, 2 *and* 0 *(*$= k-r+\lambda$*). The class graph of* D *is the complement of the line graph of* PG(3,2).

<u>PROOF</u>. Let $b_1 \subset \mathbb{F}_8$ and $b_2 \subset \mathbb{F}_8$ be non-equivalent blocks of D. Let b_1' and b_2' be the equivalent partners of b_1 and b_2, respectively. Let B_1, B_2, B_1', B_2' be the sets of points incident with b_1, b_2, b_1', b_2', respectively. Then by 6.1.1

$$|B_1 \cap B_1'| = |B_2 \cap B_2'| = 0 = k-r+\lambda\ .$$

Hence by 3.2.1.ii

$$|B_1 \cap B_2| = |B_1 \cap B_2'| = |B_1' \cap B_2| = |B_1' \cap B_2'|\ .$$

This implies

$$|(B_1 \cup B_1') \cap (B_2 \cup B_2')| = 4|B_1 \cap B_2|\ .$$

From (ii) and (iii) of 6.1.1 it follows that

$$\begin{aligned}
&|(B_1 \cup B_2') \cap (B_2 \cup B_2')| = \\
&= |\{(x,y) \in \mathbb{F}_8^2 \mid x \neq y,\ \{x,y\} \subset b_1 \text{ or } \{x,y\} \subset b_1', \\
&\qquad\qquad \{x,y\} \subset b_2 \text{ or } \{x,y\} \subset b_2'\}| = \\
&= \begin{cases} 8 & \text{if } |b_1 \cap b_2| = 2\ , \\ 12 & \text{if } |b_1 \cap b_2| \in \{1,3\}\ . \end{cases}
\end{aligned}$$

Hence 3, 2 and 0 are the intersection numbers of D. It is clear that the thirtyfive equivalence classes of blocks of D can be represented by the 4-subsets of $\mathbb{F}_8$ containing 0. Therefore, they can also be represented by all 3-subsets of $\mathbb{F}_8 - \{0\}$. Now using Result 4.2.3 we see that the class graph of D is indeed the complement of the line graph of PG(3,2). □

By the above theorem, there exists a 2-1 correspondence between the blocks of D and the lines of PG(3,2), such that two blocks have no point in common iff they correspond to the same line of PG(3,2), two blocks have two points in common iff they correspond to intersecting lines and two blocks have three points in common iff they correspond to skew lines.

6.1.4. THEOREM. D *is embeddable in a symmetric* 2 - (71,15,3) *design.*

PROOF. We extend D to D_1 with fifteen points (called *new points*), being the points of PG(3,2) and one block (*new block*). The points incident with the new block are precisely the new points. We define a new point to be incident with an old block (block of D) iff the line of PG(3,2) corresponding to that block contains that point. Now it is easily seen that D_1 is a 1 - (71,15,15) design, and that any two distinct blocks of D_1 have three points in common. This proves that the dual of D_1, and therefore D_1 itself, is a symmetric 2 - (71,15,3) design. □

From 6.1.1.iii it follows that the seven equivalence classes of blocks of type II of D correspond to seven mutually intersecting lines of PG(3,2). For these lines we may take seven lines through one point or seven lines in one plane. Let D_1 be the embedding of D in which blocks of type II correspond to lines through one point, and let D_2 be the other embedding of D. Define D_1^* and D_2^* to be the dual of D_1 and D_2, respectively. We shall show that these four $2-(71,15,3)$ designs are non-isomorhic. To achieve this we define a block B of a $2-(71,15,3)$ design to be *special* if the derived design with respect to B (i.e. the $2-(15,3,2)$ subdesign of the $2-(71,15,3)$ design, formed by the points of B and the blocks distinct from B) consists of two identical copies of a $2-(15,3,1)$ design. It is not difficult to see that the residual design with respect to a special block B (i.e. the $2-(56,12,3)$ subdesign of the $2-(71,15,3)$ design, formed by the points off B and the blocks distinct from B) satisfies the hypothesis of Theorem 3.2.4 and that its class graph is isomorphic to the complement of the block graph of the $2-(15,3,1)$ design associated to B (which clearly is quasi-symmetric, since $\lambda = 1$). From the proof of 6.1.4 it follows that the new blocks of D_1 and D_2 are special. Moreover, 6.1.1.iii implies that the new point of D_1 which is incident with all old blocks of type II, is a special block of D_1^*. By verification it turns out that these are the only special blocks of D_1, D_1^* and D_2. However, D_2^* has seven special blocks. They are the seven new points of D_2, which lie in the plane of PG(3,2) corresponding to the old blocks of type II. This already shows that

$$D_2 \not\cong D_2^*\ ,\quad D_2 \not\cong D_1\ ,\quad D_2 \not\cong D_1^*\ ,\quad D_2^* \not\cong D_1\ ,\quad D_2^* \not\cong D_1^*\ .$$

We know that any $2-(15,3,1)$ design associated to a special block of D_1 or D_2 is the design formed by the points and lines of PG(3,2). By use of 6.1.1.iii it follows that also the $2-(15,3,1)$ design associated to the special block of D_1^* is the design which comes from PG(3,2). Let D^* be the residual design of D_1^* with respect to the special block. Then the class graph of D^* is again the complement of the line graph of PG(3,2). This means that, similarly as for the design D, interchanging points and planes of PG(3,2) yields a second embedding of D^* into a $2-(71,15,3)$ design. Let D_3^* be this $2-(71,15,3)$ design and let D_3 be the dual of D_3^*. By verification it follows that D_3^* has just one special block (the one we started with), but that D_3 has seven special blocks. This already shows that $D_1^* \not\cong D_3^*$. By further investigation it turns out that the $2-(15,3,1)$ design associated to any of the seven special blocks of D_3 is again the design obtained from PG(3,2), however, the seven special blocks of D_2^* give other $2-(15,3,1)$ designs (in fact all seven of them give the second design in the list of WHITE, COLE & CUMMINGS [W3]). This proves that $D_2^* \not\cong D_3$, and therefore $D_1 \not\cong D_1^*$. Hence D_1, D_1^*, D_2, D_2^*, D_3 and D_3^* are all non-isomorphic. But there is still more. As remarked before, the $2-(15,3,1)$ design associated to one of the seven special blocks of D_3 (these seven special blocks form an orbit under the automorphism group of D_3) is the design formed by the points and lines of PG(3,2). This implies that once again we can make another 2 - (71,15,3) design by taking away the (two identical) 2 - (15,3,1) designs and putting them back again after having interchanged points and planes. Call this new design D_4, and let D_4^* be its dual. It turns out that D_4 has one special block, and that D_4^* has seven special blocks. Hence D_4 is not isomorphic to D_1, D_1^*, D_2^*, D_3 and D_4^*. In addition, D_4 is not isomorphic to D_2 and D_3^*, since otherwise D_3 would have been isomorphic to D_1 or D_1^*.

Because of time considerations we did not check whether it is possible to produce still more 2 - (71,15,3) designs by playing once again the same game with respect to a special block of D_4^*. Thus we have the following result.

<u>6.1.5. THEOREM</u>. *There exist at least eight* 2 - (71,15,3) *designs*.

The designs D_1^*, D_2^*, D_3^* and D_4^* are given explicitly in Appendix II. By taking residual designs with respect to various special blocks we obtain (at least) four non-isomorphic 2 - (56,12,3) designs which satisfy the hypothesis of Theorem 3.2.4. One of these designs has a class graph which is non-isomorphic to the class graph of the other ones.

An *oval* in a 2 - (71,15,3) design is a set S of six points such that any block has two or no points with S in common, see [A4]. Let S be an oval. It is clear that exactly twentysix blocks do not meet S. Therefore, by 3.1.1 an oval of a 2 - (71,15,3) design is equivalent to an empty subdesign with six points and twentysix blocks. From 3.1.1.ii (see also [A4]) it follows that the subdesign of the 2 - (71,15,3) design formed by the points off S and the blocks not meeting S is the dual of a 2 - (26,6,3) design. By verification it follows that the following blocks of D provide an oval in D_1^*:

$$\{1,\alpha,\alpha^2,\alpha^3\},\ \{1,\alpha,\alpha^2,\alpha^5\},\ \{0,1,\alpha,\alpha^5\}$$

$$\{1,\alpha^2,\alpha^4,\alpha^5\},\ \{\alpha,\alpha^2,\alpha^4,\alpha^5\},\ \{0,\alpha^3,\alpha^4,\alpha^6\}.$$

We conclude this section with a remark about automorphism groups. The group $2 \times G$ of order 336 is an automorphism group of D_1, D_2, D_3 and their duals. The designs D_4 and D_4^* have an automorphism group of order 48, viz. the stabilizer of a special block of D_3. F.C. Bussemaker has verified by use of a computer that the groups mentioned above are the full automorphism groups.

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6.2. SOME STRONGLY REGULAR GRAPHS

Suppose A is the adjacency matrix of a strongly regular graph G on n vertices of degree d. Furthermore, assume that A admits the following structure:

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1m} \\ \vdots & & \vdots \\ A_{m1} & \cdots & A_{mm} \end{bmatrix},$$

where A_{ij} is a square matrix of size $c := n/m$ having constant row sums equal to b_{ij}, say, for $i,j = 1,\dots,m$. From 1.2.3.iii it follows that the eigenvalues of the matrix $B := (b_{ij})$ satisfy

$$(*) \qquad \lambda_1(B) = \lambda_1(A) = d\,, \quad \lambda_i(B) \in \{\lambda_2(A),\ \lambda_n(A)\} \quad \text{for } i = 2,\dots,m\,.$$

Hence (*) yields directives for the construction of strongly regular graphs, whose adjacency matrix admits this block structure. Let us consider two special cases of this structure.

