JOURNAL OF THEORETICAL
AND APPLIED MECHANICS
42, 4, pp. 859-868, Warsaw 2004

ELASTIC BUCKLING OF A POROUS BEAM

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The work deals with the problem a straight beam of a rectangular cross-section pivoted at both ends and loaded with a lengthwise compressive force. The beam is made of an isotropic porous material. Its properties vary through thickness of the beam. The modulus of elasticity is minimal on the beam axis and assumes maximum values at its top and bottom surfaces. The principle of stationarity of the total potential energy enables one to define a system of differential equations that govern the beam stability. The system is analytically solved, which leads to an explicit expression for the critical load of the compressed beam. Results of the solution are verified on an example beam by means of the Finite Element Method (COSMOS).

Key words: buckling, porous beam, shear deformable beam

1. Displacements of a porous beam

A mathematical description of composite structures obviously includes many simplifying assumptions. Librescu and Hause (2000) provided a review of sandwich structures, paying attention to their stability problems. Vinson (1999) discussed sandwich structures made of isotropic and composite materials. Kołakowski and Kowal-Michalska (1999) presented some problems of stability of thin-walled composite structures. The above mentioned works provide descriptions of displacements (strains) in cross-sections that are based on the linear Euler-Bernoulli hypothesis. Thus, the effect of shearing due to transverse forces is omitted. A separate group includes three-layered structures, in the which shearing is taken into account. Lok and Cheng (2000) characterized properties of structures, with special attention paid to the middle layer, subject mainly to the shearing. Magnucki and Ostwald (2001) presented problems of stability and optimal shaping of three-layered structures. Displacements occurring in cross-sections of such structures (Lok, Cheng and Magnucki, Ostwald) were described using the broken-line hypothesis. Romanów (1995) assumed a hyperbolic pattern of the normal stress distribution in the cross-section of a three-layered wall. The works of Lok, Cheng and Magnucki, Ostwald and Romanów took the shearing effect into account as well. Wielgosz and Thomas (2002) discussed the results of an analytical solution, taking into consideration the shearing effect and experimental studies related to panel bending. Bart-Smith *et al.* (2001) presented the problem of bending of a sandwich structure with the middle layer made of a cellular metal.

This work is concerned with an isotropic porous beam of a rectangular cross-section pivoted at both ends and loaded with a lengthwise compressive force. Mechanical properties of the material vary through thickness of the beam. Young's modulus is minimal on the beam axis and assumes maximum values at its top and bottom surfaces. For such a case, the of Euler-Bernoulli or Timoshenko beam theories do not correctly determine displacements of the cross-section of the beam. Wang *et al.* (2000) discussed in details the effect of non-dilatational strain of middle layers on bending of beams and plates subject to various load cases.

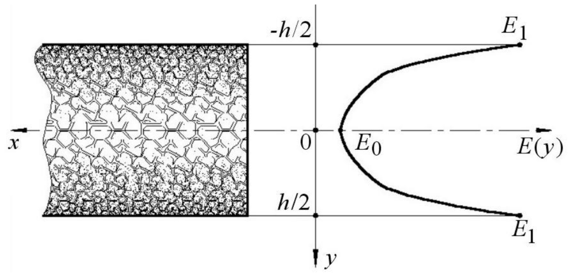

Fig. 1. Scheme of porous beam

A porous beam (Fig. 1) is a generalized sandwich beam. Its outside surfaces (top and bottom) are smooth, without pores. The material is of continuous characteristics. The beam is porous inside, with the degree of porosity varying in the transverse direction, assuming the maximum value on the beam axis. A rectangular system of coordinates is introduced, with the x-axis oriented along the beam, and the y-axis in the beam-depth direction.

