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Eigenvalues of the Laplacian of Compact Riemann Surfaces and Minimal Submanifolds.

PAUL C. YANG - SHING-TUNG YAU

Introduction.

Given a compact Riemannian manifold (M, ds^2) , the spectrum of its Laplacian is an important analytic invariant. There has been a considerable amount of work devoted to estimating the first eigenvalue λ_1 in terms of other geometric quantities associated to (M, ds^2) . We recall the known estimates valid for general Riemannian manifolds: The eigenvalue comparison theorem of Cheng [2] gives a computable sharp upper bound in terms of the diameter, and lower bound of the Ricci curvature; while Yau ([8]) derived a computable lower bound in terms of the diameter, volume and lower bound of the Ricci curvature. In the special case when M has dimension two, Hersch ([3]) has extended the method of Szegö to obtain an upper bound of λ_1 for an arbitrary metric on S^2 simply in terms of its area. Subsequently in [1], Berger suggested, having verified for flat metrics, that a similar estimate holds for the torus.

In the first part of this paper we give an affirmative answer:

THEOREM 1. *Let (M, ds^2) be an orientable Riemannian surface of genus g with area A . Then we have*

$$\lambda_1 \leq 8\pi(g + 1)A^{-1}.$$

To be more precise, we give in section 2 the following lower bound for the quantity $\sum_{i=1}^3 \lambda_i^{-1} \geq 3A/8\pi d$ for a metric Riemann surface, with a meromorphic function of degree d (holomorphic map $\pi: M \rightarrow S^2$).

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REMARKS. 1) In the case of a torus $g = 1$, Berger suggested $8\pi^2/\sqrt{3} \sim 0.9 \times 16\pi$ as an upper bound for λ_1 , we do not know whether Theorem 1 holds with this better constant. In this connection it is of interest to note that Berger also showed ([1], Proposition 4.22) that $\sum_{i=1}^6 1/\lambda_i < 6A/(8\pi^2/\sqrt{3})$ for the flat torus with lattice generated by $(1, 0)$ and $(\frac{1}{2}, \sqrt{3}/2(1 + \varepsilon))$ for ε small.

2) For manifold of dimension greater than two, an analogous upper bound for λ_1 in terms of its volume is false in general. In fact Urakawa and Tano ([7], [6] respectively) have shown that for odd dimensional spheres S^n and compact Lie groups with nontrivial commutator subgroup, the inequality $\lambda_1(g) \text{vol}^{2/n}(g) \leq C_n$ does not hold for a constant c_n independent of the metric. Thus the problem remains open only for compact non-orientable surfaces.

3) We observe two immediate consequences: 1) For holomorphic curves in CP^n , the area of the metric induced from the Fubini-Study metric is 4π times its degree, hence $\lambda_1 \leq 2$; 2) for Riemann surfaces of genus g carrying the Poincaré metric with constant curvature -1 , the Gauss-Bonnet formula and Theorem 1 yield $\lambda_1 \leq 2((g+1)/(g-1))$.

In the second part of this paper we study the eigenvalues of compact minimal submanifolds M^m of the unit sphere $S^N(1) \subset \mathbf{R}^{N+1}$. Once more we exploit the fact that the coordinate functions $\{x_\alpha\}$ are eigenfunctions and satisfy the equation $\sum_{\alpha=1}^{N+1} x_\alpha^2 = 1$ to estimate the consecutive difference of eigenvalues of M^m as in Payne, Polya and Weinberger ([5]).

THEOREM 2. *If $M^m \rightarrow S^N(1)$ is a compact minimally immersed submanifold of $S^N(1)$, then we have, setting $A_k = \sum_{i=1}^k \lambda_i$*

$$\lambda_{k+1} - \lambda_k \leq m + \frac{\sqrt{A_k^2 + 2m^2 A_k(k+1)} + A_k}{m(k+1)}.$$

We would like to thank Professor M. Berger for providing relevant references and suggestions clarifying our exposition.

