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EIGENVALUES OF HOPF MANIFOLDS

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ABSTRACT. The eigenvalues of the Laplacians Δ and $\Box$ on the Hopf manifolds are described. Some isospectral results are also given.

On a complex manifold M there are the de Rham and Dolbeault complexes, with operators d and $\bar\partial$, respectively. If we fix a hermitian metric on the manifold M, then d and $\bar\partial$ will have formal adjoints δ and $\mathfrak{d}$ with respect to the hermitian volume element. With these, we define the real Laplacian $\Delta = d\delta + \delta d$ and the complex Laplacian $\Box = \bar\partial\mathfrak{d} + \mathfrak{d}\bar\partial$. We have $\Delta = 2\Box = 2\bar\Box$ if M is Kähler.

In this note, we study the eigenvalues of the Laplacians Δ and $\Box$ on the Hopf manifolds M_α, $0 < |\alpha| < 1$, which are not Kähler. The notation $\mathrm{Sp}\,(M,A)$ will be used to denote the eigenvalues with multiplicity (spectrum) of the operator A on the manifold M. In Theorem 1, the eigenvalues of Δ and $\Box$ are described. The computation of the eigenfunctions shows that the eigenspaces of $\Box$ refine those of Δ. Some isospectral results are also given. For α in a certain domain, $\mathrm{Sp}\,(M_\alpha,\Delta)$ determines M_α up to isometry (Theorem 2). Moreover, it is shown that for "most" values of α, M_α may be determined up to isometry from either $\mathrm{Sp}\,(M_\alpha,\Delta)$ or $\mathrm{Sp}\,(M_\alpha,\Box)$ (Theorem 3).

Let W denote the n-dimensional complex vector space $\mathbf{C}^n = \{z \mid z = (z_1,\dots,z_n)\}$ minus the origin; $W = \mathbf{C}^n - \{0\}$, and let α be a complex number with $0 < |\alpha| < 1$. Consider the analytic automorphism g_α of W defined by $g_\alpha(z_1,\dots,z_n) = (\alpha z_1,\dots,\alpha z_n)$. The group G_α generated by g_α is an infinite cyclic group acting on W freely and properly discontinuously. Thus the quotient $M_\alpha = W/G_\alpha$ is an n-dimensional complex manifold, which is called a (homogeneous) Hopf manifold. It is easy to see that M_α is diffeomorphic to $S^1 \times S^{2n-1}$, where S^r denotes the standard r-sphere (cf. the proof of Lemma 3). If $n = 1$, M_α is the complex torus whose lattice is generated by 1 and $(2\pi i)^{-1}\log\alpha$, and if $n > 1$, M_α is a non-Kähler manifold. The hermitian metric

$$g = \|z\|^{-2}\sum_{i=1}^{n} dz_i\,d\bar z_i, \qquad \|z\|^2 = \sum_{i=1}^{n} z_i\bar z_i, \tag{1}$$

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on W is G_α-invariant. Hence, it induces a hermitian metric on M_α. From now on we think of M_α as a hermitian manifold with this metric. It is easy to check that if $n = 1$, M_α is a flat torus. If α is real, M_α is isometric to $S^1 \times S^{2n-1}$ as a riemannian manifold. We denote by $\mathcal{C}^\infty(M_\alpha)$ the space of complex valued smooth functions on M_α. Consider the real and complex Laplacians Δ and $\square$ induced by the hermitian metric on M_α. Since the operator $\mathfrak{d}$ maps a (p,q)-form to a $(p,q-1)$-form, we have

$$\Delta = \square + \overline{\square} \quad \text{on } \mathcal{C}^\infty(M_\alpha). \tag{2}$$

A straightforward computation [3, p. 97] shows that the complex Laplacian on $\mathcal{C}^\infty(M_\alpha)$ is

$$\square = -\|z\|^2 \sum_{i=1}^n \frac{\partial^2}{\partial z_i \partial \bar{z}_i} + (n-1) \sum_{i=1}^n \bar{z}_i \frac{\partial}{\partial \bar{z}_i}.$$