CASE 1: All diagonal entries of B are equal to r_0, say, and all off-diagonal entries of B are equal to r_1, say, that is,

$$B = (r_0 - r_1)I + r_1 J\,.$$

Hence by (*)

$$(m-1)r_1 + r_0 = d \quad \text{and} \quad r_0 - r_1 \in \{\lambda_2(A),\ \lambda_n(A)\}\,.$$

This implies

$$r_0 = (\lambda_2(A)(m-1) + d)/m\,, \quad r_1 = (d - \lambda_2(A))/m\,,$$

or

$$r_0 = (\lambda_n(A)(m-1) + d)/m\,, \quad r_1 = (d - \lambda_n(A))/m\,.$$

6.2.1. EXAMPLE. We wish to construct a strongly regular graph G with parameter set $(q^3+q^2+q+1, q^2+q, q-1, q+1)$, admitting the block structure of case 1 with $m = q^2+1$ and $c = q+1$. Then (see Appendix I) $\lambda_2(G) = q-1$, $\lambda_n(G) = -q-1$ and the formulas above yield $r_0 = q$, $r_1 = 1$. It is indeed possible to construct G, by use of this framework. To do so we define the permutation matrices P and Q of size $q+1$ by

$$P := \begin{bmatrix} 0 & I_q \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad (Q)_{ij} := \begin{cases} 1 & \text{if } i+j = q+2, \\ 0 & \text{otherwise.} \end{cases}$$

It easily follows that

$$P^{q+1} = I, \quad (P^k)^* = P^{q+1-k} = P^{-k}, \quad Q^2 = I,$$

$$P^k Q = (P^k Q)^* = QP^{-k}, \quad \sum_{i=0}^{q} P^i = J.$$

For $k = 1,\ldots,q$, define $R_k := P^k Q = QP^{-k}$. Then

$$R_k^* = R_k, \quad R_k R_\ell = P^{k-\ell}, \quad \sum_{i=0}^{q} R_i = J, \quad \sum_{i=0}^{q}\sum_{j=0}^{q} R_i R_j = (q+1)J,$$

for $k, \ell = 1,\ldots,q$. Let $p_1,\ldots,p_v$ $(v := q^2)$ be the points, and let $C_1,\ldots,C_{q+1}$ be the parallel classes of an affine plane of order q. For $i,j = 1,\ldots,q^2+1$, define the $(q+1)\times(q+1)$ matrices

$$A_{ij} := \begin{cases} J-I & \text{if } i = j, \\ I & \text{if } i \neq j,\ i = q^2+1 \text{ or } j = q^2+1, \\ R_k & \text{if } i \neq j,\ i \le q^2,\ j \le q^2, \text{ and } C_k \text{ contains the} \\ & \text{line through } p_i \text{ and } p_j. \end{cases}$$

By use of the formulas above it is straightforward to verify that the square matrix A of size q^3+q^2+q+1, built up with these block matrices A_{ij}, satisfies

$$(A+I)^2 = (q+1)J + q^2 I.$$

Therefore, A is the adjacency matrix of the desired strongly regular graph.

A strongly regular graph with the same parameters is provided by the point graph of a generalized quadrangle of order (q,q) (see Section 5.1). However, for $q > 4$ the graphs constructed above need not be geometric. Indeed, we can order the points and the parallel classes of an affine plane of order q in such a way that the graph we obtained by our construction has  as an induced subgraph. Then by 5.3.1, this graph is not geometric, and therefore non-isomorphic to any of those which come from generalized quadrangles.

As a second example of graphs admitting the block structure of case 1 we mention the eleven incidence graphs of three linked $2-(16,6,2)$ designs (see Section 4.2).

CASE 2: All diagonal entries of B are equal to r_0, say. The off-diagonal entries of B take exactly two values r_1 and r_2, say $(r_1 > r_2)$. Then by $(*)$ the $(0,1)$ matrix

$$\tilde{B} := \frac{1}{r_1 - r_2} (B - r_2(J-I) - r_0 I)$$

has just three distinct eigenvalues, one of which is simple and belongs to the eigenvector j. Hence $\tilde{B}$ is the adjacency matrix of a strongly regular graph G' with eigenvalues

$$d' := \lambda_1(G') = (d - r_2 m + r_2 - r_0) / (r_1 - r_2) \ ,$$

$$\lambda_2(G') = (\lambda_2(A) + r_2 - r_0) / (r_1 - r_2) \ ,$$

$$\lambda_m(G') = (\lambda_n(A) + r_2 - r_0) / (r_1 - r_2) \ .$$

6.2.2. EXAMPLE. We wish to construct a strongly regular graph G with parameters $(40,12,2,4)$, admitting the block structure of case 2 with $m = 10$, $c = 4$, $r_0 = r_2 = 0$, $r_1 = 2$. From $\lambda_2(G) = 2$, $\lambda_{40}(G) = -4$ and the above formulas it follows that $d' = 6$, $\lambda_2(G') = 1$, $\lambda_{10}(G') = -2$. Hence G' is the complement of the Petersen graph. For $h = 0,1$ and $i,j = 2,3,4$, define the square $(0,1)$ matrices T_{hij} of size four, by

$$(T_{hij})_{k\ell} := \begin{cases} h & \text{if } (k,\ell) \in \{(1,1),(1,\ell),(k,1),(k,\ell)\} \ , \\ 1-h & \text{otherwise.} \end{cases}$$

Then we have for $h,h' = 0,1;\ i,j,i',j' = 2,3,4$,

$$T_{hij} = T^{*}_{hji}\ ,\quad T_{hij}J = 2J\ ,\quad T_{hij} + T_{(1-h)ij} = J\ ,$$

$$T_{hij}\,T^{*}_{h'i'j'} = J \text{ if } j \neq j'\ ,\quad T_{hij}\,T^{*}_{hi'j} = 2T_{1ii'}\ ,$$

$$T_{hij}\,T_{(1-h)i'j} = 2T_{0ii'}\ .$$

With the help of these properties it is relatively easy to check that the following two matrices are adjacency matrices of strongly regular graphs with parameters (40,12,2,4).

$$\begin{bmatrix} 0 & T_{044} & T_{034} & T_{044} & T_{033} & T_{122} & T_{024} & 0 & 0 & 0 \\ T_{044} & 0 & 0 & T_{044} & T_{134} & 0 & T_{122} & T_{034} & 0 & T_{122} \\ T_{043} & 0 & 0 & T_{133} & T_{043} & T_{024} & 0 & T_{133} & T_{024} & 0 \\ T_{044} & T_{044} & T_{133} & 0 & 0 & T_{123} & 0 & T_{033} & 0 & T_{024} \\ T_{033} & T_{143} & T_{034} & 0 & 0 & 0 & T_{123} & T_{144} & T_{122} & 0 \\ T_{122} & 0 & T_{042} & T_{132} & 0 & 0 & T_{124} & 0 & T_{044} & T_{134} \\ T_{042} & T_{122} & 0 & 0 & T_{132} & T_{142} & 0 & 0 & T_{032} & T_{022} \\ 0 & T_{043} & T_{133} & T_{033} & T_{144} & 0 & 0 & 0 & T_{123} & T_{123} \\ 0 & 0 & T_{042} & 0 & T_{122} & T_{044} & T_{023} & T_{132} & 0 & T_{033} \\ 0 & T_{122} & 0 & T_{042} & 0 & T_{143} & T_{022} & T_{132} & T_{033} & 0 \end{bmatrix} \begin{bmatrix} 0 & T_{022} & T_{032} & T_{042} & T_{022} & T_{032} & T_{042} & 0 & 0 & 0 \\ T_{022} & 0 & T_{033} & T_{043} & T_{022} & 0 & 0 & T_{032} & T_{042} & 0 \\ T_{023} & T_{033} & 0 & T_{044} & 0 & T_{022} & 0 & T_{032} & 0 & T_{042} \\ T_{024} & T_{034} & T_{044} & 0 & 0 & 0 & T_{022} & 0 & T_{023} & T_{024} \\ T_{022} & T_{022} & 0 & 0 & 0 & T_{033} & T_{043} & T_{033} & T_{033} & 0 \\ T_{023} & 0 & T_{022} & 0 & T_{033} & 0 & T_{044} & T_{033} & 0 & T_{043} \\ T_{024} & 0 & 0 & T_{022} & T_{034} & T_{044} & 0 & 0 & T_{033} & T_{043} \\ 0 & T_{023} & T_{023} & 0 & T_{033} & T_{033} & 0 & 0 & T_{044} & T_{044} \\ 0 & T_{024} & 0 & T_{023} & T_{033} & 0 & T_{033} & T_{044} & 0 & T_{044} \\ 0 & 0 & T_{024} & T_{024} & 0 & T_{034} & T_{034} & T_{044} & T_{044} & 0 \end{bmatrix}$$

MATHON [M5] used the block structure of case 2 for the construction of strongly regular graphs with parameters $(pq^2\ ,\ \tfrac{1}{2}(pq^2-1)\ ,\ \tfrac{1}{4}(pq^2-5)\ ,\ \tfrac{1}{4}(pq^2-1))$ for prime powers p and q, $p \equiv 1$, $q \equiv -1 \pmod 4$. The strongly regular graph G', which provides the framework for Mathon's construction has parameters $(p\ ,\ \tfrac{1}{2}(p-1)\ ,\ \tfrac{1}{4}(p-5)\ ,\ \tfrac{1}{4}(p-1))$.

Any graph constructed in one of the examples above has the property that $p^1_{11} + 2 = p^2_{11}$. This implies that its adjacency matrix A satisfies

$$(A+I)^2 = p^2_{11}\,J + (d+1-p^2_{11})I\ ,$$

where d denotes the degree of the graph. This yields the well-known fact that $A+I$ is the incidence matrix of a symmetric $2-(n,d+1,p^2_{11})$ design. Similarly, if a strongly regular graph satisfies $p^1_{11} = p^2_{11}$, then its adjacency matrix itself is the incidence matrix of a symmetric $2-(n,d,p^1_{11})$ design. This phenomenon is behind the next example, where we derive a strongly regular graph with $p^1_{11} + 2 = p^2_{11}$ from one with $p^1_{11} = p^2_{11}$.

