

*EIGENVALUES OF THE LAPLACIAN AND CURVATURE*

BY

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**Introduction.** Let  $(M, g)$  be a closed, connected two-dimensional Riemannian manifold of genus zero with sectional curvature  $\kappa$ ,  $\kappa_0 := \min \kappa$ ,  $\kappa_1 := \max \kappa$ . Then the first positive eigenvalue  $\lambda_1$  of the Laplacian on functions fulfils the inequality

$$2\kappa_0 \leq \lambda_1 \leq 2\kappa_1,$$

and the equality on the left or the right holds iff  $(M, g)$  is isometrically diffeomorphic to a sphere (cf. [2], p. 179-180, and [9], also [16]; naturally the inequality on the left is of interest only if  $\kappa_0 > 0$ ).

Hersch's result (the right-hand inequality) suggests to investigate eigenvalues  $\lambda$  with  $\lambda > 2\kappa_1$ . The following is our main result in the two-dimensional case:

**THEOREM A.** *Let  $(M, g)$  be a closed, connected Riemannian manifold,  $\dim M = 2$ ,  $\kappa_0 > 0$ . Then  $\lambda > 2\kappa_1$  implies  $\lambda \geq 6\kappa_0$ , and  $\lambda = 6\kappa_0$  implies that the universal Riemannian covering  $(\tilde{M}, \tilde{g})$  of  $(M, g)$  is isometrically diffeomorphic to a Euclidean sphere  $S^2(\kappa_0)$  with constant curvature  $\kappa_0$ .*

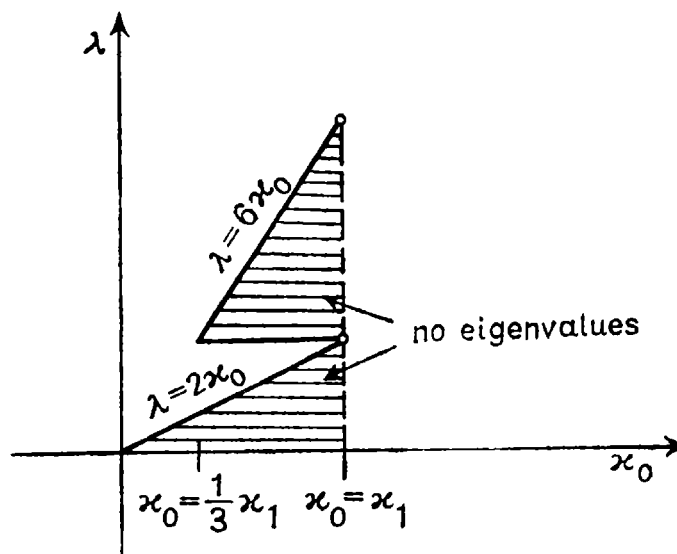
As above, the result is of interest only if  $6\kappa_0 \geq 2\kappa_1$ , i.e. for pinched manifolds with pinching constant  $\delta$ ,  $\delta \geq \frac{1}{3}$ . So especially for manifolds "not too far from a Riemannian sphere" we get additional information on relations between curvature and the distribution of small eigenvalues.

One basic tool of our proof is an integral formula for eigenfunctions on two-dimensional manifolds which we prove in Lemma 2.1. The other tool are certain systems of differential equations for eigenfunctions on spheres (Obata [13], Tanno [20], Ferus [7], Gallot [8]; for Gallot's results cf. also Tanno [20]); they suggest to "compare" eigenfunctions corresponding to the eigenvalues  $\lambda_1$  resp.  $\lambda_2$  (where  $\lambda_1 < \lambda_2$ , as we do not regard their multiplicities) with corresponding eigenfunctions and eigenvalues on spheres (cf. (2.2)-(2.5)). To extend this method to higher eigenvalues  $\lambda_p$  ( $p \geq 3$ ), one would need an analogue to (2.1) which corresponds to the

differential equations for higher eigenfunctions. Generalizing our method to higher dimensions in Section 3 we continue investigations on lower bounds for  $\lambda_2$  on Einstein spaces (Tanno [19], Simon [16]); in Section 4 we improve local results from Lange-Simon [11]. In Section 5 we finally improve results from Simon [15] on minimal submanifolds of spheres. In the two-dimensional case we get the following generalization of a result of Lawson ([12], p. 195, Proposition 3):

**THEOREM B.** *Let  $(M, g)$  be a complete, connected Riemannian manifold,  $\dim M = 2$ ; let  $\tilde{x}: M \rightarrow S^N(1)$ ,  $N > 2$ , be an isometric minimal immersion. If  $\frac{1}{3} \leq \kappa \leq 1$ , then either  $\tilde{x}(M)$  is totally geodesic ( $\kappa \equiv 1$ ) or  $\tilde{x}(M)$  is the Veronese surface in  $S^4(1)$  ( $\kappa \equiv \frac{1}{3}$ ).*

Lawson additionally assumed  $M$  to be closed and  $N = 4$ .



