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# Elastic Dislocation Theory for a Self-Gravitating Elastic Configuration with an Initial Static Stress Field

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*Summary*

The linearized equations of motion and linearized boundary and continuity conditions governing small elastic-gravitational disturbances away from equilibrium of an arbitrary, uniformly rotating, self-gravitating, perfectly elastic Earth model with an arbitrary initial static stress field are derived. The appropriate form of Rayleigh's variational principle and of the Betti reciprocal theorem and the Volterra dislocation relation for such a configuration are given. The latter is then used to derive an explicit expression for the equivalent body forces to be applied in the absence of a seismic dislocation in order to produce a dynamical response of the Earth model equivalent to that produced by the dislocation. It is found that if the initial static stress in the vicinity of the dislocation is purely hydrostatic, then a point tangential displacement dislocation has as an exactly equivalent body force the familiar double couple of moment $\mu_0 A_0 \Delta s_0$. If however the hypocentral static stress field has a deviatoric part, then additional equivalent body forces must be used properly to model a seismic dislocation. The necessary additional equivalent forces are explicitly exhibited; theoretically their existence provides a method of estimating hypocentral stresses, but the application of any such method is probably premature.

## 1. Introduction

Burridge & Knopoff (1964) have derived an explicit expression which gives the equivalent body force which must be applied to an elastic medium in the absence of a seismic fault dislocation in order to produce a radiated displacement field equivalent to that produced by the dislocation. One of their results which has since found a great deal of application is the fact that in an isotropic elastic medium, a point tangential displacement dislocation of an amount $\Delta s_0$ over a fault area $A_0$ has as an exactly equivalent body force a double couple with each couple having a net moment $\mu_0 A_0 \Delta s_0$, where $\mu_0$ is the rigidity at the source. Burridge & Knopoff (1964) treated the case of seismic dislocations in an arbitrary, inhomogeneous, anisotropic, perfectly elastic medium, but they did not consider the possibility of an initial static stress field existing in the medium. This paper represents for the most part a straightforward extension of their results for the case of a seismic dislocation in an arbitrary, uniformly rotating, self-gravitating, perfectly elastic configuration in which there exists an initial static stress field at all points.

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The linearized equations of motion and the continuity and boundary conditions governing an elastic deformation in the presence of an initial static stress have been derived and discussed extensively by Biot (1965). An independent and alternative derivation of these equations for the case of a self-gravitating configuration is given here. The derivation, following Backus (1967), proceeds by a linearization of the familiar Eulerian equations expressing the conservation of mass and momentum. The linearized partial differential equations thus obtained must be satisfied at all interior points of the initial undeformed or reference configuration; the final form is completely equivalent to the equations given by Biot (1965). The exact boundary and continuity conditions expressed on the deformed boundary of the disturbed configuration may also be linearized and expressed as boundary and continuity conditions on the corresponding undeformed boundary of the reference configuration. It is found that in the presence of an initial static stress, the linearization of the normal stress boundary condition leads not to the condition that the normal elastic stress be continuous across the undeformed boundary, but rather to the condition that the normal components of a *non-symmetric* tensor, here called the incremental pseudo-stress tensor, be continuous on the undeformed boundary. The linearized continuity conditions used here are the same as those derived by Biot (1965).

The linearized equations of motion and continuity and boundary conditions in the reference configuration are then used to derive the extension to the case of an elastic configuration with an initial static stress field of several familiar elastodynamic results. In particular, Rayleigh's variational principle is given following Backus & Gilbert (1967), and is then used to derive the appropriate form of the Betti reciprocal theorem which in turn is used to confirm the validity of the Helmholtz reciprocity theorem. The Betti reciprocal theorem is then applied to the case of an elastic configuration into which there is introduced a fault surface $\sum_0$ across which there is a tangential displacement discontinuity. Because of the relative motion of the material on one side of the fault surface with respect to the material on the other side, the linearized normal stress continuity condition across the fault surface differs from that appropriate to a welded boundary. The proper fault surface continuity conditions are utilized together with the Betti reciprocal theorem to derive the appropriate extension of the fundamental equation of elastodynamic dislocation theory, the Volterra dislocation relation. The Volterra dislocation relation is then used to obtain the equivalent body forces which should be applied to the elastic configuration in the absence of a seismic displacement dislocation in order to produce a dynamical elastic-gravitational displacement field equivalent to that produced by the dislocation.

It is shown that in the case of a tangential displacement dislocation fault in an isotropic, self-gravitating, elastic Earth model with a *purely hydrostatic* initial stress, the modified version of the Volterra dislocation relation derived here reduces exactly to the simpler version valid in the case of zero initial stress. Thus a point tangential displacement dislocation with slip-area $A_0 \Delta s_0$ remains dynamically equivalent to a double couple equivalent force with moment $\mu_0 A_0 \Delta s_0$ so long as the initial static stress in the source region is purely hydrostatic. If on the other hand there is a non-vanishing deviatoric initial stress in the source region, then additional equivalent forces are required to model a point tangential displacement dislocation. These additional equivalent forces are explicitly exhibited for an arbitrary deviatoric initial stress. Theoretically, the existence of these additional equivalent forces depending on the components of the deviatoric initial stress in the source region could allow one to determine the magnitude and orientation of the shear stresses at the source of an earthquake solely from a consideration of the geometry of the observed radiation pattern.

This paper also represents a correction to work which has been previously published by the author. Dahlen (1968) considered Rayleigh's variational principle for a self-gravitating elastic configuration with an initial anisotropic stress field, but since the wrong linearized boundary conditions were employed, some of the expressions contained in that paper are faulty. Dahlen (1971) attempted to derive the Betti reciprocal theorem and the Volterra dislocation relation for a similar situation, but once again employed the wrong linearized boundary conditions and that derivation is invalid. The errors contained in these two previous papers are corrected in this one. Fortunately, none of the final results of either of these two previous papers is affected by the modifications noted here.

## 2. Equations of motion

Consider an initial model of the Earth consisting of a self-gravitating, solid, elastic continuum occupying a volume $V$ with surface $\partial V$. Assume that this Earth model is initially in a steady-state configuration with a steady angular velocity of rotation $\mathbf{\Omega}$ about its centre of mass and with a static, in general non-isotropic, initial stress field $\mathbf{T}_0(\mathbf{r})$. Let $\rho_0(\mathbf{r})$ denote the density and $\phi_0(\mathbf{r})$ denote the gravitational potential of the body occupying the volume $V$. The equilibrium condition for this initial steady-state configuration is

$$\rho_0 \nabla(\phi_0+\psi) = \nabla\cdot\mathbf{T}_0 \tag{1}$$

together with Poisson's equation

$$\nabla^2 \phi_0 = 4\pi G\rho_0. \tag{2}$$

Here $\psi(\mathbf{r})$ is the rotational potential due to the centripetal acceleration

$$\psi(\mathbf{r}) = -\tfrac{1}{2}[\Omega^2 r^2-(\mathbf{\Omega}\cdot\mathbf{r})^2] \tag{3}$$

and $G$ is Newton's universal constant of gravitation. If one defines an initial hydrostatic pressure field $p_0(\mathbf{r})$ and an initial static stress deviator $\boldsymbol{\tau}_0(\mathbf{r})$ by the relations

$$\begin{aligned} p_0 &= -\tfrac{1}{3}\operatorname{tr}\mathbf{T}_0 \\ \boldsymbol{\tau}_0 &= \mathbf{T}_0+p_0\mathbf{I} \end{aligned} \tag{4}$$

where $\mathbf{I}$ is the second order identity tensor, then equation (1) takes the form

$$\nabla p_0+\rho_0\nabla(\phi_0+\psi) = \nabla\cdot\boldsymbol{\tau}_0. \tag{5}$$

Now if, beginning at time $t = 0$, this body is slightly disturbed by a time-dependent external body force per unit volume $\mathbf{f}(\mathbf{r}, t)$, a time-dependent particle displacement field $\mathbf{s}(\mathbf{r}, t)$ will be set up; this displacement field will change both the volume $V(t)$ and the surface $\partial V(t)$ of the body and will be accompanied by perturbations $\rho_0(\mathbf{r}, t)$ in the density, $\phi_1(\mathbf{r}, t)$ in the gravitational potential, and $\mathbf{T}_E(\mathbf{r}, t)$ in the stress tensor at a fixed location in space. The exact equations governing the motion about the equilibrium configuration are

$$(\rho_0+\rho_1)\left[\frac{D^2\mathbf{s}}{Dt^2}+2\mathbf{\Omega}\times\frac{D\mathbf{s}}{Dt}\right] = \nabla\cdot(\mathbf{T}_0+\mathbf{T}_E)-(\rho_0+\rho_1)\nabla(\phi_0+\psi+\phi_1)+\mathbf{f} \tag{6}$$

where $D/Dt$ is the substantial or Lagrangian time derivative. Neglecting terms of second order in the displacement $\mathbf{s}$ and subtracting (1) from (6), one obtains

$$\rho_0\partial t^2\mathbf{s}+2\rho_0\mathbf{\Omega}\times\partial_t\mathbf{s} = \nabla\cdot\mathbf{T}_E-\rho_0\nabla\phi_1-\rho_1\nabla(\phi_0+\psi)+\mathbf{f}. \tag{7}$$

It is useful to be able to identify individual material particles in the body $V$; the usual Lagrangian convention is to label a material particle $\mathbf{r}$ by its location $\mathbf{r}$ at time