6.2.3. EXAMPLE. We start with a description of the generalized quadrangle Q of order $(2^\ell-1,2^\ell+1)$, $\ell \in \mathbb{N}$, due to HALL [H6]. Consider AG(3,q), the three dimensional affine geometry over $\mathbb{F}_q$, with $q = 2^\ell$. Let S be a set of $m := q+2$ lines from AG(3,q) passing through one point, such that no three lines lie in one plane. Such a set exists, because it corresponds to a complete oval in the projective plane PG(2,q), which exists iff q is even, see [D3]. It is easy to prove that each plane of AG(3,q) contains two or no lines from S. The points of Q are the points of AG(3,q); the lines of Q are the lines of S, and the lines of AG(3,q) which are parallel to a line of S; a point and a line are incident in Q, iff they are incident in AG(3,q). Now it is easy to prove that Q is a generalized quadrangle of order (q-1,q+1). Let us partition the adjacency matrix A of the line graph of Q (i.e. the point graph of the dual of Q) into m^2 square block matrices of size q^2, according to the m parallel classes in Q:

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1m} \\ \vdots & & \vdots \\ A_{m1} & \cdots & A_{mm} \end{bmatrix} .$$

From the structure of Q it follows that we may arrange the lines such that

$$A_{ij} = 0 \quad \text{if } i = j\,;$$

$$A_{ij} = I_q \otimes J_q \quad \text{if } i \text{ odd and } j = 1+i, \text{ or } j \text{ odd and } i = j+1\,;$$

$$A_{ij} = \begin{bmatrix} P_{11} & \cdots & P_{1q} \\ \vdots & & \vdots \\ P_{q1} & \cdots & P_{qq} \end{bmatrix} ,$$

for all other values of i,j, where $P_{k\ell}$ is a permutation matrix of size q, for $k,\ell = 1,\ldots,q$. Now we derive a new matrix $\tilde{A}$ from A by replacing

$$\begin{bmatrix} A_{ii} & A_{i,i+1} \\ A_{i+1,i} & A_{i+1,i+1} \end{bmatrix} \text{ by } \begin{bmatrix} A_{i,i+1} & A_{ii} \\ A_{i+1,i+1} & A_{i+1,i} \end{bmatrix} ,$$

for $i = 1,3,\ldots,m-1$. Then it follows that

$$\tilde{A}^2 = A^2 = qJ + q^2 I .$$

Since $\tilde{A}$ is symmetric with all diagonal entries equal to one, the matrix

$\tilde{A} - I$ is the adjacency matrix of a strongly regular graph with parameters $(q^2(q+2), q^2+q-1, q-2, q)$. For $q = 2$ this graph is the Clebsch graph (see Section 4.1, Figure 2), but for all other values of $q = 2^\ell$ these strongly regular graphs seem to be new.

We remark that in the above example A has the block structure of case 1, whilst $\tilde{A}$ has the block structure of case 2 with the cocktailparty graph on m vertices (complete ½m-partite graph) as the underlying strongly regular graph.

The remainder of this section will be devoted to strongly regular graphs with parameters (40,12,2,4). For convenience we call such graphs 40-*graphs*. Examples 6.2.1 $(q = 3)$ and 6.2.2 provide 40-graphs. The point graph of a generalized quadrangle of order (3,3) is a 40-graph. PAYNE [P5] proved that there are exactly two generalized quadrangles of order (3,3) (one being the dual of the other). In fact, these two geometric 40-graphs are the graph of Example 6.2.1 with $q = 3$, and the second graph of Example 6.2.2. From $p^1_{11} = 2$ it follows that a subgraph of a 40-graph induced by all vertices adjacent to a given vertex, is regular of degree two, so a disjoint union of cycles. But we can say more.

<u>6.2.4. LEMMA</u>. *Let* G *be a* 40-*graph. Let* x *be a vertex of* G *and let* G_x *be the subgraph of* G *induced by the vertices adjacent to* x. *Then* G_x *is one of the following graphs:*

i. *a* 12-*cycle;*
ii. *the disjoint union of a* 9-*cycle and a triangle;*
iii. *the disjoint union of two* 6-*cycles;*
iv. *the disjoint union of a* 6-*cycle and two triangles;*
v. *the disjoint union of four triangles.*

<u>PROOF</u>. We only have to prove that the number of vertices of any component of G_x is divisible by three. If G_x is connected, there is nothing to prove. Suppose G_x has a component C of size $c < 12$. We partition A into sixteen block matrices according to: the vertex x, the vertices of C, the remaining vertices of G_x, and the vertices not adjacent to x. Then the entries of

$$B := \begin{bmatrix} 0 & c & 12-c & 0 \\ 1 & 2 & 0 & 9 \\ 1 & 0 & 2 & 9 \\ 0 & c/3 & 4-c/3 & 8 \end{bmatrix}$$

are the average row sums of the block matrices of A. It is easy to see that

$$\lambda_1(B) = 12\ ,\quad \lambda_2(B) = \lambda_3(B) = 2\ ,\quad \lambda_4(B) = -4\ .$$

On the other hand we know

$$\lambda_1(A) = 12\ ,\quad \lambda_2(A) = \lambda_3(A) = 2\ ,\quad \lambda_{40}(A) = -4\ .$$

So the eigenvalues of B interlace the eigenvalues of A tightly. Hence, by 1.2.3.ii the row sums of the block matrices are constant, so c/3 is an integer. □

We associate with a 40-graph a 5-tuple $(a_1,\dots,a_5)$, where $a_1,\dots,a_5$ denote the number of vertices x for which G_x has the form (i),...,(v), respectively, of the above lemma. Using 5.3.1 we observe that a 40-graph is the point graph of a generalized quadrangle iff its 5-tuple is (0,0,0,0,40). The first graph of Example 6.2.2 has 5-tuple (0,0,4,24,12). R. Mathon (private communication) constructed a 40-graph with 5-tuple (0,0,0,36,4).

WEISFEILER [W1] describes an algorithm for generating strongly regular graphs with a given parameter set, based on the principle of backtracking. By use of this algorithm we wrote a computer program (in Algol 60) for the construction of 40-graphs. Weisfeiler's algorithm rejects isomorphism only partially. This means that some of the produced 40-graphs may be isomorphic. We had our program run for about ten minutes. It turned out that, although we obtained about twohundred (not necessarily non-isomorphic) 40-graphs, the process of finding all 40-graphs still was in the beginning phase. For this reason there was no hope for completing the whole search. It seems that there are thousands of 40-graphs. Still we wanted to test the few hundred 40-graphs we found on isomorphism. A complete test on isomorphisms would have been too expensive. Therefore we just computed the 5-tuple of each 40-graph. It turned out that twentyone of these 40-graphs had different 5-tuples. So we found at least twentyone non-isomorphic 40-graphs. One of these is the first graph of Example 6.2.2. But none of these graphs has the

5-tuple of a geometric 40-graph or Mathon's 40-graph. So we have the following result.

6.2.5. THEOREM. *There exist at least twentyfour strongly regular graphs with parameters* (40,12,2,4).

These 40-graphs are given in Appendix II, except for the three 40-graphs that are already exhibited in the Examples 6.2.1 and 6.2.2. The first graph in the list is Mathon's 40-graph.

We noticed already that a 40-graph gives a 2 - (40,13,4) design. There is no reason why non-isomorphic graphs should lead tot non-isomorphic designs. However, it has been checked that our twentyfour 40-graphs do produce twentyfour non-isomorphic 2 - (40,13,4) designs.

An *oval* in a 2 - (40,13,4) design is a set S of four points, such that any block has at most two points in common with S, see [A4]. Easy counting arguments give that twelve blocks are disjoint from S and four blocks have exactly one point in common with S. Suppose we have a 40-graph with a coclique of size four, such that any vertex is adjacent to two or to no vertices of that coclique. Then this coclique of the 40-graph produces an oval in the corresponding 2 - (40,13,4) design. Conversely, it can be proved that any oval in a 2 - (40,13,4) design, obtained from a 40-graph, corresponds to such a coclique. We see that the two 40-graphs of Example 6.2.2 produce 2 - (40,13,4) designs with ten disjoint ovals. Also the last six 40-graphs of Appendix II supply designs with ovals. The remaining sixteen 40-graphs have no ovals.

In 4.2.2 we saw that the chromatic number of any 40-graph is at least five. Since the complement of the Petersen graph is 5-colourable, it follows that the two 40-graphs of Example 6.2.2 are 5-colourable as well.

APPENDIX I

GRAPHS AND DESIGNS

This appendix contains the basic concepts and results from the theory of graphs and designs, which are used in the present monograph. Some general references are [B5], [C12], [H8], [W6] for graphs, [D3], [H5], [H17], [R3] for designs, and [B7], [C6] for both. We shall assume knowledge of Section 1.1.

A *graph* consists of a finite non-empty set of *vertices* together with a set of *edges*, where each edge is an unordered pair of vertices (so our graphs are finite, undirected and without loops or multiple edges). The two vertices of an edge are called *adjacent* (or *joint*). A graph is *complete* if every pair of vertices is an edge. The complete graph on n vertices is denoted by K_n. A graph without edges is called *void* (or *null*). The *complement* of a graph G is the graph $\bar{G}$ on the same vertex-set as G, where any two vertices are adjacent whenever they are not adjacent in G. The *disjoint union* of a collection of graphs $G_1,\ldots,G_m$ on disjoint vertex sets is the graph whose vertex-set is the union of all vertex-sets, and whose edge-set is the union of all edge-sets of $G_1,\ldots,G_m$. A graph is *disconnected* if it is the disjoint union of two or more graphs. Any graph G is the disjoint union of one or more connected (= not disconnected) graphs, called the *components* of G.

Let G be a graph on n vertices. A sequence of distinct vertices $x_0,\ldots,x_\ell$ of G is a *path* of *length* ℓ between x_0 and x_ℓ if $\{x_{i-1},x_i\}$ is an edge for $i = 1,\ldots,\ell$. The *distance* $\rho(x,y)$ between two vertices x and y is the length of the shortest path between x and y ($\rho(x,y) = \infty$ if x and y are in distinct components of G). The *diameter* of G is the largest distance in G. A sequence of vertices $x_0,\ldots,x_\ell$ is a *circuit* of *length* ℓ if $x_1,\ldots,x_\ell$ are distinct, $x_0 = x_\ell$, $\ell > 2$ and $\{x_{i-1},x_i\}$ is an edge for $i = 1,\ldots,\ell$. The *girth* of G is the length of the shortest circuit in G. The *adjacency matrix* of G is the $n \times n$ matrix A, indexed by the vertices of G, defined by

$$(A)_{xy} = \begin{cases} 1 & \text{if } \{x,y\} \text{ is an edge,} \\ 0 & \text{otherwise.} \end{cases}$$

Obviously, $\bar{G}$ has adjacency matrix $J-A-I$. The *eigenvalues* of G are the eigenvalues of A; they are denoted by $\lambda_1(G) \geq \ldots \geq \lambda_n(G)$ (the eigenvalues are real, because A is symmetric). We easily have that

$$\lambda_1(K_n) = n-1 \ , \quad \lambda_2(K_n) = \ldots = \lambda_n(K_n) = -1 \, .$$

The *incidence matrix* N of G, whose rows are indexed by the vertices and whose columns are indexed by the edges, is defined by

$$(N)_{x,E} = \begin{cases} 1 & \text{if } x \in E \, , \\ 0 & \text{otherwise.} \end{cases}$$

