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MATHEMATICS

Eigenvalues and Eigenvectors of Matrices in Idempotent Algebra

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Abstract—The eigenvalue problem for the matrix of a generalized linear operator is considered. In the case of irreducible matrices, the problem is reduced to the analysis of an idempotent analogue of the characteristic polynomial of the matrix. The eigenvectors are obtained as solutions to a homogeneous equation. The results are then extended to cover the case of an arbitrary matrix. It is shown how to build a basis of the eigensubspace of a matrix. In conclusion, an inequality for matrix powers and eigenvalues is presented, and some extremal properties of eigenvalues are considered.

1. INTRODUCTION

In many applied problems of analysis of engineering, economical, and industrial systems based on models and methods of idempotent algebras [1–5], eigenvalues and eigenvectors of the matrix of a generalized linear operator should be determined. As usually, by an eigenvalue of a square matrix A, we mean any number λ for which a nonzero (in the sense of the idempotent algebra) vector $\mathbf{x}$ exists such that the equality

$$A \otimes \mathbf{x} = \lambda \otimes \mathbf{x}$$

holds, where the $\otimes$ symbol denotes the multiplication operation in the algebra.

As the underlying object of an idempotent algebra, one uses to take a commutative semiring endowed with a certain idempotent addition, zero, and unity. At the same time, in applied problems we often deal with an idempotent semiring such that its every nonzero element has an inverse one with respect to the multiplication. Taking into account the group properties of the multiplication, such a semiring is sometimes referred to as an idempotent semifield [4].

It should be noted that, as a rule, the methods available in the literature for solving the eigenvalue problem in an idempotent algebra are applicable only to idempotent semifields rather than to the general case of an idempotent semiring. Moreover, as a rule, these methods are different from the classical approach based, to a certain degree, on analysis of the characteristic polynomial of the matrix.

The eigenvalue problem for semifields was solved for the first time in papers [1, 6, 7], where the eigenvalues and eigenvectors were found immediately from the equation $A \otimes \mathbf{x} = \lambda \otimes \mathbf{x}$ by using methods of graph theory. Similar solutions that additionally use the properties of the power series $I \oplus A \oplus A^2 \oplus \ldots$ have been obtained in [2–4]. In a rather general form, the eigenvalue problem has been solved in papers [5, 8]. In particular, it is shown there how a solution obtained for semifields can be extended to the case of an arbitrary idempotent semiring [8].

In this paper, we propose a new approach to solving the eigenvalue problem in the case of an idempotent semifield. This approach is based on employment of certain idempotent analogues of determinants [9] and characteristic polynomials of matrices. Just as in ordinary algebra, in the case of irreducible matrices, this allows one to reduce the existence problem for an eigenvalue to the problem of determination of roots of the characteristic polynomial. In idempotent algebras, this problem can be solved easily enough. This approach allows one to derive readily the well-known expression for the eigenvalue of an irreducible matrix by solving the characteristic equation without a cumbersome proof. It happens that in this case the eigenvectors of the matrix can be found by solving a certain homogeneous equation by methods proposed in [9].

In the paper, we, first, briefly introduce basic concepts of idempotent algebras, present some auxiliary results, including the general solution to a homogeneous linear equation. Then, we consider the problems of existence and uniqueness of an eigenvalue of an irreducible matrix and determine a general expression for determining this value, as well as the general form of an eigenvector. Then, these results are generalized to the case of an arbitrary (decomposable) matrix and the problem of determination of a basis of the eigen-subspace is discussed. In conclusion, we derive a useful inequality for powers of the matrix and its eigenvalue and consider some extremal properties of eigenvalues and eigenvectors of the matrix.

2. IDEMPOTENT ALGEBRA

Let $\mathbb{X}$ be a numeric set endowed with two operations, namely, those of addition $\oplus$ and multiplication $\otimes$. We assume that $(\mathbb{X}, \oplus, \otimes)$ is an idempotent semifield, i.e., a commutative semiring with zero and unity such that the addition is idempotent and, for any nonzero element, there exists its inverse with respect to the multiplication.

Denote the zero and unit elements of the semiring $(\mathbb{X}, \oplus, \otimes)$ by symbols 0 and 1, respectively. Let $\mathbb{X}_+ = \mathbb{X}\backslash\{0\}$. Then, for any $x \in \mathbb{X}_+$, there exists an inverse element x^{-1}. Moreover, we assume that $0^{-1} = 0$.

For any $x \in \mathbb{X}$ and $y \in \mathbb{X}_+$, we introduce the power x^y in the standard way. As usually, we set $x^0 = 1$, $0^y = 0$. In what follows, all powers are considered only in the sense of the idempotent algebra. However, for simplicity, we use conventional arithmetic operations when writing expressions in exponents.

Since the addition is idempotent, a relation $\leq$ of linear order is defined on $\mathbb{X}$ by the following rule: $x \leq y$ if and only if $x \oplus y = y$. Below, the relation signs are treated only in the sense of this linear order. Note that, in accordance with this order, $x \geq 0$ for any $x \in \mathbb{X}$.

Examples of semirings of this type are $(\mathbb{R} \cup \{-\infty\}, \max, +)$, $(\mathbb{R} \cup \{+\infty\}, \min, +)$, $(\mathbb{R}_+, \max, \times)$, and $(\mathbb{R}_+ \cup \{+\infty\}, \min, \times)$, where $\mathbb{R}$ is the set of all real numbers and $\mathbb{R}_+$ is the set of nonnegative real numbers.

In particular, in the semiring $(\mathbb{R} \cup \{-\infty\}, \max, +)$, the zero is $-\infty$ and the unit is the number 0. For any $x \in \mathbb{R}$, the inverse element x^{-1} is defined which is equal to $-x$ in the usual arithmetic. For any $x, y \in \mathbb{R}$, the power x^y is defined whose value corresponds to the arithmetic product xy. The relation of order is defined in the ordinary sense.