**1. Preliminaries.** Let  $(M, g)$  be a connected Riemannian manifold of class  $C^\infty$ ,  $n = \dim M \geq 2$ , denote by  $\nabla$  the corresponding covariant differentiation and by  $g_{ij}$  resp.  $g^{ij}$  the components of the metric tensor  $g$  resp.  $g^{-1}$  in local coordinates  $(u^i)$ ; denote by  $do$  the volume element on  $M$  and by  $R^h_{ijk}$  resp.  $R_{ij}$  the components of the curvature tensor resp. the Ricci tensor (with the sign of [10], p. 201); let  $R$  denote the scalar curvature (such that  $R = 1$  on the unit sphere). As usual raising and lowering of indices are defined.

Let  $f: M \rightarrow \mathbb{R}$  be a  $C^\infty$ -function, let  $f_{ij} := \nabla_j \nabla_i f$  denote the components of the Hessian  $\text{Hess}(f)$  and denote the Laplacian by  $\Delta f := g^{ij} f_{ij}$ .

**1.1. LEMMA** ([15], (7a-b)). *Let  $f: M \rightarrow \mathbb{R}$  be a  $C^\infty$ -function. Then  $f$  fulfils the equation*

$$\frac{1}{2} \Delta(f_{ij} f^{ij}) = 2 \sum_{i < j} \kappa_{ij} (\sigma_i - \sigma_j)^2 + f^{ij} \nabla_j \nabla_i (\Delta f) + \\ + \nabla_k f_{ij} \nabla^k f^{ij} + f^{ij} f^{kl} [2 \nabla_i R_{jk} - \nabla_k R_{ij}],$$

where  $\sigma_1, \dots, \sigma_n$  are the eigenvalues of the Hessian,  $E_1, \dots, E_n$  are corresponding orthonormal eigenvectors and  $\kappa_{ij}$  is the sectional curvature of the plane  $\{E_i, E_j\}_{i \neq j}$ .

**1.2. LEMMA.** Let  $(M, g)$  be closed (compact without boundary),  $\dim M \geq 2$ . Let  $f, h: M \rightarrow \mathbf{R}$  be  $C^\infty$ -functions. Then

$$\int f_{ij} h^{ij} do - \int \Delta f \Delta h do + \int R^{ij} f_i h_j do = 0.$$

This lemma generalizes the Bochner-Lichnerowicz formula (cf. [2], p. 131).

**1.3. Remark.** Let  $(M, g)$  be closed, connected,  $\dim M \geq 2$ . Then each eigenfunction  $f$  with eigenvalue  $\lambda$  fulfils

$$(n-1)\lambda \int \|\text{grad} f\|^2 do = n \int R^{ij} f_i f_j do + \int \sum_{i < j} (\sigma_i - \sigma_j)^2 do.$$

**1.4. LEMMA [13].** Let  $M$  be a complete connected Riemannian manifold,  $\dim M = n \geq 2$ . There exist a nontrivial function  $f: M \rightarrow \mathbf{R}$ ,  $f \in C^\infty$ , and a real positive constant  $c$  which fulfil the system

$$(1.4.1) \quad n \cdot f_{ij} + c^2 f \cdot g_{ij} = 0$$

iff  $M$  is isometrically diffeomorphic to a sphere  $S^n(c^2)$  of sectional curvature  $c^2$ .

**1.5. LEMMA ([20], [7], [8]).** Let  $(M, g)$  be simply connected and complete,  $\dim M \geq 2$ . There exist a constant  $c \in \mathbf{R}$ ,  $c \neq 0$ , and a nontrivial function  $f \in C^\infty(M)$  such that

$$(1.5.1) \quad f_{ijk} + c^2 (f_j g_{ik} + f_i g_{jk} + 2f_k g_{ij}) = 0$$

iff  $(M, g)$  is isometrically diffeomorphic to an  $n$ -sphere  $S^n(c^2)$  with sectional curvature  $c^2$ .

## 2. Two-dimensional manifolds.

**2.1. LEMMA.** Let  $(M, g)$  be a closed, connected, two-dimensional Riemannian manifold. From  $\Delta f + \lambda f = 0$  it follows that

$$0 = -\frac{1}{2} \int (\lambda - 2\kappa)(\sigma_1 - \sigma_2)^2 do + \int \|\nabla_k f_{ij}\|^2 do - \int \left\{ \kappa\lambda + \frac{1}{2} (\lambda - 2\kappa)^2 \right\} \|\text{grad} f\|^2 do,$$

where  $\sigma_1$  and  $\sigma_2$  are defined in Lemma 1.1.