The graph with adjacency matrix N^*N-2I is called the *line graph* of G, denoted by L(G). The *subgraph* of G *induced* by a set S of vertices of G is the graph with vertex-set S, where two vertices are adjacent whenever they are adjacent in G (a subgraph is always an induced subgraph). Note that the adjacency matrix of a subgraph of G is a principal submatrix of A. A *clique* is a complete subgraph; a *coclique* (or *independent set of vertices*) is a void subgraph. The size of the largest clique and coclique is denoted by $\omega(G)$ and $\alpha(G)$, respectively. A *colouring* of G is a colouring of the vertices, such that adjacent vertices have different colours (i.e. a partition of the vertices into cocliques). Vertices which are coloured with the same colour form a *colour class*. G is k-*colourable* if G admits a colouring with k colours; the smallest possible value of k is the *chromatic number* of G, denoted by $\gamma(G)$. It easily follows that

$$\gamma(G) \geq \omega(G) \ , \quad \gamma(G)\alpha(G) \geq n \, .$$

If $\gamma(G) = 2$, then G is *bipartite*. By use of 1.1.2 it follows that if G is bipartite, then

$$\lambda_i(G) = -\lambda_{n+1-i}(G) \quad \text{for } i = 1,\ldots,n.$$

Conversely, $\lambda_1(G) = -\lambda_n(G)$ implies that G is void or bipartite; this follows from the Perron-Frobenius theorem on non-negative matrices (see [C12], [M3]). If G can be coloured with γ colours, such that all pairs of differently coloured vertices are edges, then G is *complete* γ-*partite* (i.e. the complement of the disjoint union of complete graphs). The complete bipartite graph ($\gamma = 2$) is denoted by $K_{\ell,m}$, where ℓ and m are the sizes of the two colour classes. By use of 1.1.2 it follows that

$$\lambda_1(K_{\ell,m}) = -\lambda_{\ell+m}(K_{\ell,m}) = \sqrt{\ell m}\,, \quad \lambda_i(K_{\ell,m}) = 0 \quad \text{for } i = 2,\dots,\ell+m-1\,.$$

The *degree* (or *valency*) of a vertex is the number of vertices adjacent to that vertex. A graph is *regular* (of degree d) if all its vertices have the same degree (equal to d).

Let G be regular of degree d. Then $Aj = dj$, hence d is an eigenvalue of G with eigenvector j. Moreover, the matrices A, J and I have a common basis of eigenvectors. By use of this we obtain

$$\lambda_1(\bar{G}) = n-d-1\,, \quad \lambda_i(\bar{G}) = -\lambda_{n+2-i}(G) - 1 \quad \text{for } i = 2,\dots,n\,.$$

If G is connected, then the Perron-Frobenius theorem yields that d is the largest eigenvalue of G with multiplicity one. Hence, $d = \lambda_1(G)$ and its multiplicity equals the number of components of G (see [B5] for an elementary proof). If G is connected and $d = 2$, then G is called an n-*cycle* (or *circuit*). G is *strongly regular* if G is not void or complete and the adjacency matrix A satisfies

$$(1) \qquad AJ = dJ\,, \quad (A-\lambda_2(G)I)(A-\lambda_n(G)I) = p_{11}^2 J\,,$$

for some number p_{11}^2. This is equivalent to requiring that A has precisely two distinct eigenvalues not belonging to the eigenvector j.

Now let G be strongly regular. Then also $\bar{G}$ is strongly regular. Computation of the diagonal entries and the row sums of both sides of the second equality of (1) yields

$$(2) \qquad d + \lambda_2(G)\lambda_n(G) = p_{11}^2\,, \quad (d-\lambda_2(G))(d-\lambda_n(G)) = p_{11}^2\, n\,.$$

Define $p_{11}^1 := \lambda_2(G) + \lambda_n(G) + p_{11}^2$. Then (1) becomes

$$AJ = dJ\,, \quad A^2 = dI + p_{11}^1 A + p_{11}^2(J-A-I)\,.$$

This reflects that G satisfies the following three properties: G is regular of degree d; for any pair of adjacent vertices there are exactly p_{11}^1 vertices adjacent to both; for any pair of non-adjacent vertices there are exactly p_{11}^2 vertices adjacent to both. The integers n, d, p_{11}^1, p_{11}^2 are the *parameters* of G. Let f be the multiplicity of the eigenvalue $\lambda_2(G)$. Then

$$0 = \text{trace } A = d + f\lambda_2(G) + (n-f-1)\lambda_n(G)\,,$$

whence

$$f = \frac{d + (n-1)\lambda_n(G)}{\lambda_n(G) - \lambda_2(G)} \; , \quad \lambda_n(G)(n - 2f - 1) = -d - f(p_{11}^1 - p_{11}^2) \; .$$

So, if $n \neq 2f + 1$, then $\lambda_n(G)$ is rational, and therefore $\lambda_2(G)$ and $\lambda_n(G)$ are integers. It is an easy exercise (see [C6]) to show that $n = 2f + 1$ implies

$$n - 1 = 2d = 4p_{11}^2 = 4(p_{11}^1 + 1) \; , \quad \lambda_2(G) = -\lambda_n(G) - 1 = -\tfrac{1}{2} + \tfrac{1}{2}\sqrt{n} \; .$$

By use of the above results and trace $A = 0$, it follows that $\lambda_2(G) \geq 0$ and $\lambda_n(G) \leq -1$. From (2) we have that $\lambda_2(G) = 0$ iff $p_{11}^2 = d$. It is easily seen that $p_{11}^2 = d$ reflects that G is a complete γ-partite graph. If $\lambda_n(G) = -1$, then $\lambda_2(\bar{G}) = 0$, hence G is the disjoint union of complete graphs and $p_{11}^2 = 0$. Conversely, by (1) $p_{11}^2 = 0$ implies $d = \lambda_2(G)$ and $\lambda_n(G) = -1$. These two families of strongly regular graphs are called *imprimitive*. Let x be a vertex of G. The two *subconstituents* of G with respect to x are the subgraphs of G induced by the vertices adjacent to x and by the vertices non-adjacent to x. The subconstituents are regular of degree p_{11}^1 and $d - p_{11}^1$, respectively. Examples of primitive (= not imprimitive) strongly regular graphs are: the pentagon (5-cycle), $L(K_m)$ for $m \geq 5$ (= *triangular graph*), $L(K_{m,m})$ for $m \geq 3$ (= *lattice graph*) and their complements. The *Petersen graph* is the complement of $L(K_5)$ and has parameters (10,3,0,1).

An *incidence structure* consists of a finite non-empty set V_1 of *points* and a finite non-empty set V_2 of *blocks*, together with a subset of $V_1 \times V_2$ of *flags*. A point and a block are *incident* if they form a flag. Often blocks are identified with the sets of points with which they are incident; if so, blocks are denoted by capitals, otherwise we use small types. An incidence structure without flags is called *empty*.

Let D be an incidence structure with v points and b blocks. An incidence structure D' formed by points and blocks of D is a *substructure* of D whenever a point and a block are incident in D' iff they are incident in D. The *incidence matrix* N of D, whose rows are indexed by the points and whose columns are indexed by the blocks, is defined by

$$(N)_{x,B} = \begin{cases} 1 & \text{if } x \in B \, , \\ 0 & \text{otherwise.} \end{cases}$$

The incidence structures with incidence matrices N^* and $J - N$ are called the *dual* and the *complement* of D, respectively. The graph with adjacency matrix $\begin{bmatrix} 0 & N \\ N^* & 0 \end{bmatrix}$ is the *incidence graph* of D. Clearly D and its dual have the same incidence graph. We call D a $t-(v,k,\lambda)$ *design* (or t-*design* with *parameters* (v,k,λ)) if all blocks have size k, and if any set of t points is contained in exactly λ blocks. A $t-(v,k,\lambda)$ design with $k < t$ or $v-k < t$ is *degenerate*. A *design* is a t-design for some t. A *subdesign* is a substructure which is a design. Note that we allow repeated blocks (two or more blocks incident with exactly the same points).

Let D be a non-degenerate $t-(v,k,\lambda)$ design. The complement of D is also a non-degenerate t-design. By elementary counting we see that for $t \geq 1$ D is also a $(t-1)-(v,k,\lambda(v-t+1)/(k-t+1))$ design. In particular, D is a $1-(v,k,r)$ design, where

$$r := \binom{v-1}{t-1} / \binom{k-1}{t-1} ;$$

r equals the number of blocks incident with any point. Counting flags yields $bk = vr$. If D is a $1-(v,k,r)$ design, then rk is the largest eigenvalue of NN^* and hence by 1.1.2 $\sqrt{rk}$ is the largest singular value of N. A 2-design is also called (*balanced incomplete*) *block design*. Now let D be a non-degenerate $2-(v,k,\lambda)$ design. In terms of the incidence matrix N this means

$$(3) \qquad N^*J = kJ \ , \quad NJ = rJ \ , \quad NN^* = \lambda J + (r-\lambda)I \ .$$

NN^* has eigenvalues $\lambda v + r - \lambda = rk$ and $r - \lambda$ of multiplicity 1 and $v - 1$, respectively. By 1.1.2 these eigenvalues are the squares of the singular values of N. From $rk \neq 0$, $r - \lambda \neq 0$, it follows that $v = \text{rank } NN^* = \text{rank } N$, hence $b \geq v$ (Fisher's inequality). If $b = v$ (i.e. $r = k$), D is called *symmetric*. Formula (3) yields $N(N^* - (\lambda/r)J) = (r-\lambda)I$. If D is symmetric, N is square, hence $(N^* - (\lambda/r)J)N = (r-\lambda)I$, i.e. $N^*N = \lambda J + (r-\lambda)I$, and therefore the dual of D is a symmetric $2-(v,k,\lambda)$ design as well. Let D be symmetric, and let B be a block of D. The subdesign formed by the points incident with B and the blocks distinct from B is a $2-(k,\lambda,\lambda-1)$ design (possibly degenerate), called the *derived design* of D with respect to B. Similarly, the subdesign formed by the points not incident with B and the blocks distinct from B is a $2-(v-k,k-\lambda,\lambda)$ design (possibly degenerate), called the *residual design* of D with respect to B. A $2-(v-k,k-\lambda,\lambda)$ design D' is *embeddable* in D if D' is a residual design of D. For example, the points and lines of PG(2,q), the projective plane of order q, form a symmetric $2-(q^2+q+1,q+1,1)$ design; the degenerate $2-(q+1,1,0)$ design is a derived design, and the affine plane of order q, which is a $2-(q^2,q,1)$ design, is a residual design.