The moduli of elasticity are defined as follows

$$E(y) = E_1[1 - e_0 \cos(\pi\eta)] \qquad G(y) = G_1[1 - e_0 \cos(\pi\eta)] \tag{1.1}$$

where

e_0	–	coefficient of beam porosity, $e_0 = 1 - E_0/E_1$
E_0, E_1	–	Young's moduli at $y = 0$ and $y = \pm h/2$, respectively
G_0, G_1	–	shear moduli (modulus of rigidity) for $y = 0$ and $y = \pm h/2$, respectively
G_j	–	relationship between the moduli of elasticity for $j = 0, 1$, $G_j = E_j/[2(1+\nu)]$
ν	–	Poisson's ratio (constant for the entire beam)
η	–	dimensionless coordinate, $\eta = y/h$
h	–	thickness of the beam.

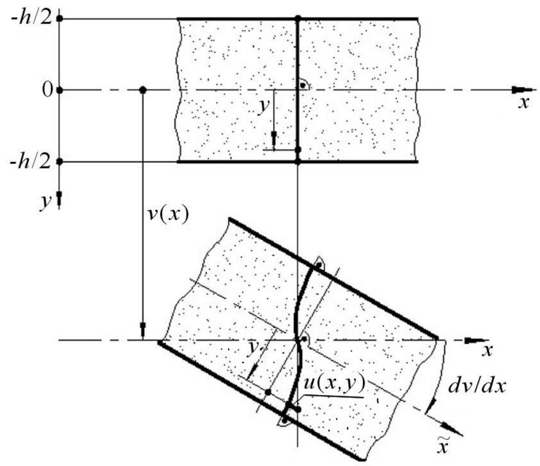

Fig. 2. Geometric model of broken-line hypothesis

The field of displacements (geometric model) in the rectangular cross-section of the beam is shown in Fig. 2. The cross-section, being initially a planar surface, becomes curved after the deformation. The surface perpendicularly intersects the top and bottom surfaces of the beam. The geometric model is similar to that obtained by making use of the broken-line hypothesis applied to three-layered structures. Such a definition of the displacement model gives a basis for adopting a field of displacements in any cross-section in the following form

$$\begin{aligned} u(x,y) &= -h\Big\{\eta\frac{dv}{dx} - \frac{1}{\pi}[\psi_1(x)\sin(\pi\eta) + \psi_2(x)\sin(2\pi\eta)\cos^2(\pi\eta)]\Big\} \\ v(x,y) &= v(x,0) = v(x) \end{aligned} \tag{1.2}$$

where

$u(x,y)$	–	longitudinal displacement along the x-axis
$v(x)$	–	deflection (displacement along the y-axis)
$\psi_1(x),\psi_2(x)$	–	dimensionless functions of displacements.

The geometric relationships, i.e. components of the strain field are

$$\begin{aligned}\varepsilon_x &= \frac{\partial u}{\partial x} = -h\Big\{\eta\frac{d^2v}{dx^2} - \frac{1}{\pi}\Big[\frac{d\psi_1}{dx}\sin(\pi\eta) + \frac{d\psi_2}{dx}\sin(2\pi\eta)\cos^2(\pi\eta)\Big]\Big\}\\ \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{dv}{dx} = \psi_1(x)\cos(\pi\eta) + \psi_2(x)[\cos(2\pi\eta) + \cos(4\pi\eta)]\end{aligned} \tag{1.3}$$

where ε_x is the normal strain along the x-axis, and γ_{xy} – the angle of shear (shear strain).

The physical relationships, according to Hooke's law are

$$\sigma_x = E(y)\varepsilon_x \qquad\qquad \tau_{xy} = G(y)\gamma_{xy} \tag{1.4}$$

Moduli of elasticity (1.1) occurring here are variable and depend on the y-coordinate.

2. Equations of stability

The field of displacements for in the thus defined problem includes three unknown functions: $v(x)$, $\psi_1(x)$ and $\psi_2(x)$. Hence, three equations are necessary for a complete description of the problem. They may be formulated on the grounds of the principle of stationarity of the total potential energy of the compressed beam

$$\delta(U_\varepsilon - W) = 0 \tag{2.1}$$

where U_ε is the energy of elastic strain

$$U_\varepsilon = \frac{ht}{2}\int_0^L\int_{-1/2}^{1/2} (\sigma_x\varepsilon_x + \tau_{xy}\gamma_{xy})\, dxd\eta$$

W is the work of the load (compressive force)

$$W = \frac{F}{2}\int_0^L \Big(\frac{dv}{dx}\Big)^2\, dx$$

and L – length of the beam, t – width of the rectangular cross-section of the beam.

