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EIGENVALUES OF THE LAPLACIAN IN TWO DIMENSIONS*

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Abstract. The eigenvalue problem for the Laplace operator in two dimensions is classical in mathematics and physics. Nevertheless, computational methods for estimating the eigenvalues are still of much current interest, particularly in applications to acoustic and electromagnetic waveguides. Although our primary interest is with the computational methods, there are a number of theoretical results on the behavior of the eigenvalues and eigenfunctions that are useful for understanding the methods and, in addition, are of interest in themselves. These results are discussed first and then the various computational methods that have been used to estimate the eigenvalues are reviewed with particular emphasis on methods that give error bounds. Some of the more powerful techniques available are illustrated by applying them to a model problem.

1. Introduction. The eigenvalue problem for the Laplace operator is

$$-\Delta u = \lambda u \quad \text{in } R \ . \tag{1.1}$$

Also known as the *Helmholtz equation* [56], it arises from separating the time variable out of the wave equation, and so occurs in many applications. Equation (1.1) is to be considered on a bounded, two-dimensional region R having boundary C with

$$u = 0 \quad \text{on } C \ . \tag{1.2}$$

Equations (1.1), (1.2) may represent the vibration of a *fixed membrane,* with the eigenvalue $\lambda = k^2$, where k is proportional to a *principal frequency of vibration,* and the eigenfunction u represents the shape of a *mode of vibration.* These are also the frequencies and modes of the *simply supported plate* of the same shape. Equations (1.1), (1.2) may also represent the propagation of a wave down a *waveguide* (either acoustical or electrical), with cross section R, where k is proportional to a *cutoff frequency* and u is called either an E-*mode* or TM-*mode* in an electrical waveguide [28].

This paper divides naturally into two parts, although we do not formally designate them this way. In § 2 through 11, we review theoretical results on the eigenvalues and eigenfunctions that have some application to numerical methods. Variational principles, inequalities, smoothness of the eigenfunctions, nodal lines, symmetry, transformations of the problem, and asymptotic formulas are discussed. We particularly note regions for which the problem has exact solutions in terms of elementary or special functions.

In § 12 through 17, we review numerical methods for estimating the eigenvalues and eigenfunctions. Rayleigh-Ritz, finite elements, intermediate methods, finite differences, point matching, *a posteriori-a priori* inequalities, Galerkin, and other methods are discussed. We have especially included references to papers containing specific numerical results for particular regions.

In § 18, we observe that the more powerful of these numerical methods become much more effective when combined with conformal transformations. This is illus-

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trated by applying Rayleigh-Ritz, intermediate methods, finite differences, and *a posteriori-a priori* inequalities to the same nontrivial waveguide problem.

2. Variational formulation and consequences. Pockels [115] first showed that (1.1), (1.2) has a spectrum of infinitely many positive eigenvalues

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots , \tag{2.1}$$

with *no finite accumulation point.* The associated eigenfunctions $u_1, u_2, u_3, \cdots$, can be normalized so they form a *complete orthonormal set* for $L_2(R)$, i.e.,

$$\int_R u_i u_j \, dxdy = \delta_{ij} , \tag{2.2}$$

where δ_{ij} is Kronecker's delta.

The eigenvalues satisfy the *minimax principle*

$$\lambda_n = \min \max \frac{\int_R [(\partial u/\partial x)^2 + (\partial u/\partial y)^2] \, dxdy}{\int_R u^2 \, dxdy} , \tag{2.3}$$

where the maximum is over all linear combinations of the form

$$u = a_1\phi_1 + a_2\phi_2 + \cdots + a_n\phi_n , \tag{2.4}$$

and the minimum is over all choices of n linearly independent continuous and piecewise-differentiable functions $\phi_1, \phi_2, \cdots, \phi_n$ vanishing on C. The ratio of quadratic forms on the right side of (2.3) is called the *Rayleigh quotient.* A good discussion of the history of the minimax and related variational principles is given in [140] and an interesting history of the membrane problem in the 19th century is given in [37].

One important consequence of the minimax principle is the *Rayleigh-Ritz method* for obtaining upper bounds for eigenvalues that will be discussed in § 12. Another consequence of the minimax principle is *domain monotonicity.* Let R' be a region that properly contains the region R,

$$R \subset R' . \tag{2.5}$$

Let $u_1, u_2, \cdots, u_n$ be the first n eigenfunctions for R and extend them as zero in $R' - R$. They are then admissible functions in the minimax principle for R' and so

$$\lambda_n' \leq \lambda_n , \tag{2.6}$$

where λ_n' is the nth eigenvalue of R'. The larger region has smaller eigenvalues.

3. Elementary solutions. We quote Rayleigh [128] who says, "The theory of the free vibrations of a membrane was first successfully considered by Poisson [116]. His theory in the case of the rectangle left little to be desired." For the rectangle $0 \leq x \leq a$, $0 \leq y \leq b$, the eigenfunctions are

$$u_{m,n} = \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right), \tag{3.1}$$

with associated eigenvalues

$$\lambda_{m,n} = \pi^2 \left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right], \qquad m,n = 1, 2, \cdots . \tag{3.2}$$

Rayleigh goes on to say, "Clebsch [27] gives the general theory of the circular membrane." For the circle of radius a, the eigenfunctions are, in polar coordinates,

$$u_{m,n} = J_m\left(\frac{j_{m,n}r}{a}\right)[A\cos m\theta + B\sin m\theta]\,, \tag{3.3}$$

with eigenvalues

$$\lambda_{m,n} = \left(\frac{j_{m,n}}{a}\right)^2\,, \quad m = 0, 1, 2, \cdots\,, \quad n = 1,2, \cdots\,, \tag{3.4}$$

where $j_{m,n}$ is the nth zero of the mth Bessel function,

$$J_m(j_{m,n}) = 0\,. \tag{3.5}$$

A table of zeros of Bessel functions is given in [2]. Note that asymptotically

$$j_{m,n} \sim \left(n + \frac{m}{2} - \frac{1}{4}\right)\pi \quad \text{as } n \to \infty\,. \tag{3.6}$$

Rayleigh also observes that the circular sector $0 \le \theta \le \pi\beta$ has eigenfunctions

$$u_{m,n} = J_{m/\beta}\left(\frac{j_{m,n}r}{a}\right)\sin\left(\frac{m\theta}{\beta}\right) \tag{3.7}$$

given in terms of Bessel functions of fractional order. When $\beta = 2$ and m is odd, these reduce to algebraic and trigonometric functions.

The annulus $a \le r \le b$ has eigenfunctions

$$u_{m,n} = \left[Y_m(k_{m,n})J_m\left(\frac{k_{m,n}r}{a}\right) - J_m(k_{m,n})Y_m\left(\frac{k_{m,n}r}{a}\right)\right][A\cos m\theta + B\sin m\theta], \tag{3.8}$$

where Y_m is the mth Bessel function of the second kind. The eigenvalues are

$$\lambda_{m,n} = \left(\frac{k_{m,n}}{a}\right)^2\,, \quad m = 0, 1, 2, \cdots\,, \quad n = 1, 2, \cdots\,, \tag{3.9}$$

where $k_{m,n}$ is the nth root of

$$Y_m(k)J_m\left(\frac{kb}{a}\right) - J_m(k)Y_m\left(\frac{kb}{a}\right) = 0\,. \tag{3.10}$$

Tables of these roots are given in [66]. Sectors of an annulus can also be treated similarly.

4. Inequalities. The *Faber-Krahn inequality* states that the circular region has the smallest λ_1 of all regions with the same area. Or, from (3.4),

$$\lambda_1 \ge \frac{2\pi}{A}j_{0,1}^2\,, \tag{4.1}$$

where A is the area of the region. This inequality was first conjectured by Rayleigh and proved independently by Faber [44] and Krahn [72]. It has been generalized to regions lying inside circular sectors in [111].

Barta's inequality [5] states that if ϕ is a positive, twice continuously differentiable function on R,

$$\inf_R\left(\frac{-\Delta\phi}{\phi}\right) \le \lambda_1 \le \sup_R\left(\frac{-\Delta\phi}{\phi}\right). \tag{4.2}$$

The inequality

$$\lambda_{n+1} \leq 3\lambda_n \tag{4.3}$$

was shown in [108], where it was conjectured that λ_2/λ_1 attains its maximum for the circular regions, i.e.,

$$\frac{\lambda_2}{\lambda_1} \leq \left(\frac{j_{1,1}}{j_{0,1}}\right)^2 = 2.5387 \cdots . \tag{4.4}$$

See [60] for related inequalities.

The eigenfunction u_1 associated with the first eigenvalue λ_1 does not change sign in R (see § 6) and may be taken to be positive. It satisfies

$$4\pi \int_R u_1^2 \, dxdy \leq \lambda_1 \left(\int_R u_1 \, dxdy\right)^2 , \tag{4.5}$$

with equality only for the circle [109]. In [110], it was shown that for convex R

$$\frac{\pi}{2A} \int_R u_1 \, dxdy \leq \max_R u_1 , \tag{4.6}$$

and

$$u_1(P) \leq \frac{\lambda_1^{1/2} d(P)}{A} \int_R u_1 \, dxdy , \tag{4.7}$$

where $d(P)$ is the distance from a point P in R to the boundary C. Generalizations for L^p norms of u_1 are in [71] and [23].

Many additional inequalities can be found in the monograph of Pólya and Szegö [118], the book of Bandle [4] with 123 references, and the excellent review article of Payne [107] with 192 references.