$t = 0$. Now $\mathbf{T}_0(\mathbf{r}, t)+\mathbf{T}_E(\mathbf{r}, t)$ is the value of the stress tensor at a fixed location $\mathbf{r}$ in space. To first order in $\mathbf{s}$, the value of the stress tensor at the fixed material particle denoted by $\mathbf{r}$ is given by

$$\mathbf{T}_0+\mathbf{T} = (1+\mathbf{s}\cdot\nabla)(\mathbf{T}_0+\mathbf{T}_E) \tag{8}$$

or neglecting the second-order term $\mathbf{s}\cdot\nabla\mathbf{T}_E$,

$$\mathbf{T} = \mathbf{T}_E+\mathbf{s}\cdot\nabla\mathbf{T}_0, \tag{9}$$

In the next section it will be shown how the incremental stress $\mathbf{T}(\mathbf{r}, t)$ at the material particle $\mathbf{r}$ may be related to the displacement $\mathbf{s}(\mathbf{r}, t)$ of that particle and of those surrounding it. It is therefore convenient to rewrite the linearized equations of motion in the form

$$\rho_0\,\partial_t^2\,\mathbf{s}+2\rho_0\,\boldsymbol{\Omega}\times\partial_t\,\mathbf{s} = -\rho_0\,\nabla\phi_1-\rho_1\,\nabla(\phi_0+\psi)+\nabla\cdot\mathbf{T}-\nabla\cdot(\mathbf{s}\cdot\nabla\mathbf{T}_0)+\mathbf{f}. \tag{10}$$

The continuity equation and Poisson's equation may also be linearized.

$$\begin{aligned}\rho_1 &= -\nabla\cdot(\rho_0\,\mathbf{s})\\ \nabla^2\phi_1 &= 4\pi G\rho_1.\end{aligned} \tag{11}$$

The equations (10) may also be written in a form in which the static stress deviator $\boldsymbol{\tau}_0(\mathbf{r})$ appears explicitly.

$$\begin{aligned}\rho_0\,\partial_t^2\,\mathbf{s}+2\rho_0\,\boldsymbol{\Omega}\times\partial_t\,\mathbf{s} = {}& -\rho_0\,\nabla\phi_1-\rho_1\,\nabla(\phi_0+\psi)\\ &-\nabla[\mathbf{s}\cdot\rho_0\,\nabla(\phi_0+\psi)]+\nabla\cdot\mathbf{T}+\nabla[\mathbf{s}\cdot(\nabla\cdot\boldsymbol{\tau}_0)]-\nabla\cdot(\mathbf{s}\cdot\nabla\boldsymbol{\tau}_0)+\mathbf{f}.\end{aligned} \tag{12}$$

The set of equations (11) combined with the equations of motion in either the form (10) or (12) must be satisfied at all interior points $\mathbf{r}$ of the undeformed or reference configuration $V$. All the time dependent first-order quantities $\mathbf{s}(\mathbf{r}, t)$, $\phi_1(\mathbf{r}, t)$, $\rho_1(\mathbf{r}, t)$, $\mathbf{f}(\mathbf{r}, t)$, $\mathbf{T}(\mathbf{r}, t)$ $\mathbf{f}(\mathbf{r}, t)$, appearing in these equations may be considered to first order in $\mathbf{s}$ to be the value of that quantity at the point $\mathbf{r}$ in the reference configuration at time $t$. If these equations are to be solved they must be completed by the addition of a constitutive relation defining the Lagrangian incremental stress $\mathbf{T}(\mathbf{r}, t)$ in terms of $\mathbf{s}(\mathbf{r}, t)$ and $\mathbf{T}_0(\mathbf{r})$.

## 3. The linearized perfectly elastic constitutive relation

Define the infinitesimal deformation tensor at the fixed material particle $\mathbf{r}$ to be $\nabla\mathbf{s}(\mathbf{r}, t)$. If the material comprising the Earth is assumed to be perfectly elastic, then the stress $\mathbf{T}(\mathbf{r}, t)$ at the material particle $\mathbf{r}$ depends only on the value of the infinitesimal deformation tensor $\nabla\mathbf{s}(\mathbf{r}, t)$ at $\mathbf{r}$ and on the entropy (or temperature) at $\mathbf{r}$. Assume that the deformation under consideration occurs isentropically and furthermore that the relation between $\mathbf{T}(\mathbf{r}, t)$ and $\nabla\mathbf{s}(\mathbf{r}, t)$ may be linearized. In that case the components of $\mathbf{T}(\mathbf{r}, t)$ relative to an arbitrary Cartesian axis system $\hat{\mathbf{x}}_1, \hat{\mathbf{x}}_2, \hat{\mathbf{x}}_3$ in the uniformly rotating reference frame may be expressed in terms of the components of $\nabla\mathbf{s}(\mathbf{r}, t)$ in the following manner

$$T_{ij} = \Gamma_{ijkl}\,\partial_k\,s_l. \tag{13}$$

The coefficients $\Gamma_{ijkl}(\mathbf{r})$ will be called the first set of linear isentropic elastic coefficients; it is clear from (13) that they are the Cartesian components of a fourth-order tensor. Since $\mathbf{T}(\mathbf{r}, t)$ is symmetric,

$$\Gamma_{ijkl} = \Gamma_{jikl} \tag{14}$$

There are thus 21 independent components $C_{ijkl}(\mathbf{r})$.

and there are only 54 independent components $\Gamma_{ijkl}(\mathbf{r})$.

Now define the strain tensor $\boldsymbol{\sigma}(\mathbf{r}, t)$ and the infinitesimal rotation tensor $\boldsymbol{\omega}(\mathbf{r}, t)$ at the material particle $\mathbf{r}$ by the usual relations

$$\left.\begin{aligned} \sigma_{ij} &= \tfrac{1}{2}(\partial_j s_i + \partial_i s_j) \\ \omega_{ij} &= \tfrac{1}{2}(\partial_j s_i - \partial_i s_j). \end{aligned}\right\} \tag{15}$$

The incremental stress $\mathbf{T}(\mathbf{r}, t)$ at the material particle $\mathbf{r}$ consists of two parts; one part may be called the elastic stress $\mathbf{E}(\mathbf{r}, t)$ and depends only on the strain $\boldsymbol{\sigma}(\mathbf{r}, t)$, the other part arises because the infinitesimal rotation $\boldsymbol{\omega}(\mathbf{r}, t)$ acts to rotate the initial static stress $\mathbf{T}_0(\mathbf{r})$. It may be shown that to first order in $\mathbf{s}$, $\mathbf{T}(\mathbf{r}, t)$ may be written in terms of $\boldsymbol{\sigma}(\mathbf{r}, t)$ and $\boldsymbol{\omega}(\mathbf{r}, t)$ in the form

$$T_{ij} = E_{ij} + \omega_{ik} T_{kj}{}^0 - T_{ik}{}^0 \omega_{kj} \tag{16}$$

where the elastic stress $\mathbf{E}(\mathbf{r}, t)$ is by assumption linearly related to the strain $\boldsymbol{\sigma}(\mathbf{r}, t)$.

$$E_{ij} = B_{ijkl} \sigma_{kl}. \tag{17}$$

The total stress at the material particle $\mathbf{r}$ is given by

$$T_{ij}{}^0 + T_{ij} = T_{ij}{}^0 + E_{ij} + \omega_{ik} T_{kj}{}^0 - T_{ik}{}^0 \omega_{kj}. \tag{18}$$

The final two terms on the right-hand side of equation (18) represent to first order in $\mathbf{s}$ the Cartesian components of the rotated static stress tensor at the material particle $\mathbf{r}$. The coefficients $B_{ijkl}(\mathbf{r})$ in (17) will be called the second set of linear isentropic elastic coefficients; these are also the Cartesian components of a fourth-order tensor. Since $\boldsymbol{\sigma}(\mathbf{r}, t)$ is symmetric, the $B_{ijkl}(\mathbf{r})$ satisfy the symmetry relations

$$B_{ijkl} = B_{jikl} = B_{ijlk} \tag{19}$$

and there are only 36 independent components $B_{ijkl}(\mathbf{r})$. The elastic coefficients $\Gamma_{ijkl}(\mathbf{r})$ may be expressed in terms of the elastic coefficients $B_{ijkl}(\mathbf{r})$ from (13), (16) and (17) by the relation

$$\Gamma_{ijkl} = B_{ijkl} + \tfrac{1}{2}(T_{ik}{}^0 \delta_{jl} + T_{jk}{}^0 \delta_{il} - T_{il}{}^0 \delta_{jk} - T_{jl}{}^0 \delta_{ik}) \tag{20}$$

where $\delta_{ij}$ are the Cartesian components of the second-order identity tensor $\mathbf{I}$.

A consideration of the second law of thermodynamics for a perfectly elastic medium allows one to deduce a further relation among the elastic coefficients $B_{ijkl}(\mathbf{r})$, namely

$$B_{ijkl} - B_{klij} = T_{kl}{}^0 \delta_{ij} - T_{ij}{}^0 \delta_{kl}. \tag{21}$$

It is thus convenient to define a third set of linear isentropic elastic coefficients $C_{ijkl}(\mathbf{r})$ by the relation

$$B_{ijkl} = C_{ijkl} + \tfrac{1}{2}(T_{kl}{}^0 \delta_{ij} - T_{ij}{}^0 \delta_{kl}). \tag{22}$$

The coefficients $C_{ijkl}(\mathbf{r})$ are also the Cartesian components of a fourth-order tensor; furthermore

$$C_{ijkl} = C_{jikl} = C_{ijlk} = C_{klij}. \tag{23}$$

The relation (16) between $\mathbf{T}(\mathbf{r}, t)$ and $\boldsymbol{\sigma}(\mathbf{r}, t)$ and $\boldsymbol{\omega}(\mathbf{r}, t)$ may finally be written in the form

$$T_{ij} = C_{ijkl}\,\sigma_{kl} + \tfrac{1}{2}(T_{kl}{}^{0}\,\sigma_{kl})\,\delta_{ij} - \tfrac{1}{2}T_{ij}{}^{0}\,\sigma_{kk} + \omega_{ik}\,T_{kj}{}^{0} - T_{ik}{}^{0}\,\omega_{kj}. \tag{24}$$

In the case of a purely hydrostatic initial stress $\mathbf{T}_0(\mathbf{r}) = -p_0(\mathbf{r})\,\mathbf{I}$, (24) reduces to

$$T_{ij} = E_{ij} = C_{ijkl}\,\sigma_{kl} \tag{25}$$

which is the familiar linearized isentropic stress-strain relation involving 21 independent elastic coefficients. Hence (24) may be rewritten in terms only of the deviatoric part $\boldsymbol{\tau}_0(\mathbf{r})$ of the initial static stress

$$T_{ij} = C_{ijkl}\,\sigma_{kl} + \tfrac{1}{2}(\tau_{kl}{}^{0}\,\sigma_{kl})\,\delta_{ij} - \tfrac{1}{2}\tau_{ij}{}^{0}\,\sigma_{kk} + \omega_{ik}\,\tau_{kj}{}^{0} - \tau_{ik}{}^{0}\,\omega_{kj}. \tag{26}$$

Equation (24) or (26) is the most general form of the linearized perfectly elastic constitutive relation which must be used in order to complete the equations of motion (11) and (10) or (12).

