Elastic demand dynamic network user equilibrium: Formulation, existence and computation

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Abstract

This paper is concerned with dynamic user equilibrium with elastic travel demand (E-DUE) when the trip demand matrix is determined endogenously. We present an infinite-dimensional variational inequality (VI) formulation that is equivalent to the conditions defining a continuous-time E-DUE problem. An existence result for this VI is established by applying a fixed-point existence theorem (Browder, 1968) in an extended Hilbert space. We present three algorithms based on the aforementioned VI and its re-expression as a *differential variational inequality* (DVI): a projection method, a self-adaptive projection method, and a proximal point method. Rigorous convergence results are provided for these methods, which rely on increasingly relaxed notions of generalized monotonicity, namely *mixed strongly-weakly monotonicity* for the projection method; *pseudomonotonicity* for the self-adaptive projection method, and *quasimonotonicity* for the proximal point method. These three algorithms are tested and their solution quality, convergence, and computational efficiency compared. Our convergence results, which transcend the transportation applications studied here, apply to a broad family of infinite-dimensional VIs and DVIs, and are the weakest reported to date.

Keywords: dynamic user equilibrium, elastic demand, variational inequality, existence, computation, convergence

1. Introductory remarks

This paper is concerned with an extension of the *simultaneous route-and-departure-time* (SRDT) dynamic user equilibrium (DUE) originally proposed in Friesz et al. (1993) and discussed subsequently by Friesz et al. (2001, 2013b, 2011); Friesz and Meimand (2014); and Friesz and Mookherjee (2006). Specifically, the model of interest herein relaxes the assumption of fixed trip volumes by considering elastic travel demands among origin-destination pairs. The extension of DUE based on a fixed trip table to the explicit consideration of elastic demand is not a straightforward matter. In particular, one has to show a dual variable associated with arrival time is equivalent to an adjoint (co-state) variable by exploiting the transversality conditions familiar from differential variational inequality (DVI) theory. The first such analysis was performed by Friesz and Meimand (2014), who employed separable demand functions for each origin-destination pair. They used a variational calculus approach, which, although correct, masks many of the measure-theoretic arguments essential to understanding the generality of a DVI representation of the elastic-demand DUE (E-DUE) problem in continuous time. By contrast, this paper not only considers nonseparable demand functions, but it also provides all measure-theoretic arguments needed to understand the DVI formulation. Furthermore, this paper presents an existence theory, three algorithms and their proofs of convergence, and numerical studies, all of which are missing from Friesz and Meimand (2014). That is to say, this paper provides the first complete mathematical and numerical analysis of the SRDT E-DUE problem.

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1.1. Dynamic user equilibrium with elastic demand: Some review

Most of the studies of DUE reported in the *dynamic traffic assignment* (DTA) literature are about dynamic user equilibrium with constant travel demand for each origin-destination pair. It is, of course, not generally true that travel demand is fixed, even for short time horizons. Arnott et al. (1993) and Yang and Huang (1997) directly consider elastic travel demand in the context of a single bottleneck. Yang and Meng (1998) extend a simple bottleneck model to a general queuing network with known elastic demand functions for each origin-destination (OD) pair. They employ a *space-time expanded network* (STEN) representation of the network loading submodel. Wie et al. (2002) study a version of the dynamic user equilibrium with elastic demand, using a complementarity formulation that requires path delays to be expressible in closed form. Szeto and Lo (2004) study dynamic user equilibrium with elastic travel demand when network loading is based on the *cell transmission model* (CTM); their formulation is discrete-time in nature and is expressed as a finite-dimensional variational inequality (VI). The VI is solved with a descent method under the assumption that the delay operator is co-coercive. Han et al. (2011) study dynamic user equilibrium with elastic travel demand for a network with a single origin-destination pair whose traffic flow dynamics are also described by CTM; the CTM is chosen to accommodate the discrete-time complementarity formulation of the user equilibrium model.

Although Friesz et al. (2011) show that analysis and computation of dynamic user equilibrium with constant travel demand is tremendously simplified by stating it as a differential variational inequality (DVI), they do not discuss how elastic demand may be accommodated within a DVI framework. Friesz and Meimand (2014) later extend the DVI formulation to an elastic demand setting, although that paper does not discuss the existence and the computation of E-DUE, which are our main focus here. Such a DVI formulation for the E-DUE problem is not a straightforward extension. In particular, the DVI presented therein has both infinite-dimensional and finite-dimensional terms. Moreover, for any given origin-destination pair, inverse travel demand corresponding to a dynamic user equilibrium depends on the terminal value of a state variable representing cumulative departures. The DVI formulation achieved in that paper is significant because it allows the still emerging theory of differential variational inequalities to be employed for the analysis and computation of solutions of the elastic-demand DUE problem when simultaneous departure time and route choices are within the purview of users, all of which constitutes a foundation problem within the field of dynamic traffic assignment.

A good review of recent insights into abstract differential variational inequality theory, including computational methods for solving such problems, is provided by Pang and Stewart (2008). Also, differential variational inequalities involving the kind of explicit, agent-specific control variables employed herein are presented in Friesz (2010).

1.2. Discussion of contributions made in this paper

In this paper, we present a unified theory and a general framework for formulating, analyzing, and computing the simultaneous route-and-departure-time (SRDT) dynamic user equilibrium with elastic demand (E-DUE). Such an analytic framework is meant to allow qualitative analyses on E-DUE to be conducted in a rigorous manner, and to accommodate any dynamic network loading model expressible by an embedded effective delay operator.

We show, using measure-theoretic argument, that a general SRDT E-DUE can be cast as an infinite-dimensional variational inequality problem. Unlike existing VI formulations in the literature, this VI is defined on an extended Hilbert space, which facilitates the analysis regarding existence and computation. As a result, the existence of E-DUE is formally established in the most general setting; that is, it incorporates both route and departure time choices of travelers, and does so without invoking the *a priori* boundedness of path departure rates [1].

This paper also makes a significant contribution to the computation of SRDT E-DUE, by proposing three different algorithms and analyzing their convergence conditions. These are achieved through the VI formulation of the E-DUE model. Regarding algorithms and computation, our paper's intent is to: (i) document how far the available mathematics can take us in assuring convergence, and (ii) illustrate what can be done computationally when proceeding heuristically by relaxing monotonicity assumptions needed to assure convergence.

In the following sections we will discuss the existence and computation of E-DUE problems in detail, while referring to the work presented in this paper and by other scholars.

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[1] We refer the reader to Han et al. (2013c) for an illustration of the subtlety of the *a priori* bound on the path departure rates when it comes to the SRDT notion of DUEs.

1.2.1. Existence of SRDT E-DUE

As commented by Han et al. (2013c), the most obvious approach to establishing existence is to convert the problem to an equivalent variational inequality problem or a fixed-point problem and then apply a version of Brouwer's famous fixed-point existence theorem. Nearly all proofs of DUE existence employ such an existence theorem, either implicitly or explicitly. One statement of Brouwer's theorem appears as Theorem 2 of Browder (1968). Approaches based on Brouwer's theorem require the set of feasible path departure rates (path flows) to be compact and convex in a topological vector space, and typically involve the *a priori* bound on path departure rates. For instance, using the *link delay model* introduced by Friesz et al. (1993), Zhu and Marcotte (2000) show that a *route choice* (RC) dynamic user equilibrium exists under certain regularity conditions. In their modeling framework, the departure rate at each origin is given as *a priori* and assumed to be bounded from above. Thus one is assured that all path departure rates are automatically uniformly bounded. Mounce (2007) uses Schauder's theorem to prove existence for the bottleneck model. Smith and Wisten (1995) prove existence of a route-choice equilibrium when the path delay operator is continuous. In Wie et al. (2002), the existence of a link-based dynamic user equilibrium is established under the assumption that the path departure rates are *a priori* bounded.

Difficulties arise in the proof of a general existence theorem from two aspects: (i) in a continuous-time setting, the set of feasible path departure rates is often not compact; and (ii) the assumption of *a priori* boundedness of path departure rates, which is usually required by a topological argument, does not arise from any behavioral argument or theory. The existence proof provided by this paper manages to overcome these two major difficulties. Regarding item (i) above, we employ successive finite-dimensional approximations of the set consisting of feasible path departure rates, which allows Brouwer's fixed-point theorem to be applied. Regarding item (ii), we propose an in-depth analysis and computation involving the path departure rates under minor assumptions on the travelers' disutility functions.

Existence result for the elastic demand case is further complicated by the fact that the corresponding VI formulation has both infinite-dimensional and finite-dimensional terms (see Theorem 3.2 below). In order to apply Browder's theorem (Browder, 1968), one needs to work in an extended Hilbert space that is a product of an infinite-dimensional space and a finite-dimensional space, and define appropriate inner product that allows compactness and weak topology to be properly defined. It is significant that our existence result for E-DUE, stated and proven in Theorem 4.5, does not rely on the *a priori* upper bound of path flows and can be established for any dynamic network loading sub-model with reasonable and weak regularity conditions.

1.2.2. Computation of SRDT E-DUE solutions

The computation of the DUE problem and its elastic-demand extension is facilitated by their equivalent mathematical formulations, such as VI, complementarity problems, fixed-point problems, and mathematical programming problems, through existing and emerging computational algorithms associated therein. The convergence of an algorithm is highly related to the property of the *path delay operator*, which is obtained through the dynamic network loading subproblem. In particular, most convergence proofs rest on certain types of continuity and generalized monotonicity.

For example, Jang et al. (2005) develop a projection-based method to solve a *route-choice* DUE problem, which is a special case of the SRDT DUE problem; the convergence of this method requires continuity and strict monotonicity of the path delay operator. Similarly, a fixed-point method developed by Friesz et al. (2011) for SRDT DUEs relies on Lipschitz continuity and strong monotonicity. Strong monotonicity is known to not hold for general networks and DNL models (Mounce and Smith, 2007), and algorithms that rely on relaxed notions of monotonicity have been proposed in the literature. Lo and Szeto (2002a) and Szeto and Lo (2004) develop an alternating direction method and a descent method for cell-based DUE problems with fixed and elastic demand, respectively; both methods require that the delay operator is co-coercive. According to Zhao and Hu (2007), a sufficient condition for co-coerciveness includes Lipschitz continuity and monotonicity (rather than strong monotonicity). Algorithms with even more relaxed convergence requirements, such as the day-to-day route swapping algorithm (Huang and Lam, 2002; Szeto and Lo, 2006; Tian et al., 2012) and the extragradient method (Long et al., 2013), are also proposed for solving DUE problems. In particular, convergence of the route swapping algorithm requires continuity and monotonicity (Mounce and Carey, 2011); and the extragradient method requires Lipschitz continuity and pseudo monotonicity for convergence.

This paper provides three new algorithms for computing E-DUE problems with convergence proofs that extend the knowledge of mathematical tools available for ensuring convergence. The first algorithm is a *projection* algorithm, which is based on the new VI formulation proposed in this paper. It relies on a minimum-norm projection operator

onto a subset of an extended Hilbert space. Notably, this new projection operator may be explicitly represented by invoking the mathematical paradigm of differential variational inequality (DVI) and linear-quadratic optimal control, which is original in the literature. Regarding convergence of the projection method, instead of relying on the well-known strong monotonicity, we propose a new condition called *mixed strongly-weakly monotonicity* (MSWM). The MSWM stipulates strong monotonicity along a subset of paths, and weak monotonicity along the rest of the paths; it is one type of component-wise monotonicity, and has been extensively studied in the literature of dynamical systems and optimization. This paper provides the first convergence result for the MSWM property. The second algorithm is a *self-adaptive projection method*, which is originally proposed by Han and Lo (2002) for solving generic variational inequalities. This method converges given that the delay operator is continuous and pseudo monotone, which is a weaker notion than (strong) monotonicity. Similar to the first computational method, this self-adaptive projection method utilizes the aforementioned explicit instantiation of the projection onto the extended Hilbert space. Finally, the third algorithm is called *proximal point method* (Konnov, 2003). It replaces the original VI problem, which may not satisfy the require monotonicity, with a sequence of regularized problems, each of which may be solved with standard algorithms due to improved regularity (monotonicity). In this paper we provide a new convergence proof for the proximal point method for a class of *quasi monotone* delay operators, and further relax the monotonicity condition compared to the existing literature. The three new algorithms are tested on two networks, which numerically demonstrates their convergence, solution quality, and computational complexity.

The key contributions made in this paper include:

- the expression of the simultaneous route-and-departure-time (SRDT) E-DUE problem as an infinite-dimensional variational inequality in an extended Hilbert space;
- a general existence result for the E-DUE when both departure time and route choices are within the purview of travelers, which does not invoke the *a priori* boundedness on the path departure rates;
- three new algorithms for computing E-DUE problems with rigorous convergence proofs that rely on increasingly relaxed monotonicity conditions on the delay operator.

The rest of this paper is organized as follows. Section 2 offers an introduction to some key concepts, notations, and mathematical background needed to present and analyze the E-DUE problem. Section 3 presents the variational inequality formulation of the E-DUE problem in an extended Hilbert space. We provide an existence theory for the E-DUE problem in Section 4. Section 5 presents the three computational methods and their convergence results. Section 6 presents a numerical study of the proposed computational methods. Finally, some concluding remarks are offered in Section 7.

2. Notation and essential background

Throughout his paper, the time interval of analysis is a single commuting period expressed as $[t_0, t_f] \subset \mathbb{R}$ where $t_f > t_0$, and both t_0 and t_f are fixed. We let $\mathcal{P}$ be the set of all paths utilized by travelers. For each $p \in \mathcal{P}$, we define the path departure rate (in vehicle per unit time), which is a function of departure time $t \in [t_0, t_f]$, $h_p(\cdot) : [t_0, t_f] \to \mathbb{R}_+$, where $\mathbb{R}_+$ denotes the set of non-negative real numbers. Each path departure rate $h_p(t)$ is interpreted as a path flow measured at the entrance of the first link of the relevant path. We next define $h(\cdot) = \{h_p(\cdot) : p \in \mathcal{P}\}$ to be a vector of departure rates, which is viewed as a vector-valued function of t, the departure time [2].

We let $L^2[t_0, t_f]$ be the space of square-integrable functions define on the interval $[t_0, t_f]$, and $L^2_+[t_0, t_f]$ be its subset consisting of non-negative functions. It is stipulated that each path departure rate is square integrable: $h_p(\cdot) \in L^2_+[t_0, t_f]$ and $h(\cdot) \in (L^2_+[t_0, t_f])^{|\mathcal{P}|}$, where $(L^2[t_0, t_f])^{|\mathcal{P}|}$ is the $|\mathcal{P}|$-fold product of the Hilbert space $L^2[t_0, t_f]$, and $(L^2_+[t_0, t_f])^{|\mathcal{P}|}$ is its subset consisting of non-negative path departure vectors. The inner product on the Hilbert space $(L^2[t_0, t_f])^{|\mathcal{P}|}$ is defined as

$$\langle h^1, h^2 \rangle = \int_{t_0}^{t_f} \left(h^1(t)\right)^T h^2(t)\,dt = \sum_{p \in \mathcal{P}} \int_{t_0}^{t_f} h^1_p(t) \cdot h^2_p(t)\,dt \tag{2.1}$$

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[2] For notation convenience and without causing any confusion, we will sometimes use h instead of $h(\cdot)$ to denote the path departure rate vectors.

where the superscript T denotes the transpose of vectors. Moreover, the norm

$$\|u\|_{L^2} = \langle u, u \rangle^{1/2} \tag{2.2}$$

is induced by the inner product (2.1).

