

Einstein's other gravity and the acceleration of the Universe

Eric V. Linder

Berkeley Lab and University of California, Berkeley, California 94720 USA
Institute for the Early Universe, Ewha Womans University, Seoul 120-750 Korea
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Spacetime curvature plays the primary role in general relativity but Einstein later considered a theory where torsion was the central quantity. Just as the Einstein-Hilbert action in the Ricci curvature scalar R can be generalized to $f(R)$ gravity, we consider extensions of teleparallel, or torsion scalar T , gravity to $f(T)$ theories. The field equations are naturally second order, avoiding pathologies, and can give rise to cosmic acceleration with unique features.

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I. INTRODUCTION

Acceleration of the cosmic expansion is one of the premier mysteries of physics and may be the clearest clue to properties of gravity beyond general relativity. Extensions to gravity have been considered by making the action a function of the spacetime curvature scalar R or other curvature invariants, by coupling this Ricci scalar to a scalar field, by introducing a vector field contribution, and by using properties of gravity in higher dimensional spacetimes. Here we take a wholly different path, avoiding the curvature completely, although our results will end up with interesting relations to each of the above mentioned theories.

Rather than use the curvature defined via the Levi-Civita connection, one could explore the opposite approach and use the Weitzenböck connection that has no curvature but instead torsion. This has the interesting property that the torsion is formed wholly from products of first derivatives of the tetrad, with no second derivatives appearing in the torsion tensor. In fact, this approach was taken by Einstein in 1928 [1,2], under the name “Fern-Parallelismus” or “distant parallelism” or “teleparallelism”. It is closely related to standard general relativity, differing only in terms involving total derivatives in the action, i.e. boundary terms.

In this paper, we investigate extensions where a scalar formed from contractions of the torsion tensor is promoted to a function of that scalar. This parallels the concept of extension of the Ricci scalar R in the Einstein-Hilbert action to a function $f(R)$, which has attracted much attention in recent years as a way to explain acceleration of the Universe [3,4]. The generalized $f(T)$ torsion theory has the advantage, however, of keeping its field equations second order due to the lack of second derivatives, unlike the fourth order equations (at least in the metric formulation) of $f(R)$ theory that can lead to pathologies.

II. COSMOLOGICAL EQUATIONS

We start with the Robertson-Walker metric for a homogeneous and isotropic space with zero spatial curvature:

$$ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j, \quad (1)$$

where a is the expansion factor. The orthonormal tetrad components $e_A(x^\mu)$ relate to the metric through

$$g_{\mu\nu} = \eta_{AB}e_\mu^A e_\nu^B, \quad (2)$$

where A, B are indices running over 0, 1, 2, 3 for the tangent space of the manifold and μ, ν are coordinate indices on the manifold, also running over 0, 1, 2, 3 (with i, j being the spatial indices).

The torsion tensor and permutations (note the different symmetry properties from the curvature case), are

$$T^\rho{}_{\mu\nu} \equiv -e_A^\rho(\partial_\mu e_\nu^A - \partial_\nu e_\mu^A), \quad (3)$$

$$K^{\mu\nu}{}_\rho \equiv -\frac{1}{2}(T^{\mu\nu}{}_\rho - T^{\nu\mu}{}_\rho - T_\rho{}^{\mu\nu}), \quad (4)$$

$$S_\rho{}^{\mu\nu} \equiv \frac{1}{2}(K^{\mu\nu}{}_\rho + \delta_\rho^\mu T^{\alpha\nu}{}_\alpha - \delta_\rho^\nu T^{\alpha\mu}{}_\alpha). \quad (5)$$

In place of the Ricci scalar for the Lagrangian density, one has the torsion scalar (also see [5])

$$T \equiv S_\rho{}^{\mu\nu} T^\rho{}_{\mu\nu}, \quad (6)$$

and the gravitational action is

$$I = \frac{1}{16\pi G} \int d^4x |e| T, \quad (7)$$

where $|e| = \det(e_\mu^A) = \sqrt{-g}$. For a more detailed derivation giving a clearer picture of the relation to general relativity, with the difference arising in boundary terms, see [6–9].