Let

$$\Delta_0 = -\sum_{i=1}^n \frac{\partial^2}{\partial z_i \partial \bar{z}_i}$$

be the standard Laplacian on $\mathbf{C}^n$. Moreover, let $\mathcal{H}_{p,q}$ be the space of harmonic polynomials of type (p,q), i.e., the polynomials f on $\mathbf{C}^n$ such that $\Delta_0 f = 0$ and $f(z) = \sum_{|\mu|=p;|\nu|=q} c_{\mu\nu} z^\mu \bar{z}^\nu$, where $\mu = (\mu_1, \ldots, \mu_n)$ and $\nu = (\nu_1, \ldots, \nu_n)$ are n-tuples of nonnegative integers, $|\mu| = \sum_{i=1}^n \mu_i$, $|\nu| = \sum_{i=1}^n \nu_i$, and $z^\mu = z_1^{\mu_1} \cdots z_n^{\mu_n}$, $\bar{z}^\nu = \bar{z}_1^{\nu_1} \cdots \bar{z}_n^{\nu_n}$.

LEMMA 1. *For $f \in \mathcal{H}_{p,q}$ and for a complex number γ,*

$$\square(\|z\|^\gamma f) = (-(\gamma/2)^2 - (p+q)\gamma/2 + q(n-1))\|z\|^\gamma f. \tag{3}$$

PROOF. Since $\Delta_0 f = 0$, we have

$$\square(\|z\|^\gamma f) = -\left(\frac{\gamma}{2}\right)^2 \|z\|^\gamma f - \frac{\gamma}{2}\|z\|^\gamma \sum_{i=1}^n z_i \frac{\partial f}{\partial z_i} + \left(n - 1 - \frac{\gamma}{2}\right)\|z\|^\gamma \sum_{i=1}^n \bar{z}_i \frac{\partial f}{\partial \bar{z}_i}.$$

Substituting

$$\sum_{i=1}^n z_i \frac{\partial f}{\partial z_i} = pf \quad \text{and} \quad \sum_{i=1}^n \bar{z}_i \frac{\partial f}{\partial \bar{z}_i} = qf$$

in the above equation, we get (3). Q.E.D.

Let ω be a complex number such that $e^{2\pi i \omega} = \alpha$. Since $|\alpha| < 1$, we have $\operatorname{Im} \omega > 0$.

DEFINITION. For nonnegative integers p and q, let $\Gamma_{p,q}$ be the set of complex numbers γ which can be expressed as $\gamma = -\gamma_1 + \bar{\gamma}_2$ with $\operatorname{Re}\gamma_1 = p$, $\operatorname{Re}\gamma_2 = -q$, and $\operatorname{Re}((\gamma_1 + \gamma_2)\bar{\omega}) \in \mathbf{Z}$.

REMARKS. 1°. $\gamma_1 + \gamma_2$ is in the lattice dual to the one generated by 1 and ω (cf. [1, p. 146]).

2°. Let $a = \mathrm{Re}\,\omega$ and $b = \mathrm{Im}\,\omega$. Then

$$\Gamma_{p,q} = \{-(p+q) + ((a(p-q)-k)/b)i \,|\, k \in \mathbf{Z}\}. \tag{4}$$

LEMMA 2. *For $f \in \mathcal{H}_{p,q}$, the function $\|z\|^\gamma f$ is G_α-invariant if and only if $\gamma \in \Gamma_{p,q}$.*

PROOF. $\|z\|^\gamma f$ is G_α-invariant if and only if $|\alpha|^\gamma \alpha^p \bar{\alpha}^q = 1$, or equivalently,

$$\begin{cases} \mathrm{Re}\,\gamma = -(p+q) \text{ and} \\ -\mathrm{Im}\,\omega \cdot \mathrm{Im}\,\gamma + (p-q)\mathrm{Re}\,\omega = k, \quad \text{for some } k \in \mathbf{Z}. \end{cases}$$

In view of (4), this is equivalent to $\gamma \in \Gamma_{p,q}$. Q.E.D.