In the semiring $(\mathbb{R}_+ \cup \{+\infty\}, \min, \times)$, the zero is $+\infty$ and the unit is the number 1. The inverse element and the power are defined in the usual sense. The relation $\leq$ specifies the order that is reverse for the conventional linear order on $\mathbb{R}_+$.

3. DEFINITIONS AND AUXILIARY RESULTS

3.1. Matrix Algebra

For any matrices $A, B \in \mathbb{X}^{m\times n}$ and $C \in \mathbb{X}^{n\times l}$ and for any number $x \in \mathbb{X}$, the operations of matrix addition and multiplication, as well as the product of a matrix by a number are defined in the usual way, i.e., for any i, j, we have

$$\{A \otimes B\}_{ij} = \{A\}_{ij} \oplus \{B\}_{ij}, \quad \{B \otimes C\}_{ij} = \bigoplus_{k=1}^{n} \{B\}_{ik} \otimes \{C\}_{kj}, \quad \{x \otimes A\}_{ij} = x \otimes \{A\}_{ij}.$$

Matrix operations $\oplus$ and $\otimes$ possess the monotonicity property; i.e., for any matrices A, B, C, and D of appropriate size, the inequalities $A \leq C$ and $B \leq D$ imply the inequalities $A \oplus B \leq C \oplus D$ and $A \otimes B \leq C \otimes D$.

As usually, a square matrix is called diagonal if all nondiagonal entries are equal to zero and is triangular if all entries above (below) the diagonal are equal to zero. A matrix whose all entries are equal to zero are referred to as a zero matrix and is denoted by the symbol 0. The matrix $I = \mathrm{diag}(1, \ldots, 1)$ is the identity matrix.

A matrix A^- is pseudoinverse [1] for matrix A if $\{A^-\}_{ij} = \{A\}_{ji}^{-1}$ for any i, j. If $A, B \in \mathbb{X}_+^{n\times m}$, then the inequality $A \leq B$ obviously implies $A^- \geq B^-$.

A square matrix is called decomposable if it can be transformed into a block-triangular matrix by interchanging its rows and columns in the same way. Otherwise, the matrix is irreducible.

3.2. Linear Vector Space

For any two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{X}^n$, where $\mathbf{a} = (a_1, \ldots, a_n)^T$ and $\mathbf{b} = (b_1, \ldots, b_n)^T$, and any number $x \in \mathbb{X}$, the following operations are defined:

$$\mathbf{a} \oplus \mathbf{b} = (a_1 \oplus b_1, \ldots, a_n \oplus b_n)^T, \quad x \otimes \mathbf{a} = (x \otimes a_1, \ldots, x \otimes a_n)^T.$$

The zero vector is the vector $\mathbf{0} = (0, \ldots, 0)^T$.

The set of vectors $\mathbb{X}^n$ endowed with the operations $\oplus$ and $\otimes$ is referred to as a generalized linear vector space (or simply, a linear vector space).

We say that a vector $\mathbf{b}$ is linearly dependent of vectors $\mathbf{a}_1, \ldots, \mathbf{a}_m$ if $\mathbf{b}$ is a linear combination of these vectors; i.e., $\mathbf{b} = x_1 \otimes \mathbf{a}_1 \oplus \ldots \oplus x_m \otimes \mathbf{a}_m$, where $x_1, \ldots, x_m \in \mathbb{X}$.

The zero vector linearly depends on any system of vectors.

Two systems of vectors $\mathbf{a}_1, \ldots, \mathbf{a}_m$ and $\mathbf{b}_1, \ldots, \mathbf{b}_k$ are equivalent if each vector of one system linearly depends on vectors of the other system.

A system of vectors $\mathbf{a}_1, \ldots, \mathbf{a}_m$ is called linearly dependent if at least one its vector linearly depends on the other vectors; otherwise, it is linearly independent.

Consider a system of nonzero vectors $\mathbf{a}_1, \ldots, \mathbf{a}_m$ and denote the matrix with columns $\mathbf{a}_1, \ldots, \mathbf{a}_m$ by A. For any vector $\mathbf{a}_k$, $k = 1, \ldots, m$, we define the set of indices of its zero entries I_k. Let $\mathbf{a}'_k$ be the vector obtained from $\mathbf{a}_k$ by deleting all such coordinates and $A'_{(k)}$ be the matrix obtained from A by deleting the column $\mathbf{a}_k$, all rows with indices $i \in I_k$, and each column with index j such that $a_{ij} \neq 0$ for at least one $i \in I_k$.

The following propositions hold [9] (also see [2]).

Lemma 1. A system of vectors $\mathbf{a}_1, \ldots, \mathbf{a}_m \in \mathbb{X}^n \setminus \{\mathbf{0}\}$ is linearly independent if and only if $(A'_{(i)} \otimes (\mathbf{a}'^{-}_i \otimes A'_{(i)})^-)^- \otimes \boldsymbol{a}'_i \neq 1$ for any $i = 1, \ldots, m$.

Corollary 1. To construct a linearly independent subsystem equivalent to a system $\mathbf{a}_1, \ldots, \mathbf{a}_m$, it is sufficient to remove successively each vector $\mathbf{a}_i$, $i = 1, \ldots, m$, from this system such that $(A''_{(i)} \otimes (\mathbf{a}'^{-}_i \otimes A''_{(i)})^-)^- \otimes \boldsymbol{a}'_i = 1$, where matrix $A''_{(i)}$ is composed of the columns of $A'_{(i)}$ that are not deleted at this time.