Following [5] we now promote T to a function, replacing it in the action by $T + f(T)$, in analogy to $f(R)$ gravity (see, e.g., [3,4]). The modified Friedmann equations of motion are (cf. [5] with different notation)

$$H^2 = \frac{8\pi G}{3}\rho - \frac{f}{6} - 2H^2 f_T, \quad (8)$$

$$(H^2)' = \frac{16\pi GP + 6H^2 + f + 12H^2 f_T}{24H^2 f_{TT} - 2 - 2f_T}, \quad (9)$$

where the Hubble expansion parameter $H = \dot{a}/a$, a prime denotes a derivative with respect to $\ln a$, ρ is the energy density and P is the pressure. Evaluating Eq. (6) for the unperturbed metric, one finds $T = -6H^2$, so one can use T and H interchangeably.

Taking a universe with only matter (so $P = 0$), we find the solution $T(a)$ in closed form:

$$a(T) = \exp\left\{-\frac{1}{3} \int_{-6H_0^2}^T \frac{d\tilde{T}}{\tilde{T}} \frac{1 + f_T + 2\tilde{T}f_{TT}}{1 - f/\tilde{T} + 2f_T}\right\}. \quad (10)$$

One can also define an effective dark energy density and equation of state

$$\rho_{de} = \frac{1}{16\pi G}(-f + 2Tf_T), \quad (11)$$

$$w = -1 + \frac{1}{3} \frac{T'}{T} \frac{f_T + 2Tf_{TT}}{f/T - 2f_T} = -\frac{f/T - f_T + 2Tf_{TT}}{(1 + f_T + 2Tf_{TT})(f/T - 2f_T)}. \quad (12)$$

From the modified Friedmann equations (8) and (9) we see that a constant f acts just like a cosmological constant, and f linear in T (i.e. $f_T = \text{constant}$) is simply a redefinition of Newton's constant G .

III. RESULTS FOR COSMIC ACCELERATION

At high redshift, we desire general relativity to hold so as to agree with primordial nucleosynthesis and cosmic microwave background constraints. Therefore we want $f/T \rightarrow 0$ at early times, $a \ll 1$. Regarding the future, Eq. (9) says an asymptotic future de Sitter state (with $H = \text{constant}$ and $w = -1$), for example, occurs when the numerator (but not the denominator) vanishes. Many functions $f(T)$ can give a de Sitter fate for the Universe; here we examine two models.

As a first example, consider a power law

$$f = \alpha(-T)^n = \alpha 6^n H^{2n}. \quad (13)$$

From Eq. (8), the dimensionless matter density today

$$\Omega_m = \frac{8\pi G \rho_m(a=1)}{3H_0^2} = 1 + \frac{f(T_0)}{6H_0^2} + 2f_T, \quad (14)$$

so $\alpha = (6H_0^2)^{1-n}(1 - \Omega_m)/(2n - 1)$. The Hubble expansion freezes in the future at the value $H_\infty = H_0(1 - \Omega_m)^{1/[2(1-n)]}$. The effective dark energy equation of state varies from $w = -1 + n$ in the past to $w = -1$ in the future. For example, solving the modified Friedmann equations numerically, for $n = 0.25$ one has $w_0 = -0.91$ and $w(a = 0.5) = -0.81$; for this form of $f(T)$ to be a viable model compared to current data one needs $n \ll 1$.

Such a functional form as Eq. (13) results in a power of H being added to the Friedmann equation and is equivalent (at least at the background level) to the phenomenological models of [10,11]. In [12] it was shown that such models behave as freezing scalar fields, and, in particular, approach a de Sitter state in the future along the curve $w' = 3w(1 + w)$. We have numerically solved the equations of motion to verify that this holds for such an $f(T)$ as well. Note that $n = 1/2$ gives the same expansion history as DGP gravity [13,14], so $f(T)$ gravity can be viewed as having some connection to higher dimension theories.