**Proof.** To use formula 1.1 for  $n = 2$ , we make the following calculations:

$$(i) \quad 2f_{ij} f^{ij} = (\sigma_1 - \sigma_2)^2 + (\Delta f)^2.$$

(ii)  $n = 2$  implies  $R_{ij} = \kappa g_{ij}$ ; the Ricci identity and  $\Delta f = -\lambda f$  give  $\Delta(f_j) = (\kappa - \lambda)f_j$ ; therefore

$$\begin{aligned} f^ij^k \nabla_i R_{jk} &= \{ \nabla_i (R_{jk} f^ij^k) \} - \nabla_i (f^ij^k) R_{jk} - f^ij^k R_{jk} \\ &= \{ \dots \} + \kappa(\lambda - \kappa) \|\text{grad} f\|^2 - \kappa f_{ij} f^{ij}. \end{aligned}$$

(iii) Analogously we have

$$f^ij^k \nabla_k R_{ij} = \{ \nabla_k (f^ij^k R_{ij}) \} - \kappa((\Delta f)^2 - \lambda \|\text{grad} f\|^2).$$

(iv) From (i), (ii) and (iii) it follows that

$$\int f^ij^k [2 \nabla_i R_{jk} - \nabla_k R_{ij}] d\sigma = - \int \kappa(\sigma_1 - \sigma_2)^2 d\sigma + \int \kappa(\lambda - 2\kappa) \|\text{grad} f\|^2 d\sigma.$$

(v) From (i) and Green's theorem we get

$$\int f^ij \nabla_j \nabla_i (\Delta f) d\sigma = -\lambda \int f_{ij} f^{ij} d\sigma = -\frac{\lambda}{2} \left[ \int (\sigma_1 - \sigma_2)^2 d\sigma + \lambda \int \|\text{grad} f\|^2 d\sigma \right].$$

The assertion follows from 1.1, (iv) and (v).

**2.2. Remark.** In the following we "compare" eigenfunctions on  $(M, g)$  and eigenfunctions on a sphere. Motivated by results of Obata [13] and Tanno [20] on the first resp. second eigenfunctions on spheres, we define for an eigenfunction  $f$  a (3, 0)-tensor  $B(f)$  by

$$(2.2.1) \quad B(f)_{ijk} := f_{ijk} + \frac{\lambda + 2\kappa}{4} g_{ij} f_k + \frac{\lambda - 2\kappa}{4} (g_{ik} f_j + g_{jk} f_i).$$

$B(f)$  vanishes identically on spheres if  $\lambda$  is the first or second eigenvalue and  $f$  a corresponding eigenfunction. Formula (2.2.1) implies

$$(2.2.2) \quad \|B(f)\|^2 = \|f_{ijk}\|^2 - \frac{1}{2} \|\text{grad} f\|^2 \left\{ \lambda^2 + \frac{1}{2} (\lambda - 2\kappa)^2 \right\},$$

and Lemma 2.1 gives

$$(2.2.3) \quad 0 = \frac{1}{2} \int (2\kappa - \lambda)(\sigma_1 - \sigma_2)^2 d\sigma + \frac{1}{4} \int \|\text{grad} f\|^2 (\lambda - 2\kappa)(\lambda + 2\kappa) d\sigma + \int \|B(f)\|^2 d\sigma.$$

**2.3. LEMMA.** *If  $(M, g)$  is a closed connected two-dimensional Riemannian manifold, then each eigenfunction  $f$  fulfils*

$$\int (\sigma_1 - \sigma_2)^2 d\sigma = \int (\lambda - 2\kappa) \|\text{grad} f\|^2 d\sigma.$$

**Proof.** Apply 1.2, 2.1 (i) and Green's theorem.

**2.4. LEMMA.** *Let  $(M, g)$  and  $f$  be as in 2.3. Then*

$$0 \geq \int \|B(f)\|^2 d\sigma + \frac{1}{4} \int \|\text{grad} f\|^2 (\lambda - 2\kappa) [4\kappa_0 - (\lambda - 2\kappa)] d\sigma.$$

**Proof.** (2.2.3) and Lemma 2.3.

**2.5. THEOREM.** *Let  $(M, g)$  be closed, connected,  $\dim M = 2$ ,  $\kappa \geq 0$ . Then  $\lambda > 2\kappa_1$  implies  $\lambda \geq 6\kappa_0$ , and  $\lambda = 6\kappa_0$  implies that the universal covering  $(\tilde{M}, \tilde{g})$  is isometrically diffeomorphic to a sphere  $S^2(\kappa_0)$ .*

**Proof.** Let  $\lambda > 2\kappa_1$ . Then  $\lambda \geq 6\kappa_0$  from 2.4. In case of equality  $\lambda = 6\kappa_0$  we get, from 2.4,

$$0 \geq \int \|B(f)\|^2 do + \frac{1}{2} \int \|\text{grad} f\|^2 (\lambda - 2\kappa)(\kappa - \kappa_0) do \geq 0,$$

which implies  $B(f) \equiv 0$  and  $(\kappa - \kappa_0)\|\text{grad} f\|^2 \equiv 0$  on  $(M, g)$ .

If  $\|\text{grad} f\| \neq 0$ , then

$$(2.5.1) \quad 0 = B(f)_{jk} = f_{jk} + \kappa_0(2g_{jk}f_k + g_{ik}f_j + g_{jk}f_i).$$

If  $G := \{p \in M \mid \text{grad} f|_p = 0\}$  is nowhere dense in  $M$ , then (2.5.1) and Tanno's result (1.5) imply the assertion. Assume there exists a non-empty open set  $V \subset G$ ; then  $\text{grad} f = 0$  on  $V$  implies  $df = 0$  and from this we get  $0 = \Delta f = -\lambda f$ , so  $f \equiv 0$  on  $V$ , i.e.  $V \subset N$ , where  $N$  is the nodal set of  $f$ . But this contradicts known results on  $N$  (cf. [3] and [5]).