5. Smoothness of eigenfunctions. The eigenfunctions have the *unique continuation property,* that is, a function cannot satisfy (1.1) in a region and vanish on an open subset of the region without vanishing identically in the region.

Each eigenfunction u_n is infinitely differentiable (C^∞) at the interior points of R [13]. At a straight segment of the boundary, u_n can be reflected as an odd function across the boundary. The resulting function satisfies (1.1) in a full neighborhood of that portion of the boundary and so is C^∞ across the boundary on straight segments. Indeed, u_n can be reflected as a C^∞ function across any part of the boundary that is C^∞.

At a corner of R formed by two straight lines meeting at an interior angle of π/m for m integer, the eigenfunction u_n can be reflected m times, resulting in a C^∞ function in a region locally bounded by a simple straight line through the point that was the corner. A final reflection shows that u_n can be extended as a C^∞ function in the neighborhood of such corners. The same is true if the π/m angle is formed by the intersection of C^∞ arcs, which become a single C^∞ arc after m reflections.

For example, the eigenfunctions of the equilateral triangle may be extended by reflection as C^∞ functions to the entire plane. This fact, plus the necessary periodicity of such an extension, led Lamé [83] to the discovery of the eigenfunctions and eigenvalues of the equilateral triangle. These are also given in the more accessible paper [112]. For the triangle $0 < y < x\sqrt{3}$, $y < (1 - x)\sqrt{3}$, the eigenvalues are

$$\lambda = \frac{16\pi^2}{27}(m^2 + n^2 - mn) , \qquad m,n = 0, \pm 1, \pm 2, \cdots , \tag{5.1}$$

where 3 divides $m + n$, $m \neq 2n$, and $n \neq 2m$. The eigenfunctions are

$$u = \sum \pm \exp\left[\left(bx + \frac{(2b - a)y}{\sqrt{3}}\right)\left(\frac{2\pi i}{3}\right)\right], \tag{5.2}$$

where (a,b) runs over $(-n,m-n)$, $(-n,-m)$, $(n-m,-m)$, $(n-m,n,)$, (m,n), $(m,m-n)$ with the $\pm$ sign alternating.

At a more general corner of R, the behavior of u_n may not be as nice. If two C^∞ arcs meet at the corner with interior angle π/α and the origin of coordinates is placed on the corner so that the lines $\theta = 0$ and $\theta = \pi/\alpha$ are tangent to the bounding curves, then in [91] it is shown that there is a constant γ_n such that

$$u_n = \gamma_n r^\alpha \sin(\alpha\theta) + o(r^\alpha) \quad \text{as } r \to 0\,. \tag{5.3}$$

If α is not an integer, the results obtained by formally differentiating are valid. When α equals an integer m, slightly better results hold: the result of formally differentiating is valid for derivatives of order $< m$; the derivatives of order $\leq 2m - 1$ are bounded; the derivatives of order $2m$ are $O(\log r)$; and those of order $n > 2m$ are $O(r^{2m-n})$.

6. Nodal lines. The set of points in R where $u_n = 0$ is the *nodal set* of u_n. By the unique continuation property, it consists of *curves* that are C^∞ in the interior of R. Where nodal lines cross, they form equal angles [31]. Also, when nodal lines intersect a C^∞ portion of the boundary, they form equal angles. Thus, a single nodal line intersects the C^∞ boundary at right angles, two intersect it at 60° angles, and so forth. (This simple observation is occasionally overlooked in drawings of nodal patterns in some papers.)

Courant's nodal line theorem [31] states that the nodal lines of the nth eigenfunction divide R into no more than $n - 1$ subregions (called *nodal domains*). In particular, u_1 has no interior nodes and so λ_1 is a simple eigenvalue (has multiplicity one). If R is convex, u_1 has convex level curves [1]. A demonstration of the nodal line theorem in [114] is based on the minimax property and unique continuation, and it is also observed that equality cannot hold for more than a finite number of n. This is a consequence of combining the Faber-Krahn inequality (4.1) for each nodal domain and *Weyl's law*, which is the asymptotic relation for the nth eigenvalue

$$\lambda_n \sim \frac{4\pi n}{A} \quad \text{as } n \to \infty\,, \tag{6.1}$$

where A is the area of R.

We remark that the nth eigenvalue λ_n of a region R is the first eigenvalue for each of its nodal domains and a higher eigenvalue for a union of nodal domains. By such reasoning, particular eigenvalues of special regions may be found. For example, the square, $0 < x,y < \pi$, has the eigenvalues $\lambda_2 = \lambda_3 = 5$, of multiplicity two, with the two-dimensional family of eigenfunctions

$$u = A \sin x \sin 2y + B \sin 2x \sin y\,. \tag{6.2}$$

For $A = 0$, u has a vertical nodal line, for $B = 0$, a horizontal nodal line, for $A = \pm B$, a diagonal nodal line, but for other values of A and B the nodal line is the curve specified by the transcendental equation

$$A \cos y + B \cos x = 0\,. \tag{6.3}$$

Thus, $\lambda = 5$ is the first eigenvalue for all of all of the regions bounded by sides of the square and the curves (6.3). For $A = 1$, $B = 2$, this looks like the lid of a grand piano (Fig. 1a).

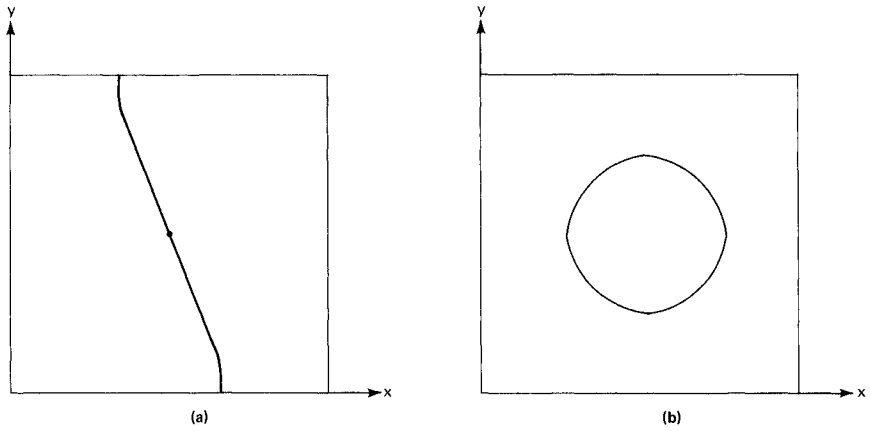

FIG. 1. *Nodal lines of eigenfunctions of the square* (a) $\cos y + 2\cos x = 0$, (b) $\sin^2 x + \sin^2 y = 3/2$.

Similarly, $\lambda_5 = \lambda_6 = 10$ with eigenfunctions

$$u = A \sin x \sin 3y + B \sin 3x \sin y , \tag{6.4}$$

which for $A = B$ has the closed, ring-like, nodal curve

$$\sin^2 x + \sin^2 y = 3/2 . \tag{6.5}$$

Thus, $\lambda = 10$ is the first eigenvalue for the region (Fig. 1b) enclosed by (6.5), as well as for the complementary region consisting of the square with a hole of this shape.

7. Symmetry. The eigenfunctions of a region possessing symmetry can be partitioned into classes according to that symmetry. If $x = 0$ is an axis of mirror symmetry for R and u is an eigenfunction of R with eigenvalue λ, so are

$$u(x,y) \pm u(-x,y) .$$

Thus, every eigenfunction of such a region may be classified as *even* (symmetric) or *odd* (antisymmetric) about the axis. Eigenfunctions in one symmetry class are orthogonal to those in the other. Regions with more than one axis of symmetry may have their eigenfunctions further partitioned. For example, the rectangle has four symmetry classes corresponding to functions that are even or odd across horizontal and vertical axes, while the square has eight symmetry classes when diagonal symmetry is also considered.

The odd eigenfunctions have the line of symmetry as a nodal line. By a completeness consideration, the odd eigenfunctions are all the eigenfunctions of the half-region. Thus, the eigenfunctions and eigenvalues of the isosceles right triangle are those of the square that are odd about a diagonal, and those of the $30° - 60° - 90°$ triangle are those of the equilateral triangle that are odd about an altitude.

A region is centro-symmetric if $(x,y) \in R$ implies $(-x,-y) \in R$. The eigenfunctions of such regions can be classified as even or odd about the center point. For example, a trapezoid is symmetric about its midpoint, but generally has no other symmetry.

A third type of symmetry is the *reflection symmetry* of Hersch [59]. An example will show the idea. Let R be the L-shaped region that is the union of three unit squares:

$[0,1] \times [0,1]$, $[1,2] \times [0,1]$, $[0,1] \times [1,2]$ (see Fig. 2). Let u satisfy $\Delta u + \lambda u = 0$ in R, let $(x,y) \in [0,1] \times [0,1]$, and consider the points $(2 - x,y)$, $(x,2 - y)$, which are the reflections in the lines $x = 1$ and $y = 1$, respectively. The function

$$f(x,y) = u(x,y) - u(2 - x,y) - u(x,2-y)$$

satisfies $-\Delta f = \lambda f$ in the basic square $[0,1] \times [0,1]$ and vanishes on its boundaries. Thus, λ is either an eigenvalue of the square, or f vanishes identically, in which case

$$u(2 - x,y) + u(x,2 - y) = u(x,y) \ . \tag{7.1}$$

This is the reflection symmetry relation. It can be used whenever a region is formed by reflecting a basic region two or more times about straight sides.