3.3. Polynomials

By a polynomial in one variable x, we mean an expression

$$P(x) = a_0 \oplus \bigoplus_{m=1}^{n} a_m \otimes x^m,$$

where numbers $a_0, \ldots, a_n \in \mathbb{X}$ are coefficients of the polynomial.

Lemma 2. If $a_0 < 1$ and $a_m \neq 0$ for at least one $m > 0$, then the equation $P(x) = 1$ has a unique solution

$$x = \left(\bigoplus_{m=1}^{n} a_m^{1/m}\right)^{-1}.$$

Proof. It is clear that function $P(x)$ is continuous and takes values both greater than unity and less than unity. Hence, a solution to the equation $P(x) = 1$ exists. Taking into account that $P(x)$ is a monotone function, we can easily verify that the solution is unique.

Let x be the solution to the equation $P(x) = 1$. Then, for any $m = 1, \ldots, n$, the inequality $a_m \otimes x^m \leq 1$ holds. This inequality is equivalent to the inequality $x^{-1} \geq a_m^{1/m}$. Summing these inequalities and taking into account that, for at least one m, we have the equality $x^{-1} = a_m^{1/m}$, we obtain $x^{-1} = a_1 \oplus \ldots \oplus a_n^{1/n}$. This implies the required solution.

3.4. Square Matrices

Let $A = (a_{ij}) \in \mathbb{X}^{n \times n}$ be an arbitrary square matrix. An integer nonnegative power of matrix A is defined in the usual way, i.e., $A^0 = E$, $A^{k+l} = A^k \otimes A^l$ for any $k, l = 0, 1, 2, \ldots$.

A number λ is an eigenvalue of matrix A if there exists a vector $\mathbf{x} \neq 0$ such that

$$A \otimes \mathbf{x} = \lambda \otimes \mathbf{x}. \tag{1}$$

Any vector $\mathbf{x} \neq 0$ satisfying this equation is referred to as an eigenvector of A corresponding to the eigenvalue λ.

The set of all eigenvectors of matrix A, which correspond to the same eigenvalue λ, together with the zero vector is a linear subspace. This subspace is referred to as the eigensubspace associated with the eigenvalue λ.

For any matrix A, we define the following functions of its entries:

$$\mathrm{tr}A = \bigoplus_{i=1}^{n} a_{ii}, \quad \mathrm{Tr}A = \bigoplus_{m=1}^{n} \mathrm{tr}A^m.$$

It is clear that, for any matrices A and B and any number x, we have

$$\mathrm{tr}(A \oplus B) = \mathrm{tr}A \oplus \mathrm{tr}B, \quad \mathrm{tr}(x \otimes A) = x \otimes \mathrm{tr}A.$$

In studying linear equations in idempotent algebra, the function $\mathrm{Tr}A$ plays the role of the determinant of the matrix in the sense that its value may be used in order to answer the question whether a homogeneous linear equation has only a trivial solution or it has other solutions as well [9].

4. HOMOGENEOUS LINEAR EQUATION

Let a matrix $A \in \mathbb{X}^{n \times n}$ be given. The equation

$$A \otimes \mathbf{x} = \mathbf{x} \tag{2}$$

is referred to as a homogeneous equation with respect to the unknown vector $\mathbf{x} \in \mathbb{X}^n$.

The solution $\mathbf{x} = 0$ to this equation is called trivial.

It is clear that all solutions to a homogeneous equation form a linear space.

To describe the solutions to Eq. (2), we introduce the following notation [3, 4, 9]. First, for any matrix A, we define the matrix

$$A^+ = E \oplus A \oplus \ldots \oplus A^{n-1}.$$

For any $i = 1, \ldots, n$, we denote the ith column of matrix A^+ by $\mathbf{a}_i^+$ and the diagonal element in the ith row of matrix A^m by a_{ii}^m.

Let $\mathrm{Tr}A = 1$. By A^* we denote the matrix with the columns

$$\mathbf{a}_i^* = \begin{cases} \mathbf{a}_i^+, & \text{if } a_{ii}^m = 1 \text{ for some } m = 1, \ldots, n, \\ 0, & \text{otherwise,} \end{cases}$$

for any $i = 1, \ldots, n$. We assume that $A^* = 0$ if $\mathrm{Tr}A \neq 1$.

4.1. Irreducible Matrices

One can readily show (see, e.g., [9]) that any vector $\mathbf{x}$ that is a nontrivial solution to the homogeneous equation (2) with an irreducible matrix A has no zero entries.

Moreover, the following result is valid [9].

Lemma 3. Let $\mathbf{x}$ be the general solution to the homogeneous equation (2) with an irreducible matrix A. Then, the following assertions hold:

(1) if $\mathrm{Tr}A = 1$, then $\mathbf{x} = A^* \otimes \mathbf{v}$ for any $\mathbf{v} \in \mathbb{X}^n$;

(2) if $\mathrm{Tr}A \neq 1$, then there is no solution other than the trivial one.

Note that, defining $A^* = 0$ in the case $\mathrm{Tr}A \neq 1$, we can always write the general solution in the form $\mathbf{x} = A^* \otimes \mathbf{v}$ regardless of the value of $\mathrm{Tr}A$.

4.2. Decomposable Matrices

Suppose that matrix A is decomposable. Making a permutation of rows together with the same permutation of columns, we can reduce this matrix to the block-triangular normal form

$$A = \begin{pmatrix} A_{11} & 0 & \dots & 0 \\ A_{21} & A_{22} & & 0 \\ \vdots & \vdots & \ddots & \\ A_{s1} & A_{s2} & \dots & A_{ss} \end{pmatrix}, \tag{3}$$

where A_{ii} is either a decomposable matrix or the zero matrix of size $n_i \times n_i$ and A_{ij} is an arbitrary matrix of size $n_i \times n_j$ for any $j < i$, $i = 1, \dots, s$, under the condition that $n_1 + \dots + n_s = n$.