The last proof was shortened by a hint of S. Tanno.

**3. Einstein spaces.** Let  $(M, g)$  be a closed, connected Einstein space,  $n = \dim M \geq 3$ . Then each eigenvalue fulfils the inequality

$$\lambda \geq nR,$$

and  $\lambda_1 = nR$  iff  $(M, g)$  is isometrically diffeomorphic to a sphere (Obata [13]), where  $R$  is the (constant) scalar curvature. Again we denote by  $\kappa_0$  the minimum of all sectional curvatures.

**3.1. THEOREM.** (a) *Let  $(M, g)$  be a closed, connected Einstein space,  $n \geq 3$ . Then there is no eigenvalue in the interval  $(nR, 2(n+2)\kappa_0 - 2R)$ , i.e.  $\lambda > nR$  implies*

$$\lambda \geq 2(n+2)\kappa_0 - 2R.$$

(b)  $\lambda = 2(n+2)\kappa_0 - 2R$  holds iff the universal covering space  $(\tilde{M}, \tilde{g})$  is isometrically diffeomorphic to a sphere  $S^n(\kappa_0)$ ,  $\kappa_0 = R$ , and  $\lambda = 2(n+1)\kappa_0$  is the second eigenvalue.

**3.2. Remarks.** 1. Part 3.1 (a) is of interest only if  $2(n+2)\kappa_0 - 2R \geq nR$ , i.e.  $2\kappa_0 \geq R$ . Theorem 3.1 improves results of S. Tanno [19] and U. Simon [16]. Furthermore, as a corollary to 3.1, one can improve Theorem 1 in [14].

2. Theorem 3.1 together with a result of Cheng ([4], Theorem 2.4) implies a result which is closely related to a result of Berger ([1], Proposition 6.4).

Let  $(M, g)$  be a closed, connected Einstein space,  $\dim M \geq 3$ . From Cheng's theorem and 3.1 it follows that if

$$n(n+4)\pi^2 \cdot \kappa_1 \leq (n+2)\kappa_0 - 2R,$$

then  $(\tilde{M}, \tilde{g})$  is isometrically diffeomorphic to a sphere.

**3 (Berger).** *If  $(n-2)/(n-1) < \kappa < 1$ , then  $(M, g)$  is a space of constant curvature.*

The assumption on the pinching constant in Remark 2 works only for small  $n$  and generally is not as good as Berger's. But the discussion of equality in 3.1 (b) suggests that it must be possible to improve Cheng's result.

### 3.3. Proof of Theorem 3.1.

(a) Using the ideas from 2.2, we define

$$B(f)_{jk} := f_{jk} + \frac{1}{n+2} (\lambda + 2R) g_{ij} f_k + \frac{1}{n+2} (\lambda - nR) (g_{ik} f_j + g_{kj} f_i),$$

which gives

$$(3.3.1) \quad \|B(f)_{jk}\|^2 = \|f_{jk}\|^2 - \frac{1}{n+2} \|\text{grad} f\|^2 \{3\lambda^2 - 4\lambda R(n-1) + 2n(n-1)R^2\}.$$

We apply Lemma 1.1 to closed Einstein spaces ( $R_{ij} = (n-1)Rg_{ij}$ ) and make the following calculations, using  $\Delta f + \lambda f = 0$ :

(A) 1.1 gives

$$0 = \int \sum_{i < j} 2\kappa_{ij} (\sigma_i - \sigma_j)^2 d\sigma - \lambda \int f_{ij} f^{ij} d\sigma + \int \nabla_k f_{ij} \nabla^k f^{ij} d\sigma;$$

(B) 1.2 gives

$$\int f_{ij} f^{ij} d\sigma = (\lambda - (n-1)R) \int \|\text{grad} f\|^2 d\sigma;$$

(C) 1.3 gives

$$\int \sum_{i < j} (\sigma_i - \sigma_j)^2 d\sigma = (n-1)(\lambda - nR) \int \|\text{grad} f\|^2 d\sigma.$$

If  $(M, g)$  is not a sphere, then  $\lambda > nR$ , and (A)-(C) imply

$$(3.3.2) \quad 0 = \int \sum_{i < j} (\sigma_i - \sigma_j)^2 \left[ 2\kappa_{ij} - \frac{\lambda(\lambda - (n-1)R)}{(n-1)(\lambda - nR)} \right] d\sigma + \int \|\nabla_k f_{ij}\|^2 d\sigma.$$

From (3.3.1) and (C) we infer for  $\lambda > nR$  that

$$(3.3.3) \quad \int \|f_{jk}\|^2 d\sigma = \int \|B(f)_{jk}\|^2 d\sigma + \frac{1}{n+2} \{3\lambda^2 - 4\lambda R(n-1) + 2n(n-1)R^2\} \times \\ \times \frac{1}{(n-1)(\lambda - nR)} \int \sum_{i < j} (\sigma_i - \sigma_j)^2 d\sigma$$

which, together with (3.3.2), gives

$$(3.3.4) \quad 0 = \int \sum_{i < j} (\sigma_i - \sigma_j)^2 \left[ 2\kappa_{ij} - \frac{\lambda + 2R}{n+2} \right] d\sigma + \int \|B(f)_{ijk}\|^2 d\sigma.$$

