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Eigenvalues of the Dirac Operator on Compact Kähler Manifolds

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Abstract: Kählerian twistor operators are introduced to get lower bounds for the eigenvalues of the Dirac operator on compact spin Kähler manifolds. In odd complex dimensions, manifolds with the smallest eigenvalues are characterized by an over determined system of differential equations similar to the Riemannian case. In these dimensions, we show the existence of a unique natural Kählerian twistor operator. It is also proved that, on a Kähler manifold with nonzero scalar curvature, the space of Riemannian twistor-spinors is trivial.

0. Introduction

In 1980, T. Friedrich [Fr 1] proved with the help of the Lichnerowicz formula [Li 1] that, on a compact Riemannian spin manifold (M^n, g), any eigenvalue λ of the Dirac operator satisfies

$$\lambda^2 \geq \frac{n}{4(n-1)} \inf_M S\,, \qquad (*)$$

where S is the scalar curvature of (M^n, g). In 1984, the author [Hi 1] improved $(*)$ by replacing the number $\inf_M S$ by a conformal invariant and showed that equality in $(*)$ implies that the manifold is non-Kähler (see [Hi 2, Li 2]). K.-D. Kirchberg investigated then the Kähler case, and showed (see [Ki 1, Ki 3]) that any eigenvalue λ of the Dirac operator on a compact Kähler spin manifold (M^{2m}, g) satisfies

$$\lambda^2 \geq \frac{m+1}{4m} \inf_M S\,, \qquad \text{if } m \text{ is odd,} \qquad (**_1)$$

and

$$\lambda^2 \geq \frac{m}{4(m-1)} \inf_M S\,, \qquad \text{if } m \text{ is even.} \qquad (**_2)$$

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Kirchberg's proof relies essentially on the decomposition of the spin bundle under the action of the Kähler form.

The aim of this paper is to show how, on a compact Kähler spin manifold, the first eigenvalues of the Dirac operator are related to the eigenvalues of the Kähler form. The main point is to show that, on Kähler manifolds with odd complex dimension, there is only one natural twistor operator, *denoted* by $P^{(1)}$, on the space of eigenspinors of the Dirac operator (see Lemma (2.6) and Lemma (2.16)). This fact makes the proofs simple and direct. We then study all possible Kählerian twistor operators and give necessary conditions to the existence of twistor-spinors, i.e. zeroes of Kählerian twistor operators (Theorem (4.30)). This approach is a first step towards the investigation of the spectral properties of the Dirac operator on compact spin quaternionic-Kähler manifolds (see [HM]), and towards the classification of Kählerian compact manifolds of odd dimension with the smallest eigenvalues. These Kähler manifolds are characterized by an over determined system of differential equations, given by the existence of a zero of the restriction on the space of eigenspinors of the twistor operator $P^{(1)}$. This situation is similar to that of the limiting-case in $(*)$.

Main Results. On a Kähler spin manifold (M^{2m}, g, J), with a Kähler form Ω, *denote* by ∇ the Levi-Civita connection, D the Dirac operator and $\widetilde{D}$ a square root of D^2 associated with the Kähler structure (see (1.4)). For any real numbers a and b, define the Kählerian twistor operator $P^{a,b} : \Gamma(T^*M \otimes \Sigma M) \to \Gamma(\Sigma M)$ by

$$P^{a,b}_X \psi = \nabla_X \psi + a\, X \cdot D\psi + b\, J(X) \cdot \widetilde{D}\psi\,.$$

A non trivial spinor field ψ is *called* Kählerian twistor-spinor if $P^{a,b}\ \psi = 0$. For $a = \frac{1}{n}$ and $b = 0$, such a spinor field is *called* a Riemannian spinor-twistor (see [Li 2]). A Killing spinor is a zero of the restriction of the Riemannian twistor operator, on the space of eigenspinors of the Dirac operator. The limiting-case in $(*)$ is characterized by the existence of a Killing spinor. A simple proof of $(**_1)$ is obtained from

Theorem A. *For any eigenspinor field ψ with $D\psi = \lambda\psi$, one has*

$$\int_M |P^{a,b}\psi|^2 = \int_M \left(p(a,b)\,\lambda^2 - \frac{1}{4}\,S\right)|\psi|^2,$$

where $p(a,b) = 1 + 2(ma-1)\,a + 2(mb-1)\,b + 4ab\,.$

We then prove

Theorem B. *On a compact Kähler spin manifold (M^{2m}, g, J) of complex dimension $m \geq 1$, assume that there exists a non-trivial spinor field ψ with $D\psi = \lambda\psi$, where*

$$\lambda^2 = \frac{m+1}{4m}\ \inf_M S\,.$$

Then, the manifold is Einstein and $\psi = \psi_+ + \psi_-$ with $\Omega \cdot \psi_\pm = \pm\, i\ \psi_\pm$, where ψ_+ and ψ_- are the half spinors associated with ψ. In particular, the complex dimension m is necessarily odd.

It should be pointed out that, under the conditions of Theorem B, Kirchberg [Ki 1] proved that the manifold is Einstein with m odd. The point here is to emphasize how the eigenvalues of the Kähler form interfere in the problem of estimating the eigenvalues of the Dirac operator. The link between the eigenvalues of the Dirac operator and the eigenvalues of the Kähler form is given by

Theorem C. *Let (M^{2m}, g, J) be a compact Kähler spin manifold of complex dimension $m \geq 1$, and λ an eigenvalue of the Dirac operator. Then, there exists a $\mu_r = m - 2r$, $0 < \mu_r \leq m$, where μ_r is the imaginary part of a generic eigenvalue of the Kähler form, such that*

$$\lambda^2 \geq \frac{m+2-\mu_r}{4(m+1-\mu_r)} \inf_M S \,. \qquad (**_r)$$

We point out that Inequality $(**_r)$ is formally the same as Inequality (70) in [Ki 3]. The main difference is that the first inequality is given for any eigenvalue while the second is valid for an eigenvalue of type r. Notice that the r.h.s. of $(**_r)$ is an increasing function of μ_r, so the inequalities $(**_1)$ and $(**_2)$ could be obtained from $(**_r)$ for $\mu_r = 1$ and $\mu_r = 2$. The following two theorems aim to compute all possible values of the real numbers a and b under the condition that there exists a Kählerian twistor-spinor.

Theorem D. *On a Kähler spin manifold (M^{2m}, g, J), with non-zero scalar curvature, the space of Riemannian twistor-spinors is reduced to zero.*

A rather indirect proof of Theorem D is given in [Ki 4]. In case the manifold is compact, this result is due to Lichnerowicz [Li 4]. We also point out that Theorem D is a strong version of Proposition 6.3 in [Hi 1].