A *partial geometry* with *parameters* (s,t,α), $s,t,\alpha \in \mathbb{N}$, is a $1-(v,s+1,t+1)$ design satisfying the following two conditions:

any two blocks have at most one point in common;

(4) for any non-incident point-block pair (x,B) the number of blocks incident with x and intersecting B equals α.

For a partial geometry we speak of lines rather than blocks. Let D denote a partial geometry with parameters (s,t,α) and incidence matrix N. The graph with adjacency matrix $NN^* - (t+1)I$ is the *point graph* of D. The *line graph* of D is the point graph of the dual of D. From the definition it readily follows that the point graph G of D is strongly regular with parameters

$$(v\,,\, s(t+1)\,,\, t(\alpha-1)+s-1\,,\, \alpha(t+1))\ .$$

By use of our identities for strongly regular graphs we obtain

$$\lambda_1(G) = s(t+1)\ ,\quad \lambda_2(G) = s-\alpha\ ,\quad \lambda_n(G) = -t-1\ ,$$

$$v = (s+1)(st+\alpha)/\alpha\ ,\quad b = (t+1)(st+\alpha)/\alpha\ .$$

Hence NN^* has the eigenvalues $(s+1)(t+1)$, 0 and $s+t+1-\alpha$ of multiplicity 1, $s(s+1-\alpha)(st+\alpha)/\alpha(s+t+1-\alpha)$ and $st(s+1)(t+1)/\alpha(s+t+1-\alpha)$, respectively. By 1.1.2 the square roots of these eigenvalues are the singular values of N. A *partial subgeometry* is a substructure, which itself is a partial geometry. Let D' be a partial subgeometry of D with parameters (s',t',α). Then the point graph of D' is an induced subgraph of the point graph of D. This can be seen as follows. Let x and y be two points of D' and suppose there exists a line L of D, which is not a line of D', incident with x and y. Let M be a line of D' incident with x. By (4) there are α lines of D' incident with x and intersecting M. Hence there are at least $\alpha+1$ such lines in D. This is a contradiction. So two points are on a line of D' iff they are on a line of D, which proves the claim. A partial geometry with parameters $(s,t,1)$ is the same as a *generalized quadrangle of order* (s,t).

APPENDIX II

TABLES

First we list the twentyone strongly regular graphs with parameters (40,12,2,4), as promised in Section 6.2. Together with the ones of Examples 6.2.1 and 6.2.2 they form the twentyfour 40-graphs of Theorem 6.2.5. The respective 5-tuples of the 40-graphs listed below are:

(0,0,0,36,4)	(8,18,2,9,3)	(16,14,4,5,1)
(12,18,6,3,1)	(4,20,4,10,2)	(18,18,0,3,1)
(12,12,0,14,2)	(9,18,0,9,4)	(18,20,0,0,2)
(6,22,6,3,3)	(18,12,9,0,1)	(27,12,0,0,1)
(0,32,0,0,8)	(0,12,0,18,10)	(0,36,0,0,4)
(0,24,0,12,4)	(8,8,0,20,4)	(6,12,12,10,0)
(2,8,12,16,2)	(10,8,18,4,0)	(0,16,6,12,6) .

2	3	4	5	6	7	8	9	10	11	12	13
1	3	4	14	15	16	17	18	19	20	21	22
1	2	4	23	24	25	26	27	28	29	30	31
1	2	3	32	33	34	35	36	37	38	39	40
1	6	7	14	17	20	23	27	31	32	37	39
1	5	7	15	18	21	24	28	29	33	35	40
1	5	6	16	19	22	25	26	30	34	36	38
1	9	10	14	17	20	25	26	30	33	35	40
1	8	10	15	18	21	23	27	31	34	36	38
1	8	9	16	19	22	24	28	29	32	37	39
1	12	13	14	17	20	24	28	29	34	36	38
1	11	13	15	18	21	25	26	30	32	37	39
1	11	12	16	19	22	23	27	31	33	35	40
2	5	8	11	15	16	23	28	30	32	36	40
2	6	9	12	14	16	24	26	31	33	37	38
2	7	10	13	14	15	25	27	29	34	35	39
2	5	8	11	18	19	24	26	31	34	35	39
2	6	9	12	17	19	25	27	29	32	36	40
2	7	10	13	17	18	23	28	30	33	37	38
2	5	8	11	21	22	25	27	29	33	37	38
2	6	9	12	20	22	23	28	30	34	35	39
2	7	10	13	20	21	24	26	31	32	36	40
3	5	9	13	14	19	21	24	25	32	35	38
3	6	10	11	15	17	22	23	25	33	36	39
3	7	8	12	16	18	20	23	24	34	37	40
3	7	8	12	15	17	22	27	28	32	35	38
3	5	9	13	16	18	20	26	28	33	36	39
3	6	10	11	14	19	21	26	27	34	37	40
3	6	10	11	16	18	20	30	31	32	35	38
3	7	8	12	14	19	21	29	31	33	36	39
3	5	9	13	15	17	22	29	30	34	37	40
4	5	10	12	14	18	22	23	26	29	33	34
4	6	8	13	15	19	20	24	27	30	32	34
4	7	9	11	16	17	21	25	28	31	32	33
4	6	8	13	16	17	21	23	26	29	36	37
4	7	9	11	14	18	22	24	27	30	35	37
4	5	10	12	15	19	20	25	28	31	35	36
4	7	9	11	15	19	20	23	26	29	39	40
4	5	10	12	16	17	21	24	27	30	38	40
4	6	8	13	14	18	22	25	28	31	38	39

9	10	15	16	17	21	22	24	28	32	35	38
6	8	9	10	14	19	23	24	32	33	34	39
4	7	12	15	16	19	23	24	30	31	35	39
3	8	10	13	17	18	20	21	30	32	37	39
7	9	12	14	18	20	21	22	28	31	34	39
2	9	12	13	15	17	18	23	31	36	37	38
3	5	10	11	14	16	18	24	33	36	37	38
2	4	11	15	18	20	22	24	29	34	35	36
1	2	5	6	11	16	20	25	30	35	37	39
1	2	4	7	12	13	22	25	29	33	38	39
7	8	9	13	15	19	21	25	28	30	33	36
3	5	6	10	17	19	21	25	29	34	35	36
4	6	10	11	16	19	22	26	28	31	34	37
2	5	7	15	17	19	22	26	29	30	32	37
1	3	6	8	11	14	21	26	29	31	38	39
1	3	7	9	13	20	23	26	29	32	34	36
1	4	6	12	14	20	24	26	28	30	33	36
4	5	6	7	8	23	25	26	28	32	35	38
2	3	11	12	13	14	20	27	28	32	35	38
4	5	8	9	16	17	19	27	29	31	33	38
1	4	5	11	12	15	23	27	32	33	34	37
1	5	8	10	13	14	23	27	30	31	35	36
2	3	6	16	18	21	22	27	28	29	30	33
1	2	3	7	8	17	25	27	28	31	34	37
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5	6	7	8	13	14	19	23	27	31	39	40
9	10	11	12	13	14	20	24	28	32	39	40
13	14	15	16	17	18	19	20	21	22	23	24
13	14	15	16	25	26	27	28	29	30	31	32
17	18	19	20	25	26	27	28	33	34	35	36
21	22	23	24	29	30	31	32	33	34	35	36

8	9	10	12	17	20	22	23	25	27	30	31
5	6	11	12	18	20	21	24	26	27	29	31
7	8	9	10	18	19	21	23	26	28	29	32
5	6	7	11	17	19	22	24	25	28	30	32
2	4	10	12	14	15	22	23	26	28	33	35
2	4	8	9	13	16	18	20	30	32	33	35
3	4	9	11	14	15	17	20	29	31	34	35
1	3	6	12	13	15	21	24	25	28	34	35
1	3	6	7	14	16	22	24	26	27	33	36
1	3	5	11	13	15	18	19	30	31	33	36
2	4	7	10	13	16	21	23	25	27	34	36
1	2	5	8	14	16	17	19	29	32	34	36
6	8	10	11	17	22	26	29	35	36	37	38
5	7	9	12	18	21	25	30	35	36	37	38
5	7	8	10	20	24	27	32	33	34	37	38
6	9	11	12	19	23	28	31	33	34	37	38
1	4	7	12	13	21	26	31	32	33	37	39
2	3	6	10	14	22	25	31	32	34	37	39
3	4	10	12	16	24	27	29	30	35	37	39
1	2	6	7	15	23	28	29	30	36	37	39
2	3	8	11	14	17	27	28	30	33	37	40
1	4	5	9	13	18	27	28	29	34	37	40
1	3	5	11	16	20	25	26	32	35	37	40
2	4	8	9	15	19	25	26	31	36	37	40
1	4	8	11	14	18	23	24	29	33	38	39
2	3	5	9	13	17	23	24	30	34	38	39
1	2	9	11	15	19	21	22	32	35	38	39
3	4	5	8	16	20	21	22	31	36	38	39
2	3	7	12	13	19	20	22	25	33	38	40
1	4	6	10	14	19	20	21	26	34	38	40
1	2	7	10	16	17	18	24	28	35	38	40
3	4	6	12	15	17	18	23	27	36	38	40
5	6	9	10	15	16	17	21	25	29	39	40
7	8	11	12	15	16	18	22	26	30	39	40
5	6	7	8	13	14	19	23	27	31	39	40
9	10	11	12	13	14	20	24	28	32	39	40
13	14	15	16	17	18	19	20	21	22	23	24
13	14	15	16	25	26	27	28	29	30	31	32
17	18	19	20	25	26	27	28	33	34	35	36
21	22	23	24	29	30	31	32	33	34	35	36