(b) If  $\lambda = 2[(n+2)\kappa_0 - R]$ , (3.3.4) gives

$$(3.3.5) \quad 0 = \int \|B(f)_{ijk}\|^2 d\sigma + \int \sum_{i < j} (\sigma_i - \sigma_j)^2 \cdot 2(\kappa_{ij} - \kappa_0) d\sigma$$

and, because of  $\kappa_{ij} - \kappa_0 \geq 0$ , we get

$$0 \equiv B(f)_{ijk} = f_{ijk} + 2\kappa_0 g_{ij} f_k + (2\kappa_0 - R)(g_{ik} f_j + g_{kj} f_i).$$

Application of Lemma 3.4 below implies  $(\tilde{M}, \tilde{g})$  to be a sphere and  $\kappa_0 = R$ .

**3.4. LEMMA.** *Let  $(M, g)$  be a connected Einstein manifold,  $\dim M \geq 3$ .*

(a) *If there exists  $f \in C^\infty(M)$  such that  $f$  fulfils*

$$(3.4.1) \quad f_{ijk} + r g_{ij} f_k + s g_{ik} f_j + t g_{jk} f_i \equiv 0 \quad (\text{with } r, s, t \in \mathbf{R})$$

*on  $M$ , then  $s = t$  and  $f$  (resp.  $g := f - c$  for a constant  $c \in \mathbf{R}$ ) is an eigenfunction ( $\Delta f + \lambda f = 0$ ) and either*

$$(3.4.2a) \quad f_{jk} + \frac{\lambda}{n} f g_{jk} \equiv 0 \text{ on } M, \quad \lambda = nR,$$

*or*

$$(3.4.2b) \quad r = 2t \quad \text{and} \quad nr + 2t = 2(n-1)R + 2r = \lambda.$$

(b) *Let  $(M, g)$  be complete and simply connected and assume that  $r \in \mathbf{R}$  fulfils  $(n+2)r > 2R > 0$ . There exists  $f \in C^\infty(M)$  which fulfils (3.4.1) iff  $(M, g)$  is isometrically diffeomorphic to a sphere and  $f$  is a first (3.4.2a) resp. a second (3.4.2b) eigenfunction*

*Proof.* (a) Because of the symmetry of (3.4.1) in  $(i, j)$  we get  $t = s$ . Formula (3.4.1) implies  $(\Delta f)_k + (nr + 2t)f_k = 0$ , so

$$(3.4.3) \quad \Delta(f - c) + \lambda(f - c) = 0, \quad \text{where } \lambda = nr + 2t.$$

As  $g = f - c$  fulfils (3.4.1), without loss of generality we assume  $c = 0$ . From (3.4.1) we get

$$f_{jku} = -r g_{jk} f_u - t(g_{ji} f_{ki} + g_{ki} f_{ji}),$$

so, on the one hand,

$$(3.4.4) \quad f_{jku} - 2f_{ijk} = r(2g_{ij} f_{ku} - g_{jk} f_u) + t(2g_{ik} f_{ju} + 2g_{jk} f_u - g_{ji} f_{ki} - g_{ki} f_{ji}).$$

The Ricci identities imply

$$(3.4.5) \quad f_{ijk} - f_{ajk} = \nabla_k(f_s R^s_{ij}),$$

$$(3.4.6) \quad f_{ijk} = f_{sj} R^s_{ikl} + f_{is} R^s_{jkl} + f_{ijk} = f_{sj} R^s_{ikl} + f_{is} R^s_{jkl} + \nabla_k(f_s R^s_{ij}) + f_{ijk},$$

so, on the other hand,

$$(3.4.7) \quad f_{jku} - 2f_{ijk} = [f_{jku} - f_{jku}] - \{f_{ijk}\} \\ = [\nabla_l(f_s R^s_{jkl})] - \{f_{sj} R^s_{ikl} + f_{is} R^s_{jkl} + \nabla_k(f_s R^s_{ij}) + f_{ijk}\}.$$

As  $R_{ij} = (n-1)Rg_{ij}$ ,  $\Delta f = -\lambda f$ , and  $\nabla^l R_{sjkl} = 0$  (from the second Bianchi identity for Einstein spaces), (3.4.4) and (3.4.7) imply (both after contraction with  $g^{ii}$ )

$$(3.4.8) \quad [\lambda - 2(R(n-1) + r)]f_{jk} + (2t - r)\lambda f g_{jk} = 0.$$

Now either both coefficients vanish, which gives (3.4.2b), or (3.4.2a) holds and then from (3.4.3) and (3.4.8) we get  $\lambda = (n+2)r - 2R$ . Using this value for  $\lambda$ , covariant differentiation of (3.4.8) and comparison with the coefficients of (3.4.1) give  $r = R$  resp.  $\lambda = nR$ .

(b) Apply Obata's [13] result in case (3.4.2a), resp. Tanno's [20] in case (3.4.2b).