1. - In this section we collect the elementary facts concerning the Laplacian operator, and recall the basic equations for minimal submanifolds of the sphere.

Given a compact m -dimensional Riemannian manifold M with metric given in local coordinates by $\sum g_{ij} dx_i dx_j$, the Laplacian operator Δ acting on functions is given by

$$g^{-1} \frac{\partial}{\partial x_i} \left(g^{ij} \frac{\partial}{\partial x_j} \right),$$

where (x_1, \dots, x_m) is a local coordinate system, g^{ij} is the inverse of the matrix g_{ij} and $g = \det(g_{ij})$, and we denote the volume element by dv . It is well-known that Δ have discrete spectrum. We list the eigenvalues as $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$, and the corresponding eigenfunctions u_k with $\Delta u_k + \lambda_k u_k = 0$ form a complete orthonormal basis for $L^2(M)$. An immediate consequence is the minimum principle:

$$(1.1) \quad \lambda_{k+1} = \inf \left(\int_M \langle \nabla \varphi, \nabla \varphi \rangle dv \right) \left(\int_M \varphi^2 dv \right)^{-1}$$

where the infimum is taken over piecewise C^1 functions $\varphi \neq 0$ satisfying

$$(1.2) \quad \int_M \varphi u_i dv = 0 \quad \text{for } 0 \leq i \leq k.$$

Similarly a simultaneous diagonalization argument involving the two quadratic forms $\int_M \langle \nabla \cdot, \nabla \cdot \rangle dv$ and $\int_M \langle \cdot, \cdot \rangle dv$ yields the following variational characterization

$$(1.3) \quad \sum_{i=1}^k \lambda_i^{-1} = \sup \sum_{i=1}^k \left(\int_M \varphi^2 dv \right) \left(\int_M |\nabla \varphi_i|^2 dv \right)^{-1}$$

where the supremum is taken over piecewise C^1 functions $\varphi_1, \dots, \varphi_k$ satisfying

$$(1.4) \quad \int_M \varphi_i dv = 0 \quad \text{for } 1 \leq i \leq k$$

and

$$(1.5) \quad \int_M \langle \nabla \varphi_i, \nabla \varphi_j \rangle = 0 \quad \text{for } i \neq j.$$

In the special case when M has dimension two, the existence of local holomorphic coordinates $z = x + iy$ simplifies the local expression for Δ : if $ds^2 = F|dz|^2$, then $\Delta = (4/F)(\partial^2/(\partial z \partial \bar{z}))$. An essential feature of the surface Laplacian is the invariance of the Dirichlet integrand: if $\tilde{ds}^2 = \lambda ds^2$ then $|\tilde{\nabla} u|^2 d\tilde{v} = |\nabla u|^2 dv$.

Suppose $x: M^m \rightarrow S^N(1)$ is an isometric minimal immersion of a Riemannian manifold M into the standard unit sphere $S^N(1)$ in Euclidean space E^{N+1} . It is well known ([4], p. 342) that the coordinate functions x_1, \dots, x_{N+1} are eigenfunctions for M^m with eigenvalue m .

2. – Theorem 1 will be an easy consequence of the following more precise:

PROPOSITION. Let (M, ds^2) be a metric Riemann surface, and suppose $\pi: M^2 \rightarrow S^2$ in a non-constant holomorphic map of degree d , then we have, denoting by A the area of M ,

$$(2.1) \quad \sum_{i=1}^k \lambda_i^{-1} \geq \frac{3A}{8\pi d}.$$

PROOF. From complex analysis, it is well known that π is a branched cover; that is there is a finite set $\{p_1, \dots, p_n\} \subset S^2$ so that $\pi: M - \pi^{-1} \cdot \{p_1, \dots, p_n\} \rightarrow S^2 - \{p_1, \dots, p_n\}$ is a covering map with d sheets; and at each singular point say $q \in \pi^{-1}(p_i)$, π can be expressed relative to local coordinates z (resp. w) around q (resp. p_i) as