Theorem E. *Let (M^{2m}, g, J) be a Kähler spin manifold with non-zero scalar curvature. Assume that, for a non-trivial spinor field ψ, $P^{a,b}\psi \equiv 0$. Then, there exists an integer r with $0 \leq r \leq m-2$, $\mu_r = m - 2r$ being the imaginary part of a generic eigenvalue of the Kähler form, such that*

$$a = b = \frac{1}{2(m+2-\mu_r)} \,, \qquad \textit{and} \qquad D^2\psi = \frac{m+2-\mu_r}{4(m+1-\mu_r)} S\,\psi \,.$$

Moreover, for $2r \neq m$, $\psi = \psi_r + \psi_{m-r}$, with $(D + i\widetilde{D})\psi_r = (D - i\widetilde{D})\psi_{m-r} = 0$, where ψ_r is an eigenspinor associated with the eigenvalue $i\,\mu_r$ of the Kähler form.

The paper is organised as follows:

Paragraph 1 is devoted to fixing up the notations and to collecting all formulas proved in [Hit, LM, Mi, Ki 1, Ki 3] which we need in this paper. In Sect. 2, we consider Kählerian twistor operators, to give a simple proof of $(**_1)$ (Theorem A) and its limiting-case (Theorem B). The key point in this section is given by Lemma (2.5). This lemma leads to $(**_1)$ without using the eigenvalues of the Kähler form.

In Sect. 3, we show how the eigenvalues of the Kähler form appear in this context and we prove $(**_2)$ and Theorem C. Lemma (3.1) and Lemma (3.3) contain the key idea of this paragraph. In the last section, we prove Theorems D and E. This paper corresponds to Chap. 3 in [Hi 3].

1. Notations and Preliminaries

Let (M^n, g, J) be a Kähler spin manifold of dimension $n = 2m$, with a Riemannian metric g and a parallel complex structure J. The Levi-Civita connection on the tangent bundle TM is denoted by ∇. The same symbol is used to denote its natural extension on the bundle of exterior forms, on the bundle of endomorphisms of the tangent bundle, and on the bundle ΣM of complex spinors. For any tangent vector fields X

and Y, the Kähler form Ω, defined by the complex structure J, satisfies the following relations

$$\Omega(X,Y) = g(X, J(Y)) = -g(Y, J(X)), \tag{1.1}$$

$$\nabla\Omega = 0, \tag{1.2}$$

$$\omega = \frac{1}{m!} \underbrace{\Omega \wedge \ldots \wedge \Omega}_{m \text{ times}}, \tag{1.3}$$

where ω is the volume element defining the orientation of (M^{2m}, g, J). The Dirac operator D acting on sections of the spin bundle is locally defined by (see [ABS], [LM])

$$D = \sum_{i=1}^{n} e_i \cdot \nabla_{e_i},$$

with $\{e_1, ..., e_n\}$ a local orthonormal basis of the tangent bundle TM. At any point x in M, we choose normal coordinates at this point so that $(\nabla e_i)(x) = 0$, for all $i \in \{1, \ldots, n\}$. All the computations in this paper will be made in such charts. Associated with J, there is an elliptic self-adjoint operator $\widetilde{D}$ (see [Hit, Mi, Ki 1]) defined locally by

$$\widetilde{D} = \sum_{i=1}^{n} J(e_i) \cdot \nabla_{e_i} = -\sum_{i=1}^{n} e_i \cdot \nabla_{J(e_i)}. \tag{1.4}$$

Denote by $(\,,\,)$ the natural Hermitian scalar product on the spin bundle. For the action of vector fields X and the Kähler form by Clifford multiplication, one can easily check the following relations:

$$\Omega = \frac{1}{2}\sum_{i=1}^{n} J(e_i) \cdot e_i = -\frac{1}{2}\sum_{i=1}^{n} e_i \cdot J(e_i), \tag{1.5}$$

$$(\Omega \cdot \psi, \varphi) = -(\psi, \Omega \cdot \varphi), \qquad \text{i.e.} \qquad \Omega^* = -\Omega, \tag{1.6}$$

$$\widetilde{D}^2 = D^2, \tag{1.7}$$

$$\widetilde{D}D + D\widetilde{D} = 0, \tag{1.8}$$

$$[X, \Omega] = 2J(X), \tag{1.9}$$

$$[D, \Omega] = 2\widetilde{D}, \tag{1.10}$$

$$[\widetilde{D}, \Omega] = -2D. \tag{1.11}$$

Moreover, the spin bundle of a Kähler spin manifold carries an anti-linear map j satisfying the relations [ABS]:

$$\nabla j = 0, \tag{1.12}$$

$$j^2 = (-1)^{\frac{m(m+1)}{2}} I_{\Sigma M}, \tag{1.13}$$

$$[X, j] = 0, \tag{1.14}$$

$$(j\psi, j\varphi) = (\varphi, \psi). \tag{1.15}$$

From the above identities one derives the following relations

$$[D, j] = 0\,, \tag{1.16}$$

$$[\widetilde{D}, j] = 0\,, \tag{1.17}$$

$$[\Omega, j] = 0\,. \tag{1.18}$$

We recall that [Ki 1] the spinor bundle ΣM of a Kähler spin manifold (M^{2m}, g, J) splits under the action of the Kähler form Ω into the direct sum

$$\Sigma M = \sum_{r=0}^{m} \Sigma_r M,$$

where $\Sigma_r M$ is the eigenbundle associated with the eigenvalue $i\mu_r = i(m-2r)$ of Ω whose rank is equal to C_m^r. According to this decomposition, any spinor field ψ can then be written as

$$\psi = \sum_{r=0}^{m} \psi_r\,, \tag{1.19}$$

with $j\psi_r = (j\psi)_{m-r}$.

2. Eigenvalue Estimate

For any real numbers a and b, define the Kählerian twistor operator
$P^{a,b} : \Gamma(T^*M \otimes \Sigma M) \to \Gamma(\Sigma M)$ by

$$P_X^{a,b}\psi = \nabla_X \psi + a\,X \cdot D\psi + b\,J(X) \cdot \widetilde{D}\psi\,. \tag{2.1}$$

A non-trivial spinor field ψ is *called* Kählerian twistor-spinor if $P^{a,b}\,\psi = 0$.
We start with some technical lemmas which are crucial for what follows.

Lemma (2.2). *For any spinor field ψ, one has*

$$\alpha(\psi) := \int_M (\Omega \cdot \widetilde{D}\psi, D\psi) = \int_M (D\psi, \Omega \cdot \widetilde{D}\psi)\,, \tag{2.3}$$

i.e., $\alpha(\psi)$ is a real number.