Here, as in all DUE modeling, the single most crucial ingredient is the path delay operator, which maps a given vector of departure rates h to a vector of path travel times. More precisely, we let

$$D_p(t, h) \qquad \forall t \in [t_0, t_f], \quad \forall p \in \mathcal{P}$$

be the path travel time of a driver departing at time t and following path p, given the departure rates associated with all the paths in the network, which is expressed by h in the expression above. We then define the path delay operator $D(\cdot)$ by letting $D(h) = \{D_p(\cdot, h) : p \in \mathcal{P}\}$, which is a vector of time-dependent path travel times $D_p(t, h)$. $D(\cdot)$ is an operator defined on $(L_+^2[t_0, t_f])^{|\mathcal{P}|}$, and maps a vector valued function $h(\cdot)$ to another vector-valued function $\{D_p(\cdot, h) : p \in \mathcal{P}\}$. In summary,

$$D : (L_+^2[t_0, t_f])^{|\mathcal{P}|} \to (L_+^2[t_0, t_f])^{|\mathcal{P}|}, \qquad h(\cdot) = \{h_p(\cdot), p \in \mathcal{P}\} \mapsto D(h) = \{D_p(\cdot, h), p \in \mathcal{P}\} \tag{2.3}$$

The *effective* path delay operator Ψ is similarly defined, except that the effective path delay contains, in addition to path travel time, also arrival penalties. As such, the effective path delay is a more general notion of "travel cost" than path delay. The effective delay operator is defined as follows.

$$\Psi : (L_+^2[t_0, t_f])^{|\mathcal{P}|} \to (L_+^2[t_0, t_f])^{|\mathcal{P}|}, \qquad h(\cdot) = \{h_p(\cdot), p \in \mathcal{P}\} \mapsto \Psi(h) = \{\Psi_p(\cdot, h), p \in \mathcal{P}\} \tag{2.4}$$

where

$$\Psi_p(t, h) = D_p(t, h) + f(t + D_p(t, h) - T_A) \qquad \forall t \in [t_0, t_f], \quad \forall p \in \mathcal{P} \tag{2.5}$$

where T_A is the desired arrival time and $T_A < t_f$. The term $f(t + D_p(t, h) - T_A)$ assesses a nonnegative penalty whenever

$$t + D_p(t, h) \neq T_A \tag{2.6}$$

since $t + D_p(t, h)$ is the clock time at which departing traffic arrives at the destination of path $p \in \mathcal{P}$. Note that, for convenience here, T_A is assumed to be independent of path or origin-destination pair. However, that assumption is easy to relax, and the consequent generalization of our model is a trivial extension.

We interpret $\Psi_p(t, h)$ as the perceived travel cost of drivers departing at time t following path p given the vector of path departure rates h. We stipulate that for all $p \in \mathcal{P}$, the function $\Psi_p(\cdot, h)$ is measurable, almost everywhere positive, and square integrable. The notation

$$\Psi(h) \doteq \{\Psi_p(\cdot, h) : p \in \mathcal{P}\} \in (L_+^2[t_0, t_f])^{|\mathcal{P}|}$$

is used to express the complete vector of effective path delays.

In order to develop an appropriate notion of minimum travel cost in the measure-theoretic context, we require the concept of *essential infimum*. In particular, for any measurable function $g : [t_0, t_f] \to \mathbb{R}$, the essential infimum of $g(\cdot)$ on $[t_0, t_f]$ is given by

$$\mathrm{essinf}\left\{g(s) : s \in [t_0, t_f]\right\} \doteq \sup\left\{x \in \mathbb{R} : meas\{s \in [t_0, t_f] : g(s) < x\} = 0\right\} \tag{2.7}$$

where *meas* represents the (Lebesgue) measure. Note that for each $x > \mathrm{essinf}\{g(s) : s \in [t_0, t_f]\}$ it must be true by definition that

$$meas\{s \in [t_0, t_f] : f(s) < x\} > 0$$

Let us define the essential infimum of the effective path delay, which depend on the path departure rate vector h,

$$v_p(h) = \mathrm{essinf}\left\{\Psi_p(t, h) : t \in [t_0, t_f]\right\} > 0 \qquad \forall p \in \mathcal{P} \tag{2.8}$$

The minimum travel cost within a given O-D pair (i, j) is thus defined as

$$v_{ij}(h) = \min\left\{v_p(h) : p \in \mathcal{P}_{ij}\right\} > 0 \qquad \forall (i, j) \in \mathcal{W} \tag{2.9}$$

By definition, $v_{ij}(h)$ is the minimum travel cost within O-D pair (i, j) for all route choices and departure time choices.

The (effective) path delay operator is a key component of analytical dynamic user equilibrium (DUE) models; it is usually not available in closed form and has to be numerically evaluated from dynamic network loading (DNL), which is a sub-problem of a complete DUE model. The DNL sub-problem aims at describing and predicting the spatial-temporal evolution of traffic flows on a network that is consistent with established route and departure time choices of travelers, by introducing appropriate dynamics to flow propagation, flow conservation, and travel delays on a network level. Any DNL must be consistent with the established path departure rates and link flow model; and it is usually performed under the *first-in-first-out* (FIFO) rule. A few link flow models commonly employed for the DNL procedure include the link delay model (Friesz et al., 1993), the Vickrey model (Vickrey, 1969; Han et al., 2013a,b), the cell transmission model (Daganzo, 1994, 1995), the link transmission model (Yperman et al., 2005; Han et al., 2015), and the Lighthill-Whitham-Richards model (Lighthill and Whitham, 1955; Richards, 1956).

Work regarding the dynamic network loading models dates back to the 1990s with a significant number of publications (Friesz et al., 2011, 2013a,b; Han, 2013; Lo and Szeto, 2002b; Nie and Zhang, 2010; Szeto, 2003; Szeto et al., 2011; Szeto and Lo, 2004, 2006) and Ukkusuri et al. (2012). Notably, despite the absence of closed-form representations of the delay operators, it has been reported that certain dynamic network loading models can be expressed as a system of *differential algebraic equations* (DAEs) or *partial differential algebraic equations* (PDAEs). Those results include: the DAE system formulation of the DNL procedure for the link delay model (Friesz et al., 2011); the DAE system formulation of the DNL procedure for the Vickrey model (Han, 2013); the DAE system formulation of the DNL procedure for the LWR-Lax model (Friesz et al., 2013b); and the PDAE/DAE system formulation of the general LWR model (Han et al., 2014).

3. Definition of SRDT E-DUE and the variational inequality formulation

3.1. Dynamic user equilibrium with elastic demand

We introduce the trip matrix $(Q_{ij} : (i, j) \in \mathcal{W})$, where each $Q_{ij} \in \mathbb{R}_+$ is the (elastic) travel demand between the origin-destination (O-D) pair $(i, j) \in \mathcal{W}$, and $\mathcal{W}$ is the set of origin-destination pairs. Note that unlike route-choice DUE models, Q_{ij} here represents total traffic volume, instead of flow that changes over time. The flow conservation constraints read

$$\sum_{p \in \mathcal{P}_{ij}} \int_{t_0}^{t_f} h_p(t)\, dt = Q_{ij} \qquad \forall (i, j) \in \mathcal{W} \tag{3.10}$$

where (3.10) consists of Lebesgue integrals, and $\mathcal{P}_{ij} \subset \mathcal{P}$ is the set of paths connecting O-D pair $(i, j) \in \mathcal{W}$. In the elastic demand case, the travel demand between O-D pair (i, j) is assumed to be expressed as the following invertible function

$$Q_{ij} = G_{ij}[v] \qquad \forall (i, j) \in \mathcal{W}$$

where $v = \{v_{ij}(h) : (i, j) \in \mathcal{W}\}$ is the vector of O-D specific minimum travel costs $v_{ij}(h)$ defined in (2.8)-(2.9). We note that Q_{ij} is the <u>unknown</u> travel demand between (i, j) that must ultimately be achieved by the end of the time horizon $t = t_f$. We will find it convenient to form the complete vector of travel demands by concatenating the O-D specific travel demands to obtain

$$Q = (Q_{ij} : (i, j) \in \mathcal{W}) = (G_{ij}[v] : (i, j) \in \mathcal{W}) \in \mathbb{R}_+^{|\mathcal{W}|}$$

which defines a mapping from v to Q that, when invertible, gives rise to the *inverse demand function*:

$$\Theta : \mathbb{R}_+^{|\mathcal{W}|} \rightarrow \mathbb{R}_{++}^{|\mathcal{W}|}, \qquad Q \mapsto v$$

where

$$v = (v_{ij} : (i, j) \in \mathcal{W}) \quad \text{and} \quad v_{ij} = \Theta_{ij}[Q] \tag{3.11}$$

Notice that the inverse demand function defined in (3.11) is non-separable in the sense that each minimum O-D travel cost v_{ij} is jointly determined by the entire vector of elastic demands $Q = (Q_{ij} : (i, j) \in \mathcal{W})$.

Accordingly, we employ the following feasible set of departure flows when the travel demand between each origin-destination pair is endogenous.

$$\widetilde{\Lambda} = \left\{ (h, Q) : h \geq 0, \quad \sum_{p \in \mathcal{P}_{ij}} \int_{t_0}^{t_f} h_p(t)\, dt = Q_{ij} \quad \forall (i, j) \in \mathcal{W} \right\} \subset \left(L^2[t_0, t_f] \right)^{|\mathcal{P}|} \times \mathbb{R}^{|\mathcal{W}|} \tag{3.12}$$

where $\left(L^2[t_0, t_f] \right)^{|\mathcal{P}|} \times \mathbb{R}^{|\mathcal{W}|}$ is the direct product of the $|\mathcal{P}|$-fold product of Hilbert spaces, and the $|\mathcal{W}|$-dimensional Euclidean space consisting of vectors of elastic travel demands.

With preceding preparation, we are in a place where the simultaneous route-and-departure-time dynamic user equilibrium with elastic demand can be rigorously defined, as follows.

Definition 3.1. (Dynamic user equilibrium with elastic demand) *A pair $(h^*, Q^*) \in \widetilde{\Lambda}$ is said to be a dynamic user equilibrium with elastic demand if for all $(i, j) \in \mathcal{W}$,*

$$h_p^*(t) > 0, \quad p \in \mathcal{P}_{ij} \implies \Psi_p(t, h^*) = \Theta_{ij}[Q^*] \qquad \text{for almost every } t \in [t_0, t_f] \tag{3.13}$$

$$\Psi_p(t, h^*) \geq \Theta_{ij}[Q^*] \qquad \text{for almost every } t \in [t_0, t_f], \quad \forall p \in \mathcal{P}_{ij} \tag{3.14}$$

3.2. The variational inequality formulation of the SRDT E-DUE problem

Experience with differential games suggests that the DUE problem with elastic demand can be expressed as a variational inequality, as shown in the theorem below.

Theorem 3.2. (E-DUE equivalent to a variational inequality) *Assume $\Psi_p(\cdot, h) : [t_0, t_f] \to \mathbb{R}_{++}$ is positive and measurable for all $p \in \mathcal{P}$ and all h such that $(h, Q) \in \widetilde{\Lambda}$. Also assume that the inverse demand function $\Theta_{ij}[\cdot]$ exists for all $(i, j) \in \mathcal{W}$. Then a pair $(h^*, Q^*) \in \widetilde{\Lambda}$ is a DUE with elastic demand (Definition 3.1) if and only if it solves the following variational inequality:*

$$\left. \begin{array}{c} \text{find } (h^*, Q^*) \in \widetilde{\Lambda} \text{ such that} \\ \displaystyle\sum_{p \in \mathcal{P}} \int_{t_0}^{t_f} \Psi_p(t, h^*)(h_p(t) - h_p^*(t))\, dt - \sum_{(i,j) \in \mathcal{W}} \Theta_{ij}[Q^*](Q_{ij} - Q_{ij}^*) \geq 0 \\ \forall (h, Q) \in \widetilde{\Lambda} \end{array} \right\} VI(\Psi, \Theta, [t_0, t_f]) \tag{3.15}$$

Proof. The proof is postponed until Appendix A. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

In the remainder of this section, we rewrite (3.15) in a more compact and generic VI form by introducing an extended Hilbert space. We consider the product space $E \doteq (L^2[t_0, t_f])^{|\mathcal{P}|} \times \mathbb{R}^{|\mathcal{W}|}$, which is a space with the induced inner product defined as follows

$$\begin{aligned} \langle X, Y \rangle_E &\doteq \sum_{i=1}^{|\mathcal{P}|} \int_{t_0}^{t_f} \xi_i(t) \cdot \eta_i(t)\, dt + \sum_{j=1}^{|\mathcal{W}|} u_j v_j \\ X &= (\xi_1(\cdot), \ldots, \xi_{|\mathcal{P}|}(\cdot), u_1, \ldots, u_{|\mathcal{W}|}) \in E \\ Y &= (\eta_1(\cdot), \ldots, \eta_{|\mathcal{P}|}(\cdot), v_1, \ldots, v_{|\mathcal{W}|}) \in E \end{aligned} \tag{3.16}$$

The inner product $\langle \cdot, \cdot \rangle_E$ immediately leads to the following norm on the space E

$$\|X\|_E = \langle X, X \rangle_E^{1/2} \tag{3.17}$$

making E a metric space. In the following proposition, we show that the inner product $\langle \cdot, \cdot \rangle_E$ and the norm $\| \cdot \|_E$ are well defined, and the resulting extended space E is indeed a Hilbert space.

Proposition 3.3. *The inner product $\langle \cdot, \cdot \rangle_E$ and norm $\| \cdot \|_E$ are well defined. In addition, the space E, equipped with $\langle \cdot, \cdot \rangle_E$ and the induced metric, is a Hilbert space.*

Proof. A well-defined inner product must satisfy, for all X, Y, $Z \in E$, the following properties:

1. symmetry, i.e. $\langle X, Y \rangle_E = \langle Y, X \rangle_E$;
2. linearity, i.e. $\langle aX, Y \rangle_E = a \langle X, Y \rangle_E$ for all $a \in \mathbb{R}$, and $\langle X + Y, Z \rangle_E = \langle X, Z \rangle + \langle Y, Z \rangle_E$;
3. positive-definiteness, i.e. $\langle X, X \rangle_E \geq 0$ and $\langle X, X \rangle_E = 0 \Rightarrow X = 0$.

It is straightforward to verify that the inner product defined in (3.16) satisfies all these three conditions, and thus is well defined. Consequently, the induced norm $\| \cdot \|_E$ is also well defined. Finally, since both $(L^2[t_0, t_f])^{|\mathcal{P}|}$ and $\mathbb{R}^{|\mathcal{W}|}$ are complete metric spaces, their product space E is also a complete metric space, and hence a Hilbert space. $\qquad \square$

The set $\widetilde{\Lambda}$ of admissible pairs (h, Q) can now be embedded in the extended space E. In view of the inverse demand function $\Theta = (\Theta_{ij}[\cdot] : (i, j) \in \mathcal{W})$, we introduce the following notation.

$$\Theta^- \doteq (-\Theta_{ij}[\cdot] : (i, j) \in \mathcal{W}) : \quad \mathbb{R}_+^{|\mathcal{W}|} \longrightarrow \mathbb{R}_{--}^{|\mathcal{W}|}$$

Consequently, we define the mapping

$$\mathcal{F} : \widetilde{\Lambda} \longrightarrow E, \qquad (h, Q) \mapsto (\Psi(h), \Theta^-[Q]) \tag{3.18}$$

where $(h, Q) \in \widetilde{\Lambda}$, $\Psi(h) \in (L_+^2[t_0, t_f])^{|\mathcal{W}|}$, $\Theta^-[Q] \in \mathbb{R}_{--}^{|\mathcal{W}|}$. Such a mapping is clearly well defined. With the preceding discussion, the VI formulation (3.15) is readily rewritten as the following variational inequality in the extended Hilbert space.

$$\left. \begin{array}{c} \text{find } X^* \in \widetilde{\Lambda} \text{ such that} \\[2mm] \langle \mathcal{F}(X^*), X - X^* \rangle_E \geq 0 \\[2mm] \forall X \in \widetilde{\Lambda} \end{array} \right\} VI(\mathcal{F}, [t_0, t_f]) \tag{3.19}$$

where $X = (h, Q)$ and $X^* = (h^*, Q^*)$. Problem (3.19) is expressed in the most generic form of a variational inequality, which allows analyses regarding solution existence and computation to be carried out in a framework well supported by the VI theory in mathematical programming.

4. Existence of SRDT E-DUE

In this section we establish the existence result for (3.19), which is an equivalent formulation of the E-DUE problem. Our proposed approach is meant to incorporate the most general dynamic network loading sub-model with minimum regularity requirements, and to yield existence of E-DUE without invoking the *a priori* upper bound on the path departure rates.

The analysis regarding solution existence for the variational inequality (3.19) is based on the following theorem that extends Brouwer's fixed point existence theorem to topological vector spaces.

Theorem 4.1. *(Browder, 1968) Let K be a compact convex subset of the locally convex topological vector space V, T a continuous (single-valued) mapping of K into V^*, where V^* is the dual space of V. Then there exists u_0 in K such that*

$$\langle T(u_0), u - u_0 \rangle \geq 0 \qquad \forall u \in K$$

Proof. See Browder (1968). $\qquad \square$

In preparation for our existence proof, we recap several key results from functional analysis that facilitate our presentation. In particular, we note the following facts without proofs. The reader is referred to Royden and Fitzpatrick (1988) for more detailed discussion on these subjects.