In general the elastic properties of an arbitrary perfectly elastic material will have to be described in terms of the 21 independent linear isentropic elastic coefficients $C_{ijkl}(\mathbf{r})$. It is customary in almost all applications to treat the material comprising the Earth as an isotropic elastic medium. In that case the elastic coefficients $C_{ijkl}(\mathbf{r})$ may be expressed in terms of only two parameters, the isentropic bulk modulus $\kappa(\mathbf{r})$ and the isentropic rigidity $\mu(\mathbf{r})$.

$$C_{ijkl} = (\kappa - \tfrac{2}{3}\mu)\,\delta_{ij}\,\delta_{kl} + \mu(\delta_{ik}\,\delta_{jl} + \delta_{il}\,\delta_{jk}). \tag{27}$$

The equations (11), (10) and (24) or alternatively (11), (12) and (26) must be satisfied at all interior points $\mathbf{r}$ of the undeformed reference configuration $V$, subject to certain boundary or continuity conditions on the undeformed boundary $\partial V$. The next section will consider the linearization of the exact elastic-gravitational boundary and continuity conditions on the deformed boundary $\partial V(t)$ in order to yield the appropriate conditions on the undeformed boundary $\partial V$.

## 4. Boundary conditions

Denote the unit outward normal to the initial undeformed surface $\partial V$ at the position of the material particle $\mathbf{r}$ by $\hat{\mathbf{n}}(\mathbf{r})$, and denote the corresponding unit outward normal to the deformed surface $\partial V(t)$ by $\hat{\mathbf{n}}(\mathbf{r}, t)$. The elastic-gravitational motion $\mathbf{s}(\mathbf{r}, t)$, $\phi_1(\mathbf{r}, t)$ of the body $V$ must satisfy the following exact continuity conditions on the deformed boundary $\partial V(t)$ at all times. First of all, the three quantities $\mathbf{s}(\mathbf{r}, t)$, $\phi_1(\mathbf{r}, t)$, and $\hat{\mathbf{n}}(\mathbf{r}, t)\cdot\nabla\phi_1(\mathbf{r}, t)$ must be continuous at all points $\mathbf{r}$ across the deformed boundary $\partial V(t)$. There is also a continuity condition on the surface stress or traction which must be given special attention in the case of an elastic configuration with an initial static stress field $\mathbf{T}_0(\mathbf{r})$.

Consider an arbitrary small simply connected surface element $dA$ centred on an arbitrary material particle $\mathbf{r}$ located on the undeformed surface $\partial V$ at time $t = 0$. At time $t$ the material particles comprising this surface element, now denoted by $dA(t)$, will have moved to a new location in space and the unit outward normal to the surface element will have changed from $\hat{\mathbf{n}}(\mathbf{r})$ at time $t = 0$ to $\hat{\mathbf{n}}(\mathbf{r}, t)$ at time $t$ (see Fig. 1). The net force $\mathscr{F}_{dA}(t)$ exerted on this surface element $dA(t)$ by the total stress field $\mathbf{T}_0(\mathbf{r}) + \mathbf{T}(\mathbf{r}, t)$ may be written in the form of a surface integral over the surface element $dA(t)$.

$$\mathscr{F}_{dA}(t) = \int_{dA(t)} dA\,\hat{\mathbf{n}}(\mathbf{r}, t)\cdot[\mathbf{T}_0(\mathbf{r}) + \mathbf{T}(\mathbf{r}, t)]. \tag{28}$$

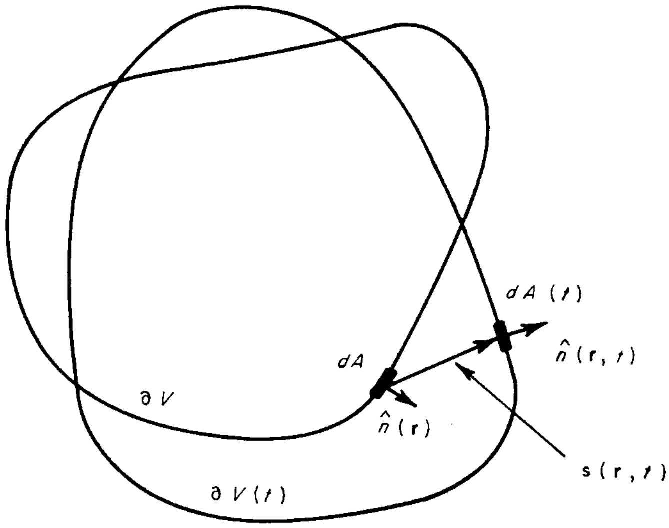


FIG. 1. Schematic diagram of the motion $\mathbf{s}(\mathbf{r}, t)$ of an infinitesimal surface element $dA(t)$.

The exact boundary condition on the surface stress or traction may be expressed by the requirement that this net force surface integral $\mathscr{F}_{dA}(t)$ must be continuous across $\partial V(t)$ when taken over any arbitary surface element $dA(t)$.

It is desirable to reduce these exact continuity conditions to corresponding continuity conditions across the undeformed boundary $\partial V$; this is done by linearization. To first order in $\mathbf{s}$, both $\mathbf{s}(\mathbf{r}, t)$ and $\phi_1(\mathbf{r}, t)$ must be continuous across the undeformed boundary $\partial V$. The linearization of the continuity condition on $\hat{\mathbf{n}}\cdot\nabla\phi_1$ gives rise to the condition that $\hat{\mathbf{n}}(\mathbf{r})\cdot\nabla\phi_1(\mathbf{r}, t)+4\pi G\rho_0(\mathbf{r})\,\hat{\mathbf{n}}(\mathbf{r})\cdot\mathbf{s}(\mathbf{r}, t)$ must be continuous across the undeformed boundary $\partial V$.

Now consider the linearization of the right-hand side of (28). To first order in $\mathbf{s}$, it may be shown that $\hat{\mathbf{n}}(\mathbf{r}, t)\, dA(t)$ on the deformed surface $\partial V(t)$ may be related to $\hat{\mathbf{n}}(\mathbf{r})\, dA$ on the corresponding undeformed surface $\partial V$ by the relation

$$n_i(\mathbf{r}, t)\, dA(t) = [(1+\partial_k s_k)\delta_{ij}-\partial_i s_j]\, n_j(\mathbf{r})\, dA. \tag{29}$$

The dilatation term in (29) arises from the stretching of the surface element $dA(t)$ and the other term arises from the deflection of the normal $\hat{\mathbf{n}}(\mathbf{r}, t)$. Using (29) and using the linearized form (24) for $\mathbf{T}(\mathbf{r}, t)$, one can transform the surface integral (28) over the patch $dA(t)$ into a surface integral over the corresponding undeformed patch $dA$. Carrying out this transformation, the net force acting on a deformed surface element $dA(t)$ may be written, to first order in $\mathbf{s}$

$$\mathscr{F}_{dA}(t) = \int_{dA} dA\, \hat{\mathbf{n}}(\mathbf{r})\cdot[\mathbf{T}_0(\mathbf{r})+\check{\mathbf{T}}(\mathbf{r})] \tag{30}$$

where

$$\check{T}_{ij} = C_{ijkl}\,\sigma_{kl}+\omega_{ik}\,T_{kj}{}^0-T_{ik}{}^0\,\omega_{kj}+\tfrac{1}{2}\sigma_{kk}\,T_{ij}{}^0+\tfrac{1}{2}T_{kl}{}^0\,\sigma_{kl}\delta_{ij}-T_{jk}{}^0\,\partial_k s_i. \tag{31}$$

Equation (30) expresses the net force on an arbitrary deformed surface element $dA(t)$ in terms of a surface integral taken over the corresponding undeformed surface element $dA$. Since $dA$ is an *arbitrary* element on the undeformed surface $\partial V$, and since $\hat{\mathbf{n}}(\mathbf{r})\cdot\mathbf{T}_0(r)$ is already continuous, the linearized surface traction continuity condition is equivalent to the point-wise condition that $\hat{\mathbf{n}}(\mathbf{r})\cdot\mathbf{\tilde{T}}(\mathbf{r}, t)$ be continuous at all points $\mathbf{r}$ of $\partial V$.

