Tôhoku Math. Journ.
25 (1973), 391-403.

EIGENVALUES OF THE LAPLACIAN OF RIEMANNIAN MANIFOLDS

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(Received December 19, 1972)

1. Introduction. Let (M, g) be a compact orientable Riemannian manifold (connected and C^∞, $\dim M = m$) with metric tensor g. By ∇ we denote the Riemannian connection and by Δ we denote the Laplacian acting on p-forms, $0 \leqq p \leqq m$. Let $\lambda_{\alpha,p}$ be eigenvalues of Δ and put

$$\text{Spec}\,(M, g; p) = \{0 \geqq \lambda_{0,p} \geqq \lambda_{1,p} \geqq \lambda_{2,p} \geqq \cdots \downarrow -\infty\}\,. \tag{1.1}$$

For $p = 0$, we denote $\text{Spec}\,(M, g; 0) = \text{Spec}\,(M, g)$ and $\lambda_{\alpha,0} = \lambda_\alpha$. Relations between $\text{Spec}\,(M, g)$ and Riemannian structures have been studied by Berger [1], Mckean-Singer [6], Patodi [10], Sakai [11], etc. Some results are listed in the first part of § 3. A useful tool is a formula of Minakshisundaram:

$$\sum_{\alpha=0}^{\infty} e^{\lambda_\alpha t} \underset{t\to 0}{\sim} \left(\frac{1}{4\pi t}\right)^{m/2} \sum_{\beta=0}^{\infty} a_\beta t^\beta\,. \tag{1.2}$$

a_0, a_1, a_2 were calculated by Berger [1] and Mckean-Singer [6]; and a_3 was calculated by Sakai [11]. In the following Theorems A, B, and D, the assumption on Spec (,) is also replaced by $\sum e^{\lambda_\alpha t}$, and more precisely in terms of a_β ($\beta = 0, 1, 2, 3$).

THEOREM A. *Let (M, g) and (M', g') be compact orientable Riemannian manifolds. Assume that* $\text{Spec}\,(M, g) = \text{Spec}\,(M', g')$. *Then $m = m'$ and*

(1) *for $2 \leqq m \leqq 5$, (M, g) is of constant curvature K, if and only if (M', g') is of constant curvature $K' = K$,*

(2) *for $m = 6$, (2-1) (M, g) is conformally flat and the scalar curvature S is constant, if and only if (M', g') is conformally flat and the scalar curvature S' is constant, $S' = S$,*

(2-2) *(M, g) is of constant curvature $K > 0$, if and only if (M', g') is of constant curvature $K' = K > 0$.*

Theorem A for $m = 2, 3$ was proved by Berger [1]. For $m = 4$, Berger's Theorem 8.1 in [1] requires an additional condition $\chi(M) = \chi(M')$, where $\chi(M)$ denotes the Euler-Poincaré characteristic of M. Our result generalizes this and furthermore it is valid for $m=5$.

By $S^m(c)$ or $(S^m(c), g_0)$ we denote a Euclidean m-sphere with constant curvature $c>0$, and by $H^m(-c)$ we denote a hyperbolic m-space with constant curvature $-c<0$.

THEOREM B. *Let (M, g) be a compact orientable Riemannian manifold, $2 \leqq m \leqq 6$. If* Spec (M, g) = Spec $(S^m(c), g_0)$, *then (M, g) is isometric to $(S^m(c), g_0)$.*

This follows from Theorem A (1) and (2-2).

Theorem B has some aspect related to Obata's theorem [8] on the first non-zero eigenvalue on Einstein spaces.

THEOREM C. *Let (M, g) be a compact orientable Riemannian manifold with $m=6$. In* (1.2), *$a_2 = a_3 = 0$ holds, if and only if (M, g) is either*

(1) *E^6/Γ_1, where Γ_1 is some discontinuous group of translations of the* 6-*dimensional Euclidean space E^6, or*

(2) *$[S^3(c) \times H^3(-c)]/\Gamma_2$, where Γ_2 is some discontinuous group of isometries of $S^3(c) \times H^3(-c)$.*

For $m \leqq 5$, Mckean-Singer [6] and Patodi [10] showed that (1) is the only case. For $m \geqq 7$, see Proposition 7 (3).

