EIGENVALUES OF THE LAPLACIAN OF A GRAPH

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William N. Anderson, Jr.

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Thomas D. Morley

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Let G be a finite undirected graph with no loops or multiple edges. We define the Laplacian matrix of G, $\Delta(G)$, by Δ_{ii} = degree of vertix i and $\Delta_{ij} = -1$ if there is an edge between vertex i and vertex j. In this paper we relate the structure of the graph G to the eigenvalues of $\Delta(G)$; in particular we prove that all the eigenvalues of $\Delta(G)$ are non-negative, less than or equal to the number of vertices, and less than or equal to twice the maximum vertex degree. Precise conditions for equality are given.

1. Introduction

Let G be a finite undirected graph with no loops or multiple edges. We define the Laplacian matrix of G, $\Delta(G)$, by Δ_{ii} = the degree of vertex i and $\Delta_{ij} = -1$ if there is an edge between vertex i and vertex j. This matrix is discussed by Harary [5]. Our name for Δ is chosen because Δ arises in numerical analysis as a discrete analog of the Laplacian operator [3]. In this paper we relate the structure of the graph G to the eigenvalues of $\Delta(G)$; in particular, we prove that all the eigenvalues of Δ are non-negative, less than or equal to the number of vertices, and less than or equal to twice the maximum vertex degree.

There is a considerable body of literature relating the eigenvalues of the adjacency matrix of a graph to its structure [6]; except for Fisher's paper [2], little seems to be known about the Laplacian.

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2. Preliminaries

Our basic graph theory reference is Harary [5]. The definitions of Δ and E, as well as Lemma 1, are taken from Chapter 13 of Harary. To define E, the vertex-edge incidence matrix, we first orient G. Then $E_{ij} = 1$ if edge j points toward vertex i, $E_{ij} = -1$ if edge j points away from vertex i, and $E_{ij} = 0$ otherwise. Let E^* denote the transpose of E.

Lemma 1. $\Delta = EE^*$.

Proof. Two distinct rows of E will have a non-zero entry in the same column if and only if an edge joins the corresponding vertices; the corresponding entry will be 1 in one row and -1 in the other, giving a product of -1. Q.E.D.

We will also need to consider the matrix N defined by $N = E^*E$. The important property of N is that if λ is a non-zero eigenvalue of Δ, then it is also an eigenvalue of N, and conversely. In fact, if $\Delta x = \lambda x$ with $\lambda \neq 0$, then $NE^*x = E^*\Delta x = \lambda E^*x$ so that λ is an eigenvalue of N with eigenvector E^*x. The matrix N of course depends on the choice of orientation; we will vary the orientation as needed. In particular, if G is a bipartite graph, we may point all edges toward vertices of one class. Then all entries of N are non-negative; in fact $N = 2I + A$, where A is the adjacency matrix of the line graph of G. Results about line graphs of bipartite graphs thus translate directly into the present context [1].

If M is a matrix, let $\rho(M)$ denote the spectral radius of M.

Let $\overline{G}$ denote the graph complementary to G. That is, $\overline{G}$ has the same set of vertices as G, and vertices v and w are joined in $\overline{G}$ if and only if they are not joined in G.

Let K_n denote the complete graph on n vertices.

3. The Global Structure of G

In this section we obtain bounds for the eigenvalues of $\Delta(G)$ in terms of the number of vertices and the number of components of G.

Lemma 2. The eigenvalues of $\Delta(K_n)$ are 0, with multiplicity 1, and n, with multiplicity n-1.

Proof: Let u be the vector with all components equal to 1; then $\Delta(K_n)u = 0$. If x is any vector orthogonal to u, it may be easily verified that $\Delta(K_n)x = nx$. Q.E.D.

Theorem 1. If the graph G has n vertices, and λ is an eigenvalue of $\Delta(G)$ then $0 \leq \lambda \leq n$. The multiplicity of 0 equals the number of components of G; the multiplicity of n is equal to one less than the number of components of $\bar{G}$.

Proof. Suppose λ is an eigenvalue of Δ. Then for some vector x, with $\|x\| = 1$, $\Delta x = \lambda x$. Thus $\lambda = (\lambda x,x) = (\Delta x,x) = (EE^{*}x,x) = \|E^{*}x\|^2$. Therefore λ is real and non-negative.

Let the vertices $v_1,\ldots,v_K$ be the vertices of a connected component of G; then the sum of the corresponding rows of E is 0, and any K-1 of these rows are independent. Therefore the nullity of E, and thus of EE^{*}, is equal to the number of components of G.