It is clear from the derivation that the surface under consideration need not necessarily be the external boundary surface $\partial V$, and that in fact all of the linearized continuity conditions must be satisfied at all undeformed welded material surfaces within the body $V$. In summary the linearized continuity conditions which must be applied at all points $\mathbf{r}$ of the undeformed boundary surface $\partial V$ (or of any internal material discontinuity surface) are

$$\left.\begin{array}{l} \mathbf{s}(\mathbf{r}, t) \text{ continuous} \\ \hat{\mathbf{n}}(\mathbf{r})\cdot\mathbf{\tilde{T}}(\mathbf{r}, t) \text{ continuous} \\ \quad \text{(note: since the external surface } \partial V \text{ is a free surface,} \\ \quad \hat{\mathbf{n}}(\mathbf{r})\cdot\mathbf{\tilde{T}}(\mathbf{r}, t) = 0 \text{ on } \partial V.) \\ \phi_1(\mathbf{r}, t) \text{ continuous} \\ \hat{\mathbf{n}}(\mathbf{r})\cdot\nabla\phi_1(\mathbf{r}, t)+4\pi G\rho_0(\mathbf{r})\,\hat{\mathbf{n}}(\mathbf{r})\cdot\mathbf{s}(\mathbf{r}, t) \text{ continuous.} \end{array}\right\} \tag{32}$$

The tensor $\mathbf{\tilde{T}}(\mathbf{r}, t)$, whose Cartesian components are given in equation (31), will be called the incremental pseudo-stress tensor; it is in fact the incremental part of the Piola–Kirchoff stress tensor which is discussed in most modern texts on continuum mechanics (e.g. Eringen 1967; Malvern 1969). Equation (31) may also be written in the more convenient form

$$\tilde{T}_{ij} = \Lambda_{ijkl}\,\partial_k\,\mathbf{s}_l \tag{33}$$

where

$$\Lambda_{ijkl} = C_{ijkl}+\tfrac{1}{2}(T_{ij}{}^0\,\delta_{kl}+T_{kl}{}^0\,\delta_{ij}+T_{ik}{}^0\,\delta_{jl}-T_{jk}{}^0\,\delta_{il}-T_{il}{}^0\,\delta_{jk}+T_{jl}{}^0\,\delta_{ik}). \tag{34}$$

The coefficients $\Lambda_{ijkl}(\mathbf{r})$ are the components of a fourth-order tensor defined at all points $\mathbf{r}$ of the undeformed reference configuration; from (34) it can be shown that these components satisfy the symmetry relation

$$\Lambda_{ijkl} = \Lambda_{klij}. \tag{35}$$

There are in general 27 independent coefficients $\Lambda_{ijkl}(\mathbf{r})$, corresponding to 21 independent $C_{ijkl}(\mathbf{r})$ and six independent $T_{ij}{}^0(\mathbf{r})$. Note that the incremental pseudo-stress tensor is *not symmetric*, in fact

$$\tilde{T}_{ij}-\tilde{T}_{ji} = T_{ik}{}^0\,\partial_k\,s_j-T_{jk}{}^0\,\partial_k\,s_i. \tag{36}$$

For the special case of a hydrostatic initial stress $\mathbf{T}_0(\mathbf{r}) = -p_0(\mathbf{r})\,\mathbf{I}$, the incremental pseudo-stress tensor takes the form

$$\tilde{T}_{ij} = C_{ijkl}\,\sigma_{kl}-p_0(\partial_k\,s_k\,\delta_{ij}-\partial_j\,s_i). \tag{37}$$

Likewise in this special case, equation (34) defining the coefficients $\Lambda_{ijkl}(\mathbf{r})$ reduces to

$$\Lambda_{ijkl} = C_{ijkl}-p_0(\delta_{ij}\,\delta_{kl}-\delta_{il}\,\delta_{jk}).$$

Since $p_0(\mathbf{r})$ is necessarily continuous across any material surface in the body $V$, the condition that $\hat{\mathbf{n}}(\mathbf{r})\cdot\mathbf{T}(\mathbf{r}, t)$ be continuous across any welded boundary reduces in the case of a purely hydrostatic initial stress to the familiar condition that $\hat{\mathbf{n}}(\mathbf{r})\cdot\mathbf{E}(\mathbf{r}, t)$ be continuous. Note however that if one wishes to compute the force exerted on an arbitrary surface element by the stress field $-p_0(\mathbf{r})\,\mathbf{I}+\mathbf{\tilde{T}}(\mathbf{r}, t)$ then one must use (30) and (37).

## 5. Alternative form of equations of motion

It is convenient to recast the linearized equations of motion, equations (10), in a form in which the incremental pseudo-stress tensor $\mathbf{\tilde{T}}(\mathbf{r}, t)$ appears explicitly. This is accomplished by merely regrouping certain terms in (10); the resulting equivalent equations are in fact simpler than (10); namely

$$\rho_0 \partial_t^2 \mathbf{s} + 2\rho_0 \mathbf{\Omega} \times \partial_t \mathbf{s} = -\rho_0 \nabla\phi_1 - \rho_0 \mathbf{s} \cdot \nabla[\nabla(\phi_0 + \psi)] + \nabla \cdot \mathbf{\tilde{T}} + \mathbf{f}. \tag{38}$$

The equations (38) are the form of the equations of motion obtained by Biot (1965) by applying Newton's second law of motion to an arbitrary material element and then transforming the resulting surface and volume integrals to equivalent integrals over the undeformed configuration of this material element by linearization.

Under the assumptions that the Earth model V is initially in a steady-state equilibrium configuration and that the material comprising the Earth model V is a perfectly elastic continuum, any small isentropic elastic-gravitational disturbance $\mathbf{s}(\mathbf{r}, t)$, $\phi_1(\mathbf{r}, t)$ away from equilibrium must satisfy the linearized equations of motion (38) and (11) together with the linearized perfectly elastic constitutive relation (33) at all interior points $\mathbf{r}$ of the undeformed reference configuration.

## 6. Rayleigh's variational principle

It is possible to obtain a variational principle from which the equations (38), (11) and (33) together with the corresponding continuity or boundary conditions (32) may be derived. Biot (1965) shows how this variational principle may be deduced from general strain energy considerations, but in order to do so he is forced to consider second order quantities in the definition of the elastic strain. The approach taken here is to derive a slightly different variational principle directly from the linearized equations of motion; in this way any consideration of second-order quantities may be avoided. The resulting variational principle applies only to the force-free situation ($\mathbf{f}(\mathbf{r}, t) = 0$), and is in fact Rayleigh's principle governing the small oscillations of an arbitrary mechanical system.

Define the Fourier transforms of the various first order quantities $\mathbf{s}(\mathbf{r}, t)$, $\phi_1(\mathbf{r}, t)$ $\rho_1(\mathbf{r}, t)$, $\mathbf{T}(\mathbf{r}, t)$, $\mathbf{\tilde{T}}(\mathbf{r}, t)$, and $\mathbf{f}(\mathbf{r}, t)$ in the usual manner, e.g.

$$\mathbf{s}(\mathbf{r}, \omega) = \int_{-\infty}^{\infty} dt \exp(-i\omega t)\, \mathbf{s}(\mathbf{r}, t). \tag{39}$$

Now for a fixed value of the frequency $\omega$, let $\mathscr{S}$ be the vector space consisting of all twice-continuously differentiable vector fields $\mathbf{s}(\mathbf{r}, \omega)$ defined throughout the undeformed earth model volume $V$. For any two members $\mathbf{s}(\mathbf{r}, \omega)$ and $\mathbf{s}'(\mathbf{r}, \omega)$ of $\mathscr{S}$ define an inner product on $\mathscr{S}$ as

$$(\mathbf{s}', \mathbf{s}) = \int_V dV [\rho_0(\mathbf{r})\, \mathbf{s}'(\mathbf{r}, \omega) \cdot \mathbf{s}^*(\mathbf{r}, \omega)] \tag{40}$$

where * denotes the complex conjugate.

Now consider the Fourier-transformed versions of the linearized equations of motion (38) and (11) and of the linearized continuity conditions (32). Since there will be little danger of confusion, the symbols $\mathbf{s}$, $\phi_1$, $\rho_1$, $\mathbf{T}$, $\mathbf{\tilde{T}}$, and $\mathbf{f}$ will be used to denote both $\mathbf{s}(\mathbf{r}, \omega)$, $\phi_1(\mathbf{r}, \omega)$, $\rho_1(\mathbf{r}, \omega)$, $\mathbf{T}(\mathbf{r}, \omega)$, $\mathbf{\tilde{T}}(\mathbf{r}, \omega)$, and $\mathbf{f}(\mathbf{r}, \omega)$ as well as $\mathbf{s}(\mathbf{r}, t)$ $\phi_1(\mathbf{r}, t)$, $\rho_1(\mathbf{r}, t)$, $\mathbf{T}(\mathbf{r}, t)$, $\mathbf{\tilde{T}}(\mathbf{r}, t)$, and $\mathbf{f}(\mathbf{r}, t)$. For brevity, the system of Fourier transformed equations (38) and (11) will be written henceforth in the convenient operator notation

$$H\mathbf{s} = \rho_0 \omega^2 \mathbf{s} - 2\rho_0 i\omega \mathbf{\Omega} \times \mathbf{s} + \mathbf{f}. \tag{41}$$

The linear integro-differential operator $H$ is defined by

$$Hs = \rho_0 \nabla\phi_1 + \rho_0\, \mathbf{s}\cdot\nabla[\nabla(\phi_0+\psi)] - \nabla\cdot\mathbf{T} \tag{42}$$

where $\phi_1(\mathbf{r}, \omega)$ and $\mathbf{T}(\mathbf{r}, \omega)$ are given in terms of $\mathbf{s}(\mathbf{r}, \omega)$ by the Fourier-transformed versions of equations (11) and (33). If the externally applied body force $\mathbf{f}(\mathbf{r}, \omega)$ is set equal to zero, then equation (41) represents an eigenvalue equation in the inner product space $\mathscr{S}$ for the linear operator $H$. The normal mode eigenfrequencies and eigenfunctions of the Earth model V may be determined by obtaining solutions to this eigenvalue problem.