LEMMA 3. *The vector subspace of $\mathcal{C}^\infty(M_\alpha)$ generated by $\bigcup_{p,q\geqslant 0}\{\|z\|^\gamma f \,|\, f \in \mathcal{H}_{p,q}, \gamma \in \Gamma_{p,q}\}$ is dense in $\mathcal{C}^\infty(M_\alpha)$.*

PROOF. Consider the unit $(2n-1)$-sphere $S^{2n-1} = \{z \in \mathbf{C}^n \,|\, \|z\| = 1\}$. The map $\varphi\colon \mathbf{R} \times S^{2n-1} \to W$ defined by $\varphi(t, z_1, \ldots, z_n) = (e^{2\pi i\omega t}z_1, \ldots, e^{2\pi i \omega t}z_n)$ is clearly a diffeomorphism. Moreover, φ induces a diffeomorphism, which we denote also by φ, from $S^1 \times S^{2n-1}$ onto M_α, where $S^1 = \mathbf{R}/\mathbf{Z}$. Take a function $\|z\|^\gamma f$ with $f \in \mathcal{H}_{p,q}$ and $\gamma \in \Gamma_{p,q}$. The pullback of $\|z\|^\gamma f$ by φ is given by

$$((\|z\|^\gamma f) \circ \varphi)(t,z) = \exp(2\pi i t(i\gamma\,\mathrm{Im}\,\omega + p\omega - q\bar{\omega}))f(z) = e^{2\pi i k t} f(z),$$

where $k = \mathrm{Re}((\gamma_1 + \gamma_2)\bar{\omega}) \in \mathbf{Z}$. On the other hand, the vector subspace of $\mathcal{C}^\infty(S^1)$ generated by the functions $\{e^{2\pi i k t}\}_{k \in \mathbf{Z}}$ is dense in $\mathcal{C}^\infty(S^1)$ and the harmonic polynomials are dense in $\mathcal{C}^\infty(S^{2n-1})$ [1, p. 160]. Q.E.D.

THEOREM 1. (i) *The eigenvalues of $\square$ on $\mathcal{C}^\infty(M_\alpha)$ are $|\gamma|^2/4 + q(n-1)$, $p, q \in \mathbf{Z}^+$, $\gamma \in \Gamma_{p,q}$.*

(ii) *The eigenvalues of Δ on $\mathcal{C}^\infty(M_\alpha)$ are $|\gamma|^2/2 + (p+q)(n-1)$, $p, q \in \mathbf{Z}^+$, $\gamma \in \Gamma_{p,q}$, where $\mathbf{Z}^+$ denotes the nonnegative integers.*

PROOF. If $\gamma \in \Gamma_{p,q}$, we have $-(\gamma/2)^2 - (p+q)\gamma/2 + q(n-1) = |\gamma|^2/4 + q(n-1)$. The theorem follows from (2), Lemmas 1–3, and Lemma A.II.1 in [1, p. 143]. Q.E.D

Now let us discuss the isospectral problem. Let $a + bi = \omega = (2\pi i)^{-1}\log\alpha$ as before. Since $a = (2\pi)^{-1}\arg\alpha$, we may assume that $|a| \leqslant \frac{1}{2}$. Note that if $|a| = \frac{1}{2}$, then α is real and M_α is isometric to $S^1 \times S^{2n-1}$.

THEOREM 2. *Suppose* (i) $b > 1/2\sqrt{2n+1}$ *or* (ii) $|a| < b\sqrt{2n+1}$. *Then* $\mathrm{Sp}\,(M_\alpha, \Delta)$ *determines M_α up to isometry, namely,* $\mathrm{Sp}\,(M_\alpha,\Delta) = \mathrm{Sp}\,(M_{\alpha'},\Delta)$ *implies M_α is isometric to $M_{\alpha'}$.*

PROOF. Let

$$a' + b'i = \omega' = (2\pi i)^{-1}\log\alpha', \qquad |a'| \leqslant \tfrac{1}{2}.$$

If $\mathrm{Sp}\,(M_\alpha,\Delta) = \mathrm{Sp}\,(M_{\alpha'},\Delta)$, then the volumes of M_α and $M_{\alpha'}$ are the same [1, Corollaire E.IV.2, p. 216], see also [4]. This implies $|\alpha| = |\alpha'|$ or equivalently $b = b'$. Consider the eigenvalues

$$\lambda(p,q,k,a^{(\nu)}) = (p+q)^2/2 + (a^{(\nu)}(p-q)-k)^2/2b^2 + (p+q)(n-1),$$

$p, q \in \mathbf{Z}^+$, $k \in \mathbf{Z}$, of Δ on $\mathcal{C}^\infty(M_{\alpha^{(\nu)}})$, $\nu = 0, 1$, $a^{(0)} = a$, $a^{(1)} = a'$. Note that if $p + q \geqslant 2$, then $\lambda(p,q,k,a^{(\nu)}) \geqslant 2n$ and that $\lambda(p,q,k,a^{(\nu)}) = 0$ if and only if $p = q = k = 0$.