Proposition 4.2. *The space $L^2[t_0, t_f]$ is a locally convex topological vector space. In addition, the $|\mathcal{P}|$-fold product of this space, denoted by $(L^2[t_0, t_f])^{|\mathcal{P}|}$, is also a locally convex topological vector space. Finally, the product space $(L^2[t_0, t_f])^{|\mathcal{P}|} \times \mathbb{R}^{|\mathcal{W}|}$ is again a locally convex topological vector space.*

Proposition 4.3. *The dual space of $(L^2[t_0, t_f])^{|\mathcal{P}|}$ has a natural isomorphism with $(L^2[t_0, t_f])^{|\mathcal{P}|}$. The dual space of the Euclidean space $\mathbb{R}^{|W|}$ consisting of columns of $|W|$ real numbers is interpreted as the space consisting of rows of $|W|$ real numbers. As a consequence, the dual space of $(L^2[t_0, t_f])^{|\mathcal{P}|} \times \mathbb{R}^{|W|}$ is again $(L^2[t_0, t_f])^{|\mathcal{P}|} \times \mathbb{R}^{|W|}$.*

Proposition 4.4. *In a metric space (and topological vector space), the notion of compactness is equivalent to the notion of sequential compactness; that is, every infinite sequence has a convergent subsequence.*

Theorem 4.1 is immediately applicable for showing that (3.19) has a solution if (1) $\mathcal{F}$ is continuous; and (2) $\widetilde{\Lambda} \subset E$ is compact. The continuity of $\mathcal{F}$ amounts to the continuity of the path delay operator Ψ, which has been shown in the literature for different link flow dynamics and network extensions. These results include the continuity result with the *link delay model* (Friesz et al., 1993) established in Han and Friesz (2012); the continuity result with the *Vickrey model* (Vickrey, 1969) established in Han et al. (2013c); and the continuity result with the LWR-Lax model (Friesz et al., 2013b) established in Bressan and Han (2013).

Unfortunately, the second condition involving compactness does not generally hold for the infinite-dimensional (continuous-time) problem we study here. To overcome such an obstacle, we proceed in a similar way as in Han et al. (2013c) by considering finite-dimensional approximations of the underlying infinite-dimensional Hilbert space. Another major hurdle that stymied many researchers is the *a priori* upper bound on the path departure rates. Such a bound is important for a topological argument that we will rely on in the proof, but does not arise from any physical or behavioral perspective of traffic modeling. In fact, as observed by Bressan and Han (2011), the equilibrium path flows could very well become unbounded or even contain Dirac-delta [3], if no additional assumptions are made regarding exogenous parameters of the differential Nash game, such as travelers' disutility functions.

The following assumptions are key to our existence result. The first assumption, **(A1)**, poses reasonable hypothesis on drivers' perceived arrival costs. In particular, (4.20) holds on the basis that the unit cost of early arrival is always less than or equal to the unit cost of elapsed travel time (Small, 1982). The reader is also referred to Han et al. (2013c) for the motivation and generality of such an assumption. The second assumption, **(A2)**, is concerned with the DNL model and can be easily verified by models such as the Vickrey model (Vickrey, 1969; Han et al., 2013a,b), the LWR-Lax model (Friesz et al., 2013b), the cell transmission model (Daganzo, 1994, 1995), the link transmission model (Yperman et al., 2005), and the Lighthill-Whitham-Richards model (Lighthill and Whitham, 1955; Richards, 1956). Assumption **(A3)** is one instantiation of the continuity of the effective delay operator Ψ, and has been shown to hold for several network loading models. These include the Vickrey model (Vickrey, 1969; Han et al., 2013a,b) and the LWR-Lax model (Friesz et al., 2013b), for which **(A3)** is established by Bressan and Han (2013); and the link delay model (Friesz et al., 1993), for which **(A3)** is shown in Han and Friesz (2012). We refer the reader to these references for detailed proofs of **(A3)**.

(A1). The function $f(\cdot)$ appearing in (2.5) is continuous on $[t_0, t_f]$ and satisfies

$$f(t_2) - f(t_1) \geq \Delta(t_2 - t_1) \qquad \forall \, t_0 \leq t_1 < t_2 \leq t_f \tag{4.20}$$

for some $\Delta > -1$

(A2). The *first-in-first-out* (FIFO) rule is obeyed on the path level. In addition, each link $a \in \mathcal{A}$ in the network has a finite exit flow capacity $M_a < \infty$.

(A3). For any sequence of departure rate vectors $\{h^n(\cdot)\}_{n \geq 1}$ that are uniformly bounded point wise by a positive constant and converge weakly to $h^* \in (L^2[t_0, t_f])^{|\mathcal{P}|}$, the corresponding effective path delays $\Psi_p(t, h^n)$ converge to $\Psi_p(t, h^*)$ uniformly for all $p \in \mathcal{P}$ and $t \in [t_0, t_f]$.

We are now ready to state and prove the main result of this section.

Theorem 4.5. (Existence of E-DUE) *Let assumptions* **(A1)**-**(A3)** *hold. If, in addition, the inverse demand function $\Theta : \mathbb{R}_+^{|W|} \to \mathbb{R}_{++}^{|W|}$ is continuous, then the E-DUE problem has a solution.*

Proof. The proof is postponed until Appendix B. $\qquad\qquad\square$

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[3]In this case, no solution in the sense of Definition 3.1 exists.

5. Computation of the E-DUE problem

Computation of the elastic demand DUE is most facilitated by the variational inequality (VI) formulation proposed in this paper. Some methods commonly seen for solving VIs or equivalent mathematical forms include the projection method or the fixed-point method, among others; see Section 1.2.2 for a review of more methods. Another method less known to the traffic research community is the *proximal point method* (PPM) (Konnov, 2003; Allevi et al., 2006), which will be detailed in this section.

We present three computational methods for the E-DUE model, while analyzing their convergence conditions based on generalized monotonicity of the delay operator. These methods are: the projection (fixed-point) method, the self-adaptive projection method, and the proximal point method.

5.1. The projection method

The projection method for solving the VI (3.19) requires the following iterative process

$$X^{k+1} = P_{\widetilde{\Lambda}}\left[X^k - \alpha\mathcal{F}(X^k)\right] \tag{5.21}$$

where $P_{\widetilde{\Lambda}}[\cdot]$ is the minimum-norm projection onto the convex set $\widetilde{\Lambda}$. Here, $X^k = (h^k, Q^k)$ belongs to the extended Hilbert space E. In order to explicitly describe this projection operator in an infinite-dimensional and extended Hilbert space, we invoke the *differential variational inequality* (DVI) formalism (Friesz and Meimand, 2014). This is done by observing that the elastic demand satisfaction constraint (3.12) can be easily rewritten as a two-point boundary value problem, leading to the following equivalent definition of the feasible set

$$\widetilde{\Lambda}_1 \doteq \left\{(h, Q) : \frac{dy_{ij}(t)}{dt} = \sum_{p\in\mathcal{P}_{ij}} h_p(t), \ y_{ij}(t_0) = 0, \ y_{ij}(t_f) = Q_{ij} \ \forall(i, j) \in \mathcal{W}\right\} \subset (L^2[t_0, t_f])^{|\mathcal{P}|} \times \mathbb{R}^{|\mathcal{W}|} \tag{5.22}$$

The next theorem establishes an explicit formula for the right hand side of (5.21), based on the DVI theory.

Theorem 5.1. *Given any* $X^k = (h^k, Q^k) \in \widetilde{\Lambda}$, *then* X^{k+1} *appearing on the left hand side of* (5.21) *can be expressed as* $X^{k+1} = (h^{k+1}, Q^{k+1})$ *where*

$$h_p^{k+1}(t) = \left[h_p^k(t) - \alpha\Psi_p(t, h^k) + Q_{ij}^k + \alpha\Theta_{ij}[Q^k] - Q_{ij}^{k+1}\right]_+ \qquad \forall p \in \mathcal{P} \tag{5.23}$$

$$Q_{ij}^{k+1} = \sum_{p\in\mathcal{P}_{ij}} \int_{t_0}^{t_f} \left[h_p^k(t) - \alpha\Psi_p(t, h^k) + Q_{ij}^k + \alpha\Theta_{ij}[Q^k] - Q_{ij}^{k+1}\right]_+ dt \qquad \forall(i, j) \in \mathcal{W} \tag{5.24}$$

In addition, such h^{k+1} *and* Q^{k+1} *are unique.*

Proof. The proof will be provided in Appendix C. $\qquad\square$

The following pseudo code summarizes the projection algorithm.

Projection algorithm

Step 0 Identify an initial feasible point $X^0 = (h^0, Q^0) \in \widetilde{\Lambda}$. Set the iteration counter $k = 0$.

Step 1 Solve the dynamic network loading problem with path departure rate vector given by h^k, and obtain the effective path delays $\Psi_p(t, h^k)$, $\forall t \in [t_0, t_f]$, $p \in \mathcal{P}$. Find Q^{k+1} and then h^{k+1} according to (5.24) and (5.23) respectively.

Step 2 Terminate the algorithm with output $X^* \approx (h^k, Q^k)$ if the relative gap

$$\frac{\left\|(h^{k+1}, Q^{k+1}) - (h^k, Q^k)\right\|_E}{\left\|(h^k, Q^k)\right\|_E} \leq \epsilon$$

where $\epsilon \in \mathbb{R}_{++}$ is a prescribed termination threshold. Otherwise, set $k = k + 1$ and repeat Step 1 and Step 2.

Convergence of the projection method typically involves continuity and monotonicity of the principal operator which, in our case, is $\mathcal{F}(\cdot)$. This further requires the Lipschitz continuity and monotonicity of the path delay operator Ψ. The continuity of the delay operator has been established in a number of references as we noted in Section 4. The monotonicity of the delay operator, on the other hand, may not hold for general networks and traffic flow models; see Mounce and Smith (2007) for a counter example. There are, however, a few studies that show the monotonicity of Ψ under specific assumptions; the reader is referred to Mounce (2006) and Perakis and Roels (2006) for further details. Since the convergence of the projection method for continuous and monotone delay operator is well known, and a generalization of the proof to the elastic demand case is straightforward, we will not elaborate these convergence conditions and repeat the proof in this paper; the reader is referred to Facchinei and Pang (2003), Mounce (2006), Mounce and Carey (in press), and Friesz et al. (2013b) for more details.

Instead, this paper proposes a new set of convergence conditions based on the property of *mixed strongly-weakly monotone* (MSWM) of the path delay operator. The MSWM assumes that a subset of the components (paths) of the operator Ψ have the strongly monotone property, and the rest of the components need only be weakly monotone. A weakly monotone operator $M(x)$ is such that

$$\langle M(x_1) - M(x_2),\, x_1 - x_2 \rangle \;\geq\; -K\|x_1 - x_2\|^2 \qquad \forall x_1,\, x_2 \qquad \text{for some } K > 0 \tag{5.25}$$

The following lemma asserts that all Lipschitz continuous operators are weakly monotone, thereby showing the generality of the weak monotonicity condition.

Lemma 5.2. *Let $U(\cdot)$ be a Lipschitz continuous map defined on a subset Ω of a topological vector space V, that is, there exists $L > 0$ such that*

$$\|U(x_1) - U(x_2)\| \;\leq\; L\|x_1 - x_2\| \qquad \forall x_1,\, x_2 \in \Omega$$

Then $U(\cdot)$ is weakly monotone on Ω.

Proof. According to the Cauchy-Schwarz inequality, we have

$$\langle U(x_1) - U(x_2),\, x_1 - x_2 \rangle \;\geq\; -\|U(x_1) - U(x_2)\| \cdot \|x_1 - x_2\| \;\geq\; -L\|x_1 - x_2\|^2$$

$\square$

Remark 5.3. *The Lipschitz continuity employed by Lemma 5.2 may be further relaxed to continuity if the set Ω is bounded.*

Theorem 5.4. (Convergence of the projection algorithm with the MSWM property) *Assume that the effective delay operator Ψ and the inverse demand function Θ are Lipschitz continuous with constants L_1 and L_2 respectively. In addition, assume the operator Ψ satisfies the MSWM conditions; i.e., there exists a non-empty set $\mathcal{P}^{sm} \subset \mathcal{P}$ ('sm' stands for strongly monotone) such that the operator is strongly monotone on these subset of paths:*

$$\sum_{p\in\mathcal{P}^{sm}} \int_{t_0}^{t_f} \left(\Psi_p(t, h^{k+1}) - \Psi_p(t, h^k) \right) \cdot \left(h_p^{k+1}(t) - h_p^k(t) \right) dt \;\geq\; K^{sm} \sum_{p\in\mathcal{P}^{sm}} \left\| h_p^{k+1} - h_p^k \right\|^2 \quad \forall k \geq 0 \tag{5.26}$$

for some $K^{sm} > 0$. For the rest of the paths, the weak monotonicity holds:

$$\sum_{q\in\mathcal{P}\backslash\mathcal{P}^{sm}} \int_{t_0}^{t_f} \left(\Psi_q(t, h^{k+1}) - \Psi_q(t, h^k) \right) \cdot \left(h_q^{k+1}(t) - h_q^k(t) \right) dt \;\geq\; -K^{wm} \sum_{q\in\mathcal{P}\backslash\mathcal{P}^{sm}} \left\| h_q^{k+1} - h_q^k \right\|^2 \quad \forall k \geq 0 \tag{5.27}$$

for some $K^{wm} > 0$ ('wm' stands for weakly monotone). In addition, assume there exists a constant $M > 0$ such that

$$M \sum_{p\in\mathcal{P}^{sm}} \left\| h_p^{k+1} - h_p^k \right\|^2 \;\geq\; \sum_{q\in\mathcal{P}\backslash\mathcal{P}^{sm}} \left\| h_q^{k+1} - h_q^k \right\|^2 \tag{5.28}$$

Finally, assume that the function $\Theta^- \doteq -\Theta$ is strongly monotone with constant $K_2 > 0$. Then the sequence $\{h^k\}$ generated by the projection algorithm converges to the solution of the VI (3.19) provided that

$$K^{sm} - K^{wm} M \;>\; 0$$

Proof. The proof is postponed until Appendix D. □

Remark 5.5. *According to Theorem 5.4, a necessary condition for the convergence of the projection algorithm is $K^{sm} - K^{wm}M > 0$. In other words, the degree of strong monotonicity associated with the subset $\mathcal{P}^{sm}$ must dominate the degree of weak monotonicity associated with the rest of the components. It is quite desirable to identify a set of such paths in a given network based on network topology, path characteristics etc., to meet the sufficient condition for the projection algorithm. Moreover, while the MSWM property may not hold universally for all points in the feasible set $\widetilde{\Lambda}$, it is more likely to be satisfied at a local level, especially when the trajectory created by the projection algorithm enters certain region.*

Furthermore, the strong monotonicity condition on $\Theta^- = -\Theta$ is satisfied by many inverse demand functions considered in the literature, which are mostly monotonically decreasing (see Table 1 for an example).

5.2. The self-adaptive projection method

The second computational method is a self-adaptive projection method proposed originally by Han and Lo (2002) for generic variational inequalities. Such a method is applied here for computing E-DUEs based on the variational inequality formulation (3.19). As we shall see, this method relies on pseudo monotonicity of the delay operator in order to converge.

We begin with some basic notations required to articulate the self-adaptive projection method. As before, $P_{\widetilde{\Lambda}}[\cdot]$ denotes the projection onto the set $\widetilde{\Lambda}$. We define the *residual*:

$$r(X; \beta) \doteq X - P_{\widetilde{\Lambda}}[X - \beta\mathcal{F}(X)] \qquad X \in \widetilde{\Lambda}, \quad \beta > 0 \tag{5.29}$$

where the projection is explicitly given by Theorem 5.1. Notice that the residual is zero if and only if X is a solution of the VI. Given $\alpha, \beta > 0$, let

$$d(X; \alpha, \beta) \doteq \alpha r(X; \beta) + \beta\mathcal{F}(X - \alpha r(X; \beta)) \tag{5.30}$$

$$g(X; \alpha, \beta) \doteq \alpha\left[r(X; \beta) - \beta\big(\mathcal{F}(X) - \mathcal{F}(X - \alpha r(X; \beta))\big)\right] \tag{5.31}$$

$$\rho(X; \alpha, \beta) \doteq \frac{\langle r(X; \beta), \, g(X; \alpha, \beta)\rangle_E}{\left\|d(X; \alpha, \beta)\right\|_E^2} \tag{5.32}$$

Self-adaptive projection algorithm

Step 0 Choose fixed parameters $\mu \in (0, 1)$, $\gamma \in (0, 2)$, $\theta > 1$, and $L \in (0, 1)$. Let $\epsilon > 0$ be the termination threshold. Identify an initial feasible point $X^0 = (h^0, Q^0) \in \widetilde{\Lambda}$ and set iteration counter $k = 0$. Let $\alpha_k = 1$.

