Einstein Metrics on Spheres

CHARLES BOYER

University of New Mexico

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History

• General Relativity.
(Einstein) Use Riemannian geometry with Lorentz signature as a theory of gravity. Reasoning: total amount of energy and momentum in the universe should equal the curvature of the universe.

Energy and momentum is represented by a symmetric 2-tensor $T_{\mu\nu}$. There are exactly two symmetric 2-tensors in the theory, the Ricci curvature, $R_{\mu\nu}$, and the (Lorentzian) metric itself $g_{\mu\nu}$. So Einstein tensor
$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} s g_{\mu\nu} = 8\pi T_{\mu\nu}$
s scalar curvature.
Later add 'cosmological constant' $\Lambda g_{\mu\nu}$ to r.h.s.
"The biggest blunder of my life." (Einstein)
But recently 'not so big a blunder'—
dark energy $\Rightarrow$ Λ small but > 0.

Riemannian manifold (M, g)

A Riemannian metric g is Einstein if $\mathrm{Ric}_g = \lambda g$

λ constant

Three cases:

(1) $\lambda > 0$,

(2) $\lambda = 0$,

(3) $\lambda < 0$.

Motivation

• Variational Principle

Normalize: (vol of $g) = 1$.

$g \mapsto \int_M s_g d\mu_g$, μ_g volume

(Hilbert) Einstein metrics are critical points.

Quadratic functionals:

$g \mapsto \int_M s_g^2 d\mu_g$ (Calabi)

Einstein metrics are critical points. Maybe Einstein metrics are distinguished.

Spheres

•History: Einstein metrics

• Round metric on S^n (Gauss-Riemann)

• Squashed metrics on S^{4n+3} (Jensen,1973)

• Homogeneous Einstein metric on S^{15} (Bourguignon and Karcher,1978).

• These are all homogeneous Einstein metrics on S^n and they are the only such metrics up to homothety
(Ziller,1982).

• Infinite sequences of inhomogeneous Einstein metrics on S^5, S^6, S^7, S^8 and S^9 (Böhm,1998). Maybe not so distinguished

Exotic Spheres

(Milnor,1956)
Spheres that are homeomorphic but not diffeomorphic to S^n. Homotopy spheres that bound a parallelizable manifold bP_{n+1} form an Abelian group. (Kervaire-Milnor)
For S^{4n+1}, $bP_{4n+2} = \mathbb{Z}_2$ if
$4n \neq 2^j - 4$ for any j.
No bP exotics $S^5, S^{13}, S^{29}, S^{61}$.

$bP_8 = \mathbb{Z}_{28}, bP_{12} = \mathbb{Z}_{992}$
$bP_{16} = \mathbb{Z}_{8128}, bP_{20} = \mathbb{Z}_{130816}$
Generally, $|bP_{4m}| =$
$2^{2m-2}(2^{2m-1} - 1)$ num $(\frac{4B_m}{m})$

- **Results** (B-,Galicki,Kollár)

$N_{SE} = \#$ of deformation classes Einstein metrics.

$\mu_{SE} = \#$ moduli of Einstein metrics.

• Each 28 diffeo types of S^7 admits hundreds of Einstein metrics, many with moduli. Largest moduli has dimension 82, standard S^7.

• All 992 diffeo types in bP_{12} and all 8128 diffeo types in bP_{16} admit Einstein metrics, i.e. on S^{11}, S^{15}.

• All elements of bP_{4n+2} admit Einstein metrics.

Our Einstein metrics are special, Sasaki-Einstein (SE)

• Both the number N_{SE} of deformation classes and the number μ_{SE} of moduli grow double exponentially with dimension.

(1) $N_{SE}(S^{13}) > 10^9$ and
$\mu_{SE}(S^{13}) = 21300113901610$
(2) $N_{SE}(S^{29}) > 5 \times 10^{1666}$ and
$\mu_{SE}(S^{29}) > 2 \times 10^{1667}$

Conjecture: Both $N_{SE}(S^{2n-1})$ and $\mu_{SE}(S^{2n-1})$ are finite.

Similar results for rational homology spheres (B-,Galicki) and other manifolds.