**4. Isometries with spheres.** Following ideas of [11] we get the following analogue to Theorem 3.1:

**4.1. THEOREM.** *Let  $(M, g)$  be a connected Einstein space,  $n = \dim M \geq 3$ , which admits  $m > 1$  linear independent eigenfunctions  $f(\alpha)$  ( $\alpha = 1, \dots, m$ ) corresponding to the same eigenvalue  $\lambda$ . Assume furthermore that*

$$(4.1.1) \quad \sum_{\alpha} f(\alpha)^2 = c,$$

where  $c$  is a positive constant.

Then either

$$(4.1.2a) \quad \lambda = nR \text{ and } f(\alpha)_{ij} + \frac{\lambda}{n} f(\alpha) g_{ij} = 0 \text{ for } \alpha = 1, \dots, m,$$

or

$$(4.1.2b) \quad \lambda \geq 2[(n+2)\kappa_0 - R].$$

Equality in (4.1.2b) holds iff

$$(4.1.2c) \quad f(\alpha)_{ijk} + \kappa_0 \{2g_{ij} f(\alpha)_k + g_{ik} f(\alpha)_j + g_{jk} f(\alpha)_i\} = 0 \text{ for } \alpha = 1, \dots, m.$$

*Proof.* Note that  $\lambda > 0$  ([11], Lemma (2.7)) and  $\lambda \geq nR$  ([11], Lemma (3.1)). Defining  $B(\alpha)_{ijk} := B(f(\alpha))_{ijk}$  in analogy to (3.3.1), we get

$$\sum_{\alpha} \|f(\alpha)_{ijk}\|^2 \\ = \sum_{\alpha} \|B(\alpha)_{ijk}\|^2 + \frac{1}{n+2} \sum_{\alpha} \|\text{grad} f(\alpha)\|^2 \{3\lambda^2 - 4(n-1)\lambda R + 2n(n-1)R^2\}.$$



This, together with [11], formulas (2.3), and Lemma 3.4, implies

$$(4.1.3) \quad -\frac{1}{2}(n-1)c\lambda\Delta R \\ = \sum_a \sum_{i < j} 2\kappa(a)_{ij}(\sigma(a)_i - \sigma(a)_j)^2 - (\lambda - (n-1)R)\lambda^2 c + \sum_a \|B(a)_{ijk}\|^2 + \\ + \frac{1}{n+2}\lambda c\{3\lambda^2 - 4(n-1)\lambda R + 2n(n-1)R^2\};$$

here  $\sigma(a)_1, \dots, \sigma(a)_n$  are the eigenvalues of  $f(a)_{ij}$  and  $\kappa(a)_{ij}$  is defined via corresponding pairs of eigenvectors (cf. Lemma 1.1). As  $n \geq 3$ , we have  $R = \text{const}$ . Introducing now the notation  $\kappa_0 := \inf \kappa$ , from [11] (formula (3.4c)) and (4.1.3) we get

$$(4.1.4) \quad (\lambda - nR)(\lambda - 2[(n+2)\kappa_0 - R]) \geq 0.$$

So  $\lambda > nR$  gives (4.1.2b). The case  $\lambda = nR$  was discussed in [11] (Theorem 3.2), while the equality  $\lambda = 2[(n+2)\kappa_0 - R]$  implies  $B(a)_{ijk} = 0$  from (4.1.3), and this together with 3.4 (a) gives (4.1.2c).

**4.2. LEMMA.** *Let  $(M, g)$  be a connected Riemannian manifold,  $\dim M = 2$ , which admits  $m > 1$  eigenfunctions  $f(a)$  ( $a = 1, \dots, m$ ) which fulfil (4.1.1). Then the curvature  $\kappa$  fulfils the following differential inequality:*

$$2\Delta\kappa \leq (\lambda - 2\kappa)(\lambda - 6\kappa).$$

*Equality holds iff  $B(a)_{ijk} \equiv 0$  for  $a = 1, \dots, m$ .*

*Proof.* We use (4.1.3), where  $n = 2$  and  $R = \kappa$ , and

$$\sum_a (\sigma(a)_1 - \sigma(a)_2)^2 = \lambda c(\lambda - 2R)$$

(cf. [11], (3.4c)).

**4.3. THEOREM.** *Let  $(M, g)$  be a complete, connected Riemannian manifold,  $\dim M = 2$ , with  $m$  linear independent eigenfunctions  $f(a)$  ( $a = 1, \dots, m$ ) corresponding to the eigenvalue  $\lambda$ , which fulfil (4.1.1). Then either  $(M, g)$  is isometrically diffeomorphic to a sphere  $S^2(\kappa)$  of curvature  $\kappa$  and*

$$\lambda = 2\kappa_0 \quad \text{or} \quad \lambda \geq 6\kappa_0.$$

*$\lambda = 6\kappa_0$  holds iff the universal covering  $(\tilde{M}, \tilde{g})$  is isometrically diffeomorphic to a sphere and  $\lambda$  is the second eigenvalue.*

The proof is similar to the proof of Theorem (5.3) in [11], but we apply Tanno's result (1.5) instead of Obata's result in [13].