Step 1 Set $\beta_k = \min\{1, \theta\alpha_k\}$. Compute the residual $r(X^k; \beta_k)$ according to (5.29). If $\left\|r(X^k; \beta_k)\right\|_E / \left\|X^k\right\|_E \leq \epsilon$, terminate the algorithm; otherwise, continue to Step 2.

Step 2 Find the smallest non-negative integer m_k such that $\alpha_{k+1} = \beta_k\mu^{m_k}$ satisfies

$$\beta_k\left\|\mathcal{F}(X^k) - \mathcal{F}(X^k - \alpha_{k+1}r(X^k; \beta_k))\right\|_E \leq L\left\|r(X^k; \beta_k)\right\|_E \tag{5.33}$$

Step 3 Compute

$$X^{k+1} = P_{\widetilde{\Lambda}}[X^k - \gamma\rho(X^k; \alpha_{k+1}, \beta_k)d(X^k; \alpha_{k+1}, \beta_k)] \tag{5.34}$$

Set $k = k + 1$ and go to Step 1.

(5.33) requires evaluation of $\mathcal{F}$ at the point $X^k - \alpha_{k+1}r(X^k; \beta_k)$. We show that this point always belongs to $\widetilde{\Lambda}$, the domain of $\mathcal{F}$. Notice that $r(X^k; \beta_k) = X^k - P_{\widetilde{\Lambda}}[X^k - \beta_k\mathcal{F}(X^k)]$, thus

$$X^k - \alpha_{k+1}r(X^k; \beta_k) = (1 - \alpha_{k+1})X^k + \alpha_{k+1}P_{\widetilde{\Lambda}}[X^k - \beta_k\mathcal{F}(X^k)] \in \widetilde{\Lambda}$$

since both X^k and $P_{\widetilde{\Lambda}}[X^k - \beta_k \mathcal{F}(X^k)]$ belong to the convex set $\widetilde{\Lambda}$.

In Step 2 of the algorithm, one is required to test a range of integers, starting from zero, in order to find the smallest integer m_k. We show below that such a procedure can always terminate within finite number of trials. Assume that $\mathcal{F}$ is a continuous operator [4] and observe that $\alpha_{k+1} \to 0$ as $m_k \to +\infty$. There must exist $N > 0$ such that for every $m_k > N$ there holds

$$\left\| \mathcal{F}(X^k) - \mathcal{F}(X^k - \alpha_{k+1} r(X^k; \beta_k)) \right\|_E \ \leq \ \frac{L\epsilon}{\beta_k} \ \leq \ \frac{L \left\| r(X^k; \beta_k) \right\|_E}{\beta_k}$$

which is (5.33). In case $m_k > 1$, the algorithm requires more than one evaluation of the operator (that is, more than one dynamic network loading procedure) within one iteration, which is less efficient than the projection algorithm. However, as we subsequently show, convergence of such an algorithm relies on a weaker notion than strong monotonicity, which the projection algorithm requires.

Definition 5.6. (Pseudo monotone) *The operator $\mathcal{F}$ is pseudo monotone if, for arbitrary X^1, $X^2 \in \widetilde{\Lambda}$, the following holds:*

$$\left\langle \mathcal{F}(X^2), X^1 - X^2 \right\rangle_E \ \geq \ 0 \quad \textit{implies} \quad \left\langle \mathcal{F}(X^1), X^1 - X^2 \right\rangle_E \ \geq \ 0 \tag{5.35}$$

By definition, pseudo monotonicity is a consequence of monotonicity, and thus is one type of generalized monotonicity (Pini and Singh, 1997). The convergence of the proposed projection algorithm requires the following property of the principal operator:

$$\langle \mathcal{F}(X), X - X^* \rangle_E \ \geq \ 0 \qquad \forall X \in \widetilde{\Lambda} \tag{5.36}$$

where X^* is a solution of the original VI (3.19). Notice that (5.36) follows from monotonicity or pseudo monotonicity of $\mathcal{F}$, and thus is weaker than these two monotonicity conditions. The following convergence proof is due to Han and Lo (2002).

Theorem 5.7. (Convergence of the self-adaptive projection method). *Assume that $\mathcal{F} : \widetilde{\Lambda} \to E$ is continuous and satisfies (5.36). Then the sequence $\{X^k\}$ generated by the self-adaptive projection algorithm converges to a solution of the VI (3.19).*

Proof. See Han and Lo (2002) for a proof. □

5.3. The proximal point method

The *proximal point method* (PPM) (Konnov, 2003) is a popular method for solving optimization problems and variational inequalities. It replaces the original problem with a sequence of regularized problems, each of which can be solved with standard algorithms due to improved regularity. The PPM is known to converge with some generalized monotonicity (Allevi et al., 2006). In this paper we apply the PPM to solve E-DUE problems while advancing our knowledge of the PPM by proposing a set of new convergence conditions weaker than those from previous studies. The proximal point method has been further developed in this paper and is summarized below.

Proximal point method

Step 0 Identify an initial feasible point $X^0 = (h^0, Q^0) \in \widetilde{\Lambda}$. Fix a large constant $a > 0$ and set a tolerance parameter $\delta > 0$. Set the iteration counter $k = 0$.

Step 1 Solve the following variational inequality for $X^{k+1} = (h^{k+1}, Q^{k+1})$:

$$\left\langle \mathcal{F}(X^{k+1}) + a(X^{k+1} - X^k), X - X^{k+1} \right\rangle_E \ \geq \ 0 \qquad \forall X \in \widetilde{\Lambda} \tag{5.37}$$

Step 2 Terminate the algorithm if $\|X^{k+1} - X^k\|_E \leq \frac{\delta}{aD}$, where D is the diameter of the set $\widetilde{\Lambda}$. Otherwise, set $k = k + 1$ and repeat Step 1 through Step 2.

---

[4]Continuity of the operator $\mathcal{F}$ is stated as one of the conditions for solution existence and algorithm convergence.

The key step of the PPM is to solve the VI (5.37), which enjoys a significantly improved regularity than the original VI problem. To see this, we rewrite $\mathcal{F}(X^{k+1}) + a(X^{k+1} - X^k)$ as $(\mathcal{F} + aI)(X^{k+1}) - aX^k$, where I is the identity map. If $\mathcal{F}$ is weakly monotone with constant $-K$ (see (5.25)), then $(\mathcal{F} + aI)(X^{k+1}) - aX^k$ is a strongly monotone operator acting on X^{k+1} provided that $a > K$. Thus, by choosing a large enough, the VI (5.37) can be solved with any existing algorithm with satisfactory convergence result.

We now present a new convergence result for the proximal point method. We begin with the articulation of the *dual formulation* of the VI (3.19).

Definition 5.8. (Dual formulation of the VI) *The dual form of the VI* (3.19), *also known as the Minty problem, is defined as follows. Find $X^d \in \widetilde{\Lambda}$ such that*

$$\left\langle \mathcal{F}(X),\, X - X^d \right\rangle_E \;\geq\; 0 \qquad \forall X \in \widetilde{\Lambda} \tag{5.38}$$

We let Y^d be the solution set of (5.38).

Lemma 5.9. *Assume that $Y^d \neq \emptyset$, and that $\widetilde{\Lambda}$ is bounded with diameter $D < \infty$. Then the sequence $\{X^k\}$ generated by the proximal point method satisfies the following: for any $k \geq 1$,*

$$\left\langle \mathcal{F}(X^{\mu(k)+1}),\, X - X^{\mu(k)+1} \right\rangle_E \;\geq\; -\frac{aD^2}{\sqrt{k+1}} \qquad \forall X \in \widetilde{\Lambda} \tag{5.39}$$

where $\mu(k) \doteq \underset{0 \leq i \leq k}{argmin} \left\| X^i - X^{i+1} \right\|_E^2 \in \{0, \ldots, k\}$.

Proof. The proof is postponed until Appendix E. $\qquad\square$

Remark 5.10. *We note that the feasible set $\widetilde{\Lambda}$ defined originally for continuous-time functions is unbounded. However, in numerical (discrete-time) computations the feasible set $\widetilde{\Lambda}$ will be bounded with a finite diameter $D < \infty$.*

As an immediate consequence of Lemma 5.9, we have the following convergence result.

Theorem 5.11. (Convergence of the proximal point method) *Assume that $Y^d \neq \emptyset$, and that $\widetilde{\Lambda}$ is bounded with diameter $D < \infty$. Then for any tolerance $\delta > 0$, there exists $R \doteq \lceil \frac{a^2 D^4}{\delta^2} \rceil - 1$, such that*

$$\left\langle \mathcal{F}(X^{\mu(R)+1}),\, X - X^{\mu(R)+1} \right\rangle_E \;\geq\; -\delta \qquad \forall X \in \widetilde{\Lambda}, \tag{5.40}$$

where $\{X^k\}$, $k \geq 0$ is the sequence generated by the proximal point method, a is the number appearing in (5.37), *and $\lceil z \rceil$ denotes the smallest integer that is larger than or equal to z. Moreover, when the PPM algorithm terminates, i.e. when $\|X^{k+1} - X^k\|_E \leq \frac{\delta}{aD}$ for the first time, then*

$$\left\langle \mathcal{F}(X^{k+1}),\, X - X^{k+1} \right\rangle_E \;\geq\; -\delta \qquad \forall X \in \widetilde{\Lambda} \tag{5.41}$$

Proof. Replacing k in (5.39) with $R = \lceil \frac{a^2 D^4}{\delta^2} \rceil - 1$ yields

$$\left\langle \mathcal{F}(X^{\mu(R)+1}),\, X - X^{\mu(R)+1} \right\rangle_E \;\geq\; -\delta \qquad \forall X \in \widetilde{\Lambda}$$

which is (5.40). To show (5.41), we invoke (E.6) in the proof of Lemma 5.9:

$$\left\langle \mathcal{F}(X^{\mu(k)+1}),\, X - X^{\mu(k)+1} \right\rangle_E \;\geq\; -aD \left\| X^{\mu(k)+1} - X^{\mu(k)} \right\|_E$$

According to the definition of $\mu(\cdot)$, the termination criterion of the PPM implies that $k = \mu(k)$. Thus we have that, when the algorithm terminates,

$$\left\langle \mathcal{F}(X^{k+1}),\, X - X^{k+1} \right\rangle_E \;\geq\; -aD \left\| X^{k+1} - X^k \right\|_E \;\geq\; -\delta$$

$\qquad\square$

Remark 5.12. *Unlike the convergence results established in Section 5.1 and 5.2, which focus on the asymptotic behavior of the sequence $\{X^k\}$ as $k \to \infty$, the convergence result developed in Theorem 5.11 is concerned with finding a solution of the approximate VI (that is, with $-\delta$ on the right hand side instead of zero) within finite iteration. Such a convergence result is quite practical for numerical computations as it estimates the number of iterations needed to achieve certain level of approximation of the original VI.*

Theorem 5.11 only requires that the dual VI (5.38) has a solution – a property subsequently referred to as *dual solvability*. Compared to the convergence conditions for the self-adaptive projection method (Section 5.2), dual solvability is weaker than the assumption (5.36) as the latter requires that the solution of the original VI must be a solution of the dual VI. In addition, Theorem 5.11 does not rely on the continuity of the principal operator. We thus conclude that the convergence conditions for the PPM are indeed weaker than the previous two methods.

In the remainder of this section, we will investigate in detail the dual solvability and provide sufficient conditions for it. One should note that if the original VI has a solution, then a sufficient condition for dual solvability is pseudo monotonicity; this is apparent from Definition 5.6. In the following presentation, we will articulate a weaker sufficient condition for dual solvability, based on the notion of semistrictly quasi monotonicity.

Definition 5.13. *The operator $\mathcal{F}$ is quasi monotone if, for arbitrary X^1, $X^2 \in \widetilde{\Lambda}$,*

$$\left\langle \mathcal{F}(X^2) , \ X^1 - X^2 \right\rangle_E > 0 \implies \left\langle \mathcal{F}(X^1) , \ X^1 - X^2 \right\rangle_E \geq 0$$

The operator $\mathcal{F}$ is semistrictly quasi monotone if it is quasi monotone and, for every X^1, $X^2 \in \widetilde{\Lambda}$,

$$\left\langle \mathcal{F}(X^2) , \ X^1 - X^2 \right\rangle_E > 0 \implies \left\langle \mathcal{F}(X^3) , \ X^1 - X^2 \right\rangle_E > 0$$

for some $X^3 \in \left\{ X : \ X = X^1 + \lambda(X^2 - X^1), \ \lambda \in (0, \frac{1}{2}) \right\}$.

The reader is referred to Konnov (1998) for a detailed discussion of quasi monotonicity. In particular, Lemma 3.1 of Konnov (1998) states that pseudo monotonicity implies semistrictly quasi monotonicity, making the latter a weaker assumption. The notion of w^*-hemicontinuity is also needed for the dual solvability.

Definition 5.14. *$\mathcal{F}$ is w^*-hemicontinuous if the function*

$$c(\lambda) \ \dot{=} \ \left\langle \mathcal{F}(X_\lambda) , \ X_\lambda - X^2 \right\rangle_E \qquad where \ \ X_\lambda = \lambda X^1 + (1 - \lambda)X^3$$

is upper semicontinuous at $\lambda = 0+$ for all X^1, X^2, $X^3 \in \widetilde{\Lambda}$ and $\lambda \in [0, 1]$.

Finally, the sufficient condition for dual solvability is summarized below.

Theorem 5.15. (Sufficient condition for dual solvability) *If $\mathcal{F}$ is continuous on $\widetilde{\Lambda}$ and is semistrictly quasi monotone, then the dual problem (5.38) has a solution.*

Proof. It is easy to verify by definition that if $\mathcal{F}$ is continuous in the strong topology, then it is w^*-hemicontinuous. Thus the conclusion follows from Theorem 4.1 of Konnov (1998). □

Remark 5.16. *The three computational methods proposed here rely on generalized monotonicity in order to converge. As we previously mentioned, most delay operators may not satisfy these generalized notions of monotonicity, with only a few exceptions (Mounce, 2006; Perakis and Roels, 2006). Thus the proper perspective on convergence of numerical algorithms for calculating DUEs on general networks is to say that almost all algorithms are presently heuristic. Exact algorithms will be created only when a fundamentally new operator class is invented, which allows non-monotonicity while also providing behavioral insights that allow convergence to be established.*
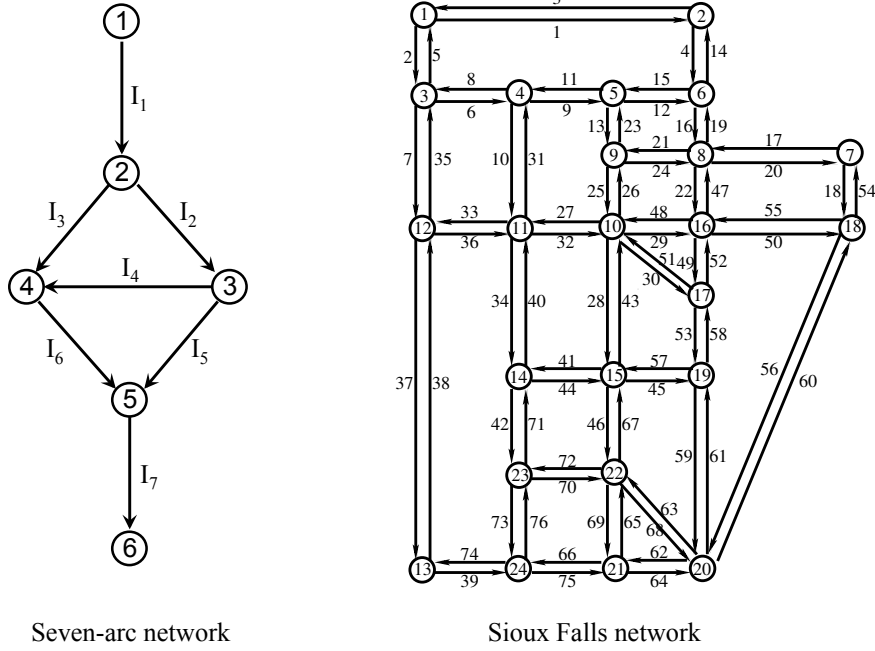
Seven-arc network   Sioux Falls network

Figure 1: The two test networks

	# of O-D pairs	# of paths	Inverse demand function
Seven-arc network	1	3	$v = -Q/2000 + 1.2$
Sioux Falls network	6	119	$v = -Q/500 + 1.6$

Table 1: Basic network information. The same inverse demand function is assumed for all the O-D pairs in the network.