Kählerian analogues are also true. Corresponding to Theorem A, we have Theorem E in §4. Corresponding to Theorem B, we have

THEOREM D. *Let (M, g, J) be a compact Kählerian manifold, $m = 2n \leqq 12$. Let $(CP^n(H), g_0, J_0)$ be a complex n-dimensional projective space with the Fubini-Study metric of constant holomorphic sectional curvature H. If* Spec (M, g, J) = Spec $(CP^n(H), g_0, J_0)$, *then (M, g, J) is holomorphically isometric to $(CP^n(H), g_0, J_0)$.*

2. **Preliminaries.** By $R = (R^i{}_{jkl})$ we denote the Riemannian curvature tensor: $R^i{}_{jkl}\partial/\partial x^i = R(\partial/\partial x^k, \partial/\partial x^l)\partial/\partial x^j$ and $R(X, Y)Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z$, $i, j, k, l = 1, \cdots, m = \dim M$. By $R_1 = (R_{jk}) = (R^r{}_{jkr})$ we denote the Ricci curvature tensor. By $S = (g^{jk}R_{jk})$ we denote the scalar curvature. For a tensor field $T = (T_{ijk})$, for example, we denote $|T|^2 = (T_{ijk}T^{ijk})$. Then we have (cf. [1], [2], [6], [11])

$$a_0 = \mathrm{Vol}\,(M) = \int dM\,, \tag{2.1}$$

$$a_1 = \frac{1}{6}\int S dM\,, \tag{2.2}$$

$$a_2 = \frac{1}{360}\int [2|R|^2 - 2|R_1|^2 + 5S^2] dM\,, \tag{2.3}$$

$$(2.4)\qquad a_3 = \frac{1}{6!}\int\Big[-\frac{1}{9}|\nabla R|^2 - \frac{26}{63}|\nabla R_1|^2 - \frac{142}{63}|\nabla S|^2 - \frac{8}{21}R^{ij}{}_{kl}R^{kl}{}_{rs}R^{rs}{}_{ij} - \frac{8}{63}R^{rs}R_r{}^{jkl}R_{sjkl} + \frac{2}{3}S|R|^2 - \frac{20}{63}R^{ik}R^{jl}R_{ijkl} - \frac{4}{7}R^i{}_jR^j{}_kR^k{}_i - \frac{2}{3}S|R_1|^2 + \frac{5}{9}S^3\Big]dM\,.$$

The following are also useful.

$$(2.5)\qquad |R|^2 - \frac{2}{m-1}|R_1|^2 \geqq 0\,,$$

$$(2.6)\qquad |R_1|^2 - \frac{1}{m}S^2 \geqq 0\,.$$

The equality in (2.5) on M implies that (M, g) is of constant curvature, and the equality in (2.6) on M implies that (M, g) is an Einstein space (cf. [1], or [2]).

The Weyl's conformal curvature tensor $C = (C^i{}_{jkl})$, $C_{ijkl} = g_{ir}C^r{}_{jkl}$, is given by (for $m \geqq 4$)

$$(2.7)\qquad C_{ijkl} = R_{ijkl} - \frac{1}{m-2}(R_{jk}g_{il} - R_{jl}g_{ik} + g_{jk}R_{il} - g_{jl}R_{ik}) + \frac{1}{(m-1)(m-2)}(g_{jk}g_{il} - g_{jl}g_{ik})S\,.$$

Then we have (cf. [14], [15])

$$(2.8)\qquad |C|^2 = |R|^2 - \frac{4}{m-2}|R_1|^2 + \frac{2}{(m-1)(m-2)}S^2 \geqq 0\,.$$

By (2.8) we have

$$(2.9)\quad 2|R|^2 - 2|R_1|^2 + 5S^2 = 2|C|^2 + \frac{2(6-m)}{m-2}|R_1|^2 + \frac{5m(m-3)+6}{(m-1)(m-2)}S^2\,.$$

3. Geometry reflected by the spectrum. Let (M, g) and (M', g') be compact orientable Riemannian manifolds. The following are known.

[i] Spec (M, g) = Spec (M', g') implies $m = m'$, Vol (M) = Vol (M') ([2], p. 215).

[ii] For $m = m' = 2$, if $a_\beta = a'_\beta$ $(\beta = 0, 1, 2)$ and S = constant, then S' is also constant and $S = S'$ (Berger: [2], p. 226).

[iii] For $m = m' = 3$, if $a_\beta = a'_\beta$ $(\beta = 0, 1, 2)$ and if (M, g) is of constant curvature K, then (M', g') is also of constant curvature K (Berger: [2], p. 228).