Proof. Using $\Omega^* = -\Omega$, $D^* = D$, and $\widetilde{D}^* = \widetilde{D}$, it is sufficient to show that

$$D\,\Omega \cdot \widetilde{D} + \widetilde{D}\,\Omega \cdot D = 0\,. \tag{2.4}$$

By (1.10) and (1.11), we get

$$D\,\Omega \cdot \widetilde{D} + \widetilde{D}\,\Omega \cdot D = (\Omega \cdot D + 2\widetilde{D})\,\widetilde{D} + (\Omega \cdot \widetilde{D} - 2D)\,D\,.$$

This yields (2.4) by using (1.7) and (1.8). □

Lemma (2.5). *For a spinor field ψ with $D\psi = \lambda\psi$, one has*

$$\alpha(\psi) = \int_M |D\psi|^2\,.$$

Proof. Since $\widetilde{D}$ is self-adjoint, $\Omega^* = -\Omega$ for the action by Clifford multiplication, and $D\psi = \lambda\psi$ one gets after using (1.11),

$$\begin{aligned}\int_M (\Omega \cdot \widetilde{D}\psi, D\psi) &= \int_M (\widetilde{D}\, \Omega \cdot \psi, D\psi) + 2\lambda^2 \int_M |\psi|^2\,, \\ &= \int_M (\widetilde{D}\, \Omega \cdot D\psi, \psi) + 2\lambda^2 \int_M |\psi|^2\,, \\ &= -\int_M (D\psi, \Omega \cdot \widetilde{D}\psi) + 2\lambda^2 \int_M |\psi|^2\,,\end{aligned}$$

which, when combined with Lemma (2.2), gives Lemma (2.5). □

Lemma (2.6). *Assume that $P^{a,b}\psi \equiv 0$ (M not necessarily compact), then*

$$(1 - a - b)\, D^2\psi = \frac{1}{4}\, S\, \psi\,. \tag{2.7}$$

Moreover, if ψ is such that $D\psi = \lambda\psi$, with $\lambda \neq 0$, then $a = b = \dfrac{1}{n+2}$.

Proof. For any $i \in \{1, ..., n\}$, we have

$$0 \equiv P^{a,b}_{e_i}\, \psi = \nabla_{e_i}\psi + a\, e_i \cdot D\psi + b\, J(e_i) \cdot \widetilde{D}\psi\,. \tag{2.8}$$

The covariant derivative with respect to e_i of the above identity gives after summing over i:

$$0 \equiv \sum_{i=1}^{n} \nabla_{e_i}\nabla_{e_i}\psi + a\, D^2\psi + b\, \widetilde{D}^2\psi\,,$$

which with the help of the Lichnerowicz formula and Eq. (1.7) gives (2.7).

Taking Clifford multiplication of (2.8) with e_i and summing over i, one gets

$$(1 - na)\, D\psi = 2b\, \Omega \cdot \widetilde{D}\psi\,. \tag{2.9}$$

Hence, by (1.8), (1.10), (2.9) and (1.7) it follows

$$\begin{aligned}(1 - na)\, \lambda^2\psi &= 2\, b\, \Omega \cdot \widetilde{D}D\psi, \\ &= -2b\Omega \cdot D\widetilde{D}\psi, \\ &= -2bD\Omega \cdot \widetilde{D}\psi + 4b\, D^2\psi, \\ &= -(1 - na + 4b)\lambda^2\psi\,,\end{aligned}$$

which yields

$$1 - na - 2b = 0\,. \tag{2.10}$$

Similarly, taking Clifford mutiplication of (2.8) with $J(e_i)$ and summing over i give

$$(1 - nb)\, \widetilde{D}\psi = -2\, a\, \Omega \cdot D\psi\,. \tag{2.11}$$

The same computation as above yields

$$1 - nb - 2a = 0\,. \tag{2.12}$$

Equations (2.10) and (2.12) give Lemma (2.6). □

Theorem (2.13). *For any eigenspinor field ψ with $D\psi = \lambda\psi$, one has*

$$\int_M |P^{a,b}\psi|^2 = \int_M (p(a,b)\,\lambda^2 - \frac{1}{4}\,S)|\psi|^2, \tag{2.14}$$

where $p(a,b) = 1 + (na-2)\,a + (nb-2)\,b + 4ab$.

Proof. Since for any $X \in TM$, ψ and $\varphi \in \Sigma M$, one has $(X\cdot\psi,\varphi) = -(\psi, X\cdot\varphi)$, one gets using (1.4) and (1.5),

$$\int_M |P^{a,b}\psi|^2 = \int_M |\nabla\psi|^2 + (na-2)a\int_M |D\psi|^2 + (nb-2)b\int_M |\widetilde{D}\psi|^2 + 4ab\,\alpha(\psi)\,. \tag{2.15}$$

Equation (2.14) is then obtained with the help of (2.15), (1.7), Lemma (2.2), Lemma (2.5), and the Lichnerowicz formula. □

It is straightforward to notice the following.

Lemma (2.16). *The polynomial p has its minimum* $\frac{n}{n+2}$ *at the point* $\left(\frac{1}{n+2}, \frac{1}{n+2}\right)$.

For $a = b = \frac{1}{n+2}$, we *denote* $P^{a,b}$ by $P^{(1)}$.

As a consequence of Theorem (2.13) and Lemma (2.16), we derive the following result of Kirchberg.

Corollary (2.17) [Ki 1]. *On a compact Kähler spin manifold (M^{2m}, g, J) of complex dimension $m \geq 1$, any eigenvalue λ of the Dirac operator satisfies*

$$\lambda^2 \geq \frac{m+1}{4m}\,\inf_M S\,. \tag{2.18}$$

Notice that for the real dimension n, one has $\frac{n+2}{n} \geq \frac{n}{n-1}$ with equality only for $n = 2$ (compare with [Hi 1]). Hence (2.18) has an interest only for $m \geq 2$.
In order to study the limiting-case of (2.18), we need a special case of the following proposition.

Proposition (2.19). *If $P^{(1)}\psi \equiv 0$, then*

$$D^2\psi = \frac{m+1}{4m}\,S\,\psi\,. \tag{2.20}$$

Moreover, if $P^{(1)}\psi \equiv 0$ for a non-trivial spinor field ψ such that $D\psi = f\psi$, where f is a non-trivial real-valued function, then the function f is constant, the manifold is Einstein, and one has

$$f^2 = \frac{m+1}{4m}\,S\,.$$

Moreover, the complex dimension m is odd.