6. Numerical examples

In this section we numerically test the three computational algorithms and illustrate their solutions. Two test networks are considered: the seven-arc network and the Sioux Falls network as shown in Figure 1. Table 1 shows the number of O-D pairs and paths in the test networks, as well as the inverse demand functions, which are assumed to be the same for all the O-D pairs. In particular, for the seven-arc network one O-D pair (1, 6) and three paths $p_1 = \{I_1, I_3, I_6, I_7\}$, $p_2 = \{I_1, I_2, I_4, I_6, I_7\}$, $p_3 = \{I_1, I_2, I_5, I_7\}$ are considered. In the Sioux Falls network six O-D pairs (1, 20), (2, 20), (3, 20), (4, 20), (5, 20) and (6, 20) are considered, among which 119 paths are utilized.

The following termination criteria are employed for the three computational algorithms proposed in this paper. For the projection method and the proximal point method, the algorithm is terminated if the relative gap

$$\frac{\|X^{k+1} - X^k\|_E}{\|X^k\|_E} \leq 10^{-5} \tag{6.42}$$

where $X^k \doteq (h^k, Q^k)$. For the self-adaptive projection method, the termination criterion is

$$\frac{\|r(X^k; \beta_k)\|_E}{\|X^k\|_E} \leq 10^{-6} \tag{6.43}$$

where the residual $r(X^k; \beta_k)$ is given in (5.29). These termination criteria will be adopted for the computations on both test networks.

Regarding the dynamic network loading sub-problem, we employ the link transmission model (Yperman et al., 2005), which is a discrete-time and simplified version of the Lighthill-Whitham-Richards model (Lighthill and

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Whitham, 1955; Richards, 1956). It is based on a triangular fundamental diagram and propagates traffic on a link through Newell's variational principle (Newell, 1993). Due to space limitation we will not elaborate this DNL procedure here and instead refer the reader to Yperman et al. (2005) for the original model and Han et al. (2015) for more elaborated discussion. All the computations reported here were performed on a standard laptop with 4 GB of RAM.

6.1. Computational results on the severn-arc network

We apply the projection method (PM), the self-adaptive projection method (SA), and the proximal point method (PP) to solve the elastic DUE problem on the seven-arc network shown in Figure 1. Under the termination criteria (6.42)-(6.43), the algorithms converge after 200 (PM), 3903 (SA), and 180 (PP) iterations. We use Figure 2 to show the left hand sides of (6.42) and (6.43) at each iteration of the algorithms. It can be seen that although all three algorithms reach the termination threshold within finite number of iterations, their convergence speeds vary. In particular, the self-adaptive projection method has a much slower yet more smooth convergence than the other two; and its relative gap is monotonically decreasing as the iteration continues. The projection method and the proximal point method show qualitatively similar convergence trends: their relative gaps decrease much more quickly than the self-adaptive method but may have some local increase and oscillation.
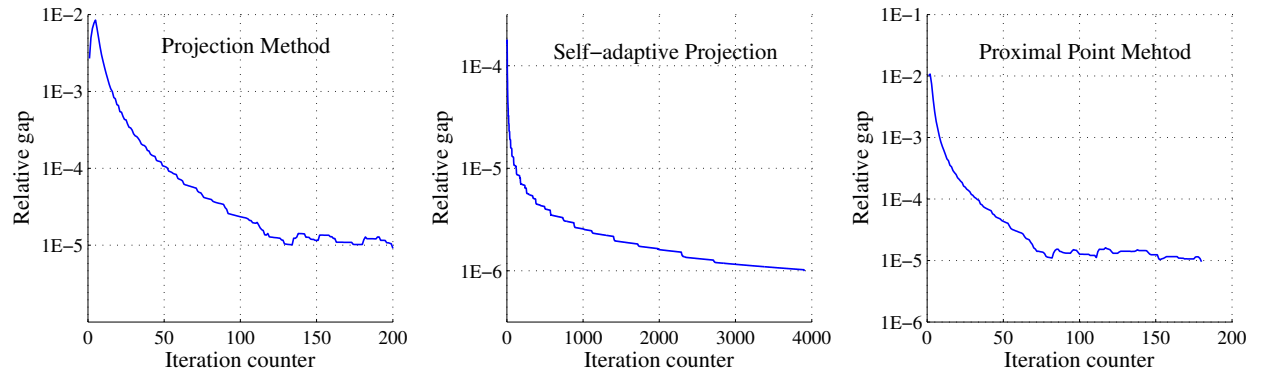

Figure 2: The seven-arc network: Relative gaps, defined in (6.42)-(6.43), at each iteration of the algorithms. Here, 1E-x means 10^{-x}.

We now turn to the solutions produces by these algorithms. Figure 3 illustrates two main quantities: the elastic demand Q^k generated at each iteration, and the drivers' average travel cost at each iteration defined as:

$$ v^k \doteq \frac{1}{Q^k} \sum_{p \in \mathcal{P}} \int_{t_0}^{t_f} \Psi_p(t, h^k) \cdot h_p^k(t) \, dt \qquad k = 0, 1, 2, \ldots $$

The reason for using the average travel cost v^k is as follows. Until an E-DUE has been found (as allowed by the prescribed tolerance), the travel costs within the same O-D pair are not equalized; thus we use the averaged travel cost to demonstrate the demand-cost relationship stipulated by the inverse demand function shown in Table 1. It can be seen from Figure 3 that both Q^k and v^k converge to a fixed value when the algorithms terminate, and they converge to a state that is consistent with the inverse demand function (see the bottom row of Figure 3).

It remains to show that the experienced travel costs within the same O-D pair for all route and departure time choices are equal and minimum. This is illustrated in Figure 4, where we show the path departure rates and the corresponding travel costs for all three paths in the network. We see that the travel costs are indeed equal and minimum whenever the departure rates are positive, which confirm the equilibrium state. Nevertheless, the equal travel costs claimed above are not exact if one inspects the pictures closely, and we use Remark 6.1 below to justify our results.

Remark 6.1. *When it comes to computation of DUEs numerically, two facts need to be realized. First, theories regarding the continuous-time formulations of E-DUE and the delay operator are exact only when the time step employed by the numerical computation is infinitely small, which is of course impossible. From this perspective, all computational results reported so far, not merely in this paper, are approximations of the DUE or E-DUE. Secondly,*
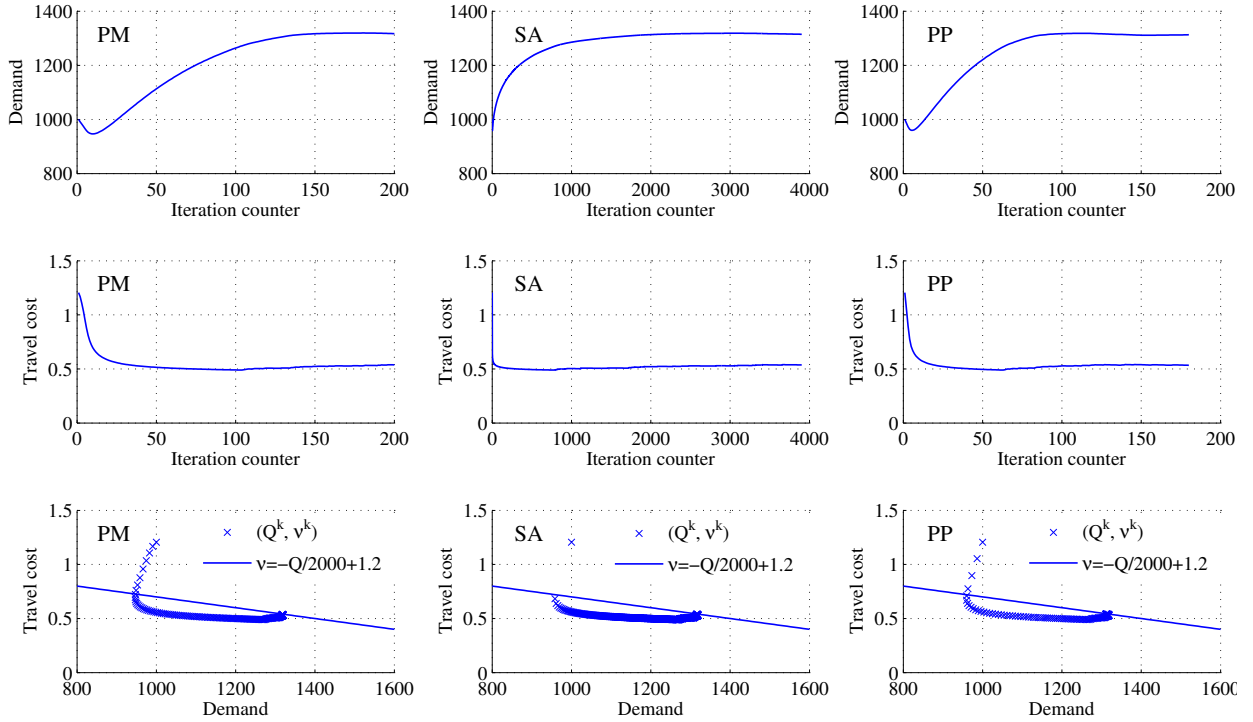
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Figure 3: The seven-arc network: The elastic demand Q^k and the average travel cost v^k at each iteration. The initial value for the demand is always set to be $Q^0 = 1000$. PM: projection method; SA: self-adaptive projection; PP: proximal point.

most computational algorithms, even when their convergence criteria are met, can only converge in the limit of an infinite number of iterations. In other words, their convergence is in the asymptotic sense. It is safe to say that all computations of DUE or E-DUE are terminated after a finite number of iterations, and thus the solutions are, again, approximations as allowed by prescribed tolerances.

6.2. Computational results on the Sioux Falls network

We test the three proposed algorithms further on the larger Sioux Falls network (see Figure 1). We start with the algorithm convergence following the same criteria (6.42)-(6.43). Figure 5 shows the relative gap at each iteration of the algorithm. The number of iterations needed for the three algorithms are: 469 (PM), 2304 (SA), and 109 (PP). Observations regarding the convergence trends are similar to the previous test network: The self-adaptive method has a slower yet more smooth convergence rate, and the relative gap is monotonically decreasing. The projection method and the proximal point method have a faster convergence but their relative gaps may have some local oscillations.

The convergence of the elastic travel demands Q_{ij}^k and the OD-specific average travel costs v_{ij}^k is demonstrated in Figure 6, which has a similar style of presentation as Figure 3 but shows information for all six O-D pairs. We can, again, confirm that Q_{ij}^k and v_{ij}^k converge to the relationship stipulated by the inverse demand function for all O-D pairs when the algorithms converge. However, there are some inaccuracies in the convergence of the self-adaptive method and the proximal point method as the trajectories of (Q_{ij}^k, v_{ij}^k) did not reach the line $v = -Q/500 + 1.6$ when the algorithms terminated. They are, again, related to the observations made in Remark 6.1, and may be overcome by tighter stopping tolerances.

6.3. Summary of algorithm performances

We summarize the performances of the three algorithms in terms of solution quality, convergence rate, computational complexity, and computational time. Table 2 shows some basic information on their computational performances.
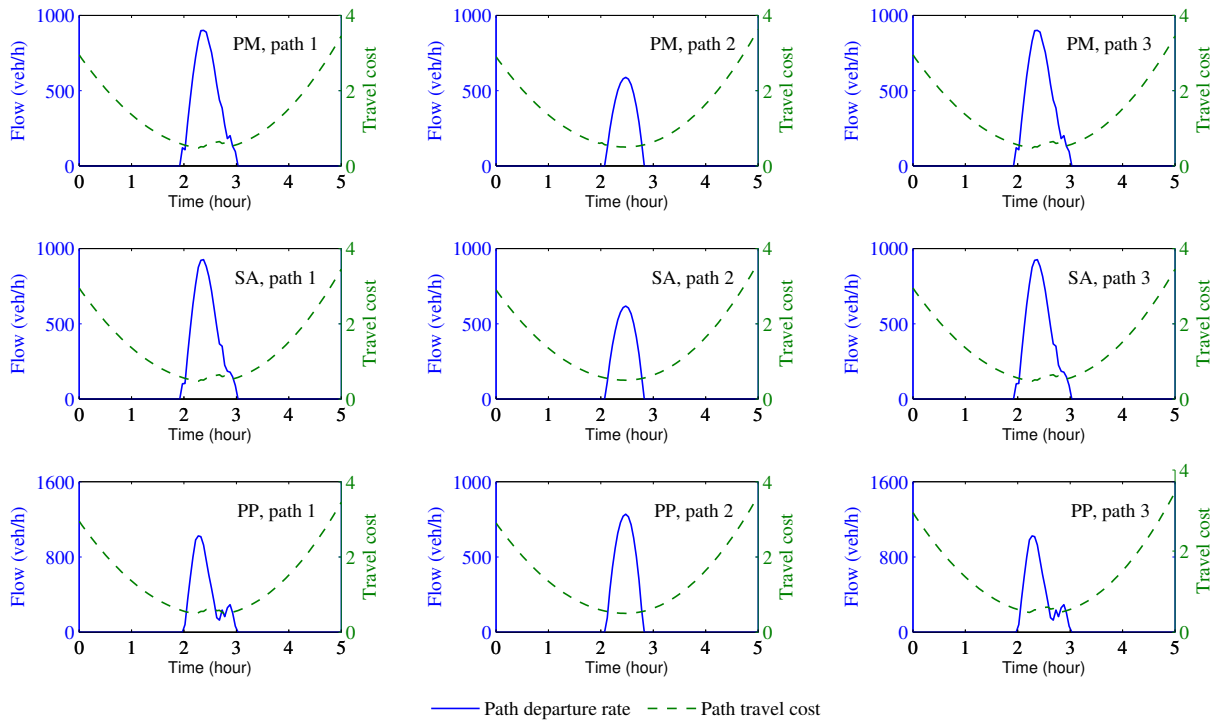
Figure 4: The seven-arc network: Path departure rates and associated travel costs for all three paths. PM: projection method; SA: self-adaptive projection; PP: proximal point.
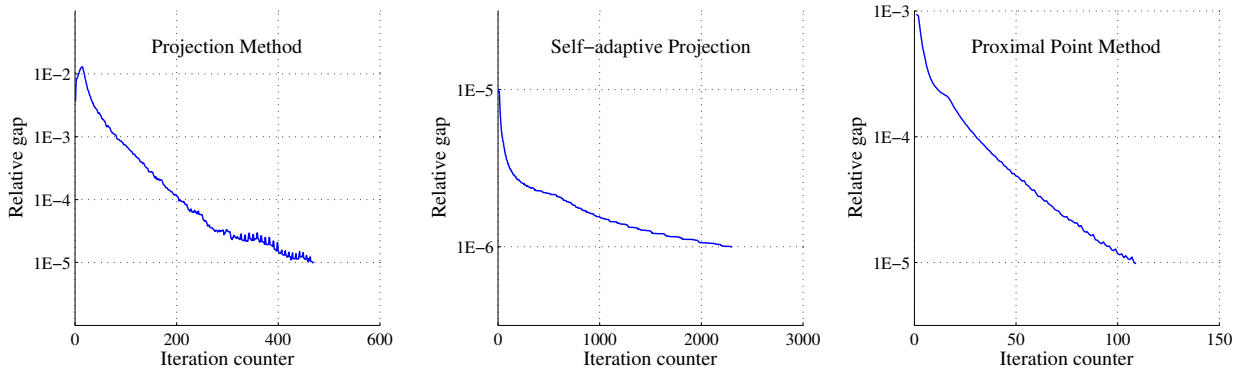

Figure 5: The Sioux Falls network: Relative gaps, defined in (6.42)-(6.43), at each iteration of the algorithms. Here, 1E-x means 10^{-x}.