[iv] For $m = m' = 4$, if $a_\beta = a'_\beta$ $(\beta = 0, 1, 2)$, $\chi(M) = \chi(M')$ and if (M, g) is of constant curvature K, then (M', g') is of constant curvature

K (Berger: [2], p. 229).

[v] Assume that (M, g) and (M', g') are Einstein spaces and $a_\beta = a'_\beta$ $(\beta = 0, 1, 2)$. Then (M, g) is of constant curvature K, if and only if (M', g') is of constant curvature K (Sakai [11], p. 599).

[vi] For $m = m' = 6$, assume that (M, g) and (M', g') are Einstein spaces. If $a_\beta = a'_\beta$ $(\beta = 0, 1, 2, 3)$ and $\chi(M) = \chi(M')$, then (M, g) is locally symmetric if and only if (M', g') is locally symmetric (Sakai [11], p. 601).

[vii] For $m \leqq 5$, assume $a_\beta = 0$ $(\beta = 1, 2)$. Then (M, g) is locally flat (for $m \leqq 3$, Mckean-Singer [6]; for $m \leqq 5$, Patodi [10]).

Concerning [iii], [iv] and [v], we have

THEOREM A′. *Let (M, g) and (M', g') be compact orientable Riemannian manifolds. Assume $a_\beta = a'_\beta$ for $\beta = 0, 1, 2$. Then,*

(1) for $m = m' \leqq 5$, (M, g) is of constant curvature K, if and only if (M', g') is of constant curvature K,

(2) for $m = m' = 6$, (M, g) is conformally flat and S is constant, if and only if (M', g') is conformally flat and S' is constant, $S = S'$.

PROOF. Since the case $m = 2, 3$ was proved in [1], assume $m \geqq 4$. By (2.3) and (2.9), $a_2 = a'_2$ is written as

$$\tag{3.1} \begin{aligned} &\int\Big[2|C|^2 + \frac{2(6-m)}{m-2}\Big(|R_1|^2 - \frac{1}{m}S^2\Big) \\ &\qquad + \Big(\frac{2(6-m)}{m(m-2)} + \frac{5m(m-3)+6}{(m-1)(m-2)}\Big)S^2\Big]dM \\ &= \int\Big[2|C'|^2 + \frac{2(6-m)}{m-2}\Big(|R'_1|^2 - \frac{1}{m}S'^2\Big) \\ &\qquad + \Big(\frac{2(6-m)}{m(m-2)} + \frac{5m(m-3)+6}{(m-1)(m-2)}\Big)S'^2\Big]dM' . \end{aligned}$$

First assume $4 \leqq m \leqq 5$ and (M', g') is of constant curvature K'. Then it is conformally flat $(C' = 0)$ and is an Einstein space $(|R'_1|^2 = S'^2/m)$. Since S' is constant, $a_0 = a'_0$ and $a_1 = a'_1$ imply $\int S^2 dM \geqq \int S'^2 dM'$. In fact, using Schwarz inequality, we have

$$\begin{aligned} \Big(\int dM\Big)\Big(\int S^2 dM\Big) &\geqq \Big(\int S dM\Big)^2 = \Big(\int S' dM'\Big)^2 \\ &= S'^2\Big(\int dM'\Big)^2 = S'^2\Big(\int dM\Big)\Big(\int dM'\Big) \\ &= \Big(\int dM\Big)\Big(\int S'^2 dM'\Big) . \end{aligned}$$

Hence, (3.1) gives $C = 0$, $|R_1|^2 = S^2/m$ and $\int S^2 dM = \int S'^2 dM'$. Consequently,

(M, g) is of constant curvature $K = K'$.

Next, assume $m = 6$, $C' = 0$ and $S' = \text{constant}$. Then, using (3.1), similarly we have $C = 0$ and $S = \text{constant}$.

REMARK. (3.1) gives a simple proof of [v].

Concerning [vii], a simple proof is (2.3) and (2.9). Patodi [10] gives a counter-example for $m = 6$. For $m = 6$, under an additional condition $a_3 = 0$, we determine (M, g).

LEMMA 1. *If $a_2 = 0$ and $m = 6$, then (M, g) is conformally flat and the scalar curvature S is vanishing.*

PROOF. This follows from (2.3) and (2.9).

Now we denote $(R_1^3) = (R^i{}_j R^j{}_k R^k{}_i)$.