Case (i). $b > 1/2\sqrt{2n+1}$. Since $|a| \leqslant \frac{1}{2}$, we have

$$\lambda(1,0,0,a) = a^2/2b^2 + n - \tfrac{1}{2} < 2a^2(2n+1) + n - \tfrac{1}{2} \leqslant \tfrac{1}{2}(2n+1) + n - \tfrac{1}{2} = 2n.$$

Similarly, we have $\lambda(1,0,0,a') < 2n$. Hence, we have

$$\lambda(1,0,0,a) = \lambda(p_1,q_1,k_1,a') \quad \text{and} \quad \lambda(1,0,0,a') = \lambda(p_2,q_2,k_2,a)$$

for some p_j, $q_j \in \mathbf{Z}^+$ and $k_j \in \mathbf{Z}$ with $0 \leqslant p_j + q_j \leqslant 1, j = 1, 2$. If $p_1 + q_1 = 1$, we have $a^2 = (a' \pm k_1)^2$. Since $|a^{(\nu)}| \leqslant \frac{1}{2}$, $\nu = 0, 1$, we get $a' = \pm a$. If $p_2 + q_2 = 1$, we also get $a' = \pm a$. If $p_j + q_j = 0, j = 1, 2$, we have $a^2 - a'^2 = k_1^2 - k_2^2$. Since $|a^2 - a'^2| \leqslant \frac{1}{2}$, we get $a' = \pm a$.

Case (ii). $0 < b \leqslant 1/2\sqrt{2n+1}$, $|a| < b\sqrt{2n+1}$. We have

$$\lambda(1,0,0,a) = \frac{a^2}{2b^2} + n - \frac{1}{2} < n + \frac{1}{2} + n - \frac{1}{2} = 2n.$$

On the other hand

$$\lambda(0,0,k,a') = \frac{k^2}{2b^2} \geqslant 2k^2(2n+1) > 2n, \quad \text{for } |k| \geqslant 1.$$

Hence, we have $\lambda(1,0,0,a) = \lambda(p_1,q_1,k_1,a')$, with $p_1 + q_1 = 1$. Thus, we get $a' = \pm a$.

If $a' = -a$, then $\alpha' = \bar{\alpha}$ and the diffeomorphism $W \to W$ defined by $(z_1, \ldots, z_n) \mapsto (\bar{z}_1, \ldots, \bar{z}_n)$ induces an isometry between M_α and $M_{\alpha'}$. Q.E.D.

THEOREM 3. *Suppose a and b are algebraically independent over* $\mathbf{Q}$. *Then* M_α *may be determined up to isometry from either* Sp (M_α, Δ) *or* Sp $(M_\alpha, \Box)$.

PROOF. We show that Sp $(M_\alpha, \Box)$ determines $|a|$ and b. The same argument with only small modifications will apply to Sp (M_α, Δ). Let α' be a complex number with $a' + ib' = (2\pi i)^{-1} \log \alpha'$, $-\frac{1}{2} \leqslant a' \leqslant \frac{1}{2}$. We shall assume that $\mathrm{Sp}(M_\alpha, \Box) = \mathrm{Sp}(M_{\alpha'}, \Box)$ and show that $|a| = |a'|$ and $b = b'$. Since the volume of M_α determines b, and the volume of M_α is determined by the asymptotic behavior of the spectrum (see, for instance, Gilkey [4]), it follows that $b = b'$. Let

$$\lambda(p,q,k,a,b) = (p+q)^2/4 + (a(p-q)-k)^2/4b^2 + (n-1)q.$$

It is easily seen that the linear span of $\{b^2\lambda | \lambda \in \mathrm{Sp}\,(M_\alpha, \Box)\}$ over Q is the linear span over $\mathbf{Q}$ of $\{1, a, a^2, b^2\}$. Thus if Sp $(M_\alpha, \Box) =$ Sp $(M_{\alpha'}, \Box)$, then $\{1, a, a^2, b^2\}$ and $\{1, a', (a')^2, b^2\}$ have the same linear span over Q. Thus there exist rationals r_j, r_j', $1 \leqslant j \leqslant 4$, such that

$$a' = r_1 + ar_2 + a^2r_3 + b^2r_4, \qquad (a')^2 = r_1' + ar_2' + a^2r_3' + b^2r_4'.$$

By the algebraic independence of a and b, it follows by squaring the left-hand equation that $r_3 = r_4 = 0$. Thus $a' = r_1 + ar_2$.