Use of symmetry classes is a valuable device in numerical calculations as it permits quadratures to be performed on basic regions, reduces multiplicities of eigenvalues, increases separation between successive eigenvalues, and specializes the form of trial functions used in Rayleigh-Ritz and other estimation procedures.

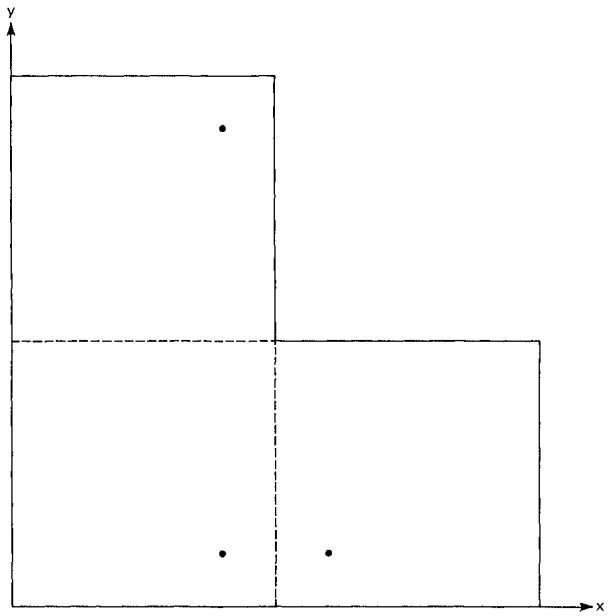

FIG. 2. *The reflection symmetry of Hersch on the L-shaped region that is the union of three squares.*

8. Multiplicities of eigenvalues. Regions with symmetries will often have eigenvalues with multiplicities greater than one. We have seen that the square of side π has eigenvalues

$$\lambda = m^2 + n^2, \qquad m,n = 1,2, \cdots \ . \tag{8.1}$$

Whenever $m \neq n$, λ will have multiplicity at least two. A little number theory allows us to say more. If

$$\lambda = 2^{\alpha} p_1^{r_1} \cdots p_k^{r_k} q_1^{s_1} \cdots q_\ell^{s_\ell} \tag{8.2}$$

is the decomposition of λ into distinct primes, where the p_i are of the form $4t + 1$ and

the q_i of the form $4t + 3$, then all of the s_i must be even and the multiplicity of λ is [55]

$$\prod_{i=1}^{k} (r_i + 1). \tag{8.3}$$

Thus, the multiplicity of 5 is two, of 25 is three, of $65 = 5 \cdot 13$ is four, and so forth. In particular, there are eigenvalues of arbitrarily large multiplicity. Eigenvalues with high multiplicity will have large families of eigenfunctions, some combinations of which will have interesting nodal patterns.

This behavior is not typical, however, of general regions. It is shown in [133], [134] that for "most" regions (a) the eigenvalues are all simple, (b) the nodal lines of the eigenfunctions do not cross, and (c) the critical points of the eigenfunctions are simple maxima or minima. Given any region, there will be arbitrarily small perturbations of it that will have these properties. Thus, multiple eigenvalues will become distinct and nodal crossings will pull apart.

9. Transformations. The eigenvalues are unchanged by translations or rotations of the region. A dilation of the form $x' = ax$, $y' = ax$, by a constant a results in the eigenvalues being *divided* by a^2,

$$\lambda' = \lambda/a^2 . \tag{9.1}$$

Relation (9.1) permits calculations to be performed on standard regions.

The affine transformation

$$x' = x + y\cos\theta , \qquad y' = y\sin\theta \tag{9.2}$$

maps the rectangle $[0,a] \times [0,b]$ onto a parallelogram with the same side lengths and interior angle θ (Fig. 3). If $-\Delta u = \lambda u$ on the parallelogram and we define $U(x,y) = u(x', y')$, then, by the chain rule, U satisfies

$$-\frac{1}{\sin^2\theta}\left[\frac{\partial^2 U}{\partial x^2} - 2\cos\theta\frac{\partial^2 U}{\partial x\partial y} + \frac{\partial^2 U}{\partial y^2}\right] = \lambda U \tag{9.3}$$

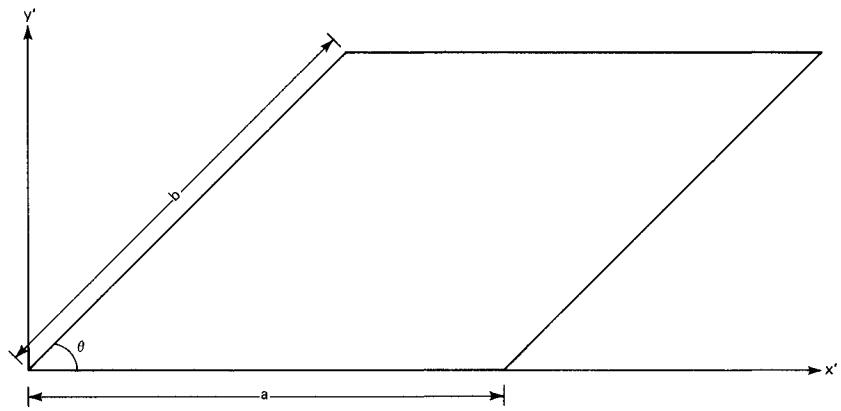

FIG. 3. *The parallelogram that is the image of a rectangle of sides a and b under an affine transformation.*

on the rectangle. Hence, solving (9.3) with $U = 0$ on the boundary of the rectangle is equivalent to solving (1.1), (1.2) on the parallelogram.

Consider an analytic function

$$w = f(z) , \tag{9.4}$$

which maps a rectangle in the z-plane onto the region R in the w-plane. If $U(z) = u(w)$, then (1.1), (1.2) is equivalent to

$$-\Delta U = \lambda \left| \frac{dw}{dz} \right|^2 U \tag{9.5}$$

on the rectangle with $U = 0$ on its boundary. Equation (9.5) represents either a membrane with nonuniform density or a waveguide with nonuniform dielectric coefficient.

We may also consider regions R that are the images of the unit circle $|z| \leq 1$ under a conformal map (9.4). Then (1.1), (1.2) is equivalent to (9.5) on the circle with $U = 0$ on its boundary. It will be shown in § 18 that the result (9.5) greatly extends the applicability of many computational methods.

10. Separation of variables. We have seen in § 3 that (1.1) can be solved by separating the variables in rectangular and polar coordinates. In [103] it is shown that the solution of (1.1) by separation of variables on a region R is equivalent to the solution of (9.5) on the rectangle by separation of variables in Cartesian coordinates, i.e.,

$$\left| \frac{dw}{dz} \right|^2 = f(x) + g(y) . \tag{10.1}$$

It is further shown that this can happen only when w is either (a) a quadratic polynomial of z, or (b) of the form $ae^z + be^{-z}$. These correspond to (a) *parabolic coordinates,* or (b) *elliptic coordinates,* the *only* coordinate systems in which (1.1) can be solved by separation of variables (rectangular and polar coordinates being special cases).

We consider elliptic coordinates first. The function

$$w = \sin z , \tag{10.2}$$

maps the rectangle $[-\pi/2, \pi/2] \times [-k, k]$ in the z-plane onto the interior of the ellipse

$$\left(\frac{\xi}{\cosh k}\right)^2 + \left(\frac{\eta}{\sinh k}\right)^2 = 1 \tag{10.3}$$

in the w-plane with slits from the foci $(\pm 1, 0)$ to the tips of the major semi-axes $(\pm \cosh k, 0)$ (see Fig. 4). Thus

$$\left| \frac{dw}{dz} \right|^2 = \frac{1}{2} (\cos 2x + \cosh 2y) . \tag{10.4}$$

Putting this in (9.5) and assuming $U = X(x)\, Y(y)$ leads to the simultaneous equations

$$-X'' = \left(a + \frac{1}{2}\lambda \cos 2x\right) X , \tag{10.5}$$

$$Y'' = \left(a - \frac{1}{2}\lambda \cosh 2y\right) Y , \tag{10.6}$$

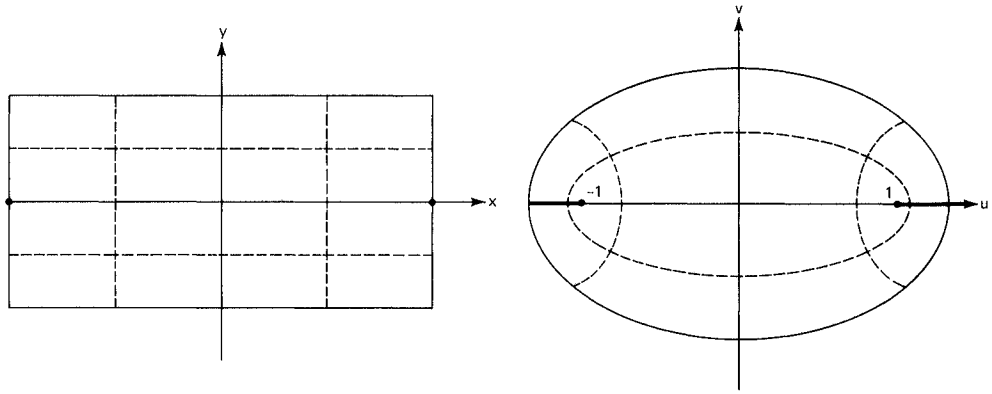

FIG. 4. *The function* $w = \sin z$ *maps rectangles in the z-plane to ellipses in the w-plane.*

where a is a separation constant. Equation (10.5) is a form of Mathieu's equation and (10.6) is an associated Mathieu equation. Mathieu [93] first found the eigenvalues of the elliptical membrane. To eliminate the effect of the slits, we associate odd solutions of (10.6) with solutions of (10.5) that satisfy

$$X\left(\pm\frac{\pi}{2}\right) = 0 \,, \tag{10.7}$$

corresponding to eigenvectors of the ellipse that are antisymmetric about the major axis. We associate even solutions of (10.6) with solutions of (10.5) that satisfy

$$X'\left(\pm\frac{\pi}{2}\right) = 0 \,, \tag{10.8}$$

giving symmetric eigenfunctions of the ellipse. In both cases, the boundary conditions for (10.6) are

$$Y(\pm k) = 0 \,. \tag{10.9}$$

Symmetry about the minor axis corresponds to symmetry in the solutions to (10.5).