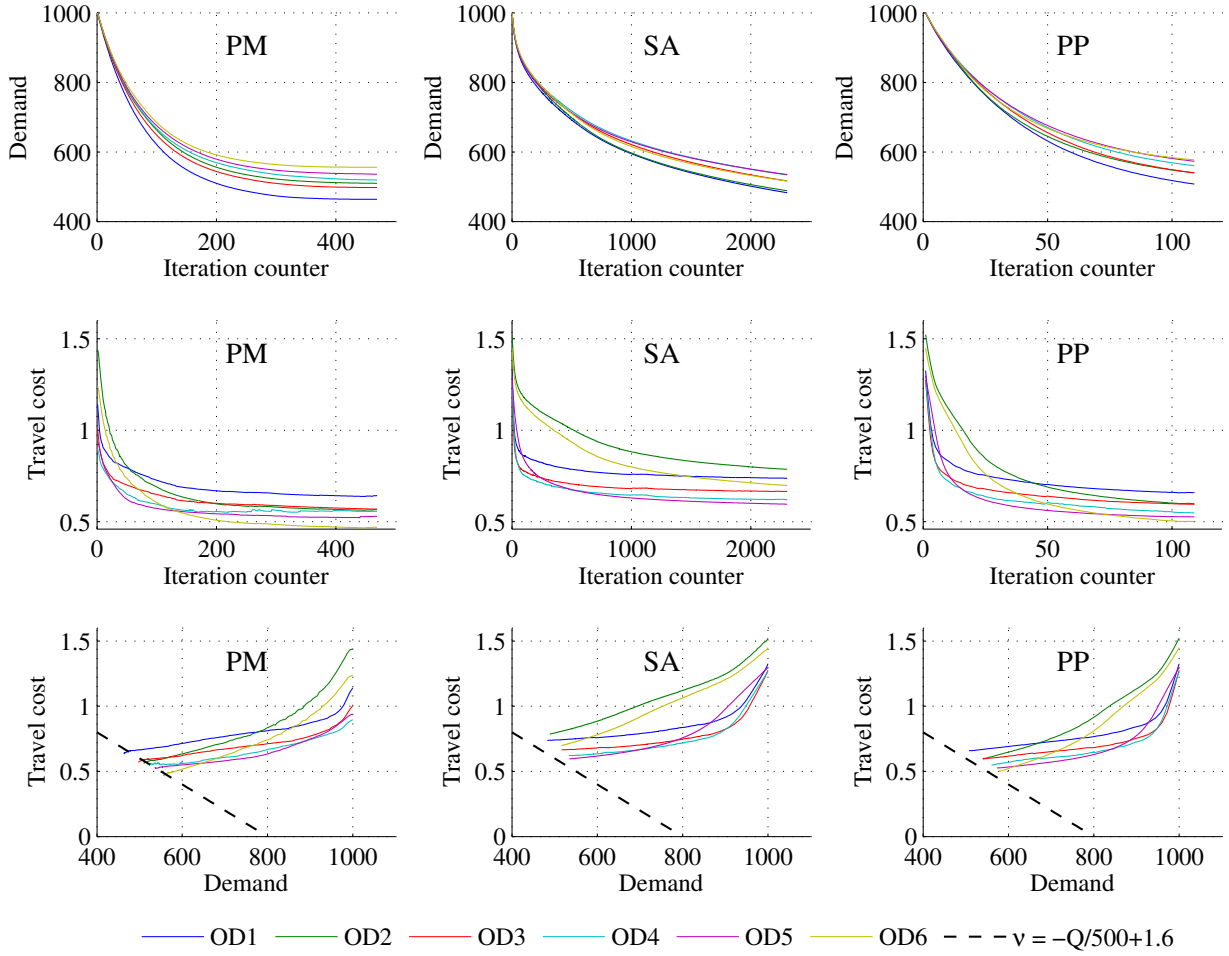|	Seven-arc			Sioux Falls		
	Iteration #	DNL #	Time	Iteration #	DNL #	Time
Projection Method	200	200	3 s	469	469	382 s
Self-Adaptive	3903	7805	108 s	2304	4,607	3810 s
Proximal Point	180	5400	74 s	109	2332	2425 s

Table 2: Summary of the three algorithms.

Figure 6: The Sioux Falls network: The elastic demands Q_{ij}^k and the average travel costs v_{ij}^k for each O-D pair (i, j) at each iteration. The initial values for the demands are always set to be $Q_{ij}^0 = 1000$ for all $(i, j) \in \mathcal{W}$. PM: projection method; SA: self-adaptive projection; PP: proximal point.

In general, the self-adaptive projection method takes much more iterations to reach a prescribed convergence threshold than the other two methods. However, one should take into account the fact that the self-adaptive method utilizes a different termination criterion than the other two; i.e. it relies on the residual function $r(X^k; \beta^k)$ while the others focus on the distance between two consecutive iterates (see (6.42) and (6.43)).

The self-adaptive method and the proximal point method require multiple dynamic network loading (DNL) procedures to be performed within a single iteration, thus end up with more DNL numbers than the projection method. In our computation, the average numbers of DNLs performed within one iteration are roughly 2 (self-adaptive) and 26 (proximal point). The proximal point method requires solving a regularized VI within each iteration and thus demands substantially more computational effort.

The projection method requires the least computational effort and yields good solution quality despite its restrictive convergence condition (based on strong monotonicity). The self-adaptive method and the proximal point method require more iterations and DNL procedures to converge, which highlights a trade-off between theoretical convergence and computational complexity. Overall, all three algorithms have successfully computed the E-DUE solutions with convergence as allowed by a prescribed stopping threshold, despite the fact that their convergence conditions have not been verified by the network performance model.

7. Conclusion

This paper address three aspects of the simultaneous route-and-departure-time dynamic user equilibrium (SRDT DUE): formulation, existence and computation. This problem is first formulated as a variational inequality problem in an extended Hilbert space. A general existence theory for the continuous-time E-DUE problem is then proposed based on the new VI formulation. This existence proof employs the most general constraints relating path departure rates to a table of elastic trip volumes, and does not invoke the *a priori* upper bounds on the path departure rates. Finally, we present three new computational algorithms: the projection algorithm, the self-adaptive projection algorithm, and the proximal point method. The first and second methods require a minimum-norm projection onto the extended Hilbert space, which can be explicitly instantiated using the differential variational inequality (DVI) formalism. Convergence proofs are provided for all three algorithms with different types of generalized monotonicity, namely the mixed strongly-weakly monotonicity, the pseudo monotonicity, and the semistrictly quasi monotonicity. These three algorithms are tested on two networks with their solution quality, convergence, and computational efficiency shown and compared. The numerical results show a clear trade-off between the generality of theoretical convergence and the computational efficiency of the algorithms. In particular, the projection method, despite its restrictive convergence condition, tends to have a faster convergence rate than the other two methods.

It should be noted that the relaxed notions of monotonicity employed by our convergence proofs are not verified against the network performance model, thus the computational methods proposed here should be considered heuristics. Presently known mathematics does not provide a means of classifying path delay operators like those intrinsic to our formulation of E-DUE. Rather a new class of operators must be discovered. The first two steps in that process of discovery are: (i) the illustration of what can be understood about convergence based on current knowledge; and (ii) the heuristic application of known algorithms when convergence cannot be rigorously assured. Again, it is our formulations that allow us to offer results directly relevant to the aforementioned two steps.

8. Acknowledgement

Appendix A. Proof of Theorem 3.2

Proof. **[Necessity].** Given a DUE solution with elastic demand $(h^*, Q^*) \in \widetilde{\Lambda}$, we easily deduce from (3.13) and (3.14) that for any $(h, Q) \in \widetilde{\Lambda}$,

$$
\sum_{p \in \mathcal{P}} \int_{t_0}^{t_f} \Psi_p(t, h^*)(h_p(t) - h_p^*(t))dt - \sum_{(i,j) \in \mathcal{W}} \Theta_{ij}[Q^*](Q_{ij} - Q_{ij}^*)
$$

$$
= \sum_{(i,j) \in \mathcal{W}} \left(\sum_{p \in \mathcal{P}_{ij}} \int_{t_0}^{t_f} \Psi_p(t, h^*)h_p(t) \, dt - \Theta_{ij}[Q^*] \cdot Q_{ij} \right)
$$

$$
- \sum_{(i,j) \in \mathcal{W}} \left(\sum_{p \in \mathcal{P}_{ij}} \int_{t_0}^{t_f} \Psi_p(t, h^*)h_p^*(t) \, dt - \Theta_{ij}[Q^*] \cdot Q_{ij}^* \right)
$$

$$
= \sum_{(i,j) \in \mathcal{W}} \left(\sum_{p \in \mathcal{P}_{ij}} \int_{t_0}^{t_f} \Psi_p(t, h^*)h_p(t) \, dt - \Theta_{ij}[Q^*] \cdot Q_{ij} \right)
$$

$$
- \sum_{(i,j) \in \mathcal{W}} \left(\sum_{p \in \mathcal{P}_{ij}} \Theta_{ij}[Q^*] \cdot \int_{t_0}^{t_f} h_p^*(t) \, dt - \Theta_{ij}[Q^*] \cdot Q_{ij}^* \right)
$$

$$
= \sum_{(i,j) \in \mathcal{W}} \left(\sum_{p \in \mathcal{P}_{ij}} \int_{t_0}^{t_f} \Psi_p(t, h^*)h_p(t) \, dt - \Theta_{ij}[Q^*] \cdot Q_{ij} \right)
$$

$$
- \sum_{(i,j) \in \mathcal{W}} \Theta_{ij}[Q^*] \cdot \left(\sum_{p \in \mathcal{P}_{ij}} \int_{t_0}^{t_f} h_p^*(t) \, dt - Q_{ij}^* \right)
$$

$$
= \sum_{(i,j) \in \mathcal{W}} \left(\sum_{p \in \mathcal{P}_{ij}} \int_{t_0}^{t_f} \Psi_p(t, h^*)h_p(t) \, dt - \Theta_{ij}[Q^*] \cdot Q_{ij} \right) \tag{A.1}
$$

Observe that within (A.1), we have, given any $(i, j) \in \mathcal{W}$, that

$$
\sum_{p \in \mathcal{P}_{ij}} \int_{t_0}^{t_f} \Psi_p(t, h^*)h_p(t) \, dt - \Theta_{ij}[Q^*] \cdot Q_{ij}
$$

$$
\geq v_{ij}(h^*) \sum_{p \in \mathcal{P}_{ij}} \int_{t_0}^{t_f} h_p(t) \, dt - \Theta_{ij}[Q^*] \cdot Q_{ij} = v_{ij}(h^*) \left(\sum_{p \in \mathcal{P}_{ij}} \int_{t_0}^{t_f} h_p(t) \, dt - Q_{ij} \right) = 0 \tag{A.2}
$$

where $v_{ij}(h^*)$ is the essential infimum of the effective path delay within O-D (i, j), and is equal to $\Theta_{ij}[Q^*]$ according to Definition 3.1.

As an immediate consequence of (A.1) and (A.2), the following inequality holds for all $(h, Q) \in \widetilde{\Lambda}$.

$$
\sum_{p \in \mathcal{P}} \int_{t_0}^{t_f} \Psi_p(t, h^*)(h_p(t) - h_p^*(t))dt - \sum_{(i,j) \in \mathcal{W}} \Theta_{ij}[Q^*](Q_{ij} - Q_{ij}^*) \geq 0 \tag{A.3}
$$

which is the desired variational inequality.

[Sufficiency]. Assume that (A.3) holds for any $(h, Q) \in \widetilde{\Lambda}$. Then h^* must be a solution of the fixed-demand DUE problem with fixed demand given by Q^*, as in this case the second term in the left hand side of (A.3) vanishes and we recover the well-known VI for the fixed demand case; see Theorem 2 of Friesz et al. (1993). By definition of fixed-demand DUE, for any $(i, j) \in \mathcal{W}$,

$$
\begin{aligned}
h_p^*(t) > 0, \; p \in \mathcal{P}_{ij} &\implies \Psi_p(t, h^*) = v_{ij}(h^*) &&\text{for almost every } t \in [t_0, t_f] \\
\Psi_p(t, h^*) &\geq v_{ij}(h^*) \quad \forall p \in \mathcal{P}_{ij} &&\text{for almost every } t \in [t_0, t_f]
\end{aligned} \tag{A.4}
$$

In order to show that (h^*, Q^*) is a DUE with elastic demand using Definition 3.1, we fix an arbitrary O-D pair $(k, l) \in \mathcal{W}$, and distinguish between two cases:

- $Q_{kl}^* > 0$. We define the following pair $(\hat{h}, \hat{Q}) \in \widetilde{\Lambda}$:

$$
\hat{h}_p(t) = \begin{cases} a\, h_p^*(t) & p \in \mathcal{P}_{kl} \\ h_p^*(t) & p \in \mathcal{P} \setminus \mathcal{P}_{kl} \end{cases} \quad \forall t \in [t_0, t_f], \qquad \hat{Q}_{ij} = \begin{cases} a\, Q_{kl}^* & (i, j) = (k, l) \\ Q_{ij}^* & (i, j) \in \mathcal{W} \setminus (k, l) \end{cases}
$$

where $a > 0$ is an arbitrary positive parameter. Substituting (h, Q) for $(\hat{h}, \hat{Q})$, the left hand side of (A.3) becomes

$$
\sum_{p \in \mathcal{P}} \int_{t_0}^{t_f} \Psi_p(t, h^*)(\hat{h}_p - h_p^*)dt - \sum_{(i, j) \in \mathcal{W}} \Theta_{ij}[Q^*](\hat{Q}_{ij} - Q_{ij}^*)
$$

$$
= \sum_{p \in \mathcal{P}_{kl}} \int_{t_0}^{t_f} \Psi_p(t, h^*)(ah_p^* - h_p^*)\, dt + \sum_{p \in \mathcal{P} \setminus \mathcal{P}_{kl}} \int_{t_0}^{t_f} \Psi_p(t, h^*)(h_p^* - h_p^*)\, dt
$$

$$
- \Theta_{kl}[Q^*]\left[aQ_{kl}^* - Q_{kl}^*\right] - \sum_{(i, j) \in \mathcal{W} \setminus (k, l)} \Theta_{ij}[Q^*](Q_{ij}^* - Q_{ij}^*)
$$

$$
= (a - 1) \sum_{p \in \mathcal{P}_{kl}} \int_{t_0}^{t_f} \Psi_p(t, h^*)\, h_p^*(t)\, dt - (a - 1)\Theta_{kl}[Q^*]\, Q_{kl}^*
$$

$$
= (a - 1)v_{kl}(h^*)Q_{kl}^* - (a - 1)\Theta_{kl}[Q^*]Q_{kl}^*
$$

We conclude from (A.3) that

$$
(a - 1)(v_{kl}(h^*) - \Theta_{kl}[Q^*])Q_{kl}^* \geq 0
$$

Since $a \in \mathbb{R}_{++}$ is arbitrary and $Q_{kl}^* > 0$, there must hold $v_{kl}(h^*) = \Theta_{kl}[Q^*]$. Thus replacing $v_{kl}(h^*)$ with $\Theta_{kl}[Q^*]$ in (A.4) yields (3.13)-(3.14).

- $Q_{kl}^* = 0$. In this case, we consider $(\hat{h}, \hat{Q}) \in \widetilde{\Lambda}$ such that $h_p(t) \equiv h_p^*(t)$ for all $t \in [t_0, t_f]$, $p \in \mathcal{P} \setminus \mathcal{P}_{kl}$. Substituting (h, Q) for $(\hat{h}, \hat{Q})$ in (A.3) yields

$$
\sum_{p \in \mathcal{P}_{kl}} \int_{t_0}^{t_f} \Psi_p(t, h^*)(h_p(t) - h_p^*(t))dt - \Theta_{kl}[Q^*]Q_{kl} \geq 0 \tag{A.5}
$$

which implies that

$$
\sum_{p \in \mathcal{P}_{kl}} \int_{t_0}^{t_f} (\Psi_p(t, h^*) - \Theta_{kl}[Q^*])\, h_p(t)\, dt \geq 0
$$

Since the vector $(h_p(t) : p \in \mathcal{P}_{kl}, t \in [t_0, t_f])$ is arbitrary, we must have that $\Psi_p(t, h^*) \geq \Theta_{kl}[Q^*]$ for any $p \in \mathcal{P}_{kl}$ and almost every $t \in [t_0, t_f]$, which is (3.14). Finally, (3.13) is trivially true since $h_p^*(t) \equiv 0$ for all $p \in \mathcal{P}_{kl}$ and almost every $t \in [t_0, t_f]$.

Combining the above two cases, we finish the proof. $\qquad \square$

Appendix B. Proof of Theorem 4.5

Proof. The proof is completed in several steps.