Suppose that matrix A is reduced to the normal form (3). The totality of rows (columns) of the matrix A corresponding to each diagonal block A_{ii} will be referred to as a horizontal (vertical) line. Note that $\mathrm{Tr}A = \mathrm{Tr}A_{11} \oplus \dots \oplus \mathrm{Tr}A_{ss}$.

By I_0 we denote the set of indices i such that the equality $\mathrm{Tr}A_{ii} = 1$ holds, and by I_1 we denote the set of indices such that $\mathrm{Tr}A_{ii} > 1$.

First, suppose that $I_1 = \emptyset$. It is clear that any matrix A can be represented in the form $A = T \oplus D$, where T is a block strictly triangular matrix and D be a block-diagonal matrix, i.e.,

$$T = \begin{pmatrix} 0 & \dots & \dots & 0 \\ A_{21} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ A_{s1} & \dots & A_{s,s-1} & 0 \end{pmatrix}, \quad D = \begin{pmatrix} A_{11} & & 0 \\ & \ddots & \\ 0 & & A_{ss} \end{pmatrix}.$$

We define the following auxiliary matrices:

$$D^+ = \mathrm{diag}(A_{11}^+, \dots, A_{ss}^+), \quad C = D^+ \otimes T, \quad D^* = \mathrm{diag}(A_{11}^*, \dots, A_{ss}^*).$$

It is easy to verify that $C^+ = I \oplus C \oplus \dots \oplus C^{s-1}$. Moreover, matrix C^+ is of the lower block-triangular form such that the size of its blocks C_{ij}^+ coincides with the size of the corresponding blocks A_{ij} of matrix A.

If $I_1 = \emptyset$, then we consider the matrix $\overline{A}$ obtained by replacing all blocks of columns $i \in I_1$ of matrix A with zero columns. We denote the blocks of matrix $\overline{A}$ by A_{ij}.

Represent matrix $\overline{A}$ in the form $\overline{A} = \overline{T} \oplus \overline{D}$, where $\overline{T}$ is a block strictly triangular matrix and $\overline{D}$ is a block-diagonal matrix. We denote

$$\overline{D}^+ = \mathrm{diag}(\overline{A}_{11}^+, \dots, \overline{A}_{ss}^+), \quad \overline{C} = \overline{D}^+ \otimes \overline{T}, \quad \overline{D}^* = \mathrm{diag}(\overline{D}_{11}^*, \dots, \overline{D}_{ss}^*),$$

where the diagonal blocks of the matrix $\overline{D}^*$ are defined for any $j = 1, \dots, s$ as follows:

$$\overline{D}_{jj}^* = \begin{cases} 0, & \text{if } j \in I_0 \text{ and } \overline{C}_{ij}^+ \neq 0 \text{ for at least one } i \in I_1, \\ \overline{A}_{jj}^*, & \text{otherwise.} \end{cases}$$

All solutions to the homogeneous equation can be found in the following way [9].

Lemma 4. Let $\mathbf{x}$ be the general solution to the homogeneous equation (2) with a matrix A presented in form (3). Then, the following assertions are valid:

(1) if $\mathrm{Tr}A < 1$, then the equation has only the trivial solution $\mathbf{x} = 0$;

(2) if $\mathrm{Tr}A = 1$, then $\mathbf{x} = C^{+} \otimes D^{*} \otimes \mathbf{v}$ for any $\mathbf{v} \in \mathbb{X}^n$;

(3) if $\mathrm{Tr}A > 1$, then $\mathbf{x} = \bar{C}^{+} \otimes \bar{D}^{*} \otimes \mathbf{v}$ for any $\mathbf{v} \in \mathbb{X}^n$; moreover, the equation has only the trivial solution $\mathbf{x} = 0$ if $I_0 = \emptyset$.

One can easily see that the general solution to (2) may be represented in the form $\mathbf{x} = \bar{C}^{+} \otimes \bar{D}^{*} \otimes \mathbf{v}$ regardless of the value of $\mathrm{Tr}A$.

Taking into account Lemma 4, one can easily verify that the following propositions hold.

Corollary 2. Equation (2) has a nontrivial solution only if $I_0 \neq \emptyset$, i.e., if $\mathrm{Tr}A_{ii} = 1$ for at least one $i = 1, \ldots, s$.

Corollary 3. Equation (2) has a nontrivial solution if and only if $\bar{D}^{*} \neq 0$, i.e., if $\bar{D}_{ii}^{*} \neq 0$ for at least one $i = 1, \ldots, s$.

Corollary 4. If $\mathrm{Tr}A = 1$, then Eq. (2) has a nontrivial solution.

4.3. Space of Solutions

Since the general solution to the homogeneous equation has the form $\mathbf{x} = B \otimes \mathbf{v}$, where $B \in \mathbb{X}^{n \times n}$ is a matrix and $\mathbf{v} \in \mathbb{X}^n$ is a general vector, the space of solutions to the equation obviously coincides with the linear hull of columns of the matrix B. However, these columns are not necessarily linear independent.

To construct a linear independent system of vectors whose linear hull coincides with the space of solutions to the equation (i.e., a basis of the solution space), it is sufficient to apply a procedure based on Lemma 1 and its corollary.

5. EIGENVALUES AND EIGENVECTORS OF A MATRIX

We show how, starting from the solution to the homogeneous equation, we can find all eigenvalues and eigenvectors of the matrix.

For any matrix A, the function $\mathrm{Tr}(\lambda^{-1} \otimes A)$ of the numerical parameter λ will be referred to as the characteristic polynomial of the matrix A and the equation

$$\mathrm{Tr}(\lambda^{-1} \otimes A) = 1 \tag{4}$$

is the characteristic equation of this matrix.