Now take the inner product of the equation (41) with another arbitrary member $\mathbf{s}'(\mathbf{r}, \omega)$ of the inner product space $\mathscr{S}$; the result may be written

$$(\mathbf{s}', \rho_0^{-1} H\mathbf{s}) = \omega^2(\mathbf{s}', \mathbf{s}) - 2\omega(\mathbf{s}', i\boldsymbol{\Omega}\times\mathbf{s}) + (\mathbf{s}', \rho_0^{-1}\mathbf{f}). \tag{43}$$

Now since any linear operator $L$ defined on an inner product space $\mathscr{S}$ may be associated with a unique bilinear functional $\mathscr{L}(\mathbf{s}', \mathbf{s})$ on $\mathscr{S}$ by the relation $\mathscr{L}(\mathbf{s}', \mathbf{s}) = (\mathbf{s}', L\mathbf{s})$, equation (43) may be thought of (in the case $\mathbf{f}(\mathbf{r}, \omega) = 0$) as an algebraic relation between bilinear functionals $\mathscr{L}(\mathbf{s}', \mathbf{s})$ defined on $\mathscr{S}$. In order to indicate that the inner product terms on (43) are in fact bilinear functionals on $\mathscr{S}$, the following notation will be used

$$\left.\begin{aligned} \mathscr{T}(\mathbf{s}', \mathbf{s}) &= (\mathbf{s}', \mathbf{s}) \\ \mathscr{W}(\mathbf{s}', \mathbf{s}) &= (\mathbf{s}', i\boldsymbol{\Omega}\times\mathbf{s}) \\ \mathscr{H}(\mathbf{s}', \mathbf{s}) &= (\mathbf{s}', \rho_0^{-1} H\mathbf{s}) \end{aligned}\right\} \tag{44}$$

Consider now the evaluation of the volume integral defining the bilinear functional $\mathscr{H}(\mathbf{s}', \mathbf{s})$.

$$\mathscr{H}(\mathbf{s}', \mathbf{s}) = \int_V dV[-s_i'\,\partial_j\, T_{ji}^* + \rho_0\, s_i'\,\partial_i\phi_1^* + \rho_0\, s_i'\, s_j^*\,\partial_i\partial_j(\phi_0+\psi)]. \tag{45}$$

An application of Gauss's theorem yields

$$\mathscr{H}(\mathbf{s}', \mathbf{s}) = \int_V dV\,[\partial_j s_i'\, T_{ji}^* + \rho_0\, s_i'\,\partial_i\phi_1^* + \rho_0\, s_i'\, s_j^*\,\partial_i\partial_j(\phi_0+\psi)] - \int_{\partial V} dA\, n_j[s_i'\, T_{ji}^*]. \tag{46}$$

Now assuming that $\nabla^2\phi_1 = 0$ in $E-V$ (where $E$ is all of space), an application of Green's first identity gives the identity

$$\int_V dV\,[\rho_0\, s_i^*\,\partial_i\phi_1'] + \int_E dV\left[\frac{1}{4\pi G}\partial_j\phi_1'\,\partial_j\phi_1^*\right] + \int_{\partial V} dA\, n_j\left[\phi_1'\left(\frac{1}{4\pi G}\partial_j\phi_1^* + \rho_0\, s_j^*\right)\right]_-^+ = 0 \tag{47}$$

where for any vector function $\mathbf{g}$, the expression $[\mathbf{g}]\pm$ denotes the jump discontinuity in $\mathbf{g}$ in going from inside $V$ to outside $V$. Adding (47) to (46) one obtains the most convenient final form for the bilinear functional $\mathscr{H}(\mathbf{s}', \mathbf{s})$

$$\mathscr{H}(\mathbf{s}', \mathbf{s}) = \mathscr{E}(\mathbf{s}', \mathbf{s}) + \mathscr{G}(\mathbf{s}', \mathbf{s}) + \Phi(\mathbf{s}', \mathbf{s}) - \mathscr{B}_1(\mathbf{s}', \mathbf{s}) \tag{48}$$

where

$$\left.\begin{aligned} \mathscr{E}(\mathbf{s}', \mathbf{s}) &= \int_V dV [\partial_i s_j' \, \mathring{T}_{ij}^*] = \int_V dV [\Lambda_{ijkl} \partial_i s_j' \partial_k s_l^*] \\ \mathscr{G}(\mathbf{s}', \mathbf{s}) &= \int_V dV [\rho_0 s_i' \partial_i \phi_1^* + \rho_0 s_i^* \partial_i \phi_1'] + \int_E dV \left[\frac{1}{4\pi G} \partial_j \phi_1' \partial_j \phi_1^*\right] \\ \Phi(\mathbf{s}', \mathbf{s}) &= \int_V dV [\rho_0 s_i' s_j \partial_i \partial_j (\phi_0 + \psi)] \end{aligned}\right\} \tag{49}$$

and where

$$\mathscr{B}_1(\mathbf{s}', \mathbf{s}) = \int_{\partial V} dA \, n_j \left\{ s_i' \, \mathring{T}_{ji}^* - \left[\phi_1' \left(\frac{1}{4\pi G} \partial_j \phi_1^* + \rho_0 s_j^*\right)\right]_-^+ \right\}. \tag{50}$$

In terms of the bilinear functionals $\mathscr{E}(\mathbf{s}', \mathbf{s})$, $\mathscr{G}(\mathbf{s}', \mathbf{s})$, $\Phi(\mathbf{s}', \mathbf{s})$, and $\mathscr{B}_1(\mathbf{s}', \mathbf{s})$ defined by equations (44), (49), and (50), equation (43) may now be rewritten

$$\omega^2 \mathscr{T}(\mathbf{s}', \mathbf{s}) - 2\omega \mathscr{W}(\mathbf{s}', \mathbf{s}) - \mathscr{E}(\mathbf{s}', \mathbf{s}) - \mathscr{G}(\mathbf{s}', \mathbf{s}) - \Phi(\mathbf{s}', \mathbf{s}) + \mathscr{B}_1(\mathbf{s}', \mathbf{s}) + (\mathbf{s}', \rho_0^{-1} \mathbf{f}) = 0. \tag{51}$$

Now consider the special case of equation (51) where the applied body force $\mathbf{f}(\mathbf{r}, \omega)$ is taken to be zero and where $\mathbf{s}'(\mathbf{r}, \omega)$ is taken to be the same as $\mathbf{s}(\mathbf{r}, \omega)$; in this case equation (51) reduces to

$$\omega^2 \mathscr{T}(\mathbf{s}, \mathbf{s}) - 2\omega \mathscr{W}(\mathbf{s}, \mathbf{s}) - \mathscr{E}(\mathbf{s}, \mathbf{s}) - \mathscr{G}(\mathbf{s}, \mathbf{s}) - \Phi(\mathbf{s}, \mathbf{s}) + \mathscr{B}_1(\mathbf{s}, \mathbf{s}) = 0. \tag{52}$$

Now if $\mathbf{s}(\mathbf{r}, \omega)$ in equation (52) is a normal mode eigenfunction of the Earth model V associated with the eigenvalue $\omega$, then $\mathbf{s}(\mathbf{r}, \omega)$ satisfies the free surface natural boundary conditions (32) on $V$, and hence $\mathscr{B}_1(\mathbf{s}, \mathbf{s}) = 0$. Thus if $\mathbf{s}(\mathbf{r}, \omega)$ is a normal mode eigenfunction, then one can write

$$\omega^2 \mathscr{T}(\mathbf{s}, \mathbf{s}) - 2\omega \mathscr{W}(\mathbf{s}, \mathbf{s}) - \mathscr{E}(\mathbf{s}, \mathbf{s}) - \mathscr{G}(\mathbf{s}, \mathbf{s}) - \Phi(\mathbf{s}, \mathbf{s}) = 0. \tag{53}$$

Equation (53) is Rayleigh's variational principle for the rotating, elastic-gravitational equilibrium configuration V. The term $\omega^2 \mathscr{T}(\mathbf{s}, \mathbf{s})$ is twice the kinetic energy of a disturbance $\mathbf{s}(\mathbf{r}, \omega) = \mathbf{s}(\mathbf{r}) \exp(i\omega t)$, while the terms $\mathscr{E}(\mathbf{s}, \mathbf{s}) + \mathscr{G}(\mathbf{s}, \mathbf{s}) + \Phi(\mathbf{s}, \mathbf{s})$ taken together are twice the elastic-gravitational potential energy of the same disturbance. The physical significance of each of these potential energy terms is clear from equations (49): the term $\mathscr{E}(\mathbf{s}, \mathbf{s})$ is twice the incremental elastic strain energy, the term $\mathscr{G}(\mathbf{s}, \mathbf{s})$ is twice the gravitational potential energy due to self-gravitation, and the term $\Phi(\mathbf{s}, \mathbf{s})$ is twice the incremental work done against the initial body force potential $\phi_0(\mathbf{r})+\psi(\mathbf{r})$. The Coriolis term $\mathscr{W}(\mathbf{s}, \mathbf{s})$ appears in (53) because of the uniform rotation $\mathbf{\Omega}$ of the Earth model V.

The variational principle which is contained in equation (53) states that if $\mathbf{s}(\mathbf{r}, \omega)$ is a normal mode eigenfunction associated with frequency $\omega$, then the bilinear functional of $\mathbf{s}$ on the left-hand side of (51) is stationary with respect to an arbitrary small variation $\delta\mathbf{s}(\mathbf{r}, \omega)$. The proof of this variational principle is immediate since all of the bilinear functionals which appear in equation (53) are Hermitian symmetric, i.e.

$$\left.\begin{aligned}\mathscr{T}(\mathbf{s}', \mathbf{s}) &= \mathscr{T}^*(\mathbf{s}, \mathbf{s}')\\ \mathscr{W}(\mathbf{s}', \mathbf{s}) &= \mathscr{W}^*(\mathbf{s}, \mathbf{s}')\\ \mathscr{E}(\mathbf{s}', \mathbf{s}) &= \mathscr{E}^*(\mathbf{s}, \mathbf{s}')\\ \mathscr{G}(\mathbf{s}', \mathbf{s}) &= \mathscr{G}^*(\mathbf{s}, \mathbf{s}')\\ \Phi(\mathbf{s}', \mathbf{s}) &= \Phi^*(\mathbf{s}, \mathbf{s}').\end{aligned}\right\} \tag{54}$$

The functionals $\mathscr{T}(\mathbf{s}, '\mathbf{s})$, $\mathscr{G}(\mathbf{s}', \mathbf{s})$, and $\Phi(\mathbf{s}', \mathbf{s})$ are Hermitian symmetric by inspection and it is a simple matter to show that $\mathscr{W}(\mathbf{s}', \mathbf{s})$ is Hermitian symmetric as well. The incremental elastic strain energy bilinear functional $\mathscr{E}(\mathbf{s}', \mathbf{s})$ is Hermitian symmetric by virtue of the symmetry relation (35). If one takes the first variation of equation (53) with respect to $\mathbf{s}$, one obtains immediately the elastic-gravitational normal mode eigenvalue equation

$$H\mathbf{s} = \rho_0\,\omega^2\,\mathbf{s} - 2\rho_0\,\omega i \mathbf{\Omega} \times \mathbf{s}. \tag{55}$$

## 7. Normal mode perturbation theory

Dahlen (1968) has given a similar derivation of Rayleigh's variational principle governing the small elastic-gravitational oscillations of a perfectly elastic, uniformly rotating Earth model with an initial static stress field $\mathbf{T}_0(\mathbf{r})$. Unfortunately that treatment contained an error in the specification of the continuity condition on the incremental surface traction, and thus the derivation of Rayleigh's principle given there is invalid. The correct treatment has been given above and the correct form of Rayleigh's principle is equation (53) where the various bilinear functions are defined by equations (49).