$$(2.2) \quad w = \pi(z) = z^j \quad \text{for some positive integer } j.$$

For a generic point $p \in S^2 - \{p_1, \dots, p_n\}$, we define a conformal metric by $ds_*^2 = \sum_{\alpha} (\pi_{\alpha}^{-1})^* ds^2$, where the sum is taken over the various sheets of the covering. In other words, ds_*^2 is a finite sum of conformal metrics $ds_{\alpha}^2 = (\pi_{\alpha}^{-1})^* ds^2$ with respect to which π is a local isometry on each corresponding sheet indexed by α . In terms of local coordinates z_{α} around each $q_{\alpha} \in \pi^{-1}(p)$, if the metric near q_{α} is given by $G(z_{\alpha})|dz|^2$, then ds_*^2 is given by

$$(2.3) \quad \sum_{\pi(z_{\alpha})=w} G(z_{\alpha}) \left| \frac{dz_{\alpha}}{dw} \right|^2 |dw|^2.$$

Near the singular points $q \in \pi^{-1}\{p_1, \dots, p_n\}$, according to the representation (2.2), the singular contribution to ds_*^2 is

$$(2.4) \quad \sum_{z_{\alpha}=w} G(z_{\alpha}) \left| \frac{dw^{1/j}}{dw} \right|^2 |dw|^2 = \sum_{z_{\alpha}=w} G(z_{\alpha}) \left(\frac{1}{j} \right)^2 |w|^{-2+2/j} |dw|^2,$$

which is clearly integrable.

LEMMA. Let $u \in C^1(S^2)$, and let dv (resp. dv_*) denote the volume element of ds^2 (resp. ds_*^2) on M (resp. S^2), we have

$$(i) \int_{S^2} u dv_* = \int_M (u \circ \pi) dv.$$

(ii) If ds_0^2 is another conformal metric on S^2 , then

$$\int_{S^2} |\nabla_0 u|^2 dv_0 = \int_{S^2} |\nabla_* u|^2 dv_* = \int_{S^2} |\nabla(u \circ \pi)|^2 dv.$$

PROOF. It suffices to prove the identities locally, i.e., for those u supported in a trivializing neighborhood U of the covering. Then (i) is an immediate consequence of the definition of ds_*^2 . The first identity in (ii) is simply the invariance of Dirichlet integrand under conformal change of metric. For the second identity, observe that for each sheet α in the local trivialization we have

$$\int_U |\nabla^* u|^2 dv^* = \int_U |\nabla_\alpha u|^2 dv^* = \int_{\pi_\alpha^{-1}(v)} |\nabla u|^2 dv$$

hence the assertion follows.

Let x_1, x_2, x_3 be the standard coordinate functions on (S^2, ds_0^2) , ds_0^2 is the standard metric.

$$(2.5) \quad \sum_1^3 x_i^2 = 1,$$

$$(2.6) \quad \int_{S^2} \langle \nabla_0 x_i, \nabla_0 x_j \rangle dv_0 = \delta_{ij} \frac{8\pi}{3}.$$

Consider the transplanted pseudometric ds_*^2 on S^2 introduced above. In order to apply the estimate of (1.13), we need to find a conformal self map φ of S^2 so that

$$\int (x_i \circ \varphi) dv_* = 0$$

for $i = 1, 2$, and 3 . The topological argument of Hersch still applies since dv_* is an integrable density on S^2 . Assuming this is done, then letting $u = (x_i \circ \varphi)$, and $v = (x_i \circ \varphi \circ \pi)$, we have by (1.3)

$$\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} \geq \sum_{i=1}^3 \frac{\int_M v_i^2 dv}{\int_M |\nabla v_i|^2 dv}.$$

In the right hand side each denominator is according to (ii) of Lemma equal to $d \cdot (8\pi/3)$, hence

$$\sum_1^3 \frac{1}{\lambda_i} \geq \frac{3}{8\pi d} \sum_1^3 \int_M v_i^2 dv = \frac{3}{8\pi d} \int_M dv = \frac{3A}{8\pi d}. \quad \text{Q.E.D.}$$

Theorem 1 follows by observing that $\lambda_1 \leq \lambda_2 \leq \lambda_3$ and that Riemann-Roch theorem ensures that each Riemann surface of genus g has at least one non-constant meromorphic function of degree $\leq g + 1$.