Step 1. Since Theorem 4.1 cannot be directly applied to obtain a solution of the infinite-dimensional VI, let us instead employ the technique from Han et al. (2013c) by considering finite-dimensional approximations of $\widetilde{\Lambda}$. More

specifically, consider, for each $n \geq 1$, the uniform partition of $[t_0, t_f]$ into n sub-intervals $I^1, \ldots, I^n$. Define the finite-dimensional subset of $\widetilde{\Lambda}$:

$$\widetilde{\Lambda}_n \doteq \left\{ (h, Q) \in \widetilde{\Lambda} : h_p(\cdot) \text{ is constant on each } I^j \quad \forall 1 \leq j \leq n, \quad \forall p \in \mathcal{P} \right\} \tag{B.1}$$

Clearly $\widetilde{\Lambda}_n \subset \widetilde{\Lambda}$, thus all the assumptions regarding $\mathcal{F}$ or Ψ continue to hold on this smaller set.

It is not restrictive to assume that there is an upper bound on the elastic demand for each origin-destination pair. That is, there exists a vector $U = (U_{ij}) \in \mathbb{R}_{++}^{|\mathcal{W}|}$ such that

$$0 \leq Q_{ij} \leq U_{ij} \qquad \forall (i, j) \in \mathcal{W}, \qquad \forall (h, Q) \in \widetilde{\Lambda}$$

We claim that $\widetilde{\Lambda}_n$ defined as such is convex and compact in $\widetilde{\Lambda}$ for each $n \geq 1$. We begin with convexity. Let $\widehat{X} = (\widehat{h}, \widehat{Q})$ and $\overline{X} = (\overline{h}, \overline{Q})$ be any two points in $\widetilde{\Lambda}_n$. Given any $\alpha \in (0, 1)$, we have that

$$\sum_{p \in \mathcal{P}_{ij}} \int_{t_0}^{t_f} \left(\alpha \widehat{h}_p(t) \, dt + (1 - \alpha) \overline{h}_p(t) \right) dt = \alpha \widehat{Q}_{ij} + (1 - \alpha) \overline{Q}_{ij} \qquad \forall (i, j) \in \mathcal{W}$$

Moreover, $\alpha \widehat{h}_p(\cdot) + (1 - \alpha) \overline{h}_p(\cdot)$ clearly remains constant on each sub-interval I^j, $j = 1, \ldots, n$, $\forall p \in \mathcal{P}$. We thus conclude that $\alpha \widehat{X} + (1 - \alpha) \overline{X} \in \widetilde{\Lambda}_n$.

Next, let us investigate compactness. Let us fix $n \geq 1$. In light of Proposition 4.4, it suffices to establish sequential compactness for $\widetilde{\Lambda}_n$. We consider an arbitrary infinite sequence $\{X^k\}_{k \geq 1} \subset \widetilde{\Lambda}_n$ where $X^k = (h^k, Q^k)$. For each $k \geq 1$ and $p \in \mathcal{P}$, let $\mu_p^k = (\mu_{p,j}^k : j = 1, \ldots, n) \in \mathbb{R}_+^n$ be such that

$$h_p^k(t) \equiv \mu_{p,j}^k \qquad \forall t \in I^j, \quad \forall j = 1, \ldots, n$$

We then define $\mu^k \in \mathbb{R}_+^{n \times |\mathcal{P}|}$ to be the concatenation of all vectors μ_p^k, $p \in \mathcal{P}$. We also notice that the sequence $\{\mu^k\}_{k \geq 1}$ are uniformly bounded by the constant

$$\max_{(i, j) \in \mathcal{W}} U_{ij} \cdot \frac{n}{t_f - t_0}$$

Thus by the Bolzano-Weierstrass theorem, there exists a convergent subsequence $\{\mu^{k'}\}_{k' \geq 1}$. It is immediately verifiable that the corresponding subsequence $\{h^{k'}\}_{k' \geq 1}$ converge uniformly on $[t_0, t_f]$ and also in the L^2 norm. Moreover, by virtue of the uniform bounds U_{ij}, $(i, j) \in \mathcal{W}$, there exists a further subsequence $\{Q^{k''}\}_{k'' \geq 1}$ that converge according to the Bolzano-Weierstrass theorem. Thus, the subsequence $\{X^{k''}\}_{k'' \geq 1}$ converges with respect to the norm $\| \cdot \|_E$.

Step 2. We fix $n \geq 1$ and consider $\widetilde{\Lambda}_n$. Due to the uniform upper bound on all the elastic demands, the departure rate vectors that are piecewise constant must be pointwise and uniformly bounded. Thus, **(A3)** immediately implies that Ψ is continuous when restricted to the set of piecewise constant path departure vectors. Combining this with the fact that Θ is continuous, we conclude that $\mathcal{F}$ defined in (3.18) is continuous on $\widetilde{\Lambda}_n$. Given this and the convexity and compactness of $\widetilde{\Lambda}_n$ established earlier, Theorem 4.1 asserts that there exists some $X^{n,*} = (h^{n,*}, Q^{n,*}) \in \widetilde{\Lambda}_n$ such that

$$\langle \mathcal{F}(X^{n,*}), X^n - X^{n,*} \rangle_E \geq 0 \qquad \forall X^n \in \widetilde{\Lambda}_n \tag{B.2}$$

As a slight reformulation of (B.2), we have

$$\sum_{p \in \mathcal{P}} \int_{t_0}^{t_f} \Psi_p(t, h^{n,*}) h_p^{n,*}(t) \, dt - \sum_{(i, j) \in \mathcal{W}} \Theta_{ij}[Q^{n,*}] Q_{ij}^{n,*} \leq \sum_{p \in \mathcal{P}} \int_{t_0}^{t_f} \Psi_p(t, h^{n,*}) h_p^n(t) \, dt - \sum_{(i, j) \in \mathcal{W}} \Theta_{ij}[Q^{n,*}] Q_{ij}^n \tag{B.3}$$

for all $(h^n, Q^n) \in \widetilde{\Lambda}_n$. In particular, for all $(h^n, Q^n) \in \widetilde{\Lambda}_n$ such that

$$\sum_{p \in \mathcal{P}_{ij}} \int_{t_0}^{t_f} h_p^n(t) \, dt = Q_{ij}^{n,*} \qquad \forall (i, j) \in \mathcal{W},$$

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inequality (B.3) becomes

$$\sum_{p\in\mathcal{P}} \int_{t_0}^{t_f} \Psi_p(t, h^{n,*}) h_p^{n,*}(t) \, dt \;\le\; \sum_{p\in\mathcal{P}} \int_{t_0}^{t_f} \Psi_p(t, h^{n,*}) h_p^{n}(t) \, dt \tag{B.4}$$

Recall that $h^{n,*}(\cdot)$ is piecewise constant, thus (B.4) implies the following:

$$h_p^{n,*}(t) > 0, \quad t \in I^k \implies \int_{I^k} \Psi_p(t, h^{n,*}) \, dt \;=\; \min_{1\le j\le n} \int_{I^j} \Psi_p(t, h^{n,*}) \, dt \tag{B.5}$$

for all $p \in \mathcal{P}_{ij}$, $(i, j) \in \mathcal{W}$.

Step 3. We assert that there must exist a constant $\mathcal{M} > 0$ such that

$$h_p^{n,*}(t) \;\le\; \mathcal{M} \qquad \forall t \in [t_0, t_f], \quad \forall p \in \mathcal{P}, \quad \forall n \ge 1 \tag{B.6}$$

Indeed, given **(A2)**, we are prompted to define the following constant

$$M^{max} \;\doteq\; \max_{a\in\mathcal{A}} M_a \;<\; +\infty$$

where $\mathcal{A}$ is the set of links in the network. Recalling the constant Δ from **(A1)**, we set $\mathcal{M}$ such that

$$\mathcal{M} \;>\; \frac{3M^{max}}{\Delta + 1}$$

We claim that (B.6) holds for such $\mathcal{M}$. Otherwise, if (B.6) fails, there must exist some $m \ge 1$, $q \in \mathcal{P}$ and $1 \le j \le m$ such that

$$h_q^{m,*}(t) \;\equiv\; \lambda \;>\; \mathcal{M} \qquad t \in I^j$$

Without losing generality, we assume that $j > 1$ and consider the interval I^{j-1}. By possibly modifying the value of the function $\Psi_q(\cdot, h^{m,*})$ at one point without changing the measure-theoretic nature of the problem, we obtain $t^* \in I^{j-1}$ such that

$$\Psi_q(t^*, h^{m,*}) \;=\; \sup_{t\in I^{j-1}} \Psi_q(t, h^{m,*})$$

We denote by $\tau_q(t, h^{m,*})$ the time of arrival at destination of driver who departs at time t along path q. According to the first-in-first-out (FIFO) principle, we deduce that $\forall t \in I^j$,

$$(t - t^j)\lambda \;\le\; \int_{t^*}^{t} h_q^{m,*}(t) \, dt \;\le\; M^{max}(\tau_q(t, h^{m,*}) - \tau_q(t^*, h^{m,*}))$$

where t^j is the left boundary of the interval I^j. We then have the following estimation:

$$\begin{aligned}
&\Psi_q(t, h^{m,*}) - \Psi_q(t^*, h^{m,*}) \\
&= D_q(t, h^{m,*}) + f(\tau_q(t, h^{m,*}) - T_A) - D_q(t^*, h^{m,*}) - f(\tau_q(t^*, h^{m,*}) - T_A) \\
&\ge \tau_q(t, h^{m,*}) - \tau_q(t^*, h^{m,*}) - (t - t^*) + \Delta(\tau_q(t, h^{m,*}) - \tau_q(t^*, h^{m,*})) \\
&= (\Delta + 1)(\tau_q(t, h^{m,*}) - \tau_q(t^*, h^{m,*})) - (t - t^*) \\
&\ge (\Delta + 1)\frac{\lambda}{M^{max}}(t - t^j) - (t - t^*) \qquad \forall t \in I^j
\end{aligned} \tag{B.7}$$
$$\tag{B.8}$$

Integrating (B.8) with respect to t over interval I^j shows the following:

$$\int_{I^j} \Psi_q(t, h^{m,*}) \, dt - (t^{j+1} - t^j)\Psi_q(t^*, h^{m,*}) \;\ge\; \frac{(t^{j+1} - t^j)^2}{2} \cdot \frac{(\Delta + 1)\lambda}{M^{max}} + (t^{j+1} - t^j) \cdot \left(t^* - \frac{t^j + t^{j+1}}{2} \right) \tag{B.9}$$

where t^j, t^{j+1} are respectively the left and right boundary of I^j. Since $t^* \in I^{j-1}$, we have that

$$t^* - \frac{t^j + t^{j+1}}{2} \geq -\frac{3}{2}(t^{j+1} - t^j)$$

With this observation, (B.9) becomes

$$\int_{I^j} \Psi_q(t, h^{m,*}) \, dt - (t^{j+1} - t^j)\Psi_q(t^*, h^{m,*}) \geq \frac{(t^{j+1} - t^j)^2}{2} \cdot \frac{(\Delta + 1)\lambda}{M^{max}} - \frac{3}{2}(t^{j+1} - t^j)^2$$

$$= \frac{(t^{j+1} - t^j)^2}{2}\left(\frac{(\Delta + 1)\lambda}{M^{max}} - 3\right) > 0$$

This implies

$$\int_{I^j} \Psi_q(t, h^{m,*}) \, dt > \int_{I^{j-1}} \Psi(t, h^{m,*}) \, dt$$

which yields contradiction to (B.5). This substantiates our claim.

Step 4. With the point-wise uniform upper bound $\mathcal{M}$ on the path departure rate vectors $h^{n,*}$ for all $n \geq 1$, by taking a subsequence if necessary, one can assume the weak convergence $h^{n,*} \to h^* \in (L^2_+[t_0, t_f])^{|\mathcal{P}|}$. Let

$$Q^*_{ij} \doteq \sum_{p \in \mathcal{P}_{ij}} \int_{t_0}^{t_f} h^*_p(t) \, dt \quad \forall (i, j) \in \mathcal{W}, \qquad Q^* \doteq (Q^*_{ij} : (i, j) \in \mathcal{W})$$

We have that for any $(i, j) \in \mathcal{W}$,

$$Q^*_{ij} = \sum_{p \in \mathcal{P}_{ij}} \int_{t_0}^{t_f} h^*_p(t) \, dt = \lim_{n \to \infty} \sum_{p \in \mathcal{P}_{ij}} \int_{t_0}^{t_f} h^{n,*}_p(t) \, dt = \lim_{n \to \infty} Q^{n,*}_{ij}$$

Therefore, we have that $X^{n,*} \doteq (h^{n,*}, Q^{n,*}) \to X^* \doteq (h^*, Q^*)$ weakly. According to **(A3)**, we have that $\Psi_p(t, h^{n,*}) \to \Psi_p(t, h^*)$ uniformly for all $t \in [t_0, t_f]$ and $p \in \mathcal{P}$, and thus $\mathcal{F}(X^{n,*}) \to \mathcal{F}(X^*)$ strongly.

Finally, we show that

$$\langle \mathcal{F}(X^*), X - X^* \rangle_E \geq 0 \qquad \forall X \in \widetilde{\Lambda}$$

Indeed, we consider any $X \in \widetilde{\Lambda}$, and choose a sequence $X^n \in \widetilde{\Lambda}_n$ such that $\|X^n - X\|_E \to 0$ as $n \to \infty$. For every n, by (B.2) one has

$$\langle \mathcal{F}(X^{n,*}), X^n - X^{n,*} \rangle_E \geq 0 \tag{B.10}$$

This leads to the following:

$$\langle \mathcal{F}(X^*), X \rangle_E - \langle \mathcal{F}(X^*), X^* \rangle_E$$

$$= \left[\langle \mathcal{F}(X^*), X \rangle_E - \langle \mathcal{F}(X^{n,*}), X^n \rangle_E \right] + \langle \mathcal{F}(X^{n,*}), X^n - X^{n,*} \rangle_E$$

$$+ \langle \mathcal{F}(X^{n,*}) - \mathcal{F}(X^*), X^{n,*} \rangle_E + \langle \mathcal{F}(X^*), X^{n,*} - X^* \rangle_E$$

$$\geq \left[\langle \mathcal{F}(X^*), X \rangle_E - \langle \mathcal{F}(X^{n,*}), X^n \rangle_E \right] + \langle \mathcal{F}(X^{n,*}) - \mathcal{F}(X^*), X^{n,*} \rangle_E + \langle \mathcal{F}(X^*), X^{n,*} - X^* \rangle_E \quad \forall n \tag{B.11}$$

By strong convergence $\mathcal{F}(X^{n,*}) \to \mathcal{F}(X^*)$ and $X^n \to X$, the term in the square brackets in (B.11) converges to zero. The second term of (B.11) converges to zero due to

$$\langle \mathcal{F}(X^{n,*}) - \mathcal{F}(X^*), X^{n,*} \rangle_E \leq \|\mathcal{F}(X^{n,*}) - \mathcal{F}(X^*)\|_E \cdot \|X^{n,*}\|_E$$

and the fact that the first factor converges to zero by strong convergence, and the second factor is uniformly bounded for all n. Finally, the third term in (B.11) converges to zero by weak convergence.

Thus, taking the limit $n \to \infty$ in (B.11) yields

$$\langle \mathcal{F}(X^*), X \rangle_E - \langle \mathcal{F}(X^*), X^* \rangle_E \geq 0$$

Since X is arbitrary, X^* is a solution of the variational inequality.