Proof. The first assertion is a consequence of Lemma (2.6). For $D\psi = f\psi$, one gets $D^2\psi = f^2\psi + df\cdot\psi$, where d is the exterior differential. Since $D^2\psi = \frac{m+1}{4m}\,S\,\psi$, the scalar product of the above equation with ψ implies that f is constant and

$$f^2 = \frac{m+1}{4m}\,S := \lambda^2\,.$$

For the other assertions, we proceed in two steps.

First Step. We show that m is necessarily odd. Equations (2.9) and (2.11) with $a = b = \frac{1}{n+2}$ imply

$$D\psi = \Omega \cdot \widetilde{D}\psi \,, \tag{2.21}$$

$$\widetilde{D}\psi = -\,\Omega \cdot D\psi \,. \tag{2.22}$$

Since $D\psi = \pm\lambda\psi$, the above equations give

$$\Omega \cdot \Omega \cdot \psi = -\psi \,. \tag{2.23}$$

By (1.19), Eq. (2.23) yields

$$\sum_{r=0}^{m} ({\mu_r}^2 - 1)\, \psi_r = 0 \,,$$

which implies that

$$\psi = \psi_{\frac{m-1}{2}} + \psi_{\frac{m+1}{2}} \,,$$

hence m is necessarily odd and ψ is the sum of two half spinors which are eigenspinors for D^2 and for Ω with eigenvalues $\pm i$.

Second Step. We prove that the manifold is Einstein. By assumption, for any vector field X one has

$$\nabla_X\psi = -\frac{1}{n+2}\,(X \cdot D\psi + J(X) \cdot \widetilde{D}\psi) \,. \tag{2.24}$$

A straightforward computation using (2.24) gives, for the spin curvature $\mathbf{R}$, the identity

$$\sum_{i=1}^{n} e_i \cdot \mathbf{R}_{X,e_i}\psi = \frac{1}{n+2}\,[(n-2)\nabla_X D\psi - X \cdot D^2\psi + 2\Omega \cdot \nabla_X \widetilde{D}\psi - J(X) \cdot D\widetilde{D}\psi - 2\nabla_{J(X)}\widetilde{D}\psi] \,.$$

On the other hand, since

$$\sum_{i=1}^{n} e_i \cdot \mathbf{R}_{X,e_i}\psi = -\frac{1}{2}\,\mathrm{Ric}(X) \cdot \psi \,,$$

where Ric is the Ricci curvature, one gets :

$$-\frac{n+2}{2}\,\mathrm{Ric}(X) \cdot \psi = (n-2)\,\nabla_X D\psi - X \cdot D^2\psi + 2\,\Omega \cdot \nabla_X \widetilde{D}\psi - J(X) \cdot D\widetilde{D}\psi - 2\,\nabla_{J(X)}\widetilde{D}\psi \,. \tag{2.25}$$

Since $D\psi = \lambda\psi$, Eq. (2.25) can be written as

$$-\frac{n+2}{2}\,\mathrm{Ric}(X) \cdot \psi = (n-2)\,\lambda\nabla_X\psi - \lambda^2 X \cdot \psi + \lambda J(X) \cdot \widetilde{D}\psi + 2\,\nabla_X(\Omega \cdot \widetilde{D}\psi) - 2\,\nabla_{J(X)}\widetilde{D}\psi.$$

By (2.24) and (2.21), it follows after simplification

$$\frac{n+2}{4}\,\mathrm{Ric}(X) \cdot \psi = \lambda^2 X \cdot \psi + \lambda\nabla_X\psi + \nabla_{J(X)}\widetilde{D}\psi \,. \tag{2.26}$$

Equation (2.24), applied to X and to $J(X)$, gives after using (1.9), (2.22) and (2.23)

$$\begin{aligned}
&\lambda\nabla_X\psi + \nabla_{J(X)}\widetilde{D}\psi \\
&= \lambda\,(\nabla_X\psi - \Omega\cdot\nabla_{J(X)}\psi)\,, \\
&= -\frac{1}{n+2}\,\lambda\,[\,X\cdot D\psi + J(X)\cdot\widetilde{D}\psi - \Omega\cdot(J(X)\cdot D\psi - X\cdot\widetilde{D}\psi)\,]\,, \\
&= -\frac{1}{n+2}\,\lambda^2\,[\,X\cdot\psi - J(X)\cdot\Omega\cdot\psi - \Omega\cdot J(X)\cdot\psi - \Omega\cdot X\cdot\Omega\cdot\psi\,]\,, \\
&= \frac{1}{n+2}\,\lambda^2\,[\,X\cdot\psi + X\cdot\Omega\cdot\Omega\cdot\psi\,]\,, \\
&= 0\,.
\end{aligned}$$

Hence for any vector field X, Eq. (2.26) yields the relation

$$\frac{n+2}{4}\ \mathrm{Ric}(X)\cdot\psi = \frac{n+2}{4n}\ S\,X\cdot\psi\,,$$

which implies that the manifold is Einstein. □

As a corollary of Proposition (2.19), one gets (comp. [Ki 1])

Theorem (2.27). *On a compact Kähler spin manifold (M^{2m}, g, J) of complex dimension $m \geq 1$, assume that there exists a non-trivial spinor field ψ with $D\psi = \lambda\psi$, where*

$$\lambda^2 = \frac{m+1}{4m}\ \inf_M\ S\,.$$

Then, the manifold is Einstein and $\psi = \psi_+ + \psi_-$ with $\Omega\cdot\psi_\pm = \pm\, i\ \psi_\pm$, where ψ_+ and ψ_- are the half spinors associated with ψ. In particular, the complex dimension m is necessarily odd.

It should be pointed out that, under the conditions of Theorem (2.27), Kirchberg [Ki 1] proved that the manifold is Einstein with m odd. The point here is to emphasize how the eigenvalues of the Kähler form interfere in the problem of estimating the eigenvalues of the Dirac operator.