LEMMA 2. *If (M, g) is conformally flat, $m \geqq 4$, and $S = \text{constant}$, then*

$$\int |\nabla R_1|^2 = \int \Big[\frac{-m}{m-2} (R_1^3) + \frac{2m-1}{(m-1)(m-2)} S |R_1|^2 - \frac{1}{(m-1)(m-2)} S^3 \Big] dM . \tag{3.2}$$

PROOF. By (2.7) and $C = 0$, R_{ijkl} is expressed by R_{jk}, g_{jk}, and S. By the second Bianchi identity for R_{ijkl} we have

$$\nabla_k R_{ij} = \nabla_j R_{ik} . \tag{3.3}$$

Then we have

$$\begin{aligned} |\nabla R_1|^2 &= \nabla_k R_{ij} \nabla^k R^{ij} = \nabla_j R_{ik} \nabla^k R^{ij} \\ &= \nabla^k (\nabla_j R_{ik} \cdot R^{ij}) - \nabla^k \nabla_j R_{ik} \cdot R^{ij} , \end{aligned} \tag{3.4}$$

where $\nabla^k \nabla_j R_{ik}$ is calculated by the Ricci identity:

$$g^{kr} \nabla_r \nabla_j R_{ik} = g^{kr} (\nabla_j \nabla_r R_{ik} - R^s{}_{ijr} R_{sk} - R^s{}_{kjr} R_{is}) . \tag{3.5}$$

Noticing that $g^{kr} \nabla_r R_{ik} = (1/2) \nabla_i S = 0$ and $g^{kr} R^s{}_{kjr} = -R^s{}_j$, we simplify (3.5). Putting the result into (3.4) and integrating, we have (3.2).

LEMMA 3. *If (M, g) is conformally flat, $m \geqq 4$, and $S = \text{constant}$, then*

$$\begin{aligned} a_3 = \frac{1}{6!} \int \Big[& \frac{24 - 26m}{63(m-2)} |\nabla R_1|^2 \\ & + \frac{1}{63(m-2)^3} [192(m-4) \\ & - 16(m-2)(m-4) - 40(m-2)^2 - 36(m-2)^3] (R_1^3) \end{aligned} \tag{3.6}$$

$$+\frac{1}{63(m-1)(m-2)^3}[576-16(m+1)(m-2)$$
$$+4(m-2)^2(52m-47)-42(m-1)(m-2)^3]S|R_1|^2$$
$$+\frac{1}{63(m-1)^2(m-2)^3}[-96m+16(m-1)(m-2)$$
$$-104(m-1)(m-2)^2+35(m-1)^2(m-2)^3]S^3\Big]dM\,.$$

PROOF. By (2.7) and $C=0$, we have

$$|\nabla R|^2=\frac{4}{m-2}|\nabla R_1|^2\,, \tag{3.7}$$

$$S|R|^2=\frac{4}{m-2}S|R_1|^2-\frac{2}{(m-1)(m-2)}S^3\,, \tag{3.8}$$

(3.9) $R^{ik}R^{jl}R_{ijkl}$
$$=\frac{2}{m-2}(R_1^3)-\frac{2m-1}{(m-1)(m-2)}S|R_1|^2+\frac{1}{(m-1)(m-2)}S^3\,,$$

(3.10) $R^{rs}R_r{}^{jkl}R_{sjkl}$
$$=\frac{2(m-4)}{(m-2)^2}(R_1^3)+\frac{2(m+1)}{(m-1)(m-2)^2}S|R_1|^2-\frac{2}{(m-1)(m-2)^2}S^3\,,$$

(3.11)
$$R^{ij}{}_{kl}R^{kl}{}_{rs}R^{rs}{}_{ij}=[\text{replacing } R^{ij}{}_{kl} \text{ by (2.7)}]R^{kl}{}_{rs}R^{rs}{}_{ij}$$
$$=\frac{-4}{m-2}R^{rs}R_r{}^{jkl}R_{sjkl}+\frac{2}{(m-1)(m-2)}S|R|^2$$
$$=-\frac{8(m-4)}{(m-2)^3}(R_1^3)-\frac{24}{(m-1)(m-2)^3}S|R_1|^2$$
$$+\frac{4m}{(m-1)^2(m-2)^3}S^3\,.$$

Substituting these into (2.4), we get (3.6).