It is obvious that the results of this paragraph improve results in [11] even if we did not formulate all the consequences of our results here (cf. especially [11], § 4).

**5. Minimal submanifolds of spheres.** Let  $\tilde{\omega}: M \rightarrow S^{N-1}(1)$  be an isometric minimal immersion,  $N-1 > n = \dim M$ .

**5.1.** If  $j: S^{N-1}(1) \rightarrow E^N$  is the canonical embedding into a Euclidean space such that the center of  $S^{N-1}(1)$  is the origin of the canonical coordinate system of  $E^N$ , then the position vector  $x$  (with respect to this coordinate system) of the immersion  $j\tilde{\omega}$  fulfils (cf. Takahashi [17])

$$(5.1.1) \quad \Delta x + nx = 0.$$

**5.2.** Each coordinate function  $x(a)$  is an eigenfunction corresponding to  $\lambda = n$ ; furthermore  $\langle x, x \rangle = 1$ , where  $\langle, \rangle$  denotes the inner product in  $E^N$ . Let  $\tilde{S}$  resp.  $S$  denote the squares of the lengths of the second fundamental forms of the immersions  $\tilde{\omega}$  resp.  $j\tilde{\omega}$ . We have

$$(5.2.1) \quad S - n = \tilde{S} = n(n-1)(1-R).$$

**5.3.** Applying Lemma 1.1 to each coordinate function, we get (cf. [15])

$$(5.3.1) \quad \frac{1}{2} \Delta \tilde{S} = \frac{1}{2} \Delta S = \sum_{a=1}^N \sum_{i < j} 2\kappa(a)_{ij} (\sigma(a)_i - \sigma(a)_j)^2 - nS + \langle x_{ijk}, x^{ijk} \rangle.$$

**5.4.** In analogy to (3.3.1) we define

$$(5.4.1) \quad \bar{X}_{ijk} := x_{ijk} + \frac{1}{n+2} (n+2R) g_{ij} x_k + \\ + \frac{n}{n+2} (1-R) (g_{ik} x_j + g_{kj} x_i)$$

which gives

$$(5.4.2) \quad \langle \bar{X}_{ijk}, \bar{X}^{ijk} \rangle = \langle x_{ijk}, x^{ijk} \rangle - \frac{n^2}{n+2} [3n - 4R(n-1) + 2(n-1)R^2].$$

Furthermore (cf. [11], (3.4c))

$$(5.4.3) \quad \sum_a \sum_{i < j} (\sigma(a)_i - \sigma(a)_j)^2 = n^2(n-1)(1-R),$$

$$(5.4.4) \quad (1-R)nS = \frac{n - (n-1)R}{n-1} \sum_a \sum_{i < j} (\sigma(a)_i - \sigma(a)_j)^2.$$

**5.5.** Assume  $\tilde{\omega}(M)$  not to be a sphere; (5.3.1) and (5.4.1)-(5.4.4) give

$$(5.5.1) \quad \frac{1}{2} \Delta \tilde{S} = \sum_a \sum_{i < j} (\sigma(a)_i - \sigma(a)_j)^2 \left\{ 2\kappa(a)_{ij} - \frac{n+2R}{n+2} \right\} + \langle \bar{X}_{ijk}, \bar{X}^{ijk} \rangle,$$

which implies the following result:

**5.6. THEOREM.** Let  $\tilde{\omega}: M \rightarrow S^{N-1}(1)$  be a minimal isometric immersion into the unit-sphere, where  $(M, g)$  is complete; assume

$$(5.6.1) \quad 2(n+2)\kappa_0 \geq n+2R.$$

Then either  $\tilde{\omega}(M)$  is totally geodesic, i.e.  $\tilde{\omega}(M)$  is a great  $n$ -sphere in  $S^{N-1}(1)$ , or  $\tilde{\omega}(M)$  is an isometric immersion (but no embedding) of an  $n$ -sphere  $S^n(\kappa_0)$ ,  $\kappa_0 = n/2(n+1)$ .

**5.7. Remark.** (a) As  $\tilde{\omega}(M) \subset S^{N-1}(1)$ , we have  $\kappa \leq 1$  and  $R \leq 1$ . If  $1 \geq \kappa \geq \frac{1}{2}$ , then (5.6.1) is fulfilled. So 5.6 improves Theorem I in [15]. The immersion of  $S^n(\kappa_0)$ ,  $\kappa_0 = n/2(n+1)$ , is one of the examples given in [6].

**5.8. THEOREM.** Let  $(M, g)$  be closed and let  $x: M \rightarrow E^N$  be an isometric immersion with parallel mean curvature vector  $\xi$  (therefore  $\langle \xi, \xi \rangle = \text{const}$ ).

If  $2n\{(n+2)\kappa_0 - R\} \geq \langle \xi, \xi \rangle$ , then  $x(M)$  is an  $n$ -sphere or  $\tilde{\omega}(M)$  is an isometric minimal immersion (but no embedding) of an  $n$ -sphere  $S^n(\kappa_0)$ ,  $\kappa_0 = \langle \xi, \xi \rangle / 2n(n+1)$ , into  $S^{N-1}(\bar{\kappa})$ ,  $\bar{\kappa} = \langle \xi, \xi \rangle / n^2$ .