5.1. Irreducible Matrices

For any matrix A and any number λ, we introduce the notation $A_\lambda = \lambda^{-1} \otimes A$ and $A_\lambda^{*} = (A_\lambda)^{*}$.

Theorem 1. A number λ is an eigenvalue of an irreducible matrix A if and only if this number is a root of the characteristic equation (4).

Proof. Represent (1) as the equation $A_\lambda \otimes \mathbf{x} = \mathbf{x}$. By virtue of Lemma 3, this equation has a solution $\mathbf{x} \neq 0$ if and only if $\mathrm{Tr}A_\lambda = \mathrm{Tr}(\lambda^{-1} \otimes A) = 1$; i.e., if λ is a root of the characteristic equation of the matrix A.

Corollary 5. For any irreducible matrix A, there exists a unique eigenvalue

$$\lambda = \bigoplus_{m=1}^{n} \mathrm{tr}^{1/m}(A^m). \tag{5}$$

Proof. Consider the characteristic polynomial of matrix A. We represent it in the form

$$\mathrm{Tr}(\lambda^{-1} \otimes A) = \bigoplus_{m=1}^{n} \mathrm{tr}(\lambda^{-m} \otimes A^m) = \bigoplus_{m=1}^{n} \lambda^{-m} \otimes \mathrm{tr}A^m.$$

Now, we make use of Lemma 2.

Corollary 6. Any eigenvector of an irreducible matrix A, which is associated with an eigenvalue λ, has the form $\mathbf{x} = A_\lambda^* \otimes \mathbf{v}$, where $\mathbf{v} \in \mathbb{X}_+^n$.

Proof. Taking into account that the eigenvector of the matrix A satisfies the equation $A_\lambda \otimes \mathbf{x} = \mathbf{x}$, by virtue of Lemma 3, we obtain the required result.

5.2. Decomposable Matrices

Suppose that A is a decomposable matrix of form (3), λ is a number, and $A_\lambda = \lambda^{-1} \otimes A$. Just as when studying the homogeneous equation, we introduce a matrix $\overline{A}_\lambda$ and represent it in the form $\overline{A}_\lambda = \overline{T}_\lambda \oplus \overline{D}_\lambda$, where $\overline{T}_\lambda$ is a block strictly triangular matrix and $\overline{D}_\lambda$ is a block-diagonal matrix. Next, we define matrices $\overline{D}_\lambda^+$, $\overline{C}_\lambda$, and $\overline{D}_\lambda^*$.

Theorem 2. Suppose that a matrix A is represented in form (3) and λ_i is an eigenvalue of the matrix A_{ii}, $i = 1, \ldots, s$. Then, the following assertions are valid:

(1) all eigenvalues of the matrix A are among the numbers $\lambda_1, \ldots, \lambda_s$;

(2) a number λ is an eigenvalue of the matrix A if and only if $\overline{D}_\lambda^* \neq 0$;

(3) the matrix A has at least one eigenvalue $\lambda = \lambda_1 \oplus \ldots \oplus \lambda_s$.

Proof. We represent (1) in the form of the equation $A_\lambda \otimes \mathbf{x} = \mathbf{x}$. By virtue of Corollary 2, the equation has a nontrivial solution only if $\mathrm{Tr}(\lambda^{-1} \otimes A_{ii}) = 1$ for some i. By Theorem 1, this means that $\lambda = \lambda_i$ is an eigenvalue of A_{ii}.

The other two assertions immediately follow from Corollaries 3 and 4.

Corollary 7. Any eigenvector of matrix A associated with an eigenvalue λ has the form $\mathbf{x} = \overline{C}_\lambda^+ \otimes \overline{D}_\lambda^* \otimes \mathbf{v}$, where $\mathbf{v} \in \mathbb{X}_+^n$.

Proof. The result immediately follows from Theorem 2 and Lemma 4.

5.3. Eigensubspace of the Matrix

As has been established above, the eigensubspace of a matrix, which corresponds to an eigenvalue of this matrix, consists of the vectors of the form $\mathbf{x} = B \otimes \mathbf{v}$, where $\mathbf{v}$ is an arbitrary vector. To find all linearly independent eigenvectors, it is sufficient to consider the columns of the matrix B and apply a procedure similar to the procedure for determination of a basis of the space of solutions of a homogeneous equation.

6. INEQUALITIES FOR POWERS OF A MATRIX

In paper [9], the following result is obtained and used for analysis of solutions to linear equations.

Lemma 5. If $\mathrm{Tr}A > 0$, then, for any integer $k \geq 0$, we have

(1) if $\mathrm{Tr}A \leq 1$, then $A^k \leq (\mathrm{Tr}A)^{(k+1)/n-1} \otimes A^+$;

(2) if $\mathrm{Tr}A > 1$, then $A^k \leq (\mathrm{Tr}A)^k \otimes A^+$.

We show that, actually, a more exact inequality holds.

Lemma 6. For any matrix A and any integer $k \geq 0$, the following inequality holds:

$$A^k \leq \bigoplus_{l=0}^{n-1} \lambda^{k-l} \otimes A^l, \qquad (6)$$

where number λ is determined by formula (5).

Proof. One can easily verify that the inequality holds for $k < n$.

Let $k \geq n$. We show that the inequality holds for each entry a_{ij}^k of matrix A^k. We represent a_{ij}^k in the form

$$a_{ij}^k = \bigoplus_{i_1=1}^{n} \ldots \bigoplus_{i_{k-1}=1}^{n} a_{ii_1} \otimes a_{i_1 i_2} \otimes \ldots \otimes a_{i_{k-1} j}.$$

Consider an arbitrary product $S_{ij} = a_{ii_1} \otimes a_{i_1 i_2} \otimes \ldots \otimes a_{i_{k-1} j}$. If zero is present among the factors a_{ii_1}, $a_{i_1 i_2}$, …, $a_{i_{k-1} j}$, then $S_{ij} = 0$. Obviously, in this case, $S_{ij} \le \lambda^{k-l} \otimes a_{ij}^l$.