3. – For general compact minimal submanifolds $M^m \rightarrow S^N(1)$ into the Euclidean unit sphere, the coordinate functions x_1, \dots, x_{N+1} are eigenfunctions M^m and with the behavior of eigenfunctions of the spherical harmonics of the standard S^N as a model we might expect suitable products of x_i with the first k eigenfunctions of M^m to give reasonable trial functions for λ_{n+1} . The proof given below is based on this idea.

PROOF OF THEOREM 2. Let $x = (x_1, \dots, x_{N+1}) : M_m \rightarrow S^N \subset \mathbf{R}^{N+1}$ be the minimal immersion. Let $\{(u_i, \lambda_i)\}_{i=1, \dots, n}$ be the normalized first $k+1$ eigenfunctions and their corresponding eigenvalues (we set $\lambda_0 = 0$, $u_0 \equiv 1/\sqrt{v}$ where $v = \text{volume}(M)$). Then

$$(3.1) \quad \Delta x_\alpha = -m x_\alpha,$$

$$(3.2) \quad \Delta u_i = -\lambda_i u_i.$$

Let

$$(3.3) \quad u_{\alpha i} = x_\alpha u_i - \sum a_{\alpha i k} u_k$$

with

$$(3.4) \quad a_{\alpha i k} = \int_M x_\alpha u_i u_k = a_{\alpha k i}$$

and we set

$$(3.5) \quad A = \sum_{\alpha, i, k} a_{\alpha k i}^2.$$

Then $u_{\alpha i}$ are orthogonal to u_0, \dots, u_k . To estimate the Rayleigh-Ritz quotient we compute

$$-\Delta u_{\alpha i} = (m + \lambda_i) x_\alpha u_i - 2 \langle \nabla x_\alpha, \nabla u_i \rangle + \sum_k a_{\alpha i k} \lambda_k u_k$$

and then integrating against $u_{\alpha i}$,

$$\int |\nabla u_{\alpha i}|^2 = - \int (\Delta u_{\alpha i}) u_{\alpha i} = (m + \lambda_i) \int u_{\alpha i}^2 - 2 \int \langle \nabla x_\alpha, \nabla u_i \rangle u_{\alpha i}.$$

Hence

$$\lambda_{k+1} \leq \frac{\int |\nabla u_{\alpha i}|^2}{\int u_{\alpha i}^2} \leq (m + \lambda_i) - 2 \frac{\int u_{\alpha i} \langle \nabla x_\alpha, \nabla u_i \rangle}{\int u_{\alpha i}^2}.$$

Obviously

$$\lambda_{k+1} - \lambda_k - m \leq \lambda_{k+1} - \lambda_i - m \leq -2 \frac{\int u_{\alpha i} \langle \nabla x_\alpha, \nabla u_i \rangle}{\int u_{\alpha i}^2}$$

hence

$$(3.6) \quad \lambda_{k+1} - \lambda_k - m \leq \frac{\sum_{\alpha, i} (-2) \int u_{\alpha i} \langle \nabla x_\alpha, \nabla u_i \rangle}{\sum_{\alpha, i} \int u_{\alpha i}^2}.$$