PROPOSITION 4. *Let (M, g) and (M', g') be compact orientable Riemannian manifolds, $m=m'=6$. Assume $a_\beta=a'_\beta$ $(\beta=0, 1, 2, 3)$. Then (M', g') is of constant curvature $K'>0$, if and only if (M, g) is also of constant curvature $K=K'>0$.*

PROOF. Assume that (M', g') is of constant curvature $K'>0$. By Theorem A′ we have $C=0$ and $S=$ constant. By (3.6) we have

$$a_3=\frac{1}{6!}\int\Big[-\frac{11}{21}|\nabla R_1|^2-\frac{2}{3}(R_1^3)+\frac{19}{105}S|R_1|^2+(^*)S^3\Big]dM\,, \tag{3.12}$$

where $(^*)$ denotes the coefficient of S^3. By (3.2) we have

(3.13) $$a_3 = \frac{1}{6!}\int\Big[-\frac{5}{63}|\nabla R_1|^2 - \frac{4}{63}S\Big(|R_1|^2 - \frac{1}{6}S^2\Big) + (**)S^3\Big]dM\,.$$

Since (M', g') is of constant curvature, we have $a_3' = (1/6!)\int(**)S'^3dM'$. Since $S = S' > 0$, $a_3 = a_3'$ implies that $\nabla R_1 = 0$ and $|R_1|^2 = S^2/6$. Hence, (M, g) is of constant curvature $K = K' > 0$. q.e.d.

Theorem A′ and Proposition 4 give a proof of Theorem A.

PROPOSITION 5. *If (M, g) is conformally flat and $S = 0$, and if $m \geqq 4$ $(m \neq 8)$ and $a_3 = 0$, then $\nabla R_1 = 0$, and hence (M, g) is either*

(1) *locally flat, or*

(2) *locally Riemannian product $S^{m/2}(c) \times H^{m/2}(-c)$.*

PROOF. By (3.2) and (3.6), we have

$$a_3 = \frac{1}{6!}\int\frac{2(m-8)(5m^2-2m-48)}{63m(m-2)^2}|\nabla R_1|^2dM\,.$$

Since $5m^2 - 2m - 48 > 0$ for $m \geqq 4$, if $m \neq 8$, we have $\nabla R_1 = 0$. If (M, g) is irreducible, then (M, g) is an Einstein space. $S = 0$ implies that (M, g) is locally flat. This is a contradiction. If (M, g) is reducible, then it is locally Riemannian product $[E^1 \times S^{m-1}(c)$, or $E^1 \times H^{m-1}(-c)$, or E^m, or $S^r(c) \times H^{m-r}(-c)]$ (cf. Kurita [4]). $S = 0$ implies that (M, g) is locally E^m or locally $S^{m/2}(c) \times H^{m/2}(-c)$.

THEOREM C′. *Let (M, g) be a compact orientable Riemannian manifold with $m = 6$. If $a_\beta = 0$ for $\beta = 2, 3$, then (M, g) is either*

(1) *E^6/Γ_1, or*

(2) *$[S^3(c) \times H^3(-c)]/\Gamma_2$.*

PROOF. This follows from Lemma 1 and Proposition 5.

REMARK. As for $\sum e^{\lambda_\alpha t}$ for $S^3(c) \times [H^3(-c)/\Gamma^*]$, cf. [10], p. 283~285.

Concerning [vii] for $m \geqq 7$, we can state

PROPOSITION 6. *Let (M, g) be a compact orientable Riemannian manifold with $a_2 = 0$, $m \geqq 7$. If*

(3.14) $$|R_1|^2 \leqq \frac{5m(m-3)+6}{2(m-1)(m-6)}S^2$$

holds on M, then (M, g) is conformally flat, and the equality holds in (3.14) on M.

PROOF. This follows from (2.3) and (2.9).

Hence, as for a_2, we can summerize the above as follows.

PROPOSITION 7. *Let* (M, g) *be a compact orientable Riemannian manifold.*

(1) *For* $2 \leqq m \leqq 5$, $a_2 \geqq 0$ *holds, equality is only for locally flat* (M, g).

(2) *For* $m = 6$, $a_2 \geqq 0$ *holds; (if* $a_3 = 0$*) equality is only for locally flat* (M, g) *or locally Riemannian product* $S^3(c) \times H^3(-c)$.

(3) *For* $m \geqq 7$, *if Ricci curvatures are non-negative (or non-positive) on* M, *then* $a_2 \geqq 0$ *holds; equality is only for locally flat* (M, g).