3. Eigenvalues Estimate for Compact Kähler Manifolds of Even Complex Dimension

It follows from Theorem (2.27) that, in the case of a compact Kähler manifold of even complex dimension, one should get a sharper estimate than (2.18). Hence, the polynomial p in Theorem (2.13) should be replaced by a polynomial q with $1 < q < p$. The only hope in this direction is to apply (2.15) to a spinor field ψ such that

$$\alpha(\psi) := \int_M (\Omega\cdot\widetilde{D}\psi, D\psi) < \int_M |D\psi|^2\,.$$

It is easy to check that (2.15) is the sum of the following two identities

$$\int_M |P^{a,b}\,\psi_\pm|^2 = \int_M |\nabla\psi_\pm|^2 + [(na-2)a + (nb-2)b]\int_M |D\psi_\pm|^2 + 4ab\,\alpha(\psi_\pm)\,,$$

where $\psi = \psi_+ + \psi_-$ is the decomposition of a spinor field as the sum of half spinors. In the limiting-case of (2.18), these half spinors are necessarily eigenvalues for the Kähler form. With these two observations in mind, the choice of the spinor field ψ becomes natural. We prove that, for m even, one can choose a spinor field ψ_r such that $D^2\psi_r = \lambda^2 \psi_r$, $\Omega \cdot \psi_r = i\, \mu_r\, \psi_r$ and $\alpha(\psi_r) \leq 0$.

We start with the following two key lemmas of this section.

Lemma (3.1). *For a non-zero constant c, define* D_c *by* $D_c = D + c\,\widetilde{D}$ *. Then,* $D_c^2 = 0$ *if and only if* $c = \pm i$*. Moreover,*

$$\Omega \cdot D_c\, \psi_r = i\, \mu_s\, D_c\, \psi_r, \qquad \textit{if} \qquad c = \pm i \quad \textit{and} \quad \mu_s = \mu_r \pm 2\,.$$

If $\ker D = 0$*, then*

$$\Omega \cdot D_c\, \psi_r = i\, \mu_s\, D_c\, \psi_r, \qquad \Longleftrightarrow \qquad c = \pm i \quad \textit{and} \quad \mu_s = \mu_r \pm 2\,.$$

Proof. The first assertion is an immediate consequence of (1.7) and (1.8). Equations (1.10) and (1.11) imply

$$\begin{aligned}
\Omega \cdot D_c\, \psi_r &= \Omega \cdot (D + c\,\widetilde{D})\psi_r,\\
&= D\, \Omega \cdot \psi_r - 2\,\widetilde{D}\psi_r + c\,(\widetilde{D}\,\Omega \cdot \psi_r + 2\,D\psi_r),\\
&= i\,\mu_r\, D_c\, \psi_r + 2c\,(D\psi_r - \frac{1}{c}\,\widetilde{D}\psi_r)\,.
\end{aligned}$$

The last equation gives the second assertion. Assume that, for a non-trivial spinor field ψ, $D_c\,\psi = 0$. Then, by (1.7) and (1.8), it follows

$$\begin{aligned}
DD_c\,\psi &= D^2\psi + c\,D\widetilde{D}\psi = 0,\\
\widetilde{D}D_c\,\psi &= c\,D^2\psi - D\widetilde{D}\psi = 0\,.
\end{aligned}$$

Hence, $c = \pm\, i$ since $D^2\psi \neq 0$. This observation combined with the above expression of $\Omega \cdot D_c\,\psi_r$ gives Lemma (3.1) after a straightforward algebraic computation. □

Denote by $D_\pm = \frac{1}{2}\,(D \pm i\widetilde{D})$. With the help of (1.16), (1.17) and Lemma (3.1), we deduce that these two operators satisfy

$$j\,D_\pm = D_\mp\, j\,, \qquad D_\pm^2 = 0\,, \qquad \text{and} \quad \Omega \cdot D_\pm\, \psi_r = i\,(\mu_r \pm 2)\, D_\pm\, \psi_r\,. \tag{3.2}$$

Lemma (3.3). *For any nonzero eigenvalue* λ *of the Dirac operator, there exists a spinor field* $\psi_r \in \Sigma_r M$ *such that* $D^2\psi_r = \lambda^2\,\psi_r$ *,* $\mu_r > 0$*, and*

$$\alpha(\psi_r) = (2 - \mu_r)\,\lambda^2 \int_M |\psi_r|^2\,. \tag{3.3}$$

Proof. By (1.10), (1.11) and (1.8), it follows that $D^2\,\Omega = \Omega \cdot D^2$, hence if $\psi = \sum_{r=0}^{m} \psi_r$ with $D^2\,\psi = \lambda^2\,\psi$, then each ψ_r satisfies $D^2\,\psi_r = \lambda^2\,\psi_r$. The antilinear isomorphism j sends $\Sigma_r M$ to $\Sigma_{m-r} M$. This allows the choice of $\mu_r \geq 0$. Take the maximum μ_r for which there exists a $\psi_r \not\equiv 0$. Such a μ_r should be positive, if not $D_+\,\psi_{\frac{m}{2}}$ and $D_-\,\psi_{\frac{m}{2}}$ should be zero which is impossible. Hence $\mu_r > 0$. Since $\mu_r > 0$ is a maximum we necessarily have $D_+\psi_r = 0$, if not $D_+\psi_r$ would be an eigenspinor field for $\mu_r + 2 > \mu_r$. Therefore, $\widetilde{D}\psi_r = i\, D\psi_r$. For $\alpha(\psi_r)$, one then gets

$$\begin{aligned}\alpha(\psi_r) &= \int_M (\widetilde{D}\,\Omega \cdot \psi_r\,, D\psi_r) + 2\int_M |D\psi_r|^2, \\ &= i\,\mu_r \int_M (\widetilde{D}\psi_r\,, D\psi_r) + 2\int_M |D\psi_r|^2, \\ &= (2-\mu_r)\int_M |D\psi_r|^2\,. \qquad \square\end{aligned}$$

As a consequence, one gets

Corollary (3.5) [Ki 3]. *On a compact Kähler spin manifold of even complex dimension, any eigenvalue of the Dirac operator satisfies*

$$\lambda^2 \geq \frac{m}{4(m-1)} \inf_M S\,. \tag{3.6}$$

Proof. Choose ψ_r as in Lemma (3.3). Equation (2.15) applied to ψ_r yields

$$\int_M |P^{a,b}\,\psi_r|^2 - 4ab\,\alpha(\psi_r) = \int_M \left(q(a,b)\,\lambda^2 - \frac{S}{4}\right)|\psi_r|^2\,, \tag{3.7}$$

where $q(a,b) = 1 + (na-2)a + (nb-2)b$. Equation (3.4) implies that $\alpha(\psi_r) \leq 0$ since for even m, $\mu_r = m - 2r \geq 2$. It is straightforward to see that the polynomial q has a minimum at $a = b = \frac{1}{n}$. The positivity of the l.h.s. of (3.7) gives (3.6). $\square$

Theorem (3.8). *Let (M^{2m}, g, J) be a compact Kähler spin manifold of complex dimension $m \geq 1$, and λ an eigenvalue of the Dirac operator. Then, there exists a $\mu_r = m - 2r$, $0 < \mu_r \leq m$, where μ_r is the imaginary part of a generic eigenvalue of the Kähler form, such that*

$$\lambda^2 \geq \frac{m+2-\mu_r}{4(m+1-\mu_r)} \inf_M S\,. \tag{3.9}$$

Notice that, since the fraction $\dfrac{m+2-\mu_r}{m+1-\mu_r}$ is strictly increasing in μ_r, Inequality (3.9) is sharper than (3.6) (obtained from (3.9) for $\mu_r = 2$) and sharper than (2.18) (obtained from (3.9) for $\mu_r = 1$). We point out that Inequality (3.9) is formally the same as Inequality (70) in [Ki 3].