8	9	10	12	18	20	21	23	26	27	29	31
5	7	10	11	17	20	22	24	25	27	30	31
6	8	9	12	17	19	22	23	25	28	30	32
5	6	7	11	18	19	21	24	26	28	29	32
2	4	10	12	14	15	22	23	26	28	33	35
3	4	9	11	13	15	18	20	30	31	33	35
2	4	8	9	14	16	17	20	29	32	34	35
1	3	7	10	13	15	21	24	25	28	34	35
1	3	6	7	14	16	22	24	26	27	33	36
1	2	5	8	13	16	18	19	30	32	33	36
2	4	6	12	13	16	21	23	25	27	34	36
1	3	5	11	14	15	17	19	29	31	34	36
6	8	10	11	17	22	26	29	35	36	37	38
5	7	9	12	18	21	25	30	35	36	37	38
5	6	8	12	20	24	27	32	33	34	37	38
7	9	10	11	19	23	28	31	33	34	37	38
2	3	7	12	13	21	26	31	32	33	37	39
1	4	6	10	14	22	25	31	32	34	37	39
3	4	10	12	16	24	27	29	30	35	37	39
1	2	6	7	15	23	28	29	30	36	37	39
1	4	8	11	14	17	27	28	30	33	37	40
2	3	5	9	13	18	27	28	29	34	37	40
1	3	5	11	16	20	25	26	32	35	37	40
2	4	8	9	15	19	25	26	31	36	37	40
2	3	8	11	14	18	23	24	29	33	38	39
1	4	5	9	13	17	23	24	30	34	38	39
1	2	9	11	15	19	21	22	32	35	38	39
3	4	5	8	16	20	21	22	31	36	38	39
1	4	7	12	13	19	20	22	25	33	38	40
2	3	6	10	14	19	20	21	26	34	38	40
1	2	6	12	16	17	18	24	28	35	38	40
3	4	7	10	15	17	18	23	27	36	38	40
5	6	9	10	15	16	17	21	25	29	39	40
7	8	11	12	15	16	18	22	26	30	39	40
5	6	7	8	13	14	19	23	27	31	39	40
9	10	11	12	13	14	20	24	28	32	39	40
13	14	15	16	17	18	19	20	21	22	23	24
13	14	15	16	25	26	27	28	29	30	31	32
17	18	19	20	25	26	27	28	33	34	35	36
21	22	23	24	29	30	31	32	33	34	35	36

8	9	10	12	18	20	22	23	25	27	29	31
5	7	10	11	17	20	21	24	26	27	30	31
6	8	9	12	17	19	21	23	26	28	30	32
5	6	7	11	18	19	22	24	25	28	29	32
2	4	10	12	14	15	22	23	26	28	33	35
3	4	9	11	13	15	18	20	30	31	33	35
2	4	8	9	14	16	17	20	29	32	34	35
1	3	7	10	13	15	21	24	25	28	34	35
1	3	6	7	14	16	22	24	26	27	33	36
1	2	5	8	13	16	18	19	30	32	33	36
2	4	6	12	13	16	21	23	25	27	34	36
1	3	5	11	14	15	17	19	29	31	34	36
6	8	10	11	17	22	26	29	35	36	37	38
5	7	9	12	18	21	25	30	35	36	37	38
5	6	8	12	20	24	27	32	33	34	37	38
7	9	10	11	19	23	28	31	33	34	37	38
2	3	7	12	13	22	25	31	32	33	37	39
1	4	6	10	14	21	26	31	32	34	37	39
3	4	10	12	16	24	27	29	30	35	37	39
1	2	6	7	15	23	28	29	30	36	37	39
2	3	8	11	14	18	27	28	29	33	37	40
1	4	5	9	13	17	27	28	30	34	37	40
1	3	5	11	16	20	25	26	32	35	37	40
2	4	8	9	15	19	25	26	31	36	37	40
1	4	8	11	14	17	23	24	30	33	38	39
2	3	5	9	13	18	23	24	29	34	38	39
1	2	9	11	15	19	21	22	32	35	38	39
3	4	5	8	16	20	21	22	31	36	38	39
1	4	7	12	13	19	20	21	26	33	38	40
2	3	6	10	14	19	20	22	25	34	38	40
1	2	6	12	16	17	18	24	28	35	38	40
3	4	7	10	15	17	18	23	27	36	38	40
5	6	9	10	15	16	17	21	25	29	39	40
7	8	11	12	15	16	18	22	26	30	39	40
5	6	7	8	13	14	19	23	27	31	39	40
9	10	11	12	13	14	20	24	28	32	39	40
13	14	15	16	17	18	19	20	21	22	23	24
13	14	15	16	25	26	27	28	29	30	31	32
17	18	19	20	25	26	27	28	33	34	35	36
21	22	23	24	29	30	31	32	33	34	35	36

Here are listed the ten systems of three linked 2 - (16,6,2) designs, as promised in Section 4.2. The incidence graphs of these systems, together with the point graph of the generalized quadrangle of order (3,5) form the eleven 4-colourable strongly regular graphs with parameters (64,18,2,6) of Theorem 4.3.1.

2 3 4 5 9 13	2 3 4 5 9 13	2 3 4 5 9 13	1 2 3 7 15 16	1 2 3 7 15 16
1 3 4 6 10 14	1 3 4 6 10 14	1 3 4 6 10 14	1 2 5 9 10 11	1 2 5 9 10 11
1 2 4 7 11 15	1 2 4 7 11 15	1 2 4 7 11 15	1 3 5 6 8 14	1 3 5 6 8 14
1 2 3 8 12 16	1 2 3 8 12 16	1 2 3 8 12 16	1 4 6 7 9 13	1 4 6 7 9 13
1 6 7 8 9 13	1 6 7 8 9 13	1 6 7 8 9 13	1 4 10 12 14 16	1 4 10 12 14 16
2 5 7 8 10 14	2 5 7 8 10 14	2 5 7 8 10 14	1 8 11 12 13 15	1 8 11 12 13 15
3 5 6 8 11 15	3 5 6 8 11 15	3 5 6 8 11 15	2 3 4 5 12 13	2 3 4 5 12 13
4 5 6 7 12 16	4 5 6 7 12 16	4 5 6 7 12 16	2 4 6 11 14 15	2 4 6 11 14 15
1 5 10 11 12 13	1 5 10 11 12 13	1 5 10 11 12 13	2 6 8 10 13 16	2 6 8 10 13 16
2 6 9 11 12 14	2 6 9 11 12 14	2 6 9 11 12 14	2 7 8 9 12 14	2 7 8 9 12 14
3 7 9 10 12 15	3 7 9 10 12 15	3 7 9 10 12 15	3 4 8 9 10 15	3 4 8 9 10 15
4 8 9 10 11 16	4 8 9 10 11 16	4 8 9 10 11 16	3 6 9 11 12 16	3 6 9 11 12 16
1 5 9 14 15 16	1 5 9 14 15 16	1 5 9 14 15 16	3 7 10 11 13 14	3 7 10 11 13 14
2 6 10 13 15 16	2 6 10 13 15 16	2 6 10 13 15 16	4 5 7 8 11 16	4 5 7 8 11 16
3 7 11 13 14 16	3 7 11 13 14 16	3 7 11 13 14 16	5 6 7 10 12 15	5 6 7 10 12 15
4 8 12 13 14 15	4 8 12 13 14 15	4 8 12 13 14 15	5 9 13 14 15 16	5 9 13 14 15 16

2 3 4 6 11 16	2 3 4 6 11 16	2 3 4 6 11 16	1 2 3 6 11 13	1 2 3 6 11 13
1 3 4 5 12 15	1 3 4 8 9 15	1 3 4 8 9 15	1 2 5 12 14 15	1 2 5 12 14 15
1 2 4 8 9 14	1 2 4 5 12 14	1 2 4 5 12 14	1 3 4 5 7 10	1 3 4 5 7 10
1 2 3 7 10 13	1 2 3 7 10 13	1 2 3 7 10 13	1 4 9 11 15 16	1 4 9 11 15 16
1 5 7 8 11 16	1 5 6 8 10 16	1 5 6 8 10 16	1 6 7 8 12 16	1 6 7 8 12 16
2 6 7 8 12 15	2 5 9 10 11 15	2 5 9 10 11 15	1 8 9 10 13 14	1 8 9 10 13 14
3 5 6 7 9 14	3 8 10 11 12 14	3 8 10 11 12 14	2 3 5 8 9 16	2 3 5 8 9 16
4 5 6 8 10 13	4 5 7 8 11 13	4 5 7 8 11 13	2 4 6 9 10 12	2 4 6 9 10 12
1 6 9 10 12 16	1 7 9 11 12 16	1 7 9 11 12 16	2 4 7 13 14 16	2 4 7 13 14 16
2 5 9 10 11 15	2 6 7 8 12 15	2 6 7 8 12 15	2 7 8 10 11 15	2 7 8 10 11 15
3 8 10 11 12 14	3 5 6 7 9 14	3 5 6 7 9 14	3 4 8 11 12 14	3 4 8 11 12 14
4 7 9 11 12 13	4 6 9 10 12 13	4 6 9 10 12 13	3 6 10 14 15 16	3 6 10 14 15 16
1 6 11 13 14 15	1 6 11 13 14 15	1 6 11 13 14 15	3 7 9 12 13 15	3 7 9 12 13 15
2 5 12 13 14 16	2 8 9 13 14 16	2 8 9 13 14 16	4 5 6 8 13 15	4 5 6 8 13 15
3 8 9 13 15 16	3 5 12 13 15 16	3 5 12 13 15 16	5 6 7 9 11 14	5 6 7 9 11 14
4 7 10 14 15 16	4 7 10 14 15 16	4 7 10 14 15 16	5 10 11 12 13 16	5 10 11 12 13 16

2 3 4 7 12 14	2 3 5 6 10 12	2 3 5 7 11 12	1 2 6 7 10 14	1 2 6 9 14 16
1 3 4 8 11 13	1 4 6 8 11 12	1 4 5 6 9 11	1 2 9 12 13 16	1 2 7 10 12 13
1 2 4 5 10 16	1 2 9 12 13 15	1 2 10 11 14 16	4 6 8 9 14 16	4 6 7 8 10 14
1 2 3 6 9 15	1 2 6 7 14 16	1 2 5 8 13 15	4 7 8 10 12 13	4 8 9 12 13 16
1 5 6 8 12 14	1 4 5 7 9 10	1 4 7 8 10 12	1 3 4 6 12 15	1 3 4 13 14 15
2 5 6 7 11 13	2 3 7 8 9 11	2 3 6 8 9 10	1 4 5 11 13 14	1 4 5 6 11 12
3 6 7 8 10 16	3 4 6 7 13 15	3 4 5 8 14 16	2 3 8 13 14 15	2 3 6 8 12 15
4 5 7 8 9 15	3 4 9 12 14 16	3 4 10 11 13 15	2 5 6 8 11 12	2 5 8 11 13 14
1 7 9 10 11 14	1 3 10 11 15 16	1 3 6 7 15 16	1 3 7 8 9 11	1 3 8 10 11 16
2 8 9 10 12 13	2 4 5 8 15 16	2 4 9 12 15 16	1 5 8 10 15 16	1 5 7 8 9 15
3 5 9 11 12 16	5 6 9 11 13 16	5 6 10 12 14 15	2 3 4 10 11 16	2 3 4 7 9 11
4 6 10 11 12 15	5 7 11 12 14 15	6 8 11 12 13 16	2 4 5 7 9 15	2 4 5 10 15 16
1 7 12 13 15 16	1 3 5 8 13 14	1 3 9 12 13 14	3 5 6 9 10 13	3 5 6 7 13 16
2 8 11 14 15 16	2 4 10 11 13 14	2 4 6 7 13 14	3 5 7 12 14 16	3 5 9 10 12 14
3 5 10 13 14 15	6 8 9 10 14 15	5 7 9 10 13 16	6 7 11 13 15 16	6 9 10 11 13 15
4 6 9 13 14 16	7 8 10 12 13 16	7 8 9 11 14 15	9 10 11 12 14 15	7 11 12 14 15 16