PROOF. (1) is [vii] or (2.9). (2) is Theorem C′. We show (3). By the assumption of Ricci curvatures we have

$$|R_1|^2 \leqq S^2 \leqq \frac{5m(m-3)+6}{2(m-1)(m-6)} S^2 . \tag{3.15}$$

Hence, (3.14) holds. Therefore $C = 0$ and (3.15) must be equalities. Thus $S = 0$ and $|R_1|^2 = 0$. This implies $R = 0$.

Next, we show

PROPOSITION 8. *Let* (M, g) *and* (M', g') *be compact orientable Riemannian manifold,* $m = m' = 4$. *Assume* $a_\beta = a'_\beta$ *for* $\beta = 0, 1,$ *and* 2. *If* (M', g') *is an Einstein space, then*

$$\chi(M) \leqq \chi(M') \tag{3.16}$$

holds. The equality holds, if and only if (M, g) *is also an Einstein space.*

PROOF. By Gauss-Bonnet formula we have (cf. [1], (8.1))

$$a_2 = \frac{8\pi^2}{45}\chi(M) + \frac{1}{120}\int [2|R_1|^2 + S^2]dM . \tag{3.17}$$

Since (M', g') is an Einstein space, $a_2 = a'_2$ is written as

$$\frac{8\pi^2}{45}\chi(M) + \frac{1}{120}\int\Big[2\Big(|R_1|^2 - \frac{1}{4}S^2\Big) + \frac{3}{2}S^2\Big]dM$$
$$= \frac{8\pi^2}{45}\chi(M') + \frac{1}{120}\int \frac{3}{2}S'^2 dM' .$$

Since $\int S^2 dM \geqq \int S'^2 dM'$ (cf. proof of Theorem A′), we have (3.16). The equality implies $|R_1|^2 = S^2/4$.

REMARK. In connection with Theorem A′ (2) and Proposition 4, the role of $a_3 = a'_3$ may be replaced by the fundamental group of M. Namely we have

PROPOSITION 9. *Let* (M, g) *and* (M', g') *be compact orientable Riemannian manifolds,* $m = m' = 6$. *Assume that* $a_\beta = a'_\beta$ *holds for* $\beta = 0, 1, 2$

and assume that (M', g') *is of constant curvature* K'. *If the fundamental group* $\pi_1(M)$ *of* M *is finite, then* (M, g) *is also of constant curvature* K' *and* $K' > 0$.

In particular, if $(M', g') = (S^6(c), g_0)$, *then* (M, g) *is isometric to* $(S^6(c), g_0)$.

This follows from Theorem A' (2) and the following fact: Let (M, g) be a compact conformally flat Riemannian manifold with finite $\pi_1(M)$, $m \geqq 3$; if S is constant, then (M, g) is of positive constant curvature (cf. Tanno [13]).

4. **Kählerian manifolds.** Let (M, g, J) be a Kählerian manifold with almost complex structure tensor $J = (J_j^i)$ and Kählerian metric tensor $g = (g_{ij})$. The complex dimension of M is $n = m/2$. Then,

$$g_{rs}J_i^rJ_j^s = g_{ij}\ , \qquad J_r^iJ_j^r = -\delta_j^i \tag{4.1}$$

and $\nabla_hJ_j^i = 0$. $J_{ij} = g_{ir}J_j^r$ is skew-symmetric. The Ricci curvature tensor satisfies

$$R_{ij}J_r^iJ_s^j = R_{rs}\ , \qquad R_{ir}J_j^r = -R_{jr}J_i^r\ . \tag{4.2}$$

The Bochner curvature tensor $B = (B^i{}_{jkl})$, $B_{ijkl} = g_{ir}B^r{}_{jkl}$, is given by (cf. [15], etc.)

$$\begin{aligned} B_{ijkl} = R_{ijkl} &- \frac{1}{m+4}(R_{jk}g_{il} - R_{jl}g_{ik} + g_{jk}R_{il} - g_{jl}R_{ik} \\ &+ R_{jr}J_k^rJ_{il} - R_{jr}J_l^rJ_{ik} + J_{jk}R_{ir}J_l^r - J_{jl}R_{ir}J_k^r - 2R_{kr}J_l^rJ_{ij} \\ &- 2R_{ir}J_j^rJ_{kl}) + \frac{1}{(m+2)(m+4)}(g_{jk}g_{il} - g_{jl}g_{ik} \\ &+ J_{jk}J_{il} - J_{jl}J_{ik} - 2J_{kl}J_{ij})S\ . \end{aligned} \tag{4.3}$$