Let $S_{ij} > 0$. We regroup the factors in the product S_{ij} as follows. First, we collect all cyclic products that consist of $m = 1$ factor. Let $\alpha_1 \ge 0$ be the number of such products. Among the other factors, we take the cyclic products consisting of $m = 2$ factors and denote their number by α_2. We continue this procedure further for all $m \le n$.

Taking into account that any cyclic product of m factors does not exceed the value $\mathrm{tr} A^m$, we obtain the inequality

$$S_{ij} \le \bigoplus_{\substack{m=1 \\ \alpha_m > 0}}^{n} \mathrm{tr}^{\alpha_m}(A^m) \otimes S'_{ij},$$

where S'_{ij} is a product that contains no cycles and consists of a number l of factors, $0 \le l < n$. Obviously, $\alpha_1 + 2\alpha_2 + \ldots + n\alpha_n + l = k$.

One can easily see that $l = 0$ if and only if $i = j$. Then, setting $S'_{ij} = 1$ if $l = 0$, we arrive at the inequality $S'_{ij} \le a_{ij}^l$.

Denote $\beta_m = m\alpha_m$. Taking into account that $\beta_1 + \ldots + \beta_n = k - l$, we obtain

$$\bigotimes_{\substack{m=1 \\ \alpha_m > 0}}^{n} \mathrm{tr}^{\alpha_m}(A^m)$$

$$= \bigotimes_{\substack{m=1 \\ \alpha_m > 0}}^{n} \mathrm{tr}^{\beta_m/m}(A^m) \le \left(\bigoplus_{m=1}^{n} \mathrm{tr}^{1/m}(A^m) \right)^{\beta_1 + \ldots + \beta_n} = \lambda^{k-l}.$$

Thus, we have the inequality $S_{ij} \le \lambda^{k-l} \otimes a_{ij}^l$. This implies that, for any $i, j = 1, \ldots, n$, we have

$$a_{ij}^k = \bigoplus_{i_1=1}^{n} \ldots \bigoplus_{i_{k-1}=1}^{n} a_{ii_1} \otimes a_{i_1 i_2} \otimes \ldots \otimes a_{i_{k-1} j} \le \bigoplus_{l=0}^{n-1} \lambda^{k-l} \otimes a_{ij}^l.$$

Corollary 8. For any integer $k \ge 0$, the following inequality holds:

$$\mathrm{tr} A^k \le \lambda^k. \tag{7}$$

Proof. One can easily see that, for $\lambda = 0$, the inequality becomes an equality. Suppose that $\lambda > 0$.

It follows from (5) that inequality (7), hence, the inequality $\lambda^{-k} \otimes \mathrm{tr} A^k \le 1$ hold if $0 \le k < n$. Then, taking into account (6), we have for all $k \ge n$

$$\mathrm{tr} A^k \le \lambda^k \otimes \bigoplus_{l=0}^{n-1} \lambda^{-l} \otimes \mathrm{tr} A^l \le \lambda^k.$$

To prove the next proposition, we make use of inequality (7).

Lemma 7. Let A be a matrix in the normal form (3), $\lambda_1, \ldots, \lambda_s$ be eigenvalues of its diagonal blocks, and $\lambda = \lambda_1 \oplus \ldots \oplus \lambda_s$. Then, the value of λ is determined by expression (5).

Proof. By virtue of inequality (7), taking into account that any integer power $m \ge 0$ of matrix A has the lower block-triangular form, we obtain

$$\lambda = \bigoplus_{i=1}^{s} \lambda_i = \bigoplus_{i=1}^{s} \bigoplus_{m=1}^{n_i} \mathrm{tr}^{1/m}(A_{ii}^m) = \bigoplus_{i=1}^{s} \bigoplus_{m=1}^{n} \mathrm{tr}^{1/m}(A_{ii}^m) = \bigoplus_{m=1}^{n} \mathrm{tr}^{1/m}(A^m).$$

On the basis of the proven proposition and Theorem 2, we may conclude that any matrix A, decomposable or irreducible, has the eigenvalue λ determined by formula (5).

7. EXTREMAL PROPERTIES OF EIGENVALUES

One extremal property of eigenvalues of an irreducible matrix was established by using the explicit form (5) of the eigenvalues in paper [10]. Now, we are going to derive this property, as well as other similar results applicable, for instance, in solving the problems of approximation of matrices [10], by a more general approach without using (5).

First, note that, for any vectors $\mathbf{x}, \mathbf{y} \in \mathbb{X}_+^n$, the following inequality holds:

$$\mathbf{x} \otimes \mathbf{y}^- \geq (\mathbf{x}^- \otimes \mathbf{y})^{-1} \otimes I. \tag{8}$$

Indeed, since $\mathbf{x}^- \otimes \mathbf{y} = x_1^{-1} \otimes y_1 \oplus \ldots \oplus x_n^{-1} \otimes y_n \geq x_i^{-1} \otimes y_i$ for any $i = 1, \ldots, n$, we have

$$\mathbf{x} \otimes \mathbf{y}^- \geq \operatorname{diag}(x_1 \otimes y_1^{-1}, \ldots, x_n \otimes y_n^{-1}) \geq (\mathbf{x}^- \otimes \mathbf{y})^{-1} \otimes I.$$

The following propositions hold, where the symbol min is considered in the sense of the relation $\leq$ on $\mathbb{X}$, which is induced by the idempotent addition.