Proof. Equation (3.7) can be written as

$$\int_M |P^{a,b}\,\psi_r|^2 = \int_M \left(q_r(a,b)\,\lambda^2 - \frac{S}{4}\right)|\psi_r|^2\,, \tag{3.10}$$

where $q_r(a,b) = 1 + (na-2)a + (nb-2)b + 4(2-\mu_r)\,ab$. For a given r, the polynomial q_r admits a minimum equal to $\dfrac{m+1-\mu_r}{m+2-\mu_r}$ at the point $a = b = \dfrac{1}{2(m+2-\mu_r)}$. $\square$

4. Kählerian Twistor-Spinors

In this section we compute all possible values of the real numbers a and b under the condition that there exists a Kählerian twistor-spinor. For this, we show that on a Kähler spin manifold with non-zero scalar curvature there are no Riemannian twistor-spinors (Theorem (4.7)). This result has been proved by Kirchberg [Ki 4] using a different method. In case the manifold is compact, this result is due to Lichnerowicz [Li 4]. We then give, on a compact Kähler spin manifold with positive scalar curvature, necessary conditions to the existence of a Kählerian twistor-spinor (Theorem (4.19)). We start with a computational lemma.

Lemma (4.1). *The following identities hold:*

$$[D, \Omega \cdot \Omega] = 4\, \widetilde{D}\, \Omega + 4\, D = 4\, \Omega \cdot \widetilde{D} - 4\, D\,, \tag{4.2}$$

$$[\widetilde{D}, \Omega \cdot \Omega] = -4\, D\, \Omega + 4\, \widetilde{D} = -4\, \Omega \cdot D - 4\, \widetilde{D}\,, \tag{4.3}$$

$$[X, \Omega \cdot \Omega] = 4\, J(X) \cdot \Omega + 4\, X\,, \tag{4.4}$$

$$[J(X), \Omega \cdot \Omega] = -4\, X \cdot \Omega + 4\, J(X)\,. \tag{4.5}$$

Proof. Equations (1.10) and (1.11) imply

$$\begin{aligned} D\, \Omega \cdot \Omega &= (\Omega \cdot D + 2\, \widetilde{D})\, \Omega\,, \\ &= \Omega \cdot (\Omega \cdot D + 2\, \widetilde{D}) + 2\, \widetilde{D}\, \Omega\,, \\ &= \Omega \cdot \Omega \cdot D + 2\, (\widetilde{D}\, \Omega + 2\, D) + 2\, \widetilde{D}\, \Omega\,, \\ &= \Omega \cdot \Omega \cdot D + 4\, \widetilde{D}\, \Omega + 4\, D\,, \\ &= \Omega \cdot \Omega \cdot D + 4\, \Omega \cdot \widetilde{D} - 4\, D\,. \end{aligned}$$

Hence (4.2). The proofs of (4.3), (4.4) and (4.5) follow similarly with the help of (1.9), (1.10) and (1.11). □

We recall that, on a Riemannian spin manifold, a spinor field ψ is *called* a Riemannian twistor-spinor if [Li 2]

$$\forall X \in TM\,, \qquad \nabla_X\, \psi + \frac{1}{n}\, X \cdot D\psi = 0\,. \tag{4.6}$$

Moreover, Killing spinors can be characterized as twistor-spinors which are also eigenspinors for the Dirac operator. In the case where the scalar curvature is a non-zero constant, a twistor-spinor is a Killing spinor. In fact, if ψ is a Riemannian twistor-spinor, then by Lemma (2.6), with $a = \frac{1}{n}$ and $b = 0$, one has $D^2\psi = \frac{n}{4(n-1)}\, S\, \psi$. If S is a non-zero constant, the limiting case in Inequality $(*)$ is achieved, hence ψ is a Killing spinor (see [Fr 1]). On a Kähler spin manifold there are no Killing spinors [Hi 1]. We now give a simple proof of the following result (comp. [Ki 4]).

Theorem (4.7). *On a Kähler spin manifold (M^{2m}, g, J), with non-zero scalar curvature, the space of Riemannian twistor-spinors is reduced to zero.*

Proof. We proceed in four steps.

First Step. For $m \neq 2$, we show that, for any Riemannian twistor-spinor ψ, $\psi = \psi_1 + \psi_{m-1}$, $D_-\, \psi_1 = 0$ and $D_+\, \psi_{m-1} = 0$.

Equation (4.6) applied to e_i gives, after taking its Clifford multiplication with $J(e_i)$ and summing over i

$$\Omega \cdot D\psi = -m\, \widetilde{D}\psi \,. \tag{4.8}$$

Equation (1.10) then implies

$$D\, \Omega \cdot \psi = -(m-2)\, \widetilde{D}\psi \,. \tag{4.9}$$

By (4.8) and (2.4) one gets

$$\begin{aligned} D\, \Omega \cdot \Omega \cdot D\psi &= -\, m\, D\, \Omega \cdot \widetilde{D}\psi \,, \\ &= m\, \widetilde{D}\, \Omega \cdot D\psi \,, \\ &= -\, m^2\, D^2\, \psi \,. \end{aligned} \tag{4.10}$$

On the other hand, Eq. (4.2) gives

$$\begin{aligned} D\, \Omega \cdot \Omega \cdot D\psi &= (\Omega \cdot \Omega \cdot D + 4\, \widetilde{D}\, \Omega + 4\, D)\, D\psi \,, \\ &= \Omega \cdot \Omega \cdot D^2\, \psi + 4\, \widetilde{D}\, \Omega \cdot D\psi + 4\, D^2 \psi \,. \end{aligned} \tag{4.11}$$