Modern papers on the eigenvalues of the ellipse are [73] and, particularly [132], where λ_n for $n = 1, \cdots, 25$ is given graphically as a function of eccentricity e, mode shapes are shown, and approximations for λ_1 and λ_2 are given as polynomials in e. A power series in e is also given for λ_1 in [67].

The problems of annular regions between confocal ellipses and sectors of ellipses bounded by orthogonal confocal hyperbolas can also be reduced to solving pairs of Mathieu equations by separation of variables. A few eigenvalues of the slit ellipse are found in [43]. Of course, those eigenvalues corresponding to modes antisymmetric about the major axis are the same as those of the full ellipse.

We now consider parabolic coordinates. The function

$$w = \frac{1}{2}z^2 \tag{10.10}$$

maps the rectangle $[0,1] \times [-1,1]$ in the z-plane to a region bounded by symmetric orthogonal parabolas confocal to the origin with a slit along the negative ξ-axis (Fig. 5). Thus,

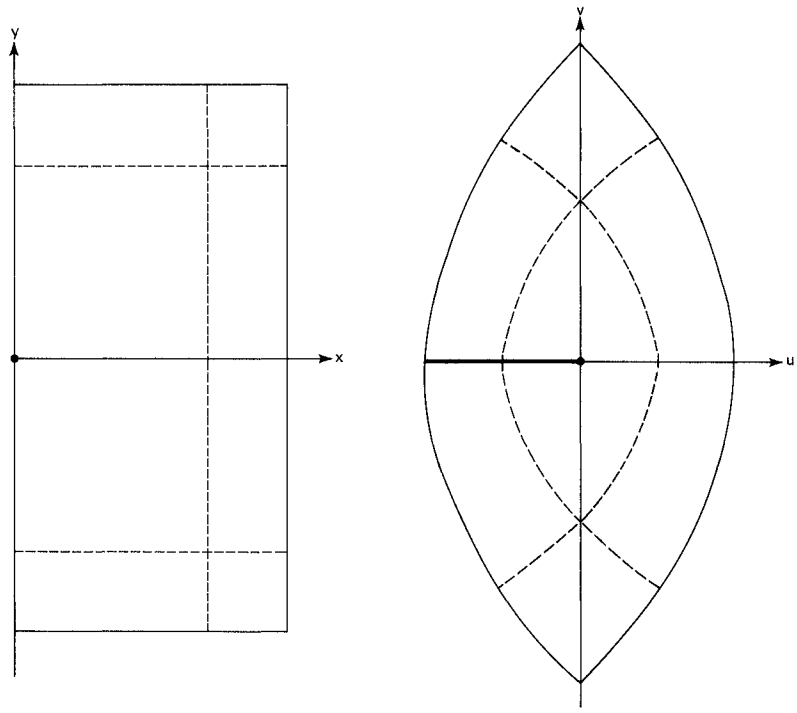

FIG. 5. *The function* $w = z^2/2$ *maps rectangles in the z-plane to confocal parabolas in the w-plane.*

$$\left|\frac{dw}{dz}\right|^2 = x^2 + y^2, \tag{10.11}$$

which, inserted in (9.5) with $U = X(x)\,Y(y)$, leads to the pair of equations

$$-X'' = (a + \lambda x^2)X\,, \tag{10.12}$$

$$Y'' = (a - \lambda y^2)Y\,, \tag{10.13}$$

with a as the separation constant. These can be solved in terms of parabolic cylinder functions [2], sometimes also called Weber functions. The effect of the slit is eliminated by symmetry considerations again. Eigenvalues of the symmetric parabolic region are given in [45] and [142]. The problem for any region bounded by four orthogonal parabolic arcs may be reduced in this way.

11. Inverse problem and asymptotics. In a now classical paper [68], Kac asked the question, "Can one hear the shape of a drum?" This is a colorful way of posing the *inverse problem*: given all the eigenvalues λ_n, can the region R be determined? A number of geometrical and topological constants associated with R can certainly be determined. Kac conjectured the asymptotic relation for the *spectral function*

$$\sum_{n=1}^{\infty} e^{-\lambda_n t} \sim \frac{A}{4\pi t} + \frac{L}{8\sqrt{\pi t}} + \frac{1}{6}(1 - r) \quad \text{as } t \to 0\,, \tag{11.1}$$

where A is the area of R, L is the length of C, r is the number of holes in R, and the factor 1/6 on the last term is for regions without corners. The first term on the right is essentially Weyl's law (6.1), the second term comes from results of [113], and the last

term follows from work in [95]. The paper [127] indicates that the nature of the corners of R may also be determinable.

A more interesting function is $N(t)$, the number of eigenvalues less than t. Courant [31] has shown

$$\left|N(t) - \frac{At}{4\pi}\right| \leq Kt^{1/2} \log t \ , \tag{11.2}$$

for K a constant. For at least some domains (in particular, polygons), the log t factor can be dropped. A conjecture due to Weyl is

$$N(t) = \frac{At}{4\pi} - \frac{Lt^{1/2}}{4\pi} + o(t^{1/2}) \ . \tag{11.3}$$

Attempts have been made to obtain results like (11.3) from the more easily obtained results like (11.1) via Tauberian theorems. Surveys on these questions are [26] with 120 references and [12] with 39 references.

Ideally, what one would like to have is an effective asymptotic expression for the nth eigenvalue λ_n itself of the form

$$\lambda_n = \frac{4\pi n}{A} + f(n) + o(1) \ . \tag{11.4}$$

This is apparently a very difficult problem. If (11.3) were true, its inversion would give

$$\lambda_n = \frac{4\pi n}{A} + \frac{L}{A}\sqrt{\frac{4\pi n}{A}} + o(n^{1/2}) \ . \tag{11.5}$$

Even for a region where the eigenvalues are known, such as the square of side π, it appears difficult to obtain an effective asymptotic formula for λ_n directly. For the square, we calculated

$$\lambda_n - \frac{4n}{\pi} - \frac{4}{\pi}\sqrt{\frac{4n}{\pi}}$$

in the vicinity of $n = 1000$, obtaining a high of $+7.25$ for $n = 980$ and a low of -4.67 for $n = 1000$.

12. Rayleigh-Ritz and finite element methods. If there were an effective asymptotic formula of the form (11.4), only a finite number of the lowest eigenvalues would need to be calculated for any given region. Most numerical techniques, in any event, are concerned mainly with the estimation of the lowest several eigenvalues.

The Rayleigh-Ritz method mentioned in § 2 is one of the oldest effective approximation methods, giving upper bounds for the eigenvalues. If u is a linear combination of n trial functions ϕ_i as in (2.4), the stationary values of the Rayleigh quotient (2.3) with respect to the coefficients a_i give upper bounds for $\lambda_1, \lambda_2, \cdots, \lambda_n$. Thus,

$$\lambda_i \leq \mu_i \ , \qquad i = 1, 2, \cdots, n \ , \tag{12.1}$$

where $\mu_1, \mu_2, \cdots, \mu_n$ are the eigenvalues of the $n \times n$ relative matrix problem

$$Ma = \mu Na \ ,$$

with

$$M_{ij} = \int_R \left(\frac{\partial \phi_i}{\partial x} \frac{\partial \phi_j}{\partial x} + \frac{\partial \phi_i}{\partial y} \frac{\partial \phi_j}{\partial y} \right) dxdy \,, \tag{12.2}$$

$$N_{ij} = \int_R \phi_i \phi_j \, dxdy \,, \tag{12.3}$$

$$a = (a_1, a_2, \cdots, a_n)'. \tag{12.4}$$

Remarkably, the basic Rayleigh-Ritz method has apparently been seldom used to estimate the eigenvalues of the Laplacian. Most applications have been to rhombical [139], [14], or parallelogram [40] regions, particularly in the equivalent form (9.3) resulting from an affine transformation.

The Rayleigh-Ritz method has sometimes been used with *finite elements* as trial functions. These are functions that have localized support. Generally, the basis of trial functions is a system of translations of elementary hill-functions such as piecewise polynomials of low order, plus, perhaps, special elements designed to conform to curved boundaries. There are many varieties of finite elements and a large literature on them. In connection with eigenvalue problems in two dimensions, many of the results are purely theoretical (see, e.g., [3], [18], and the references therein). Although the literature does not contain many references to finite-element methods applied to the two-dimensional Laplacian, they are the basis of large software packages for doing structural analysis.