1 2 3 7 15 16	1 2 3 7 15 16	1 2 3 7 15 16	1 2 3 7 15 16	2 3 4 5 9 13
1 2 5 9 10 11	1 2 5 9 10 11	1 2 5 9 10 11	1 2 5 9 10 11	1 3 4 6 10 14
1 3 5 6 8 14	1 3 5 6 8 14	1 3 5 6 8 14	1 3 5 6 8 14	1 2 4 7 11 15
1 4 6 7 9 13	1 4 6 7 9 13	1 4 6 7 9 13	1 4 6 7 9 13	1 2 3 8 12 16
1 4 10 12 14 16	1 4 10 12 14 16	1 4 10 12 14 16	1 4 10 12 14 16	1 6 7 8 9 13
1 8 11 12 13 15	1 8 11 12 13 15	1 8 11 12 13 15	1 8 11 12 13 15	2 5 7 8 10 14
2 3 4 5 12 13	2 3 4 5 12 13	2 3 4 5 12 13	2 3 4 5 12 13	3 5 6 8 11 15
2 4 6 11 14 15	2 4 6 11 14 15	2 4 6 11 14 15	2 4 6 11 14 15	4 5 6 7 12 16
2 6 8 10 13 16	2 6 8 10 13 16	2 6 8 10 13 16	2 6 8 10 13 16	1 5 10 11 12 13
2 7 8 9 12 14	2 7 8 9 12 14	2 7 8 9 12 14	2 7 8 9 12 14	2 6 9 11 12 14
3 4 8 9 10 15	3 4 8 9 10 15	3 4 8 9 10 15	3 4 8 9 10 15	3 7 9 10 12 15
3 6 9 11 12 16	3 6 9 11 12 16	3 6 9 11 12 16	3 6 9 11 12 16	4 8 9 10 11 16
3 7 10 11 13 14	3 7 10 11 13 14	3 7 10 11 13 14	3 7 10 11 13 14	1 5 9 14 15 16
4 5 7 8 11 16	4 5 7 8 11 16	4 5 7 8 11 16	4 5 7 8 11 16	2 6 10 13 15 16
5 6 7 10 12 15	5 6 7 10 12 15	5 6 7 10 12 15	5 6 7 10 12 15	3 7 11 13 14 16
5 9 13 14 15 16	5 9 13 14 15 16	5 9 13 14 15 16	5 9 13 14 15 16	4 8 12 13 14 15

1 2 3 6 11 13	1 2 3 6 11 13	1 2 3 6 11 13	1 2 3 6 11 13	2 3 4 6 11 16
1 2 5 12 14 15	1 2 5 12 14 15	1 2 5 12 14 15	1 2 5 12 14 15	1 3 4 8 9 15
1 3 4 5 7 10	1 3 4 5 7 10	1 3 4 5 7 10	1 3 4 5 7 10	1 2 4 5 12 14
1 4 9 11 15 16	1 4 9 11 15 16	1 4 9 11 15 16	1 4 9 11 15 16	1 2 3 7 10 13
1 6 7 8 12 16	1 6 7 8 12 16	1 6 7 8 12 16	1 6 7 8 12 16	1 5 6 8 10 16
1 8 9 10 13 14	1 8 9 10 13 14	1 8 9 10 13 14	1 8 9 10 13 14	2 5 9 10 11 15
2 3 5 8 9 16	2 3 5 8 9 16	2 3 5 8 9 16	2 3 5 8 9 16	3 8 10 11 12 14
2 4 6 9 10 12	2 4 6 9 10 12	2 4 6 9 10 12	2 4 6 9 10 12	4 5 7 8 11 13
2 4 7 13 14 16	2 4 7 13 14 16	2 4 7 13 14 16	2 4 7 13 14 16	1 7 9 11 12 16
2 7 8 10 11 15	2 7 8 10 11 15	2 7 8 10 11 15	2 7 8 10 11 15	2 6 7 8 12 15
3 4 8 11 12 14	3 4 8 11 12 14	3 4 8 11 12 14	3 4 8 11 12 14	3 5 6 7 9 14
3 6 10 14 15 16	3 6 10 14 15 16	3 6 10 14 15 16	3 6 10 14 15 16	4 6 9 10 12 13
3 7 9 12 13 15	3 7 9 12 13 15	3 7 9 12 13 15	3 7 9 12 13 15	1 6 11 13 14 15
4 5 6 8 13 15	4 5 6 8 13 15	4 5 6 8 13 15	4 5 6 8 13 15	2 8 9 13 14 16
5 6 7 9 11 14	5 6 7 9 11 14	5 6 7 9 11 14	5 6 7 9 11 14	3 5 12 13 15 16
5 10 11 12 13 16	5 10 11 12 13 16	5 10 11 12 13 16	5 10 11 12 13 16	4 7 10 14 15 16

1 2 6 7 10 14	1 2 6 9 14 16	1 2 6 7 10 14	1 2 6 9 14 16	2 3 7 8 9 11
1 2 9 12 13 16	1 2 7 10 12 13	1 2 9 12 13 16	1 2 7 10 12 13	1 4 5 6 9 11
4 6 8 9 14 16	4 6 7 8 10 14	4 6 8 9 14 16	4 6 7 8 10 14	1 2 10 11 14 16
4 7 8 10 12 13	4 8 9 12 13 16	4 7 8 10 12 13	4 8 9 12 13 16	1 2 9 12 13 15
1 3 4 6 12 15	1 3 4 13 14 15	1 3 4 13 14 15	1 3 4 6 12 15	1 4 7 8 10 12
1 4 5 11 13 14	1 4 5 6 11 12	1 4 5 6 11 12	1 4 5 11 13 14	2 3 5 6 10 12
2 3 8 13 14 15	2 3 6 8 12 15	2 3 6 8 12 15	2 3 8 13 14 15	3 4 9 12 14 16
2 5 6 8 11 12	2 5 8 11 13 14	2 5 8 11 13 14	2 5 6 8 11 12	3 4 10 11 13 15
1 3 8 10 11 16	1 3 7 8 9 11	1 3 8 10 11 16	1 3 7 8 9 11	1 3 5 8 13 14
1 5 7 8 9 15	1 5 8 10 15 16	1 5 7 8 9 15	1 5 8 10 15 16	2 4 6 7 13 14
2 3 4 7 9 11	2 3 4 10 11 16	2 3 4 7 9 11	2 3 4 10 11 16	5 7 9 10 13 16
2 4 5 10 15 16	2 4 5 7 9 15	2 4 5 10 15 16	2 4 5 7 9 15	5 7 11 12 14 15
3 5 6 9 10 13	3 5 6 7 13 16	3 5 6 9 11 13	3 5 6 7 13 16	1 3 6 7 15 16
3 5 7 12 14 16	3 5 9 10 12 14	3 5 7 12 14 16	3 5 9 10 12 14	2 4 5 8 15 16
6 7 11 13 15 16	6 9 10 11 13 15	6 7 11 13 15 16	6 9 10 11 13 15	6 8 9 10 14 15
9 10 11 12 14 15	7 11 12 14 15 16	9 10 11 12 14 15	7 11 12 14 15 16	6 8 11 12 13 16

Here we give the 2 - (71,15,3) designs D_1^*, D_2^*, D_3^* and D_4^*, which are constructed in Section 6.1. Together with their duals they form the eight 2 - (71,15,3) designs of Theorem 6.1.5.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	2	3	16	17	18	19	20	21	22	23	24	25	26	27
1	2	3	28	29	30	31	32	33	34	35	36	37	38	39
1	4	5	16	17	28	29	40	41	42	43	44	45	46	47
1	4	5	18	19	30	31	48	49	50	51	52	53	54	55
1	6	7	16	17	30	31	56	57	58	59	60	61	62	63
1	6	7	18	19	28	29	64	65	66	67	68	69	70	71
1	8	9	20	21	32	33	40	41	48	49	56	57	64	65
1	8	9	22	23	34	35	42	43	50	51	58	59	66	67
1	10	11	20	21	34	35	44	45	52	53	60	61	68	69
1	10	11	22	23	32	33	46	47	54	55	62	63	70	71
1	12	13	24	25	36	37	40	41	50	51	60	61	70	71
1	12	13	26	27	38	39	42	43	48	49	62	63	68	69
1	14	15	24	25	38	39	44	45	54	55	56	57	66	67
1	14	15	26	27	36	37	46	47	52	53	58	59	64	65
2	4	6	20	22	36	38	40	42	52	54	56	58	68	70
2	4	6	21	23	37	39	41	43	53	55	57	59	69	71
2	5	7	20	22	37	39	44	46	48	50	60	62	64	66
2	5	7	21	23	36	38	45	47	49	51	61	63	65	67
2	8	10	24	26	28	30	40	46	48	54	59	61	67	69
2	8	10	25	27	29	31	41	47	49	55	58	60	66	68
2	9	11	24	26	29	31	43	45	51	53	56	62	64	70
2	9	11	25	27	28	30	42	44	50	52	57	63	65	71
2	12	14	16	18	32	34	40	44	51	55	59	63	64	68
2	12	14	17	19	33	35	41	45	50	54	58	62	65	69
2	13	15	16	18	33	35	42	46	49	53	57	61	66	70
2	13	15	17	19	32	34	43	47	48	52	56	60	67	71
3	4	7	24	27	32	35	40	42	53	55	60	62	65	67
3	4	7	25	26	33	34	41	43	52	54	61	63	64	66
3	5	6	24	27	33	34	44	46	49	51	56	58	69	71
3	5	6	25	26	32	35	45	47	48	50	57	59	68	70
3	8	11	16	19	37	38	41	44	49	52	59	62	67	70
3	8	11	17	18	36	39	40	45	48	53	58	63	66	71
3	9	10	16	19	36	39	43	46	51	54	57	60	65	68
3	9	10	17	18	37	38	42	47	50	55	56	61	64	69
3	12	15	20	23	29	30	40	47	51	52	57	62	66	69
3	12	15	21	22	28	31	41	46	50	53	56	63	67	68
3	13	14	20	23	28	31	43	44	48	55	58	61	65	70
3	13	14	21	22	29	30	42	45	49	54	59	60	64	71
4	8	13	18	23	25	30	34	37	45	46	56	62	65	68
4	8	13	19	22	24	31	35	36	44	47	57	63	64	69
4	9	12	18	21	26	30	33	39	44	47	58	60	67	70
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NOTATIONS