If G has n vertices, then $\Delta(G) + \Delta(\bar{G}) = \Delta(K_n)$. If u is the vector with all components 1, then $\Delta(G)u = \Delta(\bar{G})u = \Delta(K_n)u = 0$. If $\Delta(G)x = \lambda x$ for some vector x orthogonal to u, then using Lemma 2 we have $\Delta(\bar{G})x = \Delta(K_n)x - \Delta(G)x = (n-\lambda)x$. Since the eigenvalues of $\Delta(\bar{G})$ are also non-negative, we must have $\lambda \leq n$. Moreover $\lambda = n$ if and only if $\Delta(\bar{G})x = 0$, and the dimension of the space of such vectors is one less than the nullity of $\Delta(\bar{G})$ (since all such x are orthogonal to u).

Q.E.D.

Corollary. If G has n vertices, and $\lambda = n$ is an eigenvalue of $\Delta(G)$, then G is connected.

Proof. If G were not connected, then $\bar{G}$ would be, and by the theorem n could not be an eigenvalue of $\Delta(G)$. Q.E.D.

4. The Local Structure of G

In this section we obtain an upper bound for the eigenvalues of $\Delta(G)$ in terms of vertex degrees.

Before proceeding we need to recall a few facts from the theory of non-negative matrices; our basic reference is Chapter XIII of Gantmacher [4]. Briefly, a matrix M is said to be non-negative if $M_{ij} \geq 0$ for all i and j. If M is a matrix, denote by M^+ the matrix obtained by replacing each entry of M by its absolute value. If M is irreducible, and λ is an eigenvalue of M, then $|\lambda| \leq \rho(M^+)$, with equality if and only if $M = e^{i\phi}DM^+D^{-1}$ where $D^+ = I$. For an irreducible non-negative matrix M, $\rho(M) \leq$ the maximum row sum with equality if and only if all row sums are equal.

Theorem 2. Let G be a graph. Then $\rho(\Delta(G)) \leq \text{Max}\,(\deg v + \deg w)$ where the maximum is taken over all pairs of vertices (v,w) joined by an edge of G. If G is connected, then equality holds if and only if G is bipartite and the degree is constant on each class of vertices.

Proof. We will work with the matrix N rather than Δ.

First consider a connected graph G, then N is irreducible, and thus $\rho(N) \leq \rho(N^+) \leq$ maximum row sum of N^+. But if e is an edge of G joining vertices v and w, then the row sum in the row corresponding to e is deg v + deg w. The inequality is thus established for connected graphs.

If G is bipartite, then we may orient G with all edges pointing toward the vertices in one of the two classes; thus $N(G) = N^+(G)$. Then $\rho(N)$ = max row sum if and only if all row sums are equal; i.e., if and only if the condition of the theorem holds. Equivalently, equality holds if and only if the line graph of G is regular.

If G is not bipartite, then we will show that $\rho(N) < \rho(N^+)$, so that equality cannot hold in the theorem. In fact, suppose $N = e^{i\phi} D N^+ D^{-1}$. Then since $N_{ii} = 2$, we have $2 = e^{i\phi} \cdot D_{ii} \cdot 2 \cdot D_{ii}^{-1}$, so that $e^{i\phi} = 1$. Now suppose that the edges $1, \ldots, K$ form an odd cycle (if no odd cycle exists, then G is bipartite); we may orient G so that the corresponding entries of N are -1. Then $N_{12} = -1 = D_{11} \cdot 1 \cdot D_{22}^{-1}$, so that $D_{22} = -D_{11}$; continuing around the cycle we have $D_{11} = -D_{11}$, contradicting the requirement that $D^+ = I$. Therefore, if G is not bipartite, equality cannot hold in the theorem.

If G is not connected, the inequality, and the corresponding equality statement, follow by applying the theorem to each component separately.

Q.E.D.

Corollary. Let G be a connected graph. Then $\rho(\Delta(G)) \leq$ twice the maximum vertex degree with equality if and only if G is a regular bipartite graph.

Proof. This is a special case of the theorem Q.E.D.

5. Explicit Computations

Theorems 1 and 2 were conjectured from explicit computations with eigenvalues; many of these were done on a digital computer. Some of these results are stated below; the reader may verify them without difficulty.

If G is the complete bipartite graph $K_{m,n}$, then the eigenvalues of $\Delta(G)$ are $m+n$, m, n, 0 with respective multiplicities 1, $n-1$, $m-1$, 1.

If G is the cycle with n vertices, then the eigenvalues of $\Delta(G)$ are $4\sin^2 \frac{\pi K}{n}$, $K = 1,2,\ldots,n$.

If G is the path with n vertices, the eigenvalues of $\Delta(G)$ are $4\sin^2 \frac{\pi K}{2n}$, $K = 0,1,\ldots,n-1$.

If G is the wheel with $n+1$ vertices, the eigenvalues of $\Delta(G)$ are $n+1$, 1, and $1 + 4\sin^2 \frac{\pi K}{n}$, $K = 1,2,\ldots,n-1$.

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Footnotes

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