References in the literature that apply the finite-element method to two-dimensional problems are often from engineering applications and are of limited accuracy. The finite-element method has been used in [98] for L- and H-shapes, in [20] for circular and square waveguides with quadruple ridges, in [126] for various crosses, in [65] for "bent" waveguides (L-shapes at angles other then 90°), in [82] for rectangles with rounded corners, in [32] for shapes given by polar coordinates such as portions of spirals, and in [89] for limaçons and cardioids. It is also used in the interesting paper [84] where mode shapes are followed as parameters are varied to change rectangles into ellipses via hyperellipses and into parabolas via superellipses.

13. Intermediate problems. The Rayleigh-Ritz method gives effective upper bounds on the eigenvalues. Complementary lower bounds are generally much more difficult to obtain. One method that gives lower bounds is the method of *intermediate problems*. Good general introductions to intermediate methods are [52], [49], and [140]. The idea of the intermediate method is to relate the given problem to a problem with known solution (the *base problem*) through an infinite system of constraints. When only a finite number of the constraints are applied, lower bounds for the desired solutions are obtained, and the resulting intermediate problem can be reduced to a finite-dimensional matrix calculation through either of the procedures known as *special choice* or *truncation*. The theory of intermediate problems is difficult, but its application to specific problems can be made straightforward.

In [125], the rhombical membrane was considered in the equivalent form (9.3) resulting from an affine transformation. This is written as

$$(A + T^*T)u = \lambda u \,, \tag{13.1}$$

where

$$A = -\frac{1 - \cos\theta}{\sin^2\theta}\Delta \,, \qquad T = \frac{\cos^{1/2}\theta}{\sin\theta}\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right) \,, \tag{13.2}$$

and T^* is the adjoint of T (here $T^* = -T$).

The known base problem is

$$Au = \lambda u \ , \tag{13.3}$$

which is just the rectangular membrane problem. The operator T^*T is replaced by the smaller operator T^*P_kT, where P_k is the *orthogonal projection* (in the sense of $L^2(R)$) on the set spanned by a finite number $\{p_1, p_2, \cdots, p_k\}$ of trial functions. The intermediate problem is then

$$(A + T^*P_kT)u = \lambda u \ , \tag{13.4}$$

which has eigenvalues that are lower bounds to those of (13.1). As k increases, the bounds increase.

Equation (13.4) can be reduced to a finite-dimensional problem by a *special choice* of the p_j. If $u_1, u_2, \cdots, u_n$ are the first n eigenfunctions of the base problem (13.3) and

$$Tp_j = \sum_{i=1}^{n} \beta_{ji} u_i \ , \qquad j = 1, 2, \cdots, k \ , \tag{13.5}$$

then the eigenvalues of (13.4) are the eigenvalues $\lambda_{n+1}, \lambda_{n+2}, \cdots$, of the base problem (called *persistent* eigenvalues), plus the n eigenvalues λ of the matrix equation

$$\det |\Lambda + \beta' B^{-1} \beta - \lambda I| = 0 \ , \tag{13.6}$$

where

$$\Lambda = \operatorname{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n) \ , \tag{13.7}$$

the elements of the $n \times k$ matrix β are defined by (13.5), superscript t denotes transpose, and B is the $k \times k$ *Gram matrix* of the p_j given by

$$B_{ij} = \int_R p_i p_j \, dxdy \ , \qquad i,j = 1, 2, \cdots, k \ . \tag{13.8}$$

The paper [125] has been a benchmark for the eigenvalues of the rhombus. Upper bounds were also obtained there by intermediate methods, but these have subsequently been improved by Rayleigh-Ritz methods as cited in § 12.

Intermediate methods were also used in [22] for the L-shape by embedding it in a square and using the constraints

$$\int u\phi_i \, dxdy = 0 \ , \qquad i = 1, 2, \cdots, \tag{13.9}$$

where the integration is over the complementary region. This method was also used in [33] for the regular octagon and a cross. We shall return to intermediate methods in § 18.

14. Finite differences. An old and highly popular method is *finite differences* [48]. A brief historical survey is found in [74] with 41 references.

In general, a grid or mesh is laid down on the plane, dividing it into small rectangles, squares, or triangles. At grid intersections or *nodes*, the Laplace operator Δ is approximated by a difference operator. For a square mesh of width h,

$$\begin{aligned} \Delta_h u(x,y) = h^{-2} [&u(x+h,y) + u(x,y+h) + u(x-h,y) \\ &+ u(x,y-h) - 4u(x,y)] \end{aligned} \tag{14.1}$$

is the *five-point approximation* to Δ. If u is a smooth function, the *truncation error* is

$$\Delta_h u - \Delta u = \frac{h^2}{12}\left[u_{xxxx}(x,y) + u_{yyyy}(x,y)\right] + O(h^4) , \tag{14.2}$$

as may be seen from a Taylor series expansion. Thus, the five-point operator is locally *second order* in h. Higher order difference operators and difference operators for triangular meshes can be found in [74], [76], and [69].

If the boundary C does not intersect the mesh exactly at nodes, the boundary values $u = 0$ must be transferred to nearby mesh nodes or in some way interpolated. If the mesh can be arranged so that C always intersects the mesh at nodes, we say *no boundary interpolation is required.*

The difference scheme is a system of equations of the form

$$\Delta_h U_h + \lambda_h U_h = 0 \quad \text{in } R_h , \qquad U_h = 0 \quad \text{outside } R_h , \tag{14.3}$$

where R_h is the set of mesh nodes in the interior of R. This is a matrix eigenvalue problem with the order of the matrix being the number of interior nodes.

If the region R has C^∞ boundary and the difference operator Δ_h is an $O(h^p)$ approximation to Δ over R except near the boundary where it need only be $O(h^{p-2})$, then for each n and for h sufficiently small, there is a constant C_n such that

$$|\lambda_n - \lambda_{h,n}| \leq C_n h^p . \tag{14.4}$$

See [74] for the exact details of the required form of Δ_h and related estimates for the eigenfunctions.

If the region does not have C^∞ boundary, the situation is less favorable, even when no interpolation is required at the boundary. We have seen in § 5 how the presence of corners influences the smoothness of the eigenfunctions. In the vicinity of a corner with interior angle π/α, the eigenfunction behaves like r^α, which affects the truncation error in the vicinity of the corner, and limits the rate of convergence of the discrete eigenvalues. If π/α is the largest interior angle, then [17]

$$|\lambda_n - \lambda_{h,n}| \leq C_n \max(h^p, h^{2\alpha}) . \tag{14.5}$$

Thus, for regions with reentrant corners, the rate of convergence will be less than $O(h^2)$, e.g., for a reentrant right angle, as in an L-shaped membrane, convergence will be $O(h^{4/3})$; and if the membrane has a slit, convergence will only be $O(h)$, even though no interpolation is required at the boundary.

Such considerations are very important when using the method of *Richardson extrapolation.* Let the continuous eigenvalue λ_n be related to the difference eigenvalue $\lambda_{h,n}$ by an asymptotic formula like

$$\lambda_n = \lambda_{h,n} + a_n h^p + o(h^p) , \qquad h \to 0 , \tag{14.6}$$

where a_n is some constant independent of h. Then if $\lambda_{h,n}$ is calculated for two distinct values of h, say h_1 and h_2, the extrapolated value

$$\frac{h_1^p \lambda_{h_2,n} - h_2^p \lambda_{h_1,n}}{h_1^p - h_2^p} \tag{14.7}$$

approximates λ_n to an order higher than p. Note that the value of the constant a_n need not be known. If more terms of the asymptotic expansion, e.g.,

$$\lambda_n = \lambda_{h,n} + a_n h^p + b_n h^q + o(h^q) , \tag{14.8}$$

are valid, then the extrapolation

$$\frac{(h_2^p h_3^q - h_2^q h_3^p)\lambda_1 + (h_1^q h_3^p - h_1^p h_3^q)\lambda_2 + (h_1^p h_2^q - h_1^q h_2^p)\lambda_3}{h_2^p h_3^q - h_2^q h_3^p + h_1^q h_3^p - h_1^p h_3^q + h_1^p h_2^q - h_1^q h_2^p} \tag{14.9}$$

approximates λ_n to an order higher than q. Again, a_n and b_n do not have to be known explicitly, but *the correct values of p and q must be used.* In [9], the regular hexagon is treated by a finite-difference scheme using equilateral triangular meshes. Because no extrapolation is required at the boundary, the authors assumed (14.8) holds and applied (14.9) with $p = 2$, $q = 4$. However, from § 5 and the considerations of [17], eigenvalues that are not also eigenvalues of the equilateral triangle will have no better than $p = 2$, $q = 3$ in (14.8) and (14.9), and may have an $h^3 |\log h|$ term also. Thus, the results of [9] for nontriangle eigenvalues should be recomputed. The paper [8] treats the 60° rhombus in the same way and may contain the same error. The paper [57] previously treated the regular hexagon with a triangular mesh.

The exact form of the first several terms in an asymptotic formula like (14.8) for specific regions where no boundary interpolation is required is a nice problem at about the level of a doctoral thesis.

It is possible to obtain both lower and upper bounds for the eigenvalues from finite-difference methods. By inserting piecewise linear functions on the mesh squares constructed from the difference eigenfunctions into the minimax principle, upper bounds for the continuous eigenvalues are obtained in terms of the difference eigenvalues. Conversely, inserting averages over mesh squares of the continuous eigenfunctions into the minimax principle for the finite-difference equations gives upper bounds for the difference eigenvalues in terms of the continuous eigenvalues. These ideas are developed in the chronological sequence of papers [117], [46], [47], [137], [138], [63], and [75].