A system of three equations of stability for the porous beam under compression is formulated in the following form

$$\begin{aligned}
&E_1h^3t\Big(C_1\frac{d^2v}{dx^2}-C_2\frac{d\psi_1}{dx}-C_3\frac{d\psi_2}{dx}\Big)+Fv(x)=0\\
&C_2\frac{d^3v}{dx^3}-C_4\frac{d^2\psi_1}{dx^2}-C_5\frac{d^2\psi_2}{dx^2}+\frac{1}{2(1+\nu)}\frac{1}{h^2}(C_7\psi_1+C_8\psi_2)=0\\
&C_3\frac{d^3v}{dx^3}-C_5\frac{d^2\psi_1}{dx^2}-C_6\frac{d^2\psi_2}{dx^2}+\frac{1}{2(1+\nu)}\frac{1}{h^2}(C_8\psi_1+C_9\psi_2)=0
\end{aligned} \tag{2.2}$$

where

$$\begin{aligned}
C_1&=\frac{1}{12}-\frac{\pi^2-8}{2\pi^3}e_0 & C_2&=\frac{1}{\pi^2}\Big(\frac{2}{\pi}-\frac{1}{4}e_0\Big)\\
C_3&=\frac{1}{\pi^2}\Big(\frac{3}{16}-\frac{32}{75\pi}e_0\Big) & C_4&=\frac{1}{\pi^2}\Big(\frac{1}{2}-\frac{2}{3\pi}e_0\Big)\\
C_5&=\frac{1}{\pi^2}\Big(\frac{8}{15\pi}-\frac{1}{8}e_0\Big) & C_6&=\frac{1}{\pi^2}\Big(\frac{5}{32}-\frac{128}{315\pi}e_0\Big)\\
C_7&=\frac{1}{2}-\frac{4}{3\pi}e_0 & C_8&=\frac{8}{15\pi}-\frac{1}{4}e_0\\
C_9&=1-\frac{832}{315\pi}e_0
\end{aligned}$$

Moreover, appropriate boundary conditions are formulated (for $x=0$ and $x=L$)

$$\begin{aligned}
&[v''\delta v'-v'''\delta v]\Big|_0^L=0 && [\psi_k'\delta v'-\psi_k''\delta v]\Big|_0^L=0\\
&(v''\delta\psi_k)\Big|_0^L=0 && \psi_k'\delta\psi_k)\Big|_0^L=0 \qquad k=1,2
\end{aligned} \tag{2.3}$$

The system of differential equations (2.2) may be approximately solved with the use of Galerkin's method. Hence, three unknown functions satisfying boundary conditions (2.3) are assumed in the following form

$$v(x)=v_a\sin\Big(n\pi\frac{x}{L}\Big) \qquad \psi_k(x)=\psi_{ak}\cos\Big(n\pi\frac{x}{L}\Big) \qquad k=1,2$$

where v_a, ψ_{a1}, ψ_{a2} are parameters and n is a natural number.

Substituting these functions into equations (2.2) and using Galerkin's method yields a system of three homogeneous algebraic equations

$$\mathbf{A}_{(3\times3)}\boldsymbol{X}=\mathbf{0} \tag{2.4}$$

where

$$\mathbf{A} = \begin{bmatrix} a_{11}-f & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad \boldsymbol{X} = \begin{bmatrix} -v_a \\ \psi_{a1} \\ \psi_{a2} \end{bmatrix}$$

$$a_{11} = C_1\alpha_0^2 \qquad a_{12} = C_2\alpha_0 h \qquad a_{13} = C_3\alpha_0 h$$