Now we show that $r_1 = 0$, $r_2 = \pm 1$. Observe that if

$$\mathrm{Sp}\,(M_\alpha, \square) = \mathrm{Sp}\,(M_{\alpha'}, \square)$$

then for integers (p,q,k) there exists (p',q',k') such that

$$\lambda(p,q,k,a,b) = \lambda(p',q',k',r_1 + ar_2, b). \tag{5}$$

Conversely, for (p',q',k') there exists (p,q,k). Multiplying (5) by b^2 and comparing the coefficients of 1 and a^2, we obtain

$$k^2 = ((p'-q')r_1 - k')^2, \tag{6}$$

$$(p-q)^2 = ((p'-q')r_2)^2. \tag{7}$$

Since (7) must always have integer solutions, $r_2 = \pm 1$. From equation (6) it follows that r_1 must be an integer, and so $r_1 = 0$ since $-\frac{1}{2} \leqslant a' \leqslant \frac{1}{2}$. Thus $|a|$ and b are determined. Q.E.D.

REMARKS. 1. If $n = 1$, M_α and $M_{\alpha'}$ are biholomorphic, as is well known, if and only if $\binom{\omega'}{1} = u\binom{\omega}{1}$ for some $u \in \mathrm{SL}\,(2,\mathbf{Z})$, where

$$\omega = (2\pi i)^{-1}\log\alpha \quad \text{and} \quad \omega' = (2\pi i)^{-1}\log\alpha'.$$

If $n \geqslant 2$, M_α and $M_{\alpha'}$ are biholomorphic, by Hartogs' theorem, if and only if $\alpha' = \alpha$ [2, Theorem 15.1].

2. It does not seem easy to find the dimension (multiplicity) of each eigenspace. The dimension of $\mathcal{H}_{p,q}$ can be computed as follows. Let $\mathcal{P}_{p,q}$ denote the space of polynomials on $\mathbf{C}^n$ of type (p,q) and set $r = \|z\|$. Observe that $\mathcal{H}_k$, the harmonic polynomials homogeneous of degree k, is subdivided into the spaces $\mathcal{H}_{p,q}$, i.e., $\mathcal{H}_k = \oplus_{p+q=k}\mathcal{H}_{p,q}$. This may be done since Δ_0 maps $\mathcal{P}_{p,q}$ into $\mathcal{P}_{p-1,q-1}$. By repeating the argument in Berger [1, p. 161], we may conclude that for $p \geqslant q$,

$$\mathcal{P}_{p,q} = \mathcal{H}_{p,q} \oplus r^2\mathcal{H}_{p-1,q-1} \oplus \cdots \oplus r^{2q}\mathcal{H}_{p-q,0},$$

and the summands are pairwise orthogonal in $L^2(S^{2n-1})$. Thus, $\mathcal{P}_{p,q} = \mathcal{H}_{p,q} \oplus r^2\mathcal{P}_{p-1,q-1}$ from which it follows that

$$\dim\mathcal{H}_{p,q} = \dim\mathcal{P}_{p,q} - \dim\mathcal{P}_{p-1,q-1}.$$

Since

$$\dim\mathcal{P}_{p,q} = \binom{n-1+p}{p}\binom{n-1+q}{q},$$

we get

$$\dim\mathcal{H}_{p,q} = \binom{n+p-1}{p}\binom{n+q-1}{q} - \binom{n+p-2}{p-1}\binom{n+q-2}{q-1}.$$

REFERENCES

1. M. Berger, P. Gauduchon and E. Mazet, *Le spectre d'une variété riemannienne*, Lecture Notes in Math., vol. 194, Springer-Verlag, Berlin and New York, 1971. MR **43** #8025.

2. K. Kodaira and D. C. Spencer, *On deformations of complex analytic structures*. II, Ann. of Math. (2) **67** (1958), 403–466. MR **22** #3009.

3. J. Morrow and K. Kodaira, *Complex manifolds*, Holt, Rinehart and Winston, New York, 1971. MR **46** #2080.

4. P. Gilkey, *The spectral geometry of real and complex manifolds*, Proc. Sympos. Pure Math., vol. 27, Amer. Math. Soc., Providence, R. I., 1975, pp. 265–280.

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