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Eigenvalues of the Casimir Operators of $U(n)$ and $SU(n)$ for Totally Symmetric and Antisymmetric Representations

Kozo KOBAYASHI

Department of Physics
Saitama University, Urawa

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1. $U(n)\,[SU(n)]$ is one of the most important dynamical groups in quantum mechanics and elementary particle physics. In the analysis of physical systems in terms of its dynamical group, it is required to determine some invariant operators and their eigenvalues for relevant irreducible representations, since a set of the eigenvalues specifies an irreducible representation of the dynamical group and it gives quantum numbers of the system. Formal expressions for the invariant operators of $U(n)\,[SU(n)]$ were studied by several authors[1)~3)] and a formula for the eigenvalues of the Casimir operators of any order by Perelomov and Popov.[4)] In the actual analysis of dynamical symmetries, one must know the explicit expression for the eigenvalues of invariant operators of relevance. In this paper, we shall give the explicit eigenvalues of the Casimir operators of $U(n)$ and $SU(n)$ of any order for the totally symmetric and antisymmetric representations of the most physical interest. This will be carried out in terms of the boson and fermion operator calculus.

2. We define $U(n)$ and $SU(n)$ as follows. We first introduce n linearly independent operators $a_j\,(j=1, 2, \cdots, n)$. The linear transformations with complex coefficients α_{jk} among the operators define $GL(n)$:

$$a_j \to a_j' = \sum_k \alpha_{jk} a_k . \tag{1}$$

For a_j and its hermitian conjugate $a_j^\dagger$ we define the following hermitian form H and the commutation relation (CR) or the anti-commutation relation (AR):

$$H = \sum_j a_j^\dagger a_j , \tag{2}$$

$$[a_j, a_k^\dagger]_\pm \equiv a_j a_k^\dagger \pm a_k^\dagger a_j = \delta_{jk} , \tag{3}$$

$$[a_j, a_k]_\pm = [a_j^\dagger, a_k^\dagger]_\pm = 0 . \tag{4}$$

Then $U(n)$ can be defined as a subgroup of $GL(n)$ which leaves Eqs. (2) and (3) invariant [Eq. (4) is always invariant under $GL(n)$]. When the phase of the $U(n)$ transformation is fixed so that its determinant is unity, one has a subgroup $SU(n)$. The operators obeying CR and AR are equivalent to the boson and fermion operators in quantum statistics respectively.

The generators of $U(n)$, $L_t{}^s$ are given in terms of a_j and $a_j^\dagger$ as

$$L_t{}^s = a_s^\dagger a_t . \tag{5}$$

It is easily verified that $L_t{}^s$ satisfy the true CR of $U(n)$:

$$[L_t{}^s, L_v{}^u] = \delta_{sv} L_t{}^u - \delta_{ut} L_v{}^s, \tag{6}$$

and the hermiticity condition (HC):

$$(L_t{}^s)^\dagger = L_s{}^t. \tag{7}$$

The generators of $SU(n)$, $\bar{L}_t{}^u$ are, from the tracelessness condition $\sum_s \bar{L}_s{}^s = 0$,

$$\bar{L}_t{}^s = L_t{}^s - \delta_{st} H/n . \tag{8}$$

The Casimir operator of $U(n)$ of order p is defined by

$$C^{(p)} = \sum_{s_1, s_2, \cdots s_p} L_{s_2}^{s_1} L_{s_3}^{s_2} \cdots L_{s_1}^{s_p} . \tag{9}$$

Similarly $\bar{C}^{(p)}$ of $SU(n)$ is given by taking $\bar{L}_t{}^s$ in place of $L_t{}^s$ in the above expression. By use of Eqs. (3), (4) and the relations $[H, a_j] = -a_j$, $[H, a_j^\dagger] = a_j^\dagger$, $C^{(p)}$ and $\bar{C}^{(p)}$ can be written as

$$C^{(p)} = HK^{p-1}, \tag{10}$$

$$\bar{C}^{(p)} = HK^{-1}(K+L) \times [(K+L)^{p-1} - (-H)^{p-1}], \tag{11}$$

where H is Eq. (2) and

$$K=n\pm H\mp 1$$

$$\text{for the} \begin{pmatrix}\text{boson}\\ \text{fermion}\end{pmatrix} \text{operators}, \tag{12}$$