The finite-difference method has been applied in [51] to *L*-, *T*-, and cross-shaped regions and concentric squares, in [119] and [99] to *L*-shapes, in [129] to half-*L*s, and in [35], [10], and [11] to *ridge* and *lunar* waveguides (see Fig. 6). In [77], λ_1 for the slit square is approximated by finite differences, while [19] considers this region as well as the *H*-shape.

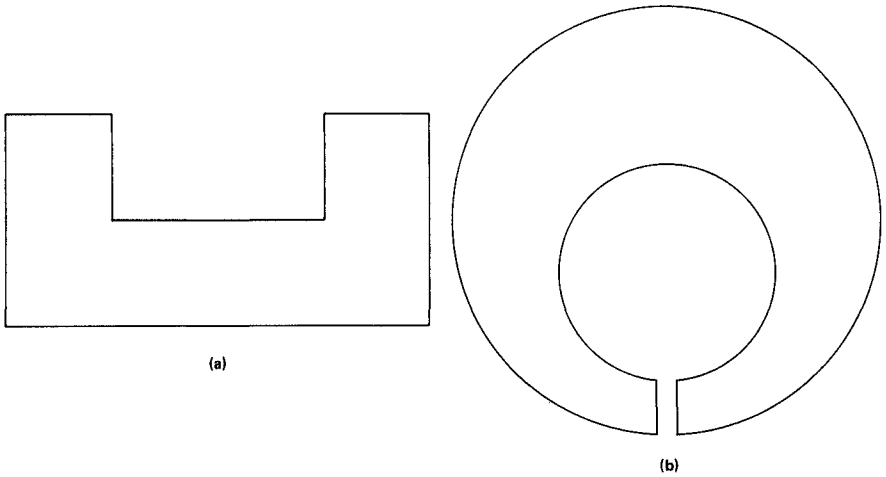

FIG. 6. (a) *Single ridge rectangular waveguide,* (b) *Lunar waveguide.*

15. Point matching. Undoubtedly, the most popular method of obtaining estimates for engineering applications is *point matching* or *collocation.* A series such as

$$u_* = \sum_{n=0}^{N-1} a_n J_n(\sqrt{\lambda} r)\cos n\theta, \tag{15.1}$$

which satisfies (1.1), is made to satisfy $u_* = 0$ at N points on the boundary C, giving a system of N homogeneous equations whose determinant must vanish for nontrivial solutions to exist. The roots λ of this determinant are approximations for the eigenvalues, and the corresponding u_* are approximations for the eigenfunctions. The method will work well or poorly according to the choice of the functions in the series. For C^∞ boundaries, most reasonable choices will work; but when corners are present, functions must be included that have the behavior indicated in § 5 or the method will converge badly even for large values of N. For a systematic method of selecting functions for point matching with convergence estimates, see [42] and [121].

We remark that requiring u_* to be small in a least-squares sense on more than N boundary points may give a smoother approximation. See also [100].

The method has been used in [29] for the regular hexagon, in [30] for triangles and parallelograms, in [36] and [39] for Ls and half-Ls, in [124] for ridge guides, in [7], [104], and [141] for circular guides with off-center circular holes, in [7] for a "Pacman" shape and a rectangular guide with semicircular ridges ("Meinke" guide, Fig. 7), in [131] for limacons, and in [136] for epicycloids.

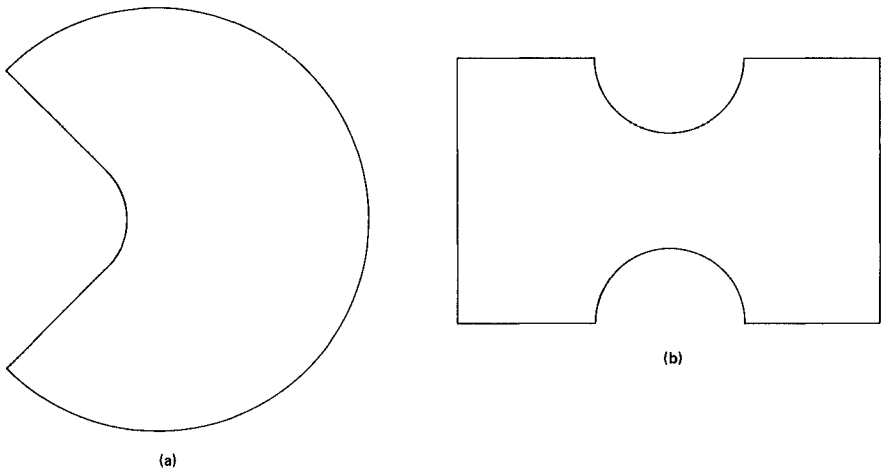

FIG. 7. (a) *Pacman,* (b) *Meinke waveguide.*

Point matching has also been used on the equivalent problem (9.5) after a conformal mapping. This has been applied in [21] to the cardioid and epitrochoid, in [54] to polygons, and in [88] to polygons with holes.

It is possible to obtain *a posteriori* error bounds on the approximations obtained by point matching. Let λ_* and u_* be the approximations obtained by point-matching and satisfying (1.1). Let

$$\epsilon = \max_C |u_*| . \tag{15.2}$$

Then in [50], it was shown that there exists an eigenvalue λ_i such that

$$\frac{|\lambda_i - \lambda_*|}{\lambda_*} \leq \frac{\sqrt{2}\epsilon + \epsilon^2}{1 - \epsilon^2}, \tag{15.3}$$

simultaneously giving a lower and upper bound on this λ_i. (Which eigenvalue has been bounded requires some additional information.) An improved version of this bound is given in [102]. This method was used in [50] to obtain estimates on the first ten eigenvalues of the L that are so precise (6 to 8 decimal places) as to have essentially disposed of this problem.

This method has also been used in [100] for rhombi and in [39] for the H-shape.

16. The method of *a posteriori-a priori* inequalities. The inequality (15.3) was generalized in [106] to λ_* and u_* no longer satisfying (1.1). Let

$$\delta = \max_R |\Delta u_* + \lambda_* u_*|, \tag{16.1}$$

and

$$\sigma = \max_R |s|, \tag{16.2}$$

where s is the *torsion function* for R satisfying

$$\Delta s = -1 \quad \text{in } R, \qquad s = 0 \quad \text{on } C. \tag{16.3}$$

Then, there is an eigenvalue λ_i such that

$$\frac{|\lambda_i - \lambda_*|}{\lambda_*} \leq \frac{\sqrt{2}\rho + \rho^2}{1 - \rho^2}, \tag{16.4}$$

where

$$\rho = \epsilon + \sigma\delta, \tag{16.5}$$

where ϵ is as defined in (15.2).

In [102] an inequality like (15.3) was given in terms of the L^2 norm. If λ_* and u_* satisfy (1.1), and w is defined by

$$\Delta w = 0 \quad \text{in } R, \qquad w = u_* \quad \text{on } C, \tag{16.6}$$

then there is an eigenvalue λ_i such that

$$\left|\frac{\lambda_i - \lambda_*}{\lambda_i}\right|^2 \leq \frac{\int_C w^2\, ds}{\int_R u_*^2\, dxdy}. \tag{16.7}$$

An estimate was also given for $\int_R (u_i - u_*)^2\, dxdy$. In [42], it was observed that w can be estimated in terms of u_* by an *a priori inequality*.

In [79], the ideas of the preceding generalizations were combined to obtain the following result. Let λ_* and u_* be approximations for an eigenvalue and eigenfunction. If w is defined by

$$\Delta w = \Delta u_* + \lambda_* u_* \quad \text{in } R, \qquad w = u_* \quad \text{on } C, \tag{16.8}$$

then there is an eigenvalue λ_i such that

$$\left| \frac{\lambda_i - \lambda_*}{\lambda_i} \right|^2 \leq \frac{\int_R w^2 \, dxdy}{\int_R u_*^2 \, dxdy} \,. \tag{16.9}$$

The proof uses Parseval's identity and integration by parts. This is an *a posteriori* inequality. It is not desirable to acutally compute the function w, so the following *a priori* inequality is used:

$$\int_R w^2 \, dxdy \leq \alpha_1 \int_R (\Delta w)^2 \, dxdy + \alpha_2 \int_C w^2 \, ds \,, \tag{16.10}$$

where α_1, α_2 are explicit constants [122]. Putting (16.10) into (16.9) and using (16.8) gives the combined *a posteriori-a priori* inequality,

$$\left| \frac{\lambda_i - \lambda_*}{\lambda_i} \right|^2 \leq \frac{\alpha_1 \int_R (\Delta u_* + \lambda_* u_*)^2 \, dxdy + \alpha_2 \int_C u_*^2 \, ds}{\int_R u_*^2 \, dxdy} \,. \tag{16.11}$$

Now, the right-hand side of (16.11) is a ratio of quadratic forms in the function u_*. Take u_* to be a linear combination of trial functions,

$$u_* = \sum_{k=1}^{n} a_k \phi_k \,, \tag{16.12}$$

and minimize the right side of (16.11) with respect to the coefficients a_k as in the Rayleigh-Ritz method. This leads to the relative matrix eigenvalue problem

$$Ma = \epsilon^2 Na \,, \tag{16.13}$$

where the $n \times n$ matrices M and N are given by

$$\begin{aligned} M_{ij} &= \alpha_1 \int_R (\Delta\phi_i + \lambda_* \phi_i)(\Delta\phi_j + \lambda_* \phi_j) \, dxdy + \alpha_2 \int_C \phi_i \phi_j ds \,, \\ N_{ij} &= \int_R \phi_i \phi_j \, dxdy \,, \end{aligned} \tag{16.14}$$

and a is the vector $(a_1, a_2, \cdots, a_n)^t$ of coefficients. If ϵ^2 is the smallest eigenvalue of (16.13), then

$$\left| \frac{\lambda_i - \lambda_*}{\lambda_i} \right| \leq \epsilon \,, \tag{16.15}$$

and if $\epsilon < 1$,

$$\frac{\lambda_*}{1 + \epsilon} \leq \lambda_i \leq \frac{\lambda_*}{1 - \epsilon} \,, \tag{16.16}$$

giving a lower and upper bound on λ_i.