$$a_{21} = C_2\alpha_0^3 \qquad a_{22} = \Big[C_4\alpha_0^2 + \frac{C_7}{2(1+\nu)}\Big]h \qquad a_{23} = \Big[C_5\alpha_0^2 + \frac{C_8}{2(1+\nu)}\Big]h$$

$$a_{31} = C_3\alpha_0^3 \qquad a_{32} = \Big[C_5\alpha_0^2 + \frac{C_8}{2(1+\nu)}\Big]h \qquad a_{33} = \Big[C_6\alpha_0^2 + \frac{C_9}{2(1+\nu)}\Big]h$$

$$\alpha_0 = n\pi\frac{h}{L}$$

and f is the dimensionless longitudal compressive force $(0 < f)$

$$f = \frac{F}{E_1 ht}$$

The condition

$$\det \mathbf{A} = 0 \tag{2.5}$$

enables determination of the dimensionless force f.

Limiting the considerations to the matrix $\mathbf{A}_{(2\times 2)}$ and taking into account condition (2.5) yields

$$f = a_{11} - \frac{a_{12}a_{21}}{a_{22}} = C_1\alpha_0^2\Big[1 - 2(1+\nu)\frac{C_2^2\alpha_0^2}{C_1(C_7+2(1+\nu)C_4\alpha_0^2)}\Big] \tag{2.6}$$

and the dimensionless critical load

$$f_{CR} = \min_n f = \pi^2\Big(\frac{h}{L}\Big)^2 C_1\Big[1 - 2(1+\nu)\frac{C_2^2}{C_1C_0}\Big] \tag{2.7}$$

for $n = 1$, where

$$C_0 = 2(1+\nu)C_4 + \frac{C_7}{\pi^2}\lambda^2$$

and $\lambda = L/h$ is the relative length of the beam.

The critical force is

$$F_{CR} = \frac{\pi^2 E_1 h^3 t}{L^2}C_1\Big[1 - 2(1+\nu)\frac{C_2^2}{C_1C_0}\Big] \tag{2.8}$$

In a particular case of a beam made of an isotropic non-porous material, the elasticity coefficients do not depend on the coordinate y $(e_0 = 0, C_1 = 1/12)$. The negligence of the transverse force effect $(C_2 = 0)$ gives the classical Euler force. Apart from varying elasticity constants, the effect of shear strain on the critical force is also taken into consideration in expression (2.8).

3. Numerical analysis of the stress state

A family of beams of the constant height $h = 100\,\text{mm}$ and width $t = 1\,\text{mm}$ was assumed. The beam lengths were: $L = 2000\,\text{mm}$, $2500\,\text{mm}$ and $5000\,\text{mm}$, the material constants: $E_1 = 2.05 \cdot 10^5\,\text{MPa}$, $e_0 = 0.99$, $\nu = 0.3$. Numerical analysis was carried out by means of the Finite Element Method – System COSMOS/M. The symmetry of the system enabled modeling of a half of the beam only by imposing suitable boundary conditions at one end for $x = 0$ (zero-displacement in the y-axis direction) and in the middle cross-section for $x = L/2$ (zero-displacement in the x-axis direction). The beam was buckled only in the xy plane. The material properties varying through thickness of the cross-section were discretized with 20 layers of constant properties. Particular layers were characterized by elasticity constants adopted according to (1.1) for points located in the middle of each of the layers (Fig. 3). For the purpose of strength analysis, the beam was subject to a transverse load of a constant intensity distributed at its whole length.