$$L=-H/n. \tag{13}$$

3. Consider the representations of $U(n)$ and $SU(n)$ by the Fock space of the operators a_j. The Fock space is formed by the simultaneous eigenvectors of the invariant operator H and the mutually commuting operators $L_1^1, L_2^2, \cdots, L_n^n$. Their respective eigenvalues h and $\nu_1, \nu_2, \cdots, \nu_n$ satisfy from Eq. (2) the relation

$$h=\nu_1+\nu_2+\cdots+\nu_n. \tag{14}$$

As is well known, the normalized eigenvector corresponding to the above set of eigenvalues $|h(\nu_1, \nu_2. \cdots, \nu_n)\rangle$ is given by

$$|h(\nu_1, \nu_2, \cdots, \nu_n)\rangle = (\nu_1!\nu_2!\cdots\nu_n!)^{-1/2} a_1^{\dagger\nu_1} a_2^{\dagger\nu_2}\cdots a_n^{\dagger\nu_n}|0\rangle. \tag{15}$$

These vectors are totally symmetric or antisymmetric according as a_j, $a_j^\dagger$ obey CR or AR. Under the action of the generators L_t^s on the Fock vectors, the eigenvalue h is invariant since it results in the creation of $a_s^\dagger$ and the annihilation of $a_t^\dagger$ by one. A subspace spanned by the Fock vectors with a fixed eigenvalue h has no invariant subspace since the creation and the annihilation work on the whole subspace. Therefore the subspace provides an irreducible representation of $U(n)$ and $SU(n)$:*) a totally symmetric representation (TSR) $[h]$ or a totally antisymmetric representation (TAR) $[1^h]$. From Eqs. (10) and (11), we can thus obtain the eigenvalues of the Casimir operator of order p:

$$C^{(p)}(h)=h(n\pm h\mp 1)^{p-1}, \tag{16}$$

$$\bar{C}^{(p)}(h)=h(n\mp 1)(n\pm h) \times[(n\mp 1)^{p-1}(n\pm h)^{p-1}-(-h)^{p-1}] \times\{n^p(n\pm h\mp 1)\}^{-1}, \tag{17}$$

where the double signs $\pm$, $\mp$ correspond to TSR and TAR respectively. The expression (16) for $C^{(p)}$ of $U(n)$ reproduces the one obtained by Perelomov and Popov [Eq. (9) in Ref. 4)]. However we should like to note that as far as we are concerned with TSR and TAR, our derivation is much simpler than theirs, and that to our knowledge, there is yet no publication of the explicit expression of the eigenvalues of $\bar{C}^{(p)}$ for $SU(n)$.

4. Finally we briefly note that if an $n\times n$ matrix representation of the Lie algebra of $U(n)$ $[SU(n)]$ is given, we can have its corresponding expression in terms of a_j, $a_j^\dagger$.[6)] Let these representation matrices be $M^{st}=(M_{jk}^{st})$ where M_{jk}^{st} is the (j,k) element of M^{st}. The matrices satisfy the same CR and HC as Eqs. (6) and (7) up to equivalence (i.e., up to a unitary transformation of M^{st}). Then the Lie algebra in terms of a_j, $a_j^\dagger$ is formed by the following formula:

$$L_t^s=a^\dagger M^{st}a\equiv\sum_{jk} a_j^\dagger M_{jk}^{st} a_k. \tag{18}$$

It is easily verified that these L_t^s satisfy indeed CR(6) and HC(7). The expression used in § 2(5) corresponds to the canonical representation: $M_{jk}^{st}=\delta_{sj}\delta_{tk}$. An equivalence transformation by a unitary matrix U, $M'^{st}=U^\dagger M^{st}U$ is nothing else than a unitary transformation of the basic operator a_j through U; in fact $L_t'^s$ obtained from the unitary-transformed M'^{st} can be written as

$$L_t'^s=a^\dagger M'^{st}a=a^\dagger U^\dagger M^{st}Ua=a'^\dagger M^{st}a', \tag{19}$$

*) The irreducibility of the representation is also seen from the fact that due to Eqs. (3) and (4), operators commutable with all the generators L_t^s are only the identity and the bilinear form H, the latter however has been exhausted in defining $U(n)$.

where a' is a unitary transformation of a by U:

$$a'=Ua. \quad (a_j'=\sum_k U_{jk}a_k) \tag{20}$$

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