The last two equations yield

$$\Omega \cdot \Omega \cdot D^2\, \psi = -(m-2)^2\, D^2\, \psi \,.$$

Since $\Omega \cdot D^2 = D^2\, \Omega$, it follows that

$$\Omega \cdot \Omega \cdot \psi = -(m-2)^2\, \psi \,. \tag{4.12}$$

Hence, for $m \neq 2$, $\psi = \psi_1 + \psi_{m-1}$, and we deduce from (4.9) that

$$D_-\, \psi_1 - D_+\, \psi_{m-1} = 0 \,. \tag{4.13}$$

For $m \neq 4$, the spinor fields $D_-\, \psi_1$ and $D_+\, \psi_{m-1}$ are eigenspinors for different eigenvalues of the Kähler form, we necessarily have

$$D_-\, \psi_1 = 0 \qquad \text{and} \qquad D_+\, \psi_{m-1} = 0 \,. \tag{4.14}$$

For $m = 4$, one has

$$D_-\, \psi_1 - D_+\, \psi_3 = 0 \,,$$

where $D_-\, \psi_1$ and $D_+\, \psi_3$ belong to the same space $\Sigma_2\, M$. By (1.8), it can be easily seen that

$$D^2\, D_\pm = D_\pm\, D^2 \,. \tag{4.15}$$

On the other hand, by Lemma (2.6) a twistor-spinor ψ satisfies

$$D^2\, \psi = \frac{n}{4(n-1)}\, S\, \psi \,,$$

and $D^2\, \Omega = \Omega \cdot D^2$, by the above relations it follows that

$$0 = D^2\, (D_-\, \psi_1) = D_-\, (D^2\, \psi_1) = D_-\, \left(\frac{2}{7}\, S\, \psi_1 \right) \,. \tag{4.16}$$

Hence $D_-\, (S\, \psi_1) = 0$, and similarly $D_+\, (S\, \psi_3) = 0$. It is easy to check that

$$D_-\, (S\, \psi_1) = (dS - i\, J(dS)) \cdot \psi_1 + S\, D_-\, \psi_1 = 0 \,. \tag{4.17}$$

$$D_+\, (S\, \psi_3) = (dS + i\, J(dS)) \cdot \psi_3 + S\, D_+\, \psi_3 = 0 \,. \tag{4.18}$$

Combining Eqs. (4.13), (4.17) and (4.18) yields

$$(dS - i\,J(dS)) \cdot \psi_1 = (dS + i\,J(dS)) \cdot \psi_3\,. \tag{4.19}$$

Clifford multiplication by dS reduces (4.19) to

$$|dS|^2\,(\psi_1 - \psi_3) + i\,dS \cdot J(dS) \cdot (\psi_1 + \psi_3) = 0\,. \tag{4.20}$$

Equation (1.9) applied twice gives

$$\forall X, \qquad [\Omega, X \cdot J(X)] = 0\,. \tag{4.21}$$

Clifford multiplication of (4. 20) by Ω implies then

$$|dS|^2(\psi_1 + \psi_3) + i\,dS \cdot J(dS) \cdot (\psi_1 - \psi_3) = 0\,. \tag{4.22}$$

Combining (4.20) and (4.22) gives

$$(|dS|^2 + i\;dS \cdot J(dS)) \cdot \psi_1 = 0\,. \tag{4.23}$$

Clifford multiplication by dS reduces (4.17) to

$$-(|dS|^2 + i\;dS \cdot J(dS)) \cdot \psi_1 + S\;dS \cdot D_-\;\psi_1 = 0\,. \tag{4.24}$$

Comparing Eqs. (4.23) and (4.24) implies

$$S\;dS \cdot D_-\;\psi_1 = 0\,.$$

The scalar curvature being not identically zero, we deduce that $D_-\;\psi_1 = 0$ or S is constant. The scalar curvature can not be constant since ψ would be a Killing spinor. This is impossible since the manifold is Kähler [Hi 2]. Finally, $D_-\;\psi_1 = 0$ and $D_+\psi_3 = 0$.

Second Step. We prove that ψ_1 and ψ_{m-1} are twistor-spinors themselves. The Kähler form Ω being parallel, Clifford multiplication of (4.6) by $\Omega \cdot \Omega$ gives

$$\nabla_X\,(\Omega \cdot \Omega \cdot (\psi_1 + \psi_{m-1})) + \frac{1}{n}\;\Omega \cdot \Omega \cdot X \cdot D(\psi_1 + \psi_{m-1}) = 0\,.$$

Since $D\psi_1 \in \Sigma_0 M$ and $D\psi_{m-1} \in \Sigma_m M$, it follows by (4.4)

$$-(m-2)^2\;\nabla_X\psi + \frac{1}{n}\;(X \cdot \Omega \cdot \Omega - 4J(X) \cdot \Omega - 4X) \cdot (D\psi_1 + D\psi_{m-1}) = 0\,,$$

$$-(m-2)^2\;\nabla_X\psi - \frac{m^2+4}{n}\;X \cdot D\psi - \frac{i\,4m}{n}\;J(X) \cdot (D\psi_1 - D\psi_{m-1}) = 0\,,$$

which by (4.6) reduces to the identity

$$\forall X \in TM\,, \qquad \nabla_X\psi - \frac{i}{n}\;J(X) \cdot (D\psi_1 - D\psi_{m-1}) = 0\,. \tag{4.25}$$

Hence, Eqs. (4.6) and (4.25) imply

$$\forall X \in TM\,, \qquad (X + i\;J(X)) \cdot D\psi_1 + (X - i\;J(X)) \cdot D\psi_{m-1} = 0\,. \tag{4.26}$$

Clifford multiplication of (4.26) by Ω, with the help of (1.9), $D\psi_1 \in \Sigma_0 M$ and $D\psi_{m-1} \in \Sigma_m M$, yields

$$\forall X \in TM\,, \qquad i\,(m+2)\,[(X+i\;J(X))\cdot D\psi_1 - (X - i\;J(X))\cdot D\psi_{m-1}] = 0\,. \tag{4.27}$$

Equations (4.26) and (4.27) give

$$\forall X \in TM\,, \qquad (X+i\,J(X))\cdot D\psi_1 = 0 \quad \text{and} \quad (X-i\,J(X))\cdot D\psi_{m-1} = 0\,. \quad (4.28)$$