7.1. Irreducible Matrices

Lemma 8. Let A be an irreducible matrix and λ its eigenvalue. Then, the following equalities hold:

$$\min_{\mathbf{x} \in \mathbb{X}_+^n} \mathbf{x}^- \otimes A \otimes \mathbf{x} = \lambda, \tag{9}$$

$$\min_{\mathbf{x} \in \mathbb{X}_+^n} (A \otimes \mathbf{x})^- \otimes \mathbf{x} = \lambda^{-1}. \tag{10}$$

Moreover, the minimum is attained on any eigenvector of matrix A.

Proof. Let $\mathbf{x}_0$ be an eigenvector of matrix A associated with the eigenvalue λ. First, we prove (9). Using inequality (8), we obtain

$$\mathbf{x}^- \otimes A \otimes \mathbf{x} = \mathbf{x}^- \otimes A \otimes \mathbf{x} \otimes \mathbf{x}_0^- \otimes \mathbf{x}_0 \geq \mathbf{x}^- \otimes A \otimes \mathbf{x}_0 \otimes (\mathbf{x}^- \otimes \mathbf{x}_0)^{-1} = \lambda.$$

It remains to verify that $\mathbf{x}^- \otimes A \otimes \mathbf{x} = \lambda$ for $\mathbf{x} = \mathbf{x}_0$.

To prove (10), we apply inequality (8) in the form $\mathbf{x} \otimes \mathbf{x}^- \geq I$. We have

$$(A \otimes \mathbf{x})^- \otimes \mathbf{x} \geq (A \otimes \mathbf{x}_0 \otimes \mathbf{x}_0^- \otimes \mathbf{x})^- \otimes \mathbf{x} = (\mathbf{x}_0^- \otimes \mathbf{x})^{-1} \otimes (A \otimes \mathbf{x}_0)^- \otimes \mathbf{x} = \lambda^{-1}.$$

Taking into account that $(A \otimes \mathbf{x}_0)^- \otimes \mathbf{x}_0 = \lambda^{-1}$, we obtain the required result.

7.2. Decomposable Matrices

We are going to extend the results of Lemma 8 to the case of decomposable matrices. First, consider assertion (9).

Let a matrix A be of form (3). Introduce a block-triangular matrix $\hat{A}$ with blocks $\hat{A}_{ij} = \lambda_i^{-1} \otimes A_{ij}$ for any $i = 1, \ldots, s, j = 1, \ldots, i$, where λ_i is an eigenvalue of matrix A_{ii}. Assign $\lambda = \lambda_1 \oplus \ldots \oplus \lambda_s$.

Just as above, we represent $\hat{A}$ in the form $\hat{A} = \hat{T} \oplus \hat{D}$ and define matrices

$$\hat{D}^+ = \operatorname{diag}(\hat{A}_{11}^+, \ldots, \hat{A}_{ss}^+), \quad \hat{C} = \hat{D}^+ \otimes \hat{T}, \quad \hat{D}^* = \operatorname{diag}(\hat{A}_{11}^*, \ldots, \hat{A}_{ss}^*).$$

Lemma 9. Let A be a matrix represented in form (3) and $\lambda_i > 0$ for any $i = 1, \ldots, s$. Then, the equality

$$\min_{\mathbf{x} \in \mathbb{X}_+^n} \mathbf{x}^- \otimes A \otimes \mathbf{x} = \lambda$$

holds and the minimum is attained at $\mathbf{x} = \hat{C}^+ \otimes \hat{D}^* \otimes \mathbf{v}$ for all $\mathbf{v} \in \mathbb{X}_+^n$.

Proof. Let $\mathbf{x}_i \neq 0$ denote a vector of size n_i, $i = 1, \dots, s$. By virtue of (9), for any vector $\mathbf{x} = (\mathbf{x}_1^T, \dots, \mathbf{x}_s^T)^T$, we have the inequalities

$$\mathbf{x}^- \otimes A \otimes \mathbf{x} = \bigoplus_{i=1}^{s} \bigoplus_{j=1}^{i} \mathbf{x}_i^- \otimes A_{ij} \otimes \mathbf{x}_j \geq \bigoplus_{i=1}^{s} \mathbf{x}_i^- \otimes A_{ii} \otimes \mathbf{x}_i \geq \bigoplus_{i=1}^{s} \lambda_i = \lambda.$$

Suppose that vector $\mathbf{x} \in \mathbb{X}_+^n$ is a nontrivial solution to the homogeneous equation $\hat{A} \otimes \mathbf{x} = \mathbf{x}$. We show that, for this choice of vector $\mathbf{x}$, both sides of the inequality are equal to each other.

Consider the equation corresponding to the ith horizontal line of matrix $\hat{A}$

$$\mathbf{x}_i = \bigoplus_{j=1}^{i} \hat{A}_{ij} \otimes \mathbf{x}_j = \lambda_i^{-1} \otimes \bigoplus_{j=1}^{i} A_{ij} \otimes \mathbf{x}_j.$$

Multiplying both sides of this equation by $\lambda_i \otimes \mathbf{x}_i^-$ from the left, we obtain

$$\bigoplus_{j=1}^{i} \mathbf{x}_i^- \otimes A_{ij} \otimes \mathbf{x}_j = \lambda_i.$$

This implies that

$$\mathbf{x}^- \otimes A \otimes \mathbf{x} = \bigoplus_{i=1}^{s} \bigoplus_{j=1}^{i} \mathbf{x}_i^- \otimes A_{ij} \otimes \mathbf{x}_j = \bigoplus_{i=1}^{s} \lambda_i = \lambda.$$

Applying Lemma 4 to the equation $\hat{A} \otimes \mathbf{x} = \mathbf{x}$ and taking into account that $\operatorname{Tr}\hat{A} = 1$, we obtain a solution to the equation in the form $\mathbf{x} = \hat{C}^+ \otimes \hat{D}^* \otimes \mathbf{v}$ for a general $\mathbf{v} \in \mathbb{X}_+^n$.