$|B|$ is given by (cf. [15])

$$|B|^2 = |R|^2 - \frac{16}{m+4}|R_1|^2 + \frac{8}{(m+2)(m+4)}S^2\ . \tag{4.4}$$

A Kählerian manifold (M, g, J), $m \geqq 4$, is of constant holomorphic sectional curvature H, if and only if

$$R_{ijkl} = \frac{H}{4}(g_{jk}g_{il} - g_{jl}g_{ik} + J_{jk}J_{il} - J_{ik}J_{jl} - 2J_{ij}J_{kl}) \tag{4.5}$$

holds. Then R_{jk} and S are given by

$$R_{jk} = \frac{m+2}{4}Hg_{jk}\ , \qquad S = \frac{m(m+2)}{4}H \tag{4.6}$$

A Kählerian manifold (M, g, J) is of constant holomorphic sectional curvature, if and only if $B=0$ and $|R_1|^2 = S^2/m$.

PROPOSITION 10. *Let (M, g, J) and (M', g', J') be compact Kählerian manifolds. Assume $a_\beta = a'_\beta$ for $\beta = 0, 1$, and 2. Then,*

(1) *for $m = m' \leqq 10$, (M, g, J) is of constant holomorphic sectional curvature, if and only if (M', g', J') is of constant holomorphic sectional curvature, $H = H'$,*

(2) *for $m = m' = 12$, $B = 0$ and $S =$ constant, if and only if $B' = 0$ and $S' =$ constant, $S = S'$.*

PROOF. By (4.4), we have

$$a_2 = \frac{1}{360}\int\Big[2|B|^2 + \frac{2(12-m)}{m+4}\Big(|R_1|^2 - \frac{1}{m}S^2\Big) + \frac{5m^2+8m+12}{m(m+2)}S^2\Big]dM\,. \tag{4.7}$$

Then the proof is completed in a way similar to that of Theorem A'.

LEMMA 11. (Matsumoto [5]) *If the Bochner curvature tensor $B = 0$ (more generally, parallel) and $S =$ constant, then the Ricci curvature tensor R_1 is parallel (and (M, g) is locally symmetric).*

LEMMA 12. *If $\nabla R_1 = 0$, then*

$$R_{rjks}R^{rs} = R_{jr}R^r{}_k\,. \tag{4.8}$$

PROOF. By $0 = (\nabla_s\nabla_k R_{rj} - \nabla_k\nabla_s R_{rj})g^{rs}$ and the Ricci identity, we have (4.8).

LEMMA 13. *If $B = 0$ and $\nabla R_1 = 0$, then*

$$m(R_1^3) = \frac{2(m+1)}{m+2}S|R_1|^2 - \frac{1}{m+2}S^3\,. \tag{4.9}$$

PROOF. Transvect (4.8) with R^{jk} and use (4.3). Then, using (4.2), we have (4.9).

LEMMA 14. *If $B = 0$ and $\nabla R_1 = 0$, then*

$$S|R|^2 = \frac{16}{m+4}S|R_1|^2 - \frac{8}{(m+2)(m+4)}S^3\,, \tag{4.10}$$

$$R^{ik}R^{jl}R_{ijkl} = -(R_1^3)\,, \tag{4.11}$$

$$R^{rs}R_{rjkl}R_s{}^{jkl} = \frac{16}{m+4}(R_1^3) - \frac{8}{(m+2)(m+4)}S|R_1|^2\,, \tag{4.12}$$

$$(4.13)\qquad R^{ij}{}_{kl}R^{kl}{}_{rs}R^{rs}{}_{ij} = -\frac{16(m+12)}{(m+4)^2}(R_1^3) + \frac{8(m+20)}{(m+2)(m+4)^2}S|R_1|^2 - \frac{32}{(m+2)^2(m+4)^2}S^3 .$$

PROOF. (4.10) follows from (4.4). (4.11) follows from (4.8). (4.12) is calculated as follows: First write down $R^{rs}R_{rjkl}$ using (4.3). Next, transvect it with $R_s{}^{jkl}$ and use (4.2) and well known identities:

$$(4.14)\qquad R_{ijkl}J^{kl} = 2J_i^r R_{rj} , \quad R_{ijkl}J_r^k J_s^l = R_{ijrs} , \quad \text{etc} .$$

For example, $R_s{}^{jkl}R_{kt}J_l^t R^s{}_h J_j^h$ ($= -2(R_1^3)$) is calculated by the first Bianchi identity and the above relations.