We now prove that ψ_1 and ψ_{m-1} are necessarily Riemannian twistor-spinors. For this, we take Clifford multiplication of (4.6) by Ω and use (1.9) with (4.28) to get, for any vector field X:

$$\begin{aligned}
& i\,(m-2)\,\nabla_X(\psi_1 - \psi_{m-1}) \\
&\quad + \frac{1}{n}\,\Omega\cdot X\cdot(D\psi_1 + D\psi_{m-1}) = 0\,, \\
& i\,(m-2)\,\nabla_X(\psi_1 - \psi_{m-1}) \\
&\quad + \frac{1}{n}\,(X\cdot\Omega - 2J(X))\cdot(D\psi_1 + D\psi_{m-1}) = 0\,, \\
& i\,(m-2)\,\nabla_X(\psi_1 - \psi_{m-1}) \\
&\quad + \frac{i\,m}{n}\,X\cdot(D\psi_1 - D\psi_{m-1}) - \frac{2}{n}\,J(X)\cdot(D\psi_1 + D\psi_{m-1}) = 0\,, \\
& i\,(m-2)\,\nabla_X(\psi_1 - \psi_{m-1}) \\
&\quad + \frac{i\,m}{n}\,X\cdot(D\psi_1 - D\psi_{m-1}) - \frac{2\,i}{n}\,X\cdot(D\psi_1 - D\psi_{m-1}) = 0\,,
\end{aligned}$$

which gives, since $m \neq 2$, the relation

$$\forall X \in TM\,, \qquad \nabla_X\,(\psi_1 - \psi_{m-1}) + \frac{1}{n}\,X\cdot(D\psi_1 - D\psi_{m-1}) = 0\,. \qquad (4.29)$$

Equations (4.29) and (4.6) necessarily imply that ψ_1 and ψ_{m-1} are twistor-spinors.

Third Step. For $m \neq 2$, we show that $\psi = 0$. The spinor field ψ_1 being a twistor-spinor, it follows

$$\begin{aligned}
\Omega\cdot\nabla_X\psi_1 + \frac{1}{n}\,\Omega\cdot X\cdot D\psi_1 &= 0\,, \\
i\,m\,\nabla_X\psi_1 + \frac{1}{n}\,\Omega\cdot X\cdot D\psi_1 &= 0\,, \\
i\,m\left(-\frac{1}{n}\,X\cdot D\psi_1\right) + \frac{1}{n}\,\Omega\cdot X\cdot D\psi_1 &= 0\,,
\end{aligned}$$

which shows that

$$\forall X \in TM\,, \qquad X\cdot D\psi_1 \in \Sigma_0 M\,.$$

Since the rank of the space $\Sigma_0 M$ is equal to one, while Clifford multiplication with a non-zero vector field is an isomorphism of the spin bundle, this is impossible unless $D\psi_1 = 0$. Therefore, the spinor field $D\psi_1 = 0$, and $\psi_1 = 0$ by the Lichnerowicz Theorem. The same argument holds for ψ_{m-1}.

Fourth Step. For $m = 2$, we prove that the above result is also true. By (4.9), we have $\Omega\cdot\psi = 0$. The argument in the third step shows that, for all vector fields X,

$$\Omega\cdot X\cdot D\psi = 0\,,$$

hence the rank of Σ_1 is at least equal to $2m$, which contradicts the fact that this rank is equal to m. □

We now use Theorem (4.7) to prove the following.

Theorem (4.30). *Let (M^{2m}, g, J) be a Kähler spin manifold with non-zero scalar curvature. Assume that, for a non-trivial spinor field ψ, $P^{a,b}\psi \equiv 0$. Then, there exists an integer r with $0 \leq r \leq m-2$, $\mu_r = m - 2r$ being the imaginary part of a generic eigenvalue of the Kähler form, such that*

$$a = b = \frac{1}{2(m+2-\mu_r)}\,, \qquad \text{and} \qquad D^2\psi = \frac{m+2-\mu_r}{4(m+1-\mu_r)}\, S\,\psi\,. \tag{4.31}$$

Moreover, for $2r \neq m$, $\psi = \psi_r + \psi_{m-r}$, with

$$D_+\,\psi_r = D_-\,\psi_{m-r} = 0\,. \tag{4.32}$$

Proof. First we prove that $a = b$. Assume that $1 - n(a+b) = 0$. Then by (2.7), the spinor field ψ is a Riemannian twistor-spinor (see [Li 2] or (3.7)), which is impossible by Theorem (4.7). If $a = \frac{1}{n}$, Eq. (2.9) implies $b = 0$. Moreover, by (2.11), if $b = \frac{1}{n}$, then $a = 0$. In both cases $1 - n(a+b) = 0$, hence $ab \neq 0$. Equations (2.9) and (2.11) give

$$(1-na)\,D^2\psi = 2b\,D\,\Omega\cdot\widetilde{D}\psi\,, \tag{4.33}$$

$$(1-nb)\,\widetilde{D}^2\psi = -2a\,\widetilde{D}\,\Omega\cdot D\psi\,. \tag{4.34}$$

From (4.33), (4.34) and (2.4), since $\widetilde{D}^2 = D^2$, it follows that

$$(a-b)\,(1-n(a+b)) = 0\,.$$

Hence $a = b$.

We then prove that, for $2r \neq m$, $\psi = \psi_r + \psi_{m-r}$ with $0 \leq r \leq m-2$. By Eqs. (2.9) and (2.11) one has

$$\Omega\cdot\Omega\cdot D\psi = -\frac{(1-na)^2}{4a^2}\,D\psi\,,$$

which with the help of (4.2) and (2.9) yields

$$\Omega\cdot\Omega\cdot D^2\psi + \left(\frac{na-1}{2a}+2\right)^2 D^2\psi = 0\,.$$

The scalar curvature being non-zero, by (2.7) $D^2\psi \neq 0$, hence there exists an integer r with $0 \leq r \leq m$, such that, for $2r \neq m$, $\psi = \psi_r + \psi_{m-r}$, and $\mu_r = \frac{na-1}{2a} + 2$, i.e., $a = \frac{1}{2(m+2-\mu_r)}$.

By (1.11), Eq. (2.9) gives

$$\widetilde{D}\,\Omega\cdot\psi = -\,\mu_r\,D\psi\,,$$

$$i\,\mu_r\,(\widetilde{D}\psi_r - \widetilde{D}\psi_{m-r}) = -\,\mu_r\,(D\psi_r + D\psi_{m-r})\,.$$

If $\mu_r \neq 0$, the above equation is equivalent to

$$D_+\,\psi_r + D_-\,\psi_{m-r} = 0\,. \tag{4.35}$$

The same computation using (1.10) and (2.11) gives

$$D_+\,\psi_r - D_-\,\psi_{m-r} = 0\,. \tag{4.36}$$

Equations (4.35) and (4.36) show that

$$D_+ \psi_r = D_- \psi_{m-r} = 0 . \tag{4.37}$$

If $r = m$, then $D_- \psi_r = 0$, which contradicts (4.37). If $r = m - 1$, then ψ is a twistor-spinor, which is impossible by Theorem (4.7). Hence, $0 \leq r \leq m - 2$. □

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