In conclusion, consider assertion (10). For matrix A in form (3), by I we denote the set of indices i such that $A_{ij} = 0$ for any $j = 1, \dots, i-1$. We assume that the set I contains index 1.

Denoting the eigenvalue of matrix A_{ii} by λ_i for any $i = 1, \dots, s$, we define the quantity

$$\tilde{\lambda}^{-1} = \bigoplus_{i \in I} \lambda_i^{-1}.$$

We introduce a lower block-triangular matrix $\tilde{A}$ in the following way. For any $i = 1, \dots, s$, we set $\tilde{A}_{ij} = \tilde{\lambda}^{-1} \otimes A_{ij}$ if $j < i$ and

$$\tilde{A}_{ii} = \begin{cases} \lambda_i^{-1} \otimes A_{ii}, & \text{if } i \in I, \\ 0, & \text{if } i \notin I. \end{cases}$$

We represent the matrix $\tilde{A}$ in the form $\tilde{A} = \tilde{T} \oplus \tilde{D}$ and define matrices

$$\tilde{D}^+ = \operatorname{diag}(\tilde{A}_{11}^+, \dots, \tilde{A}_{ss}^+), \quad \tilde{C} = \tilde{D}^+ \otimes \tilde{T}, \quad \tilde{D}^* = \operatorname{diag}(\tilde{A}_{11}^*, \dots, \tilde{A}_{ss}^*).$$

Lemma 10. Let A be a matrix represented in form (3) and $\lambda_i > 0$ for any $i = 1, \dots, s$. Then, the equality

$$\min_{\mathbf{x} \in \mathbb{X}_+^n} (A \otimes \mathbf{x})^- \otimes \mathbf{x} = \tilde{\lambda}^{-1}$$

holds and the minimum is attained at $\mathbf{x} = \tilde{C}^+ \otimes \tilde{D}^* \otimes \mathbf{v}$ for all $\mathbf{v} \in \mathbb{X}_+^n$.

Proof. It is clear that, for any vector $\mathbf{x} \in \mathbb{X}_+^n$, by virtue of (10), we have

$$(A \otimes \mathbf{x})^- \otimes \mathbf{x} = \bigoplus_{i=1}^{s} \left(\bigoplus_{j=1}^{i} A_{ij} \otimes \mathbf{x}_j \right)^- \otimes \mathbf{x}_i \geq \bigoplus_{i \in I} (A_{ii} \otimes \mathbf{x}_i)^- \otimes \mathbf{x}_i \geq \bigoplus_{i \in I} \lambda_i^{-1} = \tilde{\lambda}^{-1}.$$

We show that, if vector $\mathbf{x} \neq 0$ satisfies the homogeneous equation $\tilde{A} \otimes \mathbf{x} = \mathbf{x}$, then the equality $(A \otimes \mathbf{x})^- \otimes \mathbf{x} = \tilde{\lambda}^{-1}$ holds.

Indeed, for any $i \in I$, the equation $\tilde{A} \otimes \mathbf{x} = \mathbf{x}$ implies the equality

$$\mathbf{x}_i = \bigoplus_{j=1}^{i} \tilde{A}_{ij} \otimes \mathbf{x}_j = \lambda_i^{-1} \otimes A_{ii} \otimes \mathbf{x}_i.$$

Taking pseudoinverses of all parts of the equality and multiplying the results by $\lambda_i^{-1} \otimes \mathbf{x}_i$ from the right, we arrive at the equality

$$\left(\bigoplus_{j=1}^{i} A_{ij} \otimes \mathbf{x}_j \right)^- \otimes \mathbf{x}_i = (A_{ii} \otimes \mathbf{x}_i)^- \otimes \mathbf{x}_i = \lambda_i^{-1}.$$

In the case $i \notin I$, we have

$$\mathbf{x}_i = \bigoplus_{j=1}^{i} \tilde{A}_{ij} \otimes \mathbf{x}_j = \tilde{\lambda}^{-1} \otimes \bigoplus_{j=1}^{i-1} A_{ij} \otimes \mathbf{x}_j \leq \tilde{\lambda}^{-1} \otimes \bigoplus_{j=1}^{i} A_{ij} \otimes \mathbf{x}_j.$$

This implies the inequality

$$\left(\bigoplus_{j=1}^{i} A_{ij} \otimes \mathbf{x}_j \right)^- \otimes \mathbf{x}_i \leq \tilde{\lambda}^{-1}.$$

Then, for the vector $\mathbf{x}$ considered, we have the inequality

$$(A \otimes \mathbf{x})^- \otimes \mathbf{x} = \bigoplus_{i \in I} \left(\bigoplus_{j=1}^{i} A_{ij} \otimes \mathbf{x}_j \right)^- \otimes \mathbf{x}_i \oplus \bigoplus_{i \notin I} \left(\bigoplus_{j=1}^{i} A_{ij} \otimes \mathbf{x}_j \right)^- \otimes \mathbf{x}_i \leq \tilde{\lambda}^{-1}.$$

Since the opposite inequality is always valid, we conclude that $(A \otimes \mathbf{x})^- \otimes \mathbf{x} = \tilde{\lambda}^{-1}$.

It is clear that $\mathrm{Tr}\,\tilde{A} = 1$. Hence, by Lemma 4, the homogeneous equation $\tilde{A} \otimes \mathbf{x} = \mathbf{x}$ has a solution $\mathbf{x} = \tilde{C}^+ \otimes \tilde{D}^* \otimes \mathbf{v}$ for all $\mathbf{v} \in \mathbb{X}_+^n$.

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