The initial guess λ_* for an eigenvalue can be improved as follows. Once the vector of coefficients $(a_1, a_2, \cdots, a_n)^t$ satisfying (16.13) is found, hold u_* fixed in (16.11)

as given by (16.12) and minimize the right side of (16.11) with respect to λ_*. This gives

$$\lambda_* = \frac{-\int_R u_* \Delta u_* dxdy}{\int_R u_*^2 dxdy} \tag{16.17}$$

With this new value for λ_*, the entire process can be iterated.

A bound on the approximation of the associated eigenfunction u_i by u_* can also be given [80]:

$$\int_R (u_i - u_*)^2 \, dxdy \leq 2(1 - \sqrt{1 - c^2\epsilon^2}) \, , \tag{16.18}$$

where

$$c = \max \left(\frac{\lambda_{i-1}}{\lambda_* - \lambda_{i-1}} \, , \frac{\lambda_{i+1}}{\lambda_{i+1} - \lambda_*} \right) . \tag{16.19}$$

Advantages of the method of *a posteriori-a priori* inequalities are: (a) the trial functions used do not need to satisfy any boundary condition or, indeed, any auxiliary conditions at all other than be twice differentiable, (b) application of the method is like the Rayleigh-Ritz method, and (c) two-sided bounds are given. A disadvantage is that only one eigenvalue at a time is estimated.

The method was used in [79] for rhombi. We will return to the method of *a posteriori-a priori* inequalities in § 18.

17. Galerkin and other numerical methods. In the *Galerkin method,* a series

$$u_* = \sum_{i=1}^{n} a_i \phi_i \tag{17.1}$$

is assumed for the solution, where the functions ϕ_i are usually required to satisfy the boundary condition $\phi_i = 0$ on C. The coefficients a_i are determined by requiring

$$\int_R (\Delta u_* + \lambda u_*) \psi_i \, dxdy = 0 \, , \qquad i = 1,2, \cdots , n \, , \tag{17.2}$$

for some functions ψ_i (which usually coincide with the ϕ_i). As in the method of point matching, this is a system of n homogeneous equations whose determinant must vanish for nontrivial solutions to exist. The roots λ of this determinant are approximations for the eigenvalues, and the corresponding u_* approximate the eigenfunctions. Also, as with point matching, the success of the method is largely dependent on a good choice of the ϕ_i.

Galerkin methods have been used in [86], [24], [25] for trapezoids and in [135] for circles with off-center holes. Galerkin methods applied to the equivalent problem (9.5), resulting from a conformal mapping, have been used in [61] for the circle with off-center holes, in [96] for Meinke, hexagonal, and lunar guides, in [97] for ridge and lunar guides, in [87] for the four-leaf lemniscate, the octagon, and a flattened circle, and in [88] and [90] for polygons with holes.

The method of *field-matching* is applied to regions that are the unions of simpler regions for which solutions are known, such as rectangles. A series solution is assumed for each subregion in functions appropriate to that subregion. The series are then re-

quired to match at the interfaces, giving an infinite set of equations for the coefficients. This set of equations is then truncated and solved. The method seems very cumbersome. It has been used in [53] for a rectangle with a centered, rectangular hole and in [130] and [92] for various crosses.

Perturbation methods can be used for regions that differ only slightly from a region with known solutions. The desired solution is expanded as a power series in a small parameter governing the perturbation. This method is used in [120] for epicycloids and in [41] for circles and squares with holes.

In [62], the Barta inequality (4.2) is used to estimate λ_1 for rhombi.

We mention the very useful review paper [105] on numerical results with 77 references, and other reviews [94] with 150 references, [34] with 35 references, and [123] with 79 references. (Some of the references of the last three are on other topics, however.)

18. Conformal mapping combined with other methods. We have seen examples of the point-matching and Galerkin methods applied to the equivalent problem (9.5) connected to the original problem by the conformal map (9.4). The problem (9.5) on a rectangle is in a particularly convenient form for using the method of intermediate problems, and is also convenient for Rayleigh-Ritz, for finite differences using square meshes to fit the boundary exactly, and also for the method of *a posteriori-a priori* inequalities. For some reason, no one has previously used any of these methods on the equivalent problem (9.5) obtained from a conformal map.

Let us do a simple but nontrivial problem by all of these methods. Consider the conformal map

$$w = \tan\left(\frac{z}{2}\right), \tag{18.1}$$

which maps the square S: $[-\pi/2, \pi/2] \times [-\pi/2, \pi/2]$ in the z-plane onto the region R in the w-plane bounded by arcs of the unit circle and a pair of orthogonal circles (see Fig. 8). If $z = x + iy$, $w = u + iv$, then

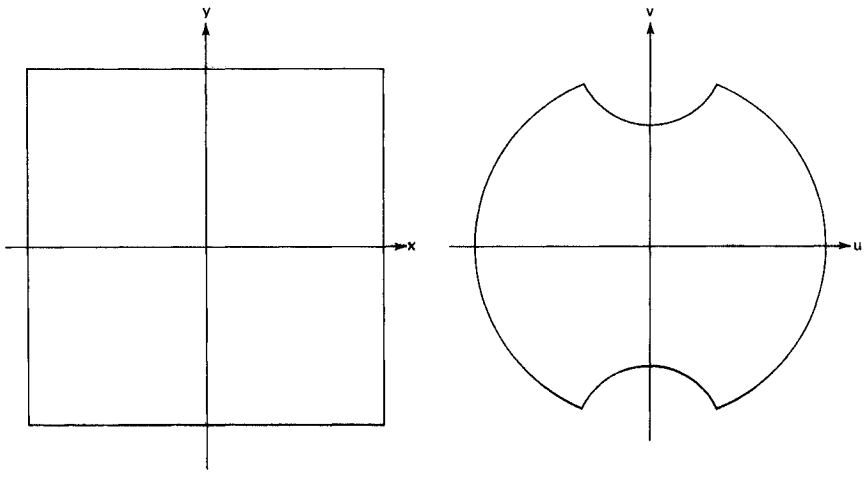

FIG. 8. *The function* $w = \tan(z/2)$ *maps a square in the z-plane to a circular waveguide with circular ridges.*

$$u = \frac{\sin x}{\cos x + \cosh y}, \qquad v = \frac{\sinh y}{\cos x + \cosh y}, \tag{18.2}$$

and the boundary of R is given by

$$u^2 + v^2 = 1, \qquad \operatorname{sech}\frac{\pi}{2} \leq |u| \leq 1, \tag{18.3}$$

and

$$u^2 + \left(v \pm \coth\frac{\pi}{2}\right)^2 = \operatorname{csch}^2\frac{\pi}{2}, \qquad |u| \leq \operatorname{sech}\frac{\pi}{2}. \tag{18.4}$$

This represents a circular waveguide with circular ridges.

Since

$$\frac{dw}{dz} = \frac{1}{2}\sec^2\left(\frac{z}{2}\right),$$

we have

$$\left|\frac{dw}{dz}\right| = \frac{1}{\cos x + \cosh y}, \tag{18.5}$$

and (1.1) on R is equivalent to

$$-\Delta U = \lambda\sigma U \quad \text{in } S, \tag{18.6}$$

where

$$\sigma = (\cos x + \cosh y)^{-2}. \tag{18.7}$$

Now,

$$\max_S \sigma = 1,$$

so we can write (18.6) in the form

$$AU = \lambda(I - T^2)U, \tag{18.8}$$

where A is $-\Delta$ and T is multiplication by

$$\frac{\sqrt{(\cos x + \cosh y)^2 - 1}}{\cos x + \cosh y}.$$

If we drop the positive operator T^2, we have the base problem

$$AU = \lambda U, \tag{18.9}$$

which is just the problem (1.1) on the square, with known eigenvalues giving lower bounds for the desired eigenvalues. To improve these bounds, we consider the intermediate problem

$$AU = \lambda(I - TP_kT)U, \tag{18.10}$$

where P_k is the operator of orthogonal projection on a set spanned by the linearly independent trial functions $\{p_1, p_2, \cdots, p_k\}$. Then (18.10) also gives lower bounds that tend to the exact eigenvalues as k is increased.