$\square$

Appendix C. Proof of Theorem 5.1

Proof. According to the definition of the minimum-norm projection, at the k-th iteration, we need to solve the following minimization problem:

$$X^{k+1} = \underset{X \in \widetilde{\Lambda}_1}{\operatorname{argmin}} \left\{ \frac{1}{2} \left\| X^k - \alpha \mathcal{F}(X^k) - X \right\|^2 \right\}, \tag{C.1}$$

which is recognized as a linear-quadratic optimal control problem:

$$X^{k+1} = \left(h^{k+1},\, y^{k+1}(t_f) \right) = \underset{X=(h,\,y(t_f))}{\operatorname{argmin}} \int_{t_0}^{t_f} \frac{1}{2} \sum_{(i,j) \in \mathcal{W}} \sum_{p \in \mathcal{P}_{ij}} \left[h_p^k(t) - \alpha \Psi_p(t,\, h^k) - h_p(t) \right]^2 dt$$

$$+ \frac{1}{2} \sum_{(i,j) \in \mathcal{W}} \left(y_{ij}^k(t_f) - \alpha \Theta_{ij}^-[y^k(t_f)] - y_{ij}(t_f) \right)^2 \tag{C.2}$$

subject to

$$\frac{dy_{ij}(t)}{dt} = \sum_{p \in \mathcal{P}_{ij}} h_p(t) \qquad \forall (i,\, j) \in \mathcal{W} \tag{C.3}$$

$$y_{ij}(t_0) = 0 \qquad \forall (i,\, j) \in \mathcal{W} \tag{C.4}$$

$$h \geq 0 \tag{C.5}$$

The Hamiltonian for the above optimal control problem is

$$H = \frac{1}{2} \sum_{(i,j) \in \mathcal{W}} \sum_{p \in \mathcal{P}_{ij}} \left[h_p^k(t) - \alpha \Psi_p(t,\, h^k) - h_p(t) \right]^2 + \sum_{(i,j) \in \mathcal{W}} \lambda_{ij}(t) \sum_{p \in \mathcal{P}_{ij}} h_p(t),$$

for which the adjoint equations are

$$\frac{d\lambda_{ij}(t)}{dt} = -\frac{\partial H}{\partial y_{ij}} = 0 \qquad \forall (i,\, j) \in \mathcal{W}, \quad \forall t \in [t_0,\, t_f]; \tag{C.6}$$

and the transversality conditions read: for all $(i,\, j) \in \mathcal{W}$,

$$\lambda_{ij}(t_f) = \frac{1}{2} \frac{\partial \sum_{(k,l) \in \mathcal{W}} \left(y_{kl}^k(t_f) - \alpha \Theta_{kl}^-[y^k(t_f)] - y_{kl}(t_f) \right)^2}{\partial y_{ij}(t_f)} = -y_{ij}^k(t_f) + \alpha \Theta_{ij}^-[y^k(t_f)] + y_{ij}(t_f) \tag{C.7}$$

According to the minimum principle, we enforce the following minimization problem

$$\min_h H \quad \text{s.t.} \quad -h \leq 0,$$

for which the Kuhn-Tucker conditions are (we denote the K-T point by h^{k+1})

$$-\left[h_p^k(t) - \alpha \Psi_p(t,\, h^k) - h_p^{k+1}(t) \right] + \lambda_{ij}(t) = \rho_p(t) \qquad \forall (i,\, j) \in \mathcal{W}, \quad p \in \mathcal{P}_{ij}, \quad \forall t \in [t_0,\, t_F] \tag{C.8}$$

$$\rho_p(t) \cdot h_p^{k+1}(t) = 0 \qquad \forall (i,\, j) \in \mathcal{W}, \quad p \in \mathcal{P}_{ij}, \quad \forall t \in [t_0,\, t_F] \tag{C.9}$$

$$\rho_p(t) \geq 0 \qquad \forall (i,\, j) \in \mathcal{W}, \quad p \in \mathcal{P}_{ij}, \quad \forall t \in [t_0,\, t_F] \tag{C.10}$$

Thus, the optimality conditions for system (C.2)-(C.5) are:

$$\lambda_{ij}(t) \equiv -y_{ij}^k(t_f) + \alpha \Theta_{ij}^-[y^k(t_f)] + y_{ij}^{k+1}(t_f) \qquad \forall (i,\, j) \in \mathcal{W} \tag{C.11}$$

$$0 \le h_p^{k+1}(t) = \arg\left\{\frac{\partial H}{\partial h_p} = 0\right\} = \arg\left\{-\left(h_p^k(t) - \alpha\Psi_p(t, h^k) - h_p(t)\right) + \lambda_{ij} = 0\right\}$$

$$= \arg\left\{-\left(h_p^k(t) - \alpha\Psi_p(t, h^k) - h_p(t)\right) - y_{ij}^k(t_f) + \alpha\Theta_{ij}^-[y^k(t_f)] + y_{ij}^{k+1}(t_f) = 0\right\} \tag{C.12}$$

In other words, given $(i, j) \in \mathcal{W}$ and $p \in \mathcal{P}_{ij}$, $h_p^{k+1}(\cdot)$ is determined as

$$h_p^{k+1}(t) = \left[h_p^k(t) - \alpha\Psi_p(t, h^k) + y_{ij}^k(t_f) - \alpha\Theta_{ij}^-[y^k(t_f)] - y_{ij}^{k+1}(t_f)\right]_+$$

$$= \left[h_p^k(t) - \alpha\Psi_p(t, h^k) + Q_{ij}^k + \alpha\Theta_{ij}[Q^k] - Q_{ij}^{k+1}\right]_+ \tag{C.13}$$

where

$$Q_{ij}^k \equiv y_{ij}^k(t_f) = \sum_{p \in \mathcal{P}_{ij}} \int_{t_0}^{t_f} h_p^k(t)\,dt \qquad \forall(i, j) \in \mathcal{W}, \quad \forall k,$$

and $[x]_+ \doteq \max\{0, x\}$ for $x \in \mathbb{R}$. Notice that for all $(i, j) \in \mathcal{W}$, Q_{ij}^{k+1} must satisfy

$$Q_{ij}^{k+1} = \sum_{p \in \mathcal{P}_{ij}} \int_{t_0}^{t_f} h_p^{k+1}(t)dt = \sum_{p \in \mathcal{P}_{ij}} \int_{t_0}^{t_f} \left[h_p^k(t) - \alpha\Psi_p(t, h^k) + Q_{ij}^k + \alpha\Theta_{ij}[Q^k] - Q_{ij}^{k+1}\right]_+ dt \tag{C.14}$$

Finally, we show the existence and uniqueness of $Q_{ij}^{k+1} \in \mathbb{R}_+$ that satisfies (C.14). We slightly rewrite (C.14) as

$$\sum_{p \in \mathcal{P}_{ij}} \int_{t_0}^{t_f} \left[h_p^k(t) - \alpha\Psi_p(t, h^k) + Q_{ij}^k + \alpha\Theta_{ij}[Q^k] - Q_{ij}^{k+1}\right]_+ dt - Q_{ij}^{k+1} = 0 \tag{C.15}$$

Two cases may arise:

(i) $h_p^k(t) - \alpha\Psi_p(t, h^k) + Q_{ij}^k + \alpha\Theta_{ij}[Q^k] \le 0$ for almost every $t \in [t_0, t_f]$ and all $p \in \mathcal{P}_{ij}$. Clearly $Q_{ij}^{k+1} = 0$ is the only solution to (C.15) in this case.

(ii) $h_p^k(t) - \alpha\Psi_p(t, h^k) + Q_{ij}^k + \alpha\Theta_{ij}[Q^k] > 0$ for some $p \in \mathcal{P}_{ij}$ and for $t \in \mathcal{B} \subset [t_0, t_f]$ where $\mathcal{B}$ is a set with positive measure. We call the left hand side of (C.15) $f(Q_{ij}^{k+1})$, which is a continuous function of Q_{ij}^{k+1}. According to the hypothesis, the following hold

$$f(0) > 0, \qquad f(Q_{ij}^{k+1}) < 0 \quad \text{if} \quad Q_{ij}^{k+1} \text{ is very large}$$

Therefore, by the Intermediate Value Theorem, there must exist at least one value of Q_{ij}^{k+1} such that $f(Q_{ij}^{k+1})$ vanishes. The uniqueness of such a solution follows by observing that $f(\cdot)$, as a function of Q_{ij}^{k+1}, is strictly decreasing.

Therefore, $X^{k+1} = (h^{k+1}, Q^{k+1})$ given by (C.13) and (C.14) is unique. $\qquad\square$

Appendix D. Proof of Theorem 5.4

Proof. Using the non-expansiveness property of the projection operator, we have that

$$\left\|P_{\widetilde{\Lambda}}[X^{k+1} - \alpha\mathcal{F}(X^{k+1})] - P_{\widetilde{\Lambda}}[X^k - \alpha\mathcal{F}(X^k)]\right\|_E^2$$

$$\le \left\|X^{k+1} - \alpha\mathcal{F}(X^{k+1}) - X^k + \alpha\mathcal{F}(X^k)\right\|_E^2$$

$$= \left\|X^{k+1} - X^k\right\|_E^2 - 2\alpha\left\langle\mathcal{F}(X^{k+1}) - \mathcal{F}(X^k), X^{k+1} - X^k\right\rangle_E + \alpha^2\left\|\mathcal{F}(X^{k+1}) - \mathcal{F}(X^k)\right\|_E^2 \tag{D.1}$$

Recalling that $X^{k+1} = (h^{k+1}, Q^{k+1})$ and $X^k = (h^k, Q^k)$, we deduce from the hypotheses that

$$
\begin{aligned}
& \left\langle \mathcal{F}(X^{k+1}) - \mathcal{F}(X^k), X^{k+1} - X^k \right\rangle_E \\
= {} & \left\langle \Psi(h^{k+1}) - \Psi(h^k), h^{k+1} - h^k \right\rangle + \left\langle \Theta^-[Q^{k+1}] - \Theta^-[Q^k], Q^{k+1} - Q^k \right\rangle \\
= {} & \sum_{p \in \mathcal{P}^{sm}} \left\langle \Psi_p(h^{k+1}) - \Psi_p(h^k), h_p^{k+1} - h_p^k \right\rangle + \sum_{q \in \mathcal{P} \backslash \mathcal{P}^{sm}} \left\langle \Psi_q(h^{k+1}) - \Psi_q(h^k), h_q^{k+1} - h_q^k \right\rangle \\
& + \left\langle \Theta^-[Q^{k+1}] - \Theta^-[Q^k], Q^{k+1} - Q^k \right\rangle \\
\geq {} & K^{sm} \sum_{p \in \mathcal{P}^{sm}} \left\| h_p^{k+1} - h_p^k \right\|^2 - K^{wm} \sum_{q \in \mathcal{P} \backslash \mathcal{P}^{sm}} \left\| h_q^{k+1} - h_q^k \right\|^2 + K_2 \left\| Q^{k+1} - Q^k \right\|^2 \\
\geq {} & K^{sm} \sum_{p \in \mathcal{P}^{sm}} \left\| h_p^{k+1} - h_p^k \right\|^2 - K^{wm} M \sum_{p \in \mathcal{P}^{sm}} \left\| h_p^{k+1} - h_p^k \right\|^2 + K_2 \left\| Q^{k+1} - Q^k \right\|^2 \\
= {} & (K^{sm} - K^{wm} M) \sum_{p \in \mathcal{P}^{sm}} \left\| h_p^{k+1} - h_p^k \right\|^2 + K_2 \left\| Q^{k+1} - Q^k \right\|^2 \\
\geq {} & \frac{K^{sm} - K^{wm} M}{M + 1} \left\| h^{k+1} - h^k \right\|_{L^2}^2 + K_2 \left\| Q^{k+1} - Q^k \right\|^2
\end{aligned}
\tag{D.2}
$$

Combining (D.1) and (D.2) yields

$$
\begin{aligned}
& \left\| P_{\widetilde{\Lambda}}[X^{k+1} - \alpha \mathcal{F}(X^{k+1})] - P_{\widetilde{\Lambda}}[X^k - \alpha \mathcal{F}(X^k)] \right\|_E^2 \\
\leq {} & \left\| X^{k+1} - X^k \right\|_E^2 - 2\alpha \left(\frac{K^{sm} - K^{wm} M}{M + 1} \left\| h^{k+1} - h^k \right\|_{L^2}^2 + K_2 \left\| Q^{k+1} - Q^k \right\|^2 \right) \\
& + \alpha^2 \left(\left\| \Psi(h^{k+1}) - \Psi(h^k) \right\|_{L^2}^2 + \left\| \Theta^-[Q^{k+1}] - \Theta^-[Q^k] \right\|^2 \right) \\
\leq {} & \left\| X^{k+1} - X^k \right\|_E^2 - 2\alpha \left(\frac{K^{sm} - K^{wm} M}{M + 1} \left\| h^{k+1} - h^k \right\|_{L^2}^2 + K_2 \left\| Q^{k+1} - Q^k \right\|^2 \right) \\
& + \alpha^2 \left(L_1^2 \left\| h^{k+1} - h^k \right\|_{L^2}^2 + L_2^2 \left\| Q^{k+1} - Q^k \right\|^2 \right) \\
\leq {} & \left\| X^{k+1} - X^k \right\|_E^2 - 2\alpha \min \left\{ \frac{K^{sm} - K^{wm} M}{M + 1}, K_2 \right\} \left\| X^{k+1} - X^k \right\|_E^2 + \alpha^2 \max \left\{ L_1^2, L_2^2 \right\} \left\| X^{k+1} - X^k \right\|_E^2 \\
= {} & \left(1 - 2\alpha \min \left\{ \frac{K^{sm} - K^{wm} M}{M + 1}, K_2 \right\} + \alpha^2 \max \left\{ L_1^2, L_2^2 \right\} \right) \cdot \left\| X^{k+1} - X^k \right\|_E^2
\end{aligned}
$$

Thus, by setting

$$
1 - 2\alpha \min \left\{ \frac{K^{sm} - K^{wm} M}{M + 1}, K_2 \right\} + \alpha^2 \max \left\{ L_1^2, L_2^2 \right\} < 1
$$

that is,

$$
\alpha < \frac{2 \min \left\{ \frac{K^{sm} - K^{wm} M}{M + 1}, K_2 \right\}}{\max \left\{ L_1^2, L_2^2 \right\}}
\tag{D.3}
$$

we obtain the contracting property of the map $X^k \mapsto P_{\widetilde{\Lambda}}[X^k - \alpha \mathcal{F}(X^k)]$. The convergence follows from the contracting mapping theorem (Rudin, 2006). □

Appendix E. Proof of Lemma 5.9

Proof. Let $X^d \in Y^d$. We have that for any $k \geq 0$,

$$
\left\| X^k - X^d \right\|_E^2 = \left\| X^{k+1} - X^k + X^d - X^{k+1} \right\|_E^2 = \left\| X^{k+1} - X^k \right\|_E^2 + \left\| X^d - X^{k+1} \right\|_E^2 + 2 \left\langle X^{k+1} - X^k, X^d - X^{k+1} \right\rangle_E
\tag{E.1}
$$

Taking $X = X^d$ in (5.37), and combining this with (5.38) yields

$$0 \geq \left\langle \mathcal{F}(X^{k+1}), X^d - X^{k+1} \right\rangle_E \geq -a \left\langle X^{k+1} - X^k, X^d - X^{k+1} \right\rangle_E \tag{E.2}$$

(E.1) and (E.2) together yield that for all $k \geq 0$,

$$\left\| X^{k+1} - X^k \right\|_E^2 \leq \left\| X^k - X^d \right\|_E^2 - \left\| X^{k+1} - X^d \right\|_E^2 \tag{E.3}$$

Summing up (E.3) over different values of k yields

$$(k+1) \min_{0 \leq i \leq k} \left\| X^{i+1} - X^i \right\|_E^2 \leq \sum_{i=0}^{k} \left\| X^{i+1} - X^i \right\|_E^2 \leq \left\| X^0 - X^d \right\|_E^2 - \left\| X^{k+1} - X^d \right\|_E^2 \tag{E.4}$$

For each $k \geq 0$, we introduce the notation $\mu(k) \doteq \operatorname{argmin}_{0 \leq i \leq k} \left\| X^{i+1} - X^i \right\|_E^2$. Then (E.4) implies

$$(k+1) \left\| X^{\mu(k)+1} - X^{\mu(k)} \right\|_E^2 \leq \left\| X^0 - X^d \right\|_E^2 \leq D^2$$

and thus

$$\left\| X^{\mu(k)+1} - X^{\mu(k)} \right\|_E^2 \leq \frac{1}{k+1} D^2 \tag{E.5}$$

where D denotes the diameter of the feasible set $\widetilde{\Lambda}$. By invoking (5.37), we have

$$\left\langle \mathcal{F}(X^{\mu(k)+1} + a(X^{\mu(k)+1} - X^{\mu(k)}), \, X - X^{\mu(k)+1} \right\rangle_E \geq 0 \qquad \forall X \in \widetilde{\Lambda},$$

which immediately leads to

$$
\begin{aligned}
\left\langle \mathcal{F}(X^{\mu(k)+1}), \, X - X^{\mu(k)+1} \right\rangle_E &\geq -a \left\langle X^{\mu(k)+1} - X^{\mu(k)}, \, X - X^{\mu(k)+1} \right\rangle_E \\
&\geq -a \left\| X^{\mu(k)+1} - X^{\mu(k)} \right\|_E \cdot \left\| X - X^{\mu(k)+1} \right\|_E \\
&\geq -a \left\| X^{\mu(k)+1} - X^{\mu(k)} \right\|_E \cdot D \\
&\geq -a \sqrt{\frac{1}{k+1} D^2} \cdot D \qquad \forall X \in \widetilde{\Lambda}
\end{aligned}
\tag{E.6}
$$

$\square$

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