Dahlen (1968) proceeded to utilize Rayleigh's principle in order to compute the small changes in the eigenfrequencies $\omega$ and normal mode eigenfunctions $\mathbf{s}(\mathbf{r}, \omega)$ of an Earth model V as a result of specified small changes in the parameters describing the Earth model. The correct form of Rayleigh's principle should be used to do this rather than the incorrect version given in the previous paper. For this purpose, Rayleigh's principle as given in equations (53) and (49) is not a particularly convenient one; it is necessary to first rearrange the various terms slightly. Some applications of Gauss's theorem are all that is needed to show that the sum of potential energy bilinear forms $\mathscr{E}(\mathbf{s}', \mathbf{s})+\mathscr{G}(\mathbf{s}', \mathbf{s})+\Phi(\mathbf{s}', \mathbf{s})$ may be rewritten

$$\mathscr{E}(\mathbf{s}', \mathbf{s})+\mathscr{G}(\mathbf{s}', \mathbf{s})+\Phi(\mathbf{s}', \mathbf{s}) = \mathscr{V}(\mathbf{s}', \mathbf{s})+\Psi(\mathbf{s}', \mathbf{s})+\mathscr{P}(\mathbf{s}', \mathbf{s})+\mathscr{B}_0(\mathbf{s}', \mathbf{s})$$

where

$$\left.\begin{aligned}
\mathscr{V}(\mathbf{s}',\mathbf{s}) &= \int_V dV[C_{ijkl}\,\sigma_{ij}'\,\sigma_{kl}^* + \rho_0\, s_i'\,\partial_i\phi_1^* + \rho_0\, s_i^*\,\partial_i\phi_1' \\
&\quad + \tfrac{1}{2}\rho_0\,\partial_j\phi_0(s_i'\,\partial_i s_j^* + s_i^*\,\partial_i s_j' - s_j'\,\partial_i s_i^* - s_j^*\,\partial_i s_i') \\
&\quad + \rho_0\, s_i'\, s_j^*\,\partial_i\partial_j\phi_0] + \int_E dV\left[\frac{1}{4\pi G}\partial_j\phi_1'\,\partial_j\phi_1^*\right] \\
\Psi(\mathbf{s}',\mathbf{s}) &= \int_V dV\,[\tfrac{1}{2}\rho_0\,\partial_j\psi(s_i'\,\partial_i s_j^* + s_i^*\,\partial_i s_j' - s_j'\,\partial_i s_i^* - s_j^*\,\partial_i s_i') \\
&\quad + \rho_0\, s_i'\, s_j^*\,\partial_i\partial_j\psi] \\
\mathscr{P}(\mathbf{s}',\mathbf{s}) &= \int_V dV[\tfrac{1}{2}\tau_{ij}{}^0(s_k'\,\partial_j\partial_k s_i^* + s_k^*\,\partial_j\partial_k s_i' - s_i'\,d_j\partial_k s_k^* \\
&\quad - s_i^*\,\partial_j\partial_k s_k' + \partial_i s_k'\,\partial_j s_k^* - \partial_k s_i'\,\partial_k s_j^*)] \\
\mathscr{B}_0(\mathbf{s}',\mathbf{s}) &= \int dA\, n_j[\tfrac{1}{2}T_{ij}{}^0(s_i'\,\partial_k s_k^* + s_i^*\,\partial_k s_k' - s_k'\,\partial_k s_i^* - s_k^*\,\partial_k s_i')].
\end{aligned}\right\} \quad (57)$$

Equation (51) may thus be rewritten as

$$\omega^2\mathscr{T}(\mathbf{s}',\mathbf{s}) - 2\omega\mathscr{W}(\mathbf{s}',\mathbf{s}) - \mathscr{V}(\mathbf{s}',\mathbf{s}) - \Psi(\mathbf{s}',\mathbf{s}) - \mathscr{P}(\mathbf{s}',\mathbf{s}) \\ - \mathscr{B}_0(\mathbf{s}',\mathbf{s}) + \mathscr{B}_1(\mathbf{s}',\mathbf{s}) + (\mathbf{s}', \rho_0{}^{-1}\mathbf{f}) = 0 \quad (58)$$

and Rayleigh's principle equation (53) may be rewritten as

$$\omega^2\mathscr{T}(\mathbf{s},\mathbf{s}) - 2\omega\mathscr{W}(\mathbf{s},\mathbf{s}) - \mathscr{V}(\mathbf{s},\mathbf{s}) - \Psi(\mathbf{s},\mathbf{s}) - \mathscr{P}(\mathbf{s},\mathbf{s}) = 0. \quad (59)$$

The various terms in this rearranged version of Rayleigh's principle have now been grouped in a manner which is convenient for normal mode perturbation theory. Note that the bilinear functional $\Psi(\mathbf{s}',\mathbf{s})$ contains all terms linear in the centrifugal potential $\psi(\mathbf{r})$, while the bilinear functional $\mathscr{P}(\mathbf{s}',\mathbf{s})$ contains all terms linear in the deviatoric initial stress $\boldsymbol{\tau}_0(\mathbf{r})$. If $\partial V$ is a free surface then $\hat{\mathbf{n}}\cdot\mathbf{T}_0(\mathbf{r}) = 0$ on $\partial V$ and $\mathscr{B}_0(\mathbf{s}',\mathbf{s}) = 0$.

The correction to the normal mode perturbation theory is now straightforward. The final result is that equation (17) in Dahlen (1968) defining the first-order perturbation matrix elements $R_{ij}$ should be replaced by

$$R_{ij} = \delta\mathscr{V}(\mathbf{s}_i,\mathbf{s}_j) - \omega^2\,\delta\mathscr{T}(\mathbf{s}_i,\mathbf{s}_j) + 2\omega\mathscr{W}(\mathbf{s}_i,\mathbf{s}_j) + \Psi(\mathbf{s}_i,\mathbf{s}_j) + \mathscr{P}(\mathbf{s}_i,\mathbf{s}_j) \quad (60)$$

where $\mathscr{W}(\mathbf{s}',\mathbf{s})$ is defined in equation (44), $\Psi(\mathbf{s}',\mathbf{s})$ and $\mathscr{P}(\mathbf{s}',\mathbf{s})$ are defined in equations (57), and $\delta\mathscr{V}(\mathbf{s}',\mathbf{s})$ and $\delta\mathscr{T}(\mathbf{s}',\mathbf{s})$ are given by

$$\left.\begin{aligned}
\delta\mathscr{T}(\mathbf{s}',\mathbf{s}) &= \int_V dV[\delta\rho_0\, s_i'\, s_i^*] \\
\delta\mathscr{V}(\mathbf{s}',\mathbf{s}) &= \int_V dV[\delta C_{ijkl}\,\sigma_{ij}'\,\sigma_{kl}^* \\
&\quad + \delta\rho_0\, s_i'\, s_j^*\,\partial_i\partial_j\phi_0 + \rho_0\, s_i'\, s_j^*\,\partial_i\partial_j\delta\phi_0 + \delta\rho_0(s_i'\,\partial_i\phi_1^* \\
&\quad + s_i^*\,\partial_i\phi_1' + \tfrac{1}{2}(\delta\rho_0\,\partial_j\phi_0 + \rho_0\,\partial_j\delta\phi_0) \\
&\qquad \times (s_i'\,\partial_i s_j^* + s_i^*\,\partial_i s_j' - s_j'\,\partial_i s_i^* - s_j^*\,\partial_i s_i')].
\end{aligned}\right\} \quad (61)$$

Note that all of the bilinear functionals $\delta\mathcal{T}(\mathbf{s}',\mathbf{s})$, $\delta\mathcal{V}(\mathbf{s}',\mathbf{s})$, $\mathcal{W}(\mathbf{s}',\mathbf{s})$, $\Psi(\mathbf{s}',\mathbf{s})$, and $\mathcal{P}(\mathbf{s}',\mathbf{s})$ as now defined are by inspection Hermitian symmetric separately. The incorrect expressions for the matrix elements $R_{ij}$ given in Dahlen (1968) are correct for the special case of a hydrostatic initial stress $\mathbf{T}_0(\mathbf{r}) = -p_0(\mathbf{r})\,\mathbf{I}$, and hence the treatment of the eigenfrequency perturbations due to the Earth's rotation and ellipticity is unaffected by the correction pointed out here.