Since $a_{\alpha ik}$ is symmetric in i and k we have

$$\sum a_{\alpha ik} \langle \nabla x_\alpha, \nabla (u_k u_i) \rangle = \sum 2a_{\alpha ik} u_k \langle \nabla x_\alpha, \nabla u_i \rangle,$$

so that

$$\begin{aligned} (3.7) \quad & \sum (-2) \int u_{\alpha i} \langle \nabla x_\alpha, \nabla u_i \rangle \\ &= \sum \int (-2) x_\alpha u_i \langle \nabla x_\alpha, \nabla u_i \rangle + \sum \int 2a_{\alpha ik} u_k \langle \nabla x_\alpha, \nabla u_i \rangle \\ &= \sum \int (-1) \langle \nabla x_\alpha^2, u_i \nabla u_i \rangle + \sum \int a_{\alpha ik} \langle \nabla x_\alpha, \nabla (u_k u_i) \rangle \\ &= \sum a_{\alpha ik} \int (-\Delta x_\alpha) u_i u_k \quad \left(\sum_\alpha x_\alpha^2 \equiv 1 \right) \\ &= m \sum a_{\alpha ik} \int x_\alpha u_i u_k \quad (\Delta x_\alpha = -m x_\alpha) \\ &= mA \end{aligned}$$

and

$$\begin{aligned} (3.8) \quad & \sum \int u_{\alpha i}^2 = \sum_{\alpha, i} \int (x_\alpha^2 u_i^2 - 2x_\alpha u_i \sum_k a_{\alpha ik} u_k + \sum_{k, l} a_{\alpha ik} a_{\alpha il} u_k u_l) \\ &= \sum_i \int u_i^2 - 2A + A \\ &= (k+1) - A. \end{aligned}$$

The Schwarz inequality gives

$$(3.9) \quad \begin{aligned} \sum \left| \int u_{\alpha i} \langle \nabla x_{\alpha}, \nabla u_i \rangle \right| &\leq \sum \left[\int (u_{\alpha i})^2 \right]^{\frac{1}{2}} \left[\int \langle \nabla x_{\alpha}, \nabla u_i \rangle^2 \right]^{\frac{1}{2}} \\ &\leq \left[\sum \int u_{\alpha i}^2 \right]^{\frac{1}{2}} \left[\sum \int \langle \nabla x_{\alpha}, \nabla u_i \rangle^2 \right]^{\frac{1}{2}} \leq \left[\sum \int u_{\alpha i}^2 \right]^{\frac{1}{2}} \left[\sum_i \int |\nabla u_i|^2 \right]^{\frac{1}{2}} \\ &\text{(since } \sum_{\alpha} \langle \nabla x_{\alpha}, \nabla u_i \rangle^2 \equiv |\nabla u_i|^2) \\ &\leq \left[\sum \int u_{\alpha i}^2 \right]^{\frac{1}{2}} \left[\sum_1^k \lambda_i \right]^{\frac{1}{2}}. \end{aligned}$$

Applying (3.9) to (3.6) we obtain, setting $A_k = \sum_1^k \lambda_i$

$$(3.10) \quad \lambda_{k+1} - \lambda_k - m \leq \frac{\sum (-2) \int u_{\alpha i} \langle \nabla x_{\alpha}, \nabla u_i \rangle}{\left[\sum \int u_{\alpha i} \langle \nabla x_{\alpha}, \nabla u_i \rangle \right]^2} \left[\sum_1^k \lambda_i \right] \leq \frac{2A_k}{mA}.$$

Combining (3.6), (3.7), (3.8) and (3.10) we have

$$\lambda_{k+1} - \lambda_k - m \leq \min \left[\frac{2A_k}{mA}, \frac{mA}{(k+1) - A} \right].$$

Consider the expression in the bracket as functions of A , and observe that the first is decreasing while the second is increasing in A for $A \in (0, k+1)$. Then both must be less than their common value which occurs at

$$A = \frac{\sqrt{A_k^2 + 2m^2 A_k(k+1)} - A_k}{m^2} < k+1$$

and for this A the common value is

$$\frac{\sqrt{A_k^2 + 2m^2 A_k(k+1)} + A_k}{m(k+1)}$$

as claimed.

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