Ingredients of Proof

1. Contact geometry. Sasakian metrics
2. Differential topology. diffeomorphism types
3. Singularity theory. Links of isolated hypersurface singularities
4. Algebraic geometry. algebraic orbifolds
5. Analysis. Monge-Ampère deformations

• **Contact Manifold**(compact)

A **contact 1-form** η such that

$$\eta \wedge (d\eta)^n \neq 0.$$

defines a **contact structure**

$$\eta' \sim \eta \iff \eta' = f\eta$$

for some $f \neq 0$, take $f > 0$. or equivalently a codimension 1 subbundle $\mathcal{D} = \mathrm{Ker}\ \eta$ of TM.
$(\mathcal{D}, d\eta)$ symplectic vector bundle

Unique vector field ξ, called the **Reeb vector field**, satisfying

$$\xi \rfloor \eta = 1, \qquad \xi \rfloor d\eta = 0.$$

The **characteristic foliation** $\mathcal{F}_\xi$ each leaf of $\mathcal{F}_\xi$ passes through any nbd U at most k times $\iff$ **quasi-regular**, $k = 1 \leftrightarrow$ regular, otherwise **irregular**

Contact bundle $\mathcal{D} \longrightarrow$ choose **almost complex structure** J extend to Φ with $\Phi\xi = 0$

Get a compatible metric

$$g = d\eta \circ (\Phi \otimes \mathbb{1} + \eta \otimes \eta)$$

Quadruple $\mathcal{S} = (\xi, \eta, \Phi, g)$ called **contact metric structure**

Definition: The structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ is **K-contact** if $\pounds_\xi g = 0$ (or $\pounds_\xi \Phi = 0$). It is **Sasakian** if in addition $(\mathcal{D}, J)$ is integrable.
Note: Here we work entirely with Sasakian structures.

Geometry of Links

$\mathbb{C}^{n+1}$ coord's $\mathbf{z} = (z_0, \ldots, z_n)$
weighted $\mathbb{C}^*$-action

$$(z_0, \ldots, z_n) \mapsto (\lambda^{w_0} z_0, \ldots, \lambda^{w_n} z_n),$$

weight vector $\mathbf{w} = (w_1, \cdots, w_n)$ with $w_j \in \mathbb{Z}^+$
and
$\gcd(w_0, \ldots, w_n) = 1$.
f weighted homogeneous polynomial
$f(\lambda^{w_0} z_0, \ldots, \lambda^{w_n} z_n) = \lambda^d f(z_0, \ldots, z_n)$
$d \in \mathbb{Z}^+$ is **degree** of f.
$0 \in \mathbb{C}^{n+1}$ isolated singularity.
link L_f defined by

$$L_f = f^{-1}(0) \cap S^{2n+1},$$

S^{2n+1} unit sphere in $\mathbb{C}^{n+1}$

Special Case: Brieskorn-Pham poly. (BP)

$$f(z_0, \ldots, z_n) = z_0^{a_0} + \cdots + z_n^{a_n}$$

$a_i w_i = d, \ \forall i$.

Brieskorn-Pham Graph Thm:

For $\mathbf{a} = (a_0, \ldots, a_n)$ integers $\geq 2 \Rightarrow$ a graph $G(\mathbf{a})$ whose vertices are a_i. And a_i is connected to a_j if
$\gcd(a_i, a_j) > 1$.

Link L_f is a homology sphere $\iff$
(1): $G(\mathbf{a})$ contains at least two isolated points, or
(2): $G(\mathbf{a})$ has an odd # of vertices and a_i, a_j, $\gcd(a_i, a_j) = 2$ if $\gcd(a_i, a_j) > 1$.

Determine the diffeomorphism type:
(1): If $\dim \equiv 3 \bmod 4$: given by Hirzebruch signature of manifold that L_f bounds. Combinatorical formula (Brieskorn)
(2): If $\dim \equiv 1 \bmod 4$: $G(\mathbf{a})$ has one isolated point a_k such that $a_k \equiv \pm 3 \bmod 8$ gives Kervaire sphere. $a_k \equiv \pm 1 \bmod 8$ gives standard sphere.

Fact: L_f has natural structure with commutative diagram: $S^{2n+1}_{\mathbf{w}}$ weight sphere
$\mathbb{P}_{\mathbb{C}}(\mathbf{w})$ weighted projective space

$$\begin{array}{ccc} L_f & \longrightarrow & S^{2n+1}_{\mathbf{w}} \\ \Big\downarrow \pi & & \Big\downarrow \\ \mathcal{Z}_f & \longrightarrow & \mathbb{P}_{\mathbb{C}}(\mathbf{w}), \end{array}$$

horizontal arrows: Sasakian and Kählerian embeddings.
vertical arrows: orbifold Riemannian submersions.