## 8. The reciprocal and reciprocity relations

Consider again in the Fourier-transform domain an elastic-gravitational response $\mathbf{s}(\mathbf{r},\omega)$, $\phi_1(\mathbf{r},\omega)$ of the Earth model V to an externally applied body force $\mathbf{f}(\mathbf{r},\omega)$. The equations of motion relating $\mathbf{s}(\mathbf{r},\omega)$ and $\phi_1(\mathbf{r},\omega)$ to $\mathbf{f}(\mathbf{r},\omega)$ are equations (41) which are rewritten here

$$H\mathbf{s} = \rho_0\,\omega^2\,\mathbf{s} - 2\rho_0\,\omega i\boldsymbol{\Omega}\times\mathbf{s} + \mathbf{f}. \tag{41}$$

Consider at the same time the response $\mathbf{s}'(\mathbf{r},\omega)$, $\phi_1'(\mathbf{r},\omega)$ of the same Earth model V to another external body force $\mathbf{f}'(\mathbf{r},\omega)$.

$$H\mathbf{s}' = \rho_0\,\omega^2\,\mathbf{s}' - 2\rho_0\,\omega i\boldsymbol{\Omega}\times\mathbf{s}' + \mathbf{f}'. \tag{62}$$

Now taking the inner product of $\mathbf{s}'(\mathbf{r},\omega)$ with equation (41) and the inner product of $\mathbf{s}(\mathbf{r},\omega)$ with equation (62), one obtains respectively

$$\omega^2\,\mathcal{T}(\mathbf{s}',\mathbf{s}) - 2\omega\mathcal{W}(\mathbf{s}',\mathbf{s}) - \mathcal{E}(\mathbf{s}',\mathbf{s}) - \mathcal{G}(\mathbf{s}',\mathbf{s}) - \Phi(\mathbf{s}',\mathbf{s}) + \mathcal{B}_1(\mathbf{s}',\mathbf{s}) + (\mathbf{s}',\rho_0^{-1}\,\mathbf{f}) = 0 \tag{63}$$

$$\omega^2\,\mathcal{T}(\mathbf{s},\mathbf{s}') - 2\omega\mathcal{W}(\mathbf{s},\mathbf{s}') - \mathcal{E}(\mathbf{s},\mathbf{s}') - \mathcal{G}(\mathbf{s},\mathbf{s}') - \Phi(\mathbf{s},\mathbf{s}') \quad \mathcal{B}_1(\mathbf{s},\mathbf{s}') + (\mathbf{s},\rho_0^{-1}\,\mathbf{f}') = 0. \tag{64}$$

Now taking the complex conjugate of equation (64) and utilizing the fact that $\mathcal{T}(\mathbf{s}',\mathbf{s})$, $\mathcal{W}(\mathbf{s}',\mathbf{s})$, $\mathcal{E}(\mathbf{s}',\mathbf{s})$, $\mathcal{G}(\mathbf{s}',\mathbf{s})$, and $\Phi(\mathbf{s}',\mathbf{s})$ are all Hermitian symmetric bilinear forms, one obtains the result

$$(\mathbf{s}',\rho_0^{-1}\,\mathbf{f}) + \mathcal{B}_1(\mathbf{s}',\mathbf{s}) = (\mathbf{s},\rho_0^{-1}\,\mathbf{f}')^* + \mathcal{B}_1(\mathbf{s},\mathbf{s}')^*. \tag{65}$$

Equation (65) is the Betti reciprocal theorem (Love 1927) for a uniformly rotating, self-gravitating elastic configuration with an arbitrary initial static stress field $\mathbf{T}_0(r)$. Written out in full, equation (65) is

$$\int_V dV\,[s_i' f_i^*] + \int_{\partial V} dA\, n_j \left\{ s_i'\,\tilde{T}_{ji}^* - \left[\phi_1' \left(\frac{1}{4\pi G}\,\partial_j\phi_1^* + \rho_0\, s_j^*\right)\right]_-^+ \right\}$$

$$= \int_V dV\,[s_i^* f_i'] + \int_{\partial V} dA\, n_j \left\{ s_i^*\,\tilde{T}_{ji}' - \left[\phi_1^* \left(\frac{1}{4\pi G}\,\partial_j\phi_1' + \rho_0\, s_j'\right)\right]_-^+ \right\}. \tag{66}$$

The Betti reciprocal theorem, given here in the frequency domain, is a global relation between any two possible but different solutions to the dynamic elastic-gravitational equations of motion (38).

Consider now the special case where $\partial V$ is a free surface so that both $\mathcal{B}_1(\mathbf{s}',\mathbf{s})$ and $\mathcal{B}_1(\mathbf{s},\mathbf{s}')$ are equal to zero. Furthermore take the body force $\mathbf{f}(\mathbf{r},t)$ to be a unit point force acting at time $s$ and point $\mathbf{x} = (x_1, x_2, x_3)$ in the $\hat{\mathbf{x}}_p$ direction, and take $\mathbf{f}'(\mathbf{r},t)$ to be a unit impulsive point force acting at time $s'$ and point $\mathbf{x}' = (x_1', x_2', x_3')$

in the $\hat{\mathbf{x}}_q$ direction; i.e.

$$\left.\begin{aligned}\mathbf{f}(\mathbf{r},t)&=\hat{\mathbf{x}}_p\,\delta(\mathbf{r}-\mathbf{x})\,\delta(t-s)\\ \mathbf{f}'(\mathbf{r},t)&=\hat{\mathbf{x}}_q\,\delta(\mathbf{r}-\mathbf{x}')\,\delta(t-s')\end{aligned}\right\}\tag{67}$$

where $\delta(\mathbf{r}-\mathbf{x})$ denotes $\delta(r_1-x_1)\,\delta(r_2-x_2)\,\delta(r_3-x_3)$. Use the notation $s_p{}^q(\mathbf{x},s;\mathbf{x}',s')$ to denote the $\hat{\mathbf{x}}_p$ component of the elastic-gravitational displacement response at the point $(\mathbf{x},s)$ due to an instantaneous point force acting in the $\hat{\mathbf{x}}_q$ direction at the point $(\mathbf{x}',s')$. Using the convolution theorem to transform to the time domain, equation (66) for this special case of the free response of the Earth model V to a unit impulsive point force takes the form

$$s_p{}^q(\mathbf{x},s;\mathbf{x}',s')=s_q{}^p(\mathbf{x}',-s';\mathbf{x},-s)^*.\tag{68}$$

Equation (68) is a special case of Helmholtz's reciprocity theorem in classical mechanics (Whittaker 1936), and the above argument merely confirms that it remains valid in the initially stressed, self-gravitating Earth under consideration.

## 9. Introduction of a fault surface: continuity conditions

Now assume that one wishes to determine the elastic-gravitational disturbance $\mathbf{s}(\mathbf{r},t)$, $\phi_1(\mathbf{r},t)$ in $V$ produced not by an externally applied body force but rather produced by a prescribed tangential displacement discontinuity across a fault surface $\Sigma_0$ embedded in $V$. The introduction of a prescribed tangential displacement discontinuity into an otherwise perfectly elastic Earth model V is intended of course to serve as a kinematical model of the earthquake faulting process. When there is an initial static stress field $\mathbf{T}_0(r)$ in the Earth model V, it is necessary to use some care in deriving the physically meaningful continuity conditions across the fault surface $\Sigma_0$.

Consider then the introduction of a fault surface $\Sigma_0$ with boundary curve $\partial\Sigma_0$ into the initially undeformed or reference configuration $V$. Points (in the reference configuration) located on this fault surface $\Sigma_0$ will be denoted by $\mathbf{r}_0$; the unit normal to $\Sigma_0$ at a point $\mathbf{r}_0$ will be denoted by $\hat{\mathbf{n}}_0(\mathbf{r}_0)$ (the usual convention will be followed and $\hat{\mathbf{n}}_0$ is taken to point out of the positive side of $\Sigma_0$). The fault surface $\Sigma_0$ may or may not intersect the surface $\partial V$ of the earth model (i.e. in set-theoretic language, $\partial\Sigma_0\cap\partial V$ may or may not be empty). Since $\Sigma_0$ is a fault surface, the elastic-gravitational displacement $\mathbf{s}(\mathbf{r}_0,t)$ will be double-valued on $\Sigma_0$. The jump discontinuity in displacement $[\mathbf{s}(\mathbf{r}_0,t)]_-^+$ will be taken to be an arbitrarily prescribed function on $\Sigma_0$, except that $[\mathbf{s}(\mathbf{r}_0,t)]_-^+$ must be zero on the boundary $\partial\Sigma_0-\partial\Sigma_0\cap\partial V$. This paper will furthermore only consider *tangential* displacement dislocations across $\Sigma_0(\hat{\mathbf{n}}_0(\mathbf{r}_0)\cdot[\mathbf{s}(\mathbf{r}_0,t)]_-^+=0$ for all points $\mathbf{r}_0$ on $\Sigma_0)$.