Another fine tuning for the power law models is that one has a similar condition to $f(R)$ gravity in that the factor f_T , acting to rescale Newton's constant, should be small. The condition is not as sensitive as in $f(R)$ gravity, because there R changes with scale so solar system and galactic constraints impose tight bounds on f_R . For $f(T)$ theories, T is much less scale dependent (of order $(k/H)^2 \Phi^2$ [15]) so the time variation of G gives the main limit. This again imposes $n \ll 1$.

To keep the variation of the gravitational coupling small within $f(R)$ theory, [16] adopted an exponential dependence on the curvature scalar. Here we explore a similar exponential dependence on the torsion scalar as an example. We take the form

$$f = -\alpha T(1 - e^{pT_0/T}), \quad (15)$$

where $\alpha = (1 - \Omega_m)/[1 - (1 - 2p)e^p]$. Note that there is only one parameter, p , besides the value of the matter density today, Ω_m .

Figure 1 illustrates the behavior of the equation of state for several values of p . At high redshift the model acts like

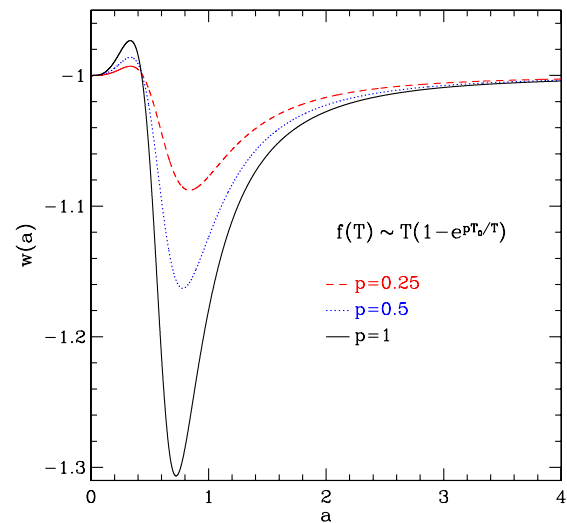


FIG. 1 (color online). Effective dark energy equation of state is plotted vs scale factor for the exponential $f(T)$ model of Eq. (15). The model acts like a cosmological constant at high redshift and in the future, crossing $w = -1$ in between but in the opposite manner from $f(R)$ gravity.

Λ CDM, then it deviates first to $w > -1$ then $w < -1$: crossing $w = -1$ in the opposite sense from $f(R)$ gravity—and asymptotically approaches a de Sitter fate. The parameter p mainly controls the amplitude of the deviation from $w = -1$. No fine-tuning is needed, with $p \lesssim 1$ allowed by current cosmological observations of the expansion history.

IV. SUMMARY AND CONCLUSIONS

The class of $f(T)$ gravity theories is an intriguing generalization of Einstein’s “new general relativity,” taking a curvature-free approach and instead using a connection with torsion. It is analogous to the $f(R)$ extension of the Einstein-Hilbert action of standard general relativity, but has the advantage of second order field equations. We have also seen that it can be related to the form of modifications to the Friedmann equations due to higher dimensional gravity theories such as DGP.

It is also related to scalar-tensor gravity. Writing the gravitational action as

$$S = \int d^4x |e| \{T + f(T) + (T - A)[1 + f_A(A)]\}, \quad (16)$$

one can view the last term as a Lagrange multiplier term and find an equivalent scalar-tensor theory with $A = T$ and an effective potential

$$V_{\text{eff}}(\psi) = \frac{T}{1 + f_T} - \frac{T + f}{(1 + f_T)^2}, \quad (17)$$

$$\psi = -\ln(1 + f_T). \quad (18)$$

Furthermore, Einstein originally introduced teleparallelism to obtain a vector field component of the field equations [2,17], intending to unify gravity and electromagnetism. Recently, interest has grown in vector fields, “Einstein aether theories,” as a way to obtain cosmic acceleration [18]. These theories can also give modifications to the field equations involving functions of H^2 , i.e. T (see, e.g., [19]). Indeed they can be viewed as closely related to torsion theories (see [7,8] for details).

Thus, torsion theories can unify a number of interesting extensions of gravity beyond general relativity. In investigating the nature of gravitation, we may find that Einstein presaged the acceleration of the Universe not only through the cosmological constant but through a generalization of “Einstein’s other gravity.”

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