L_f is Sasaki-Einstein (SE) $\iff$ $\mathcal{Z}_f$ is Kähler-Einstein (KE)

Question: When do we have SE or KE metrics?
1. $c_1^{orb}(\mathcal{Z}) > 0$ (easy)
2. solve Monge-Ampère equation (hard)

$$\frac{\det(g_{i\bar{j}} + \frac{\partial^2 \phi}{\partial z_i \partial \bar{z}_j})}{\det(g_{i\bar{j}})} = e^{f - t\phi}.$$

Tian: uniform boundedness

$$\int_{\mathcal{Z}} e^{-\gamma t \phi_{t_0}} \omega_0^n < +\infty$$

Many people Yau, Tian, Siu, Nadel, and most recently by Demailly and Kollár in orbifold category.

alebraic geometry of **orbifolds**:
local uniformizing covers
branch divisor: $\mathbb{Q}$-divisor

$$\Delta := \sum (1 - \frac{1}{m_j}) D_j$$

canonical orbibundle

$$K_{\mathcal{Z}}^{orb} = K_{\mathcal{Z}} + \sum (1 - \frac{1}{m_j})[D_j],$$

ramification index: m_j
Kawamata log terminal or **klt** For every $s \geq 1$ and holomorphic section $\tau_s \in H^0(\mathcal{Z}, \mathcal{O}((K_{\mathcal{Z}}^{orb})^{-s})$ there is $\gamma > \frac{n}{n+1}$ such that $|\tau_s|^{-\gamma/s} \in L^2(\mathcal{Z})$.

Theorem 2: $c_1^{orb}(\mathcal{Z}) > 0$, klt $\Rightarrow$ Sasaki-Einstein metric.

Sasaki-Einstein metrics

Positivity $\Rightarrow$ $I = (\sum w_i - d) > 0$

klt estimates for L_f

$$d(\sum w_i - d) < \frac{n}{n-1}\mathrm{min}_{i,j} w_i w_j.$$

BP polyn: (better)

$$1 < \sum_{i=0}^{n} \frac{1}{a_i} < 1 + \frac{n}{n-1} \min_i \{\frac{1}{a_i}, \frac{1}{b_i b_j}\}.$$

a_i BP exponents and

$$b_i = \mathrm{gcd}(a_i, \mathrm{lcm}(a_j \mid j \neq i))$$

$\exists$ other estimates. Positivity plus a klt estimate $\Rightarrow$ SE metric

To determine the moduli μ_{SE} add monomials $z_{i_1}^{b_{i_1}} \cdots z_{i_k}^{b_{i_k}}$ such that $\sum_j b_{i_j} = d$ to BP polynomial. Divide by equivalence of $\mathfrak{Aut}(\mathcal{Z}_f)$.

Why double exponential growth?

Reason for growth: Sylvester's sequence determined by $c_{k+1} = 1 + c_0 \cdots c_k$ begins as $2, 3, 7, 43, 1807, 3263443, 10650056950807, \ldots$

N_{SE}: sequences $\mathbf{a} = (a_0 = c_0, \ldots, a_{n-1} = c_{n-1}, a_n)$ with $c_{n-1} < a_n < c_0 \cdots c_{n-1}$ give SE metrics. Use prime number theorem.

μ_{SE}: sequences $\mathbf{a} = (a_0 = c_0, \ldots, a_{n-1} = c_{n-1}, a_n)$ where $a_n = (c_{n-1} - 2)c_{n-1}$. Polynomial f contains $G(z_{n-1}, z_n^{c_{n-1}-2})$. Again by prime number theorem gives double exponential growth.

Conjecture: All elements of bP_{2n} admit SE metrics.

Estimate of Lichnerowicz $\Rightarrow$ if
$I = (\sum w_i - d) > n\ \mathrm{min}_i w_i$ then $\nexists$ SE metrics.
Only applies to KE orbifolds!
(Gauntlett,Martelli,Sparks,Yau)
(Ghigi,Kollár) class of SE metrics where bound is sharp. $\times 10$ more on spheres.