To reduce the solution of (18.10) to a finite-dimensional matrix problem, we introduce the *truncation* of the operator A. The truncation $A^{(n)}$ of order n agrees with A on the span of the first n eigenvalues of A, $U_1, U_2, \cdots, U_n$, and is simply multiplication by λ_{n+1} on the orthogonal complement of $U_1, U_2, \cdots, U_n$. Thus,

$$A^{(n)} = AQ_n + \lambda_{n+1}(I - Q_n) , \tag{18.11}$$

where Q_n is the orthogonal projection onto the span of $U_1, U_2, \cdots, U_n$. Since

$$(A^{(n)}U, U) \leq (AU, U),$$

the problem

$$A^{(n)}U = \lambda(I - TP_kT)U \tag{18.12}$$

also has eigenvalues that are lower bounds for the eigenvalues of (18.8). These bounds will increase to the true eigenvalues as both n and k are increased.

Now, problem (18.12) can always be solved as a relative matrix eigenvalue problem with no restriction on the choice of the p_i. To see this, put

$$U = \sum_{i=1}^{n} a_i U_i + \sum_{j=1}^{k} b_j Tp_j$$

into (18.12). Then $n + k$ eigenvalues of (18.12) are found from the partitioned relative matrix problem

$$\begin{bmatrix} \Lambda & (\Lambda - \lambda_{n+1}I)E^t \\ 0 & \lambda_{n+1}B \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \lambda \begin{bmatrix} I & 0 \\ -E & B - C \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} , \tag{18.13}$$

where Λ is the diagonal matrix of the first n eigenvalues of A as in (13.7), B is the $k \times k$ Gram matrix of the p_j as in (13.8), the $k \times k$ matrix C is given by

$$C_{ij} = \int_R Tp_i Tp_j \, dxdy , \tag{18.14}$$

and the $k \times n$ matrix E is given by

$$E_{ij} = \int_R U_i Tp_j \, dxdy . \tag{18.15}$$

Equation (18.12) also has λ_{n+1} as an eigenvalue of infinite multiplicity.

For the present problem, the choice

$$p_j = \sqrt{(\cos x + \cosh y)^2 - 1} \ (\cos x + \cosh y) U_j \tag{18.16}$$

was made, whereby all of the required integrals in B, C, and E are elementary. The matrix problem (18.13) was symmetrized and solved using the standard matrix packages LINPACK [38] and EISPACK [15], [16] on an IBM sytem 3033. The results for the projection and truncation sizes $(k,n) = (10,20), (20,60), (40,80)$ are shown in Table 1 for even-even eigenvalues, i.e., those associated with eigenfunctions symmetric about both axes.

TABLE 1

Even-even eigenvalues λ_ν of the region in Fig. 8. Lower bounds by (n, k)-order truncation and upper bounds by n-order Ritz.

	Truncation of order (n,k)			Ritz of order n		
ν	(10,20)	(20,60)	(40,80)	144	64	16
1	7.555	7.5685	7.56942	7.57012	7.5716	7.588
2	27.79	29.0615	29.10971	29.11951	29.1332	29.41
3	42.77	44.7421	44.8341	44.8425	44.8493	44.98
4	49.8	66.43	67.802	67.863	67.91	69.21
5		73.18	74.417	74.511	74.57	76.97
6		91.7	104.34	104.65	104.70	107.3
7		97.2	122.75	123.29	123.43	128.9
8			135.68	137.13	137.53	151.
9			145.97	147.12	147.18	162.
10			167.21	177.21	177.50	
11			184.67	192.98	193.43	
12			200.45	212.81	213.6	

To obtain complementary upper bounds, Rayleigh-Ritz was used. Equation (18.6) is in a very convenient form to apply Rayleigh-Ritz. The minimax principle for (18.6) is

$$\lambda_n = \min \max \frac{\int_S \left[\left(\partial U/\partial x\right)^2 + \left(\partial U/\partial y\right)^2 \right] dxdy}{\int_S \sigma U^2 \, dxdy} \,. \tag{18.17}$$

For trial functions, we used

$$\phi_j = (\cos x + \cosh y)\, U_j \,, \tag{18.18}$$

making the Rayleigh-Ritz equation a standard eigenvalue problem

$$Ma = \mu a \,, \tag{18.19}$$

with M given by (12.2). Results of Rayleigh-Ritz problems of order $n = 16, 64, 144$ are shown also in Table 1 for the even-even symmetry class. We see that the average between the lower and upper bounds has less than 0.005% relative error for λ_1. This method has also been used in [78] for lunar and eccentric annular waveguides.

Equation (18.6) is also in a convenient form to use the finite-difference method [64]. Let $h = \pi/n$ be the width of a square mesh laid down on S so the boundary of S falls on mesh lines. Then we approximate (18.6) by

$$-\Delta_h U_h = \lambda_h \sigma_h U_h \quad \text{on } S_h \,, \qquad U_h = 0 \quad \text{on } C_h \,, \tag{18.20}$$

where Δ_h is the five-point operator (14.1), σ_h is the restriction of σ to mesh nodes, S_h and C_h are the sets of mesh nodes in S and its boundary C. No interpolation is required at the boundary. For even-even symmetry, we can work with just the points in the first quadrant, reducing the order of the matrix by a factor of nearly four. Again, EISPACK was used. Results for $n = 21, 27, 33$ are given in Table 2.

TABLE 2
Even-even eigenvalues λ_ν of the region in Fig. 8 by finite differences with mesh sizes $h = \pi/n$ and Richardson extrapolation.

ν \ n	21	27	33	Extrapolated
1	7.552838	7.559379	7.562726	7.569578
2	28.61367	28.80899	28.90936	29.11569
3	44.0487	44.3597	44.5180	44.8402
4	65.655	66.540	66.980	67.855
5	71.509	72.641	73.239	74.504
6	99.59	101.56	102.56	104.63
7	116.18	119.04	120.46	123.25
8	124.94	129.47	131.92	137.18
9	139.10	142.27	143.87	147.10
10	162.46	168.19	171.13	177.20
11	175.12	182.18	185.75	192.97
12	185.31	196.67	202.56	214.80

The corners of the original region R are formed by orthogonal circular arcs. Reflecting the eigenfunctions in the circular arcs shows that the eigenfunctions u_n are C^∞ at the corners. Thus, the eigenfunctions U_n are also C^∞ functions at the corners of S. Since there is no interpolation at the boundary, (14.8) will hold with $p = 2$, $q = 4$. Thus, we may use the Richardson extrapolation formula (14.9). The extrapolated values for the eigenvalues are also given in Table 2. We see good agreement with the bounds of Table 1, particularly for the lower eigenvalues, although some of the higher extrapolated difference eigenvalues exceed the upper bounds.

Moler [101] has applied his method of point matching [50], [100] to this problem. Since the eigenfunctions here are C^∞, this method works quite well also. He obtained

$$\lambda_1 = 7.5695769 \ .$$

Finally, we used the method of *a posteriori-a priori* inequalities on the problem (18.6). The appropriate *a posteriori* inequality is

$$\min_i \left| \frac{\lambda_i - \lambda_*}{\lambda_i} \right|^2 \leq \frac{\int_S \sigma w^2 \, dxdy}{\int_S \sigma u_*^2 \, dxdy} \ , \tag{18.21}$$

where w satisfies

$$\Delta w = \Delta u_* + \lambda_* \sigma u_* \quad \text{in } S, \qquad w = u_* \quad \text{on } C \ . \tag{18.22}$$

The appropriate *a priori* inequality is

$$\int_S \sigma w^2 \, dxdy \leq \alpha_1 \int_S (\Delta w)^2 \, dxdy + \alpha_2 \int_C w^2 \, ds \ , \tag{18.23}$$

for α_1, α_2 explicit constants. This leads to

$$\frac{\lambda_*}{1+\epsilon} \leq \lambda_i \leq \frac{\lambda_*}{1-\epsilon} \ , \tag{18.24}$$

where ϵ^2 is the smallest eigenvalue of

$$Ma = \epsilon^2 Na \ , \tag{18.25}$$

and

$$M_{ij} = \alpha_1 \int_S (\Delta\phi_i + \lambda_* \sigma\phi_i)(\Delta\phi_j + \lambda_* \sigma\phi_j)\, dxdy + \alpha_2 \int_C \phi_i \phi_j \, ds \ ,$$

$$N_{ij} = \int_S \sigma\phi_i \phi_j \, dxdy \ . \tag{18.26}$$

The trial functions ϕ used for the even-even symmetry class were simply

$$\cos kx \cos ly \ , \qquad k,l = 0,1,2, \cdots , 10 \ . \tag{18.27}$$

Some of the integrals in the matrices M and N were computed by using 32-point Gaussian quadrature. The smallest eigenvalue ϵ^2 of (18.25) is easily found by inverse iteration. The resulting lower and upper bounds are given in Table 3. This method has also been used in [81] for the limaçon and cardioid.

TABLE 3

Bounds for even-even eigenvalues λ_ν of the region in Fig. 8 by the method of a posteriori-a priori inequalities (121 polynomial trial functions).

ν	Lower	Upper
1	7.5694	7.5704
2	29.092	29.128
3	44.825	44.854
4	67.682	68.019
5	74.381	74.620
6	101.82	107.59

Thus, we have seen that all of the methods, finite differences, *a posteriori-a priori* inequalities, intermediate problems using truncation, and Rayleigh-Ritz, give very satisfactory results on the equivalent problem resulting from the conformal mapping. The last three methods give bounds, and the last two methods used together give very good estimates. None of the methods is difficult to use in practice. Many conformal transformations are available (see, e.g., [70]) that lead to regions of interest. The conformal mapping approach coupled with the above methods is a very useful technique, which should have many further applications.

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