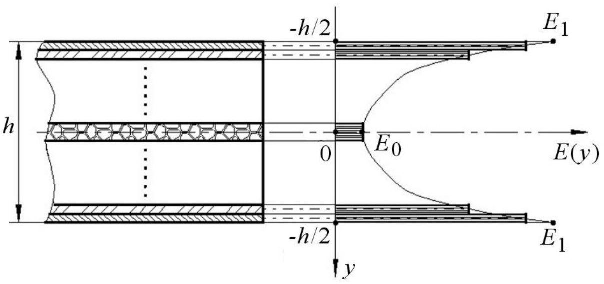

Fig. 3. Discretization of material properties

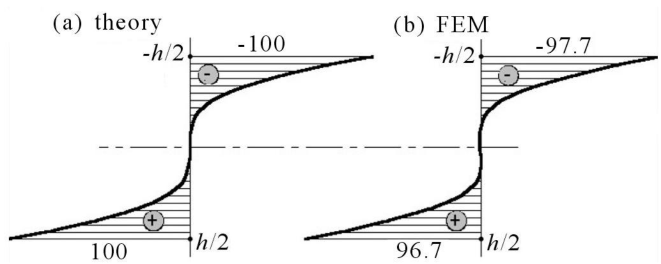

Fig. 4. Normal stress at cross-section: (a) theory, (b) FEM

Figure 4 presents an example of normal stress distribution in points located in the middle cross-section of the beam. The theoretical distribution

(Fig. 5a) was determined from Hooke's law (1.4) based on the adopted formulas of moduli of elasticity (1.1) and the broken-line hypothesis assumed for determination of displacements (1.2). The stress distribution patterns obtained analytically and numerically (FEM) are very similar, which seems to confirm the justness of the broken-line hypothesis.

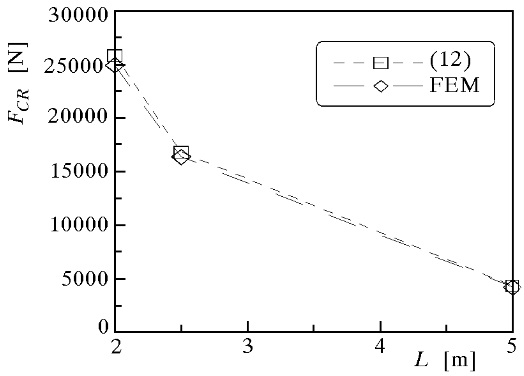

Fig. 5. Critical load as function of beam length

4. Numerical analysis of buckling

The critical loads determined on the grounds of the analytical solution to equation (2.8) for a family of beams are specified in Table 1. Moreover, the critical loads are determined by means of FEM. The subspace Iteration algorithm was applied. Values of the loads are shown in Table 1. The comparison of the solutions obtained with both methods shows that the error does not exceed 4 percent for the beam of the length $L = 5000$ mm (Fig. 5).

Table 1. Values of critical loads

L [m]	F_{CR} [N]	
	Eq. (2.8)	FEM
2.0	25808	24915
2.5	16795	16346
5.0	4295	4166

5. Conclusions

The above proposal of analytical description of the field of strains in a beam properties varying through thickness is a generalization of an approach to multi-layered composite beams. The linear Euler-Bernoulli hypothesis for beams subject to bending makes a particular case of the description. The general solution to three equations of stability enable one define a simple formula for the critical load of the beam. The critical loads obtained from the analytical and numerical (FEM) solutions are similar, with the maximum difference not exceeding 4 percent.

The work was presented at the 10th Symposium of Structure Stability in 2003 (Zakopane).

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Wyboczenie sprężyste belki porowatej

Streszczenie

Przedmiotem pracy jest prosta belka o przekroju prostokątnym, podparta przegubowo na obu końcach, obciążona wzdłużną siłą ściskającą. Belka wykonana jest z materiału izotropowego porowatego. Właściwości tego materiału są zmienne na wysokości belki. Na osi belki moduł sprężystości jest najmniejszy, natomiast na powierzchniach górnej i dolnej największy. Z zasady stacjonarności całkowitej energii potencjalnej wyznaczono układ równań różniczkowych stateczności belki. Układ ten rozwiązano analitycznie i zapisano w postaci zamkniętej wyrażenie na obciążenie krytyczne ściskanej belki. Wyniki tego rozwiązania zweryfikowano dla przykładowej belki za pomocą metody elementów skończonych (System COSMOS).

Manuscript received January 6, 2004; accepted for print March 30, 2004