A^*	hermitian transpose of matrix A.
$(A)_{ij}$	ij-th entry of matrix A
(b_{ij})	matrix with entries b_{ij}.
$\lambda_1(A) \geq \ldots \geq \lambda_n(A)$	eigenvalues (if real) of matrix A.
$A \otimes B$	Kronecker product of matrices A and B.
$\mathrm{diag}(a_1,\ldots,a_n)$	diagonal matrix with diagonal entries $a_1,\ldots,a_n$.
I or I_n	$n \times n$ identity matrix.
J	all-one matrix.
J_n	$n \times n$ all-one matrix.
j	all-one vector.
$\|u\|$	length of vector u.
$< u_1,\ldots,u_n >$	linear span of vectors $u_1,\ldots,u_n$.
$< u_1,\ldots,u_n >^{\perp}$	orthogonal complement of $< u_1,\ldots,u_n >$.
$\bar{G}$	complement of graph G.
$\alpha(G)$	size of largest clique of G.
$\gamma(G)$	chromatic number of G.
$\omega(G)$	size of largest coclique of G.
$\lambda_1(G) \geq \ldots \geq \lambda_n(G)$	eigenvalues of G.
$f_i(G)$	multiplicity of eigenvalue $\lambda_i(G)$.
$L(G)$	line graph of G.
$D(G,G_1)$	incidence structure formed by G and G_1, see p. 17.
K_n	complete graph on n vertices.
$K_{\ell,m}$	complete bipartite graph on $\ell + m$ vertices.
$\rho(x,y)$	distance between vertices x and y, see p. 79.
$p_{ij}(x,y)$ or p_{ij}^k	intersection numbers of a graph, see p. 51 and p. 81.
$\lambda(e_1,e_2)$	distance between elements of an n-gon, see p. 52.
$p_{ijk}(L,x,y)$	see p. 58.
$\mathbb{Z}$	integers.
$\mathbb{N}$	positive integers.
$\mathbb{F}_q$	field with q elements.
$PG(n,q)$	n-dimensional projective geometry over $\mathbb{F}_q$.
$AG(n,q)$	n-dimensional affine geometry over $\mathbb{F}_q$.
δ_{ij}	1 if $i = j$; 0 if $i \neq j$.
$\lfloor x \rfloor$	lower integer part of real number x.
$\lceil x \rceil$	upper integer part of real number x.

INDEX

SAMENVATTING

Matrices en eigenwaarden zijn al vaak benut bij het bestuderen van eindige grafen en incidentiestructuren (designs). Dit gebeurt ook in dit proefschrift. Het uitgangspunt is een stelling over de eigenwaarden van gepartitioneerde matrices. Het toepassen van deze stelling op matrices geassocieerd met grafen of designs levert grenzen voor de grootte van deelgrafen, de afmetingen van subdesigns en de doorsnijdingsgetallen van designs. Het geval dat deze grenzen worden bereikt wordt ook behandeld. De genoemde stelling geeft bovendien aanleiding tot een nieuw bewijs voor de ongelijkheden van A.J. Hoffman, betreffende de eigenwaarden van gepartitioneerde matrices. Deze ongelijkheden leveren grenzen voor het kleuringsgetal van een graaf. Mede dankzij die grenzen is het mogelijk alle 4-kleurbare sterk reguliere grafen te bepalen.

De behandeling van het bovenstaande gebeurt in de eerste vier hoofdstukken. Hoofdstuk 5 behandelt de ongelijkheid $t \leq s^3$, alsook het geval van gelijkheid, voor veralgemeende zeshoeken van de orde (s,t), $s \neq 1$. Dit gebeurt met behulp van eigenwaardetechnieken. Dezelfde methode toegepast op veralgemeende vierhoeken levert nieuwe bewijzen voor een aantal bekende stellingen.

Met de hulp van eigenwaardemethoden kan men richtlijnen opstellen voor het construeren van designs en grafen met speciale eigenschappen. Langs deze weg worden er in Hoofdstuk 6 nieuwe 2-designs en sterk reguliere grafen geconstrueerd.

CURRICULUM VITAE

De schrijver van dit proefschrift werd op 29 december 1948 te Eindhoven geboren. In 1967 behaalde hij zijn H.B.S.-b diploma aan het Van der Puttlyceum te Eindhoven. Daarna studeerde hij aanvankelijk natuurkunde en later wiskunde aan de Technische Hogeschool te Eindhoven. In oktober 1975 behaalde hij het diploma wiskundig ingenieur. Gedurende de laatste 2 jaren van zijn studie was hij student-assistent bij het rekencentrum van de hogeschool. Zijn afstudeerwerk, dat werd verricht onder leiding van Prof.dr. J.J. Seidel, bestond uit het onderzoeken van relaties tussen sterke grafen en 2-designs. Sinds zijn afstuderen werkt hij aks wetenschappelijk ambtenaar aan de Technische Hogeschool te Eindhoven bij Prof.dr. J.J. Seidel.

STELLINGEN

behorende bij het proefschrift

Eigenvalue techniques in design and
graph theory

door
W.H. Haemers

STELLINGEN

1

Stel men wil boodschappen overseinen die gevormd worden met letters uit een alfabet A van n letters genummerd van 1 t/m n, waarvan sommige paren letters met elkaar verwarbaar zijn. Beschouw een n×n matrix M over een willekeurig lichaam, die voldoet aan $(M)_{i,i} \neq 0$ voor $i = 1, \ldots, n$ en $(M)_{i,j} = 0$ als de i^e en j^e letter van A niet verwarbaar zijn. Dan is de Shannon capaciteit van A, die een maat is voor de hoeveelheid foutloos overseinbare informatie, ten hoogste gelijk aan de rang van M.

Ref.: Willem Haemers, An upperbound for the Shannon capacity of a graph, Proc. Conf. Algebraic Methods in Graph Theory, Szeged, 1978 (te verschijnen).

2

Het antwoord op de vragen van Probleem 1, 2 en 3 uit [2] is ontkennend.

Ref.: [1] Willem Haemers, On some problems of Lovász concerning the Shannon capacity of a graph, IEEE Trans. Information Theory 25 (1979) 231 - 232.

[2] László Lovász, On the Shannon capacity of a graph, IEEE Trans. Information Theory 25 (1979) 1 - 7.

3

Zoals bekend verscheen een speciaal geval van de stelling van Turán reeds als opgave 28 in het tiende deel van "Wiskundige Opgaven met de Oplossingen" (1910). De aldaar afgedrukte oplossing is echter onjuist.

4

Er bestaat geen sterk reguliere graaf met 49 hoekpunten en graad 16.

Ref.: F.C. Bussemaker, W. Haemers, R. Mathon & H.A. Wilbrink, The non-existence of a strongly regular graph with parameters (49, 16, 3, 6) (in voorbereiding).

5

Het is bekend dat de verbindingsmatrix van een sterk reguliere graaf soms tevens de incidentiematrix van een symmetrisch 2-design is. Het is mogelijk dat op deze manier twee niet isomorfe grafen isomorfe 2-designs opleveren. Dit is echter niet mogelijk als een van de grafen een automorfismegroep heeft van oneven orde. Met de resultaten van het onderstaande rapport volgt hieruit dat er tenminste 15531 niet isomorfe 2-(36, 15, 6) designs zijn.

Ref.: F.C. Bussemaker, R. Mathon & J.J. Seidel, Tables of two-graphs, T.H.-rapport (in voorbereiding).

6

Voor het bestaan van een 3-($\frac{1}{2}k^2-\frac{1}{2}k+1$,k,2)design is nodig dat k-3 een twaalfvoud is.

7

Als er voor $n > 1$ een projectief vlak van de orde $2^{2n}-1$ bestaat, dan bestaat er een "near-square λ-linked design" met $2^{4n}-2^{2n}$ punten en $\lambda = 2^{4n-2} - 3.2^{3n-2} + - 2^{n-1}+1$, en is dus het antwoord op de eerste vraag van het Probleem uit §9 van onderstaand artikel, bevestigend.

Ref.: D.R. Woodall, Square λ-linked designs, Proc. London Math. Soc. 20 (1970) 669 - 687.

8

Laat $A = \begin{bmatrix} O & B \\ B^* & C \end{bmatrix}$ (B^* is der hermitisch geconjungeerde van B) een hermitische matrix zijn met grootste eigenwaarde λ^+ en kleinste eigenwaarde λ^-. Veronderstel B heeft afmetingen m×n en gemiddelde rijsom r. Dan geldt

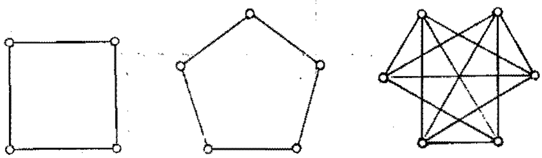
$$-\lambda^+\lambda^- \geq r^2m/n.$$

9

Beschouw een rechthoekig veld, betegeld met een eindig aantal rechthoekige tegels. Een tegel heet zuiderbuur van een andere tegel als de noordkant van de eerste op dezelfde lijn ligt als de zuidkant van de andere tegel. Een noord-zuid pad is een rijtje tegels waarbij elke tegel (behalve de eerste) zuiderbuur is van zijn voorganger. Analoog is een oost-west pad gedefinieerd. Er geldt nu dat elk paar tegels op een noord-zuid pad of op een oost-west pad ligt.

10

De enige samenhangende planaire sterk reguliere grafen zijn:



11

Meetkundig inzicht is een van de meest toepasbare dingen die men zich binnen de wiskunde kan verwerven. Daarom is het treurig dat het gewijzigde onderwijsprogramma het ruimtelijk voorstellingsvermogen van de doorsnee middelbare-school-verlater in tien jaar tijd met ongeveer een dimensie heeft doen dalen.

Eindhoven, 30 oktober 1979 W.H. Haemers.