**Proof.** Cf. [15], Theorem II (in [15], without loss of generality, the constant  $\langle \xi, \xi \rangle$  was chosen to be  $\langle \xi, \xi \rangle = n^2$ ).

Especially for  $n = 2$  we get from (5.5.1) Theorem B which was formulated in the introduction. Naturally there is an analogue to Theorem 5.8:

**5.9. THEOREM.** Let  $(M, g)$  be closed,  $\dim M = 2$ , and  $x: M \rightarrow E^N$  be an isometric immersion with parallel mean curvature vector and  $12\kappa_0 \geq \langle \xi, \xi \rangle$ .

Then either  $x(M)$  is a great sphere  $S^2(\bar{\kappa}) \subset S^4(\bar{\kappa})$ ,  $\bar{\kappa} = \langle \xi, \xi \rangle / 4$ , or  $x(M)$  is a Veronese surface of constant curvature  $\hat{\kappa} = \langle \xi, \xi \rangle / 12$  in  $S^4(\bar{\kappa})$ .

**Addendum.** As a result of discussion with K. Voss and P. Buser (Geometrietagung Oberwolfach 1978) we get the following corollary to Theorem A:

**THEOREM C.** Let  $(M, g)$  be closed, simply connected,  $\dim M = 2$ , and  $\kappa_0 \geq \frac{1}{3}\kappa_1 > 0$ . Then

(a) There are exactly three eigenvalues  $\lambda_1 \leq \lambda_2 \leq \lambda_3$  (counted with their multiplicity) in the interval  $[2\kappa_0, 2\kappa_1]$ , and  $\lambda_4 \geq 6\kappa_0 \geq 2\kappa_1$ .

(b) If  $\lambda_1 = 2\kappa_0$  or  $\lambda_3 = 2\kappa_1$ , then  $(M, g)$  is isometrically diffeomorphic to a sphere. If  $\lambda_4 = 6\kappa_0$ , then  $(M, g)$  is isometrically diffeomorphic to  $S^2(\kappa_0)$ .

**Proof.** (a) Let  $(M, g)$  be given as above. For simplicity, assume that  $\kappa_0 \geq \frac{1}{3}$  and  $\kappa_1 = 1$ . By the existence theorem of Weyl<sup>(1)</sup> one can realize the Riemannian manifold as an ovaloid in the Euclidean space

<sup>(1)</sup> Cf., e.g., L. Nirenberg, *The Weyl and Minkowski problems in differential geometry in the large*, Communications on Pure and Applied Mathematics 6 (1953), p. 337-394.

$E^3$ , choosing  $M = S^2(1)$  and considering the curvature  $\kappa$  as a given function of the outer normal  $\xi$  of  $S^2(1)$ . Define a one-parameter family  $(S^2(1), g(t))$  of ovaloids with metric  $g(t)$  in the following way:

For each  $t \in [0, 1]$  there exists an ovaloid with curvature  $\kappa(t)$ , given as a function of the outer normal, where

$$\frac{1}{\kappa(t)} = (1-t)\frac{1}{\kappa} + t, \quad t \in [0, 1].$$

As

$$\int \frac{1}{\kappa} \xi d\omega = 0 \quad \text{and} \quad \int \xi d\omega = 0 \quad \text{on } S^2(1),$$

we get

$$\int \frac{1}{\kappa(t)} \xi d\omega = 0 \quad \text{for each } t.$$

The metric  $g(t)$  changes continuously with  $t$  and so do  $\lambda_1(t)$ ,  $\lambda_2(t)$ ,  $\lambda_3(t)$ . As  $\lambda_1(0) = \lambda_2(0) = \lambda_3(0) = 2\kappa_0 = 2\kappa_1$  on the unit sphere, from Theorem A we get  $2\kappa_0(t) \leq \lambda_1(t)$ ,  $\lambda_2(t)$ ,  $\lambda_3(t) \leq 2\kappa_1(t)$  for  $t \in [0, 1]$ , which especially holds for  $t = 1$ . Thus

$$(M, g(1)) = (M, g).$$

(b) Assume that an eigenvalue  $\lambda$  fulfils  $\lambda = 2\kappa_1$ . Then 2.4 implies

$$0 \geq \int \|B(f)\|^2 d\sigma + \int \|\text{grad} f\|^2 (\kappa_1 - \kappa)(2\kappa_0 + \kappa - \kappa_1) d\sigma \geq 0.$$

As  $G := \{p \in M \mid \text{grad} f|_p = 0\}$  is nowhere dense in  $M$  (cf. 2.5), we get

$$(\kappa_1 - \kappa)(2\kappa_0 + \kappa - \kappa_1) = 0 \quad \text{on } M,$$

which together with  $\kappa_1 \geq \kappa \geq \frac{1}{2}\kappa_1$  gives

$$(\kappa_1 - \kappa)(\kappa - \kappa_0) = 0 \quad \text{on } M.$$

But this is possible only if  $\kappa = \text{const}$  on  $M$ . Therefore  $\lambda_1 = 2\kappa_0$ , which implies the assertion.

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