(4.13) is calculated as follows (using (4.2), (4.8), (4.14))

$$\begin{aligned}&[\text{replacing } R_{ijkl} \text{ by (4.3)}]\ R^{ij}{}_{rs}R^{rskl}\\ &= \frac{1}{m+4}[-8R^{rs}R_{rjkl}R_s{}^{jkl} - 16(R_1^3)]\\ &\quad - \frac{1}{(m+2)(m+4)}[-4|R|^2 - 8|R_1|^2]S .\end{aligned}$$

Substituting (4.10) and (4.12), we have (4.13).

LEMMA 15. *If $B = 0$ and $\nabla R_1 = 0$, then*

$$(4.15)\qquad a_3 = \frac{1}{6!}\int\left[\left(\frac{128(3m^2+40m+64)}{63m(m+2)(m+4)^2} - \frac{32(m+1)}{63m(m+2)} + \frac{2(12-m)}{3(m+4)}\right)S|R_1|^2 + \langle * \rangle S^3\right]dM .$$

PROOF. In (2.4) we substitute (4.10), $\cdots$, (4.13) and next eliminate (R_1^3) using (4.9).

PROPOSITION 16. *Let (M, g, J) and (M', g', J') be compact Kählerian manifolds, $m = m' = 12$. Assume $a_\beta = a'_\beta$ for $\beta = 0, 1, 2$, and 3. Then (M', g', J') is of constant holomorphic sectional curvature $H' \neq 0$, if and only if (M, g, J) is of constant holomorphic sectional curvature $H = H'$.*

PROOF. Assume that (M', g', J') is of constant holomorphic sectional curvature $H' \neq 0$. By proposition 10 and Lemmas 11 and 15, $a_3 = a'_3$ is written as

$$(4.16)\qquad \frac{1}{6!}\int\left[\frac{1}{147}S\left(|R_1|^2 - \frac{1}{12}S^2\right) + \langle ** \rangle S^3\right]dM = \frac{1}{6!}\int \langle ** \rangle S'^3 dM' .$$

$S = S' \neq 0$ gives $|R_1|^2 = S^2/12$. Hence, (M, g, J) is of constant holomorphic sectional curvature H and $H = H'$.

By $CD^n(-H)$ we denote a simply connected complex space form (of constant holomorphic sectional curvature $-H < 0$) of complex dimension n.

PROPOSITION 17. *Let (M, g, J) be a compact Kählerian manifold with dimension $m = 2n \leqq 12$. $a_2 = 0$ holds good, if and only if (M, g, J) is either*

(1) *CE^n/Γ_3, where Γ_3 is some discontinuous group of automorphisms of the complex n-dimensional Euclidean space CE^n, or*

(2) *$m = 2n = 12$ and $[CP^3(H) \times CD^3(-H)]/\Gamma_4$, where Γ_4 is some discontinuous group of automorphisms of $CP^3(H) \times CD^3(-H)$.*

PROOF. The case $m = 2n \leqq 10$ is clear from (4.7). For $m = 2n = 12$, by (4.7) we have $B = 0$ and $S = 0$. By Lemma 11, we have $\nabla R_1 = 0$. Since $S = 0$, (M, g, J) is not irreducible. Hence, it is reducible and locally $[CE^6$ or $CP^r(H) \times CD^s(-H), r + s = 6]$ (cf. Takagi-Watanabe [12]). $S = 0$ gives $r = s = 3$.

Finally we combine Proposition 10 and Proposition 16.

THEOREM E. *Let (M, g, J) and (M', g', J') be compact Kählerian manifolds, $m = m' \leqq 12$. Assume* Spec (M, g, J) = Spec (M', g', J').

(1) *For $m \leqq 10$, (M, g, J) is of constant holomorphic sectional curvature H, if and only if (M', g', J') is of constant holomorphic sectional curvature $H' = H$.*

(2) *For $m = 12$, (M, g, J) is of constant holomorphic sectional curvature $H \neq 0$, if and only if (M', g', J') is of constant holomorphic sectional curvature $H' = H$.*

By Theorem E we have Theorem D in the introduction.

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