The fault surface $\Sigma_0$ is thus a surface embedded in $V$ upon which the jump discontinuity in tangential displacement $[\mathbf{s}(\mathbf{r}_0,t)]_-^+$ is taken to be a prescribed function. Consider now the other continuity conditions which must be imposed on the fault surface $\Sigma_0$. Since the displacement discontinuity is purely tangential, the continuity conditions involving the incremental gravitational potential $\phi_1(\mathbf{r}_0,t)$ will be the same as those appropriate to a welded boundary, equations (32). However, because of the relative motion $[\mathbf{s}(\mathbf{r}_0,t)]_-^+$ of the material on one side of the fault surface $\Sigma_0$ with respect to the material on the other side, the normal stress continuity condition across $\Sigma_0$ differs from that appropriate to a welded boundary. Let $dA_0{}^+$ be an arbitrary patch centred on a point $\mathbf{r}_0{}^-$ on the positive side of $\Sigma_0$, and take $dA_0{}^-$ (centred on $\mathbf{r}_0{}^-$) to be such that at time $t$, $dA_0{}^+(t)=dA_0{}^-(t)$ (see Fig. 2). Linearization of the

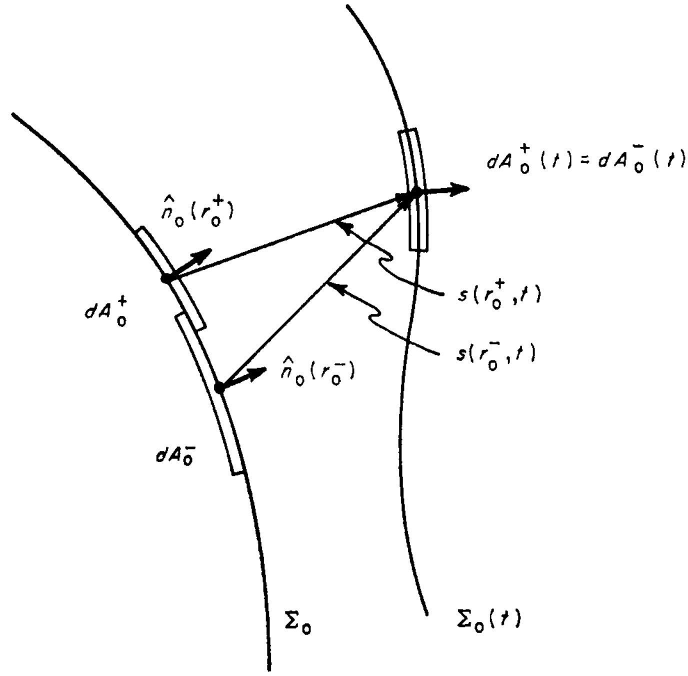


FIG. 2. Schematic diagram of the motion $\mathbf{s}(\mathbf{r}_0^+, t)$ and $\mathbf{s}(\mathbf{r}_0^-, t)$ of surface elements $dA_0^+$ $(t)$ and $dA_0^-$ $(t)$ on either side of a fault surface $\Sigma_0$ $(t)$. The patches $dA^{0+}$ $(t)$ and $dA_0^-$ $(t)$ are chosen to coincide at time $t$, so that $\mathbf{r}_0^+ + \mathbf{s}(\mathbf{r}_0^+, t) = \mathbf{r}_0^- + \mathbf{s}(\mathbf{r}_0^-, t)$.

exact continuity condition at time $t$ leads to the condition

$$\left.\begin{aligned} &\int_{dA_0^+} dA_0\, \hat{\mathbf{n}}_0(\mathbf{r}_0^+)\cdot[\mathbf{T}_0(\mathbf{r}_0^+)+\tilde{\mathbf{T}}(\mathbf{r}_0^+, t)] \\ &= \int_{dA_0^-} dA_0\, \hat{\mathbf{n}}_0(\mathbf{r}_0^-)\cdot[\mathbf{T}_0(\mathbf{r}_0^-)+\tilde{\mathbf{T}}(\mathbf{r}_0^-, t)]. \end{aligned}\right\} \tag{69}$$

In order to obtain a continuity condition, it is first necessary to transform the surface integral over the patch $dA_0^-$ to a surface integral over the *arbitrary* patch $dA_0^+$. The resulting continuity condition is most conveniently written as a relation giving the discontinuity in the normal pseudo-stress tensor $[\hat{\mathbf{n}}_0(\mathbf{r}_0)\cdot\tilde{\mathbf{T}}(\mathbf{r}_0, t)]_-^+$ in terms of the prescribed displacement discontinuity $[\mathbf{s}(\mathbf{r}_0, t)]_-^+$. This relation is

$$[\hat{\mathbf{n}}_0(\mathbf{r}_0)\cdot\tilde{\mathbf{T}}(\mathbf{r}_0, t)]_-^+ = \nabla_\Sigma{}^0 \cdot\{[\mathbf{s}(\mathbf{r}_0, t)]_-^+\, \hat{\mathbf{n}}_0(\mathbf{r}_0)\cdot\mathbf{T}_0(\mathbf{r}_0)\} \tag{70}$$

where $\nabla_\Sigma{}^0$ is the surface gradient operator on the surface $\Sigma_0$, defined by

$$\nabla_\Sigma{}^0 = \nabla_0 - \hat{\mathbf{n}}_0(\mathbf{r}_0)[\hat{\mathbf{n}}_0(\mathbf{r}_0)\cdot\nabla_0]. \tag{71}$$

The quantity on the right-hand side of equation (70) is the surface divergence of the second-order tangent tensor field $[\mathbf{s}(\mathbf{r}_0, t)]_-^+\, \hat{\mathbf{n}}_0(\mathbf{r}_0)\cdot\mathbf{T}_0(\mathbf{r}_0)$ defined on $\Sigma_0$. Written in terms of Cartesian components with respect to the uniformly rotating Cartesian axis system $\hat{\mathbf{x}}_1, \hat{\mathbf{x}}_2, \hat{\mathbf{x}}_3$, equation (70) is

$$[n_j{}^0\, \tilde{T}_{ji}]_-^+ = [\partial_k{}^0 - n_k{}^0\, n_l{}^0\, \partial_l{}^0]\{[s_k]_-^+\, n_j\, T_{ij}{}^0\}. \tag{72}$$

Note that because of the appearance of the surface gradient operator $\nabla_\Sigma{}^0$, the discontinuity $[\hat{\mathbf{n}}_0(\mathbf{r}_0)\cdot\mathbf{T}(\mathbf{r}_0,t)]_-^+$ is completely determined in terms of the prescribed displacement dislocation $[\mathbf{s}(\mathbf{r}_0,t)]_-^+$ on $\Sigma_0$.

In summary, the continuity conditions which must be applied across the fault surface $\Sigma_0$ are:

$$\left.\begin{aligned}
&[\mathbf{s}(\mathbf{r}_0,t)]_-^+ \text{ prescribed (except that } \hat{\mathbf{n}}_0(\mathbf{r}_0)\cdot[\mathbf{s}(\mathbf{r}_0,t)]_-^+ = 0 \text{ and}\\
&\qquad\qquad\qquad [\mathbf{s}(\mathbf{r}_0,t)]_-^+ = 0 \text{ on } \partial\Sigma_0 - \partial\Sigma_0 \cap \partial V)\\
&[\phi_1(\mathbf{r}_0,t)]_-^+ = 0\\
&[\hat{\mathbf{n}}_0(\mathbf{r}_0)\cdot\nabla\phi_1(\mathbf{r}_0,t) + 4\pi G\rho_0(\mathbf{r}_0)\,\hat{\mathbf{n}}_0(\mathbf{r}_0)\cdot\mathbf{s}(\mathbf{r}_0,t)]_-^+ = 0\\
&[\hat{\mathbf{n}}_0(\mathbf{r}_0)\cdot\mathbf{T}(\mathbf{r}_0,t)]_-^+ = \nabla_\Sigma{}^0\cdot\{[\mathbf{s}(\mathbf{r}_0,t)]_-^+\,\hat{\mathbf{n}}_0(\mathbf{r}_0)\cdot\mathbf{T}_0(\mathbf{r}_0)\}
\end{aligned}\right\} \quad (73)$$

## 10. The Volterra dislocation relation

The continuity conditions, equations (73), will be taken to be a complete kinematical description of the earthquake faulting process in an otherwise perfectly elastic Earth model V. A representation theorem for this boundary value problem can be obtained through a specific application of the Betti reciprocal relation (66). First note that the derivation of equation (66) is virtually unchanged if the volume $V$ and surface $\partial V$ of integration are not the actual volume $V$ and surface $\partial V$ of the Earth model. For this specific application consider a volume of integration $V$ minus a small interior volume element $V_0$ with surface $\partial V_0$. Denote the unit inward normal of $V_0$ (the unit outward normal of $V - V_0$; see Fig. 3) by $\hat{\mathbf{n}}_0$. The material properties of the elastic-gravitational continuum comprising $V$ are assume to be continuous across $\partial V_0$. For this volume of integration, equation (66) in the frequency domain is altered to

$$\left.\begin{aligned}
&\int_{V-V_0} dV\,[s_i' f_i^*] + \int_{\partial V} dA\,n_j\left\{s_i'\,T_{ji}^* - \left[\phi_1'\left(\frac{1}{4\pi G}\partial_j\phi_1^* + \rho_0 s_j^*\right)\right]_-^+\right\}\\
&\qquad + \int_{\partial V_0} dA\,n_j^0\left\{s_i'\,T_{ji}^* + \phi_1'\left(\frac{1}{4\pi G}\partial_j\phi_1^* + \rho_0 s_j^*\right)\right\}\\
&= \int_{V-V_0} dV\,[s_i^* f_i'] + \int_{\partial V} dA\,n_j\left\{s_i^*\,T_{ji}' - \left[\phi_1^*\left(\frac{1}{4\pi G}\partial_j\phi_1' + \rho_0 s_j'\right)\right]_-^+\right\}\\
&\qquad + \int_{\partial V_0} dA\,n_j^0\left\{s_i^*\,T_{ji}' + \phi_1^*\left(\frac{1}{4\pi G}\partial_j\phi_1' + \rho_0 s_j'\right)\right\}.
\end{aligned}\right\} \quad (74)$$

The fact that $\nabla^2\phi_1 = 4\pi G\rho_1$ instead of zero inside of $V_0$ necessitates a slight alteration in the use of Green's first identity in equation (47), and this gives rise to the lack of a $[\;]_-^+$ in the surface integrals over $\partial V_0$ in (74). Now let the volume $V_0$ shrink to zero in such a way that the surface $\partial V_0$ collapses upon itself to become the fault surface $\Sigma_0$ with the unit normal $\hat{\mathbf{n}}_0$ (see Fig. 3). If at the same time the external surface $\partial V$ is assumed to be a free surface, equation (74) reduces to

$$\begin{aligned}
&\int_V dV\,[s_i' f_i^*] - \int_{\Sigma_0} dA_0\,n_j^0\left[s_i'\,T_{ji}^* + \phi_1'\left(\frac{1}{4\pi G}\partial_j\phi_1^* + \rho_0 s_j^*\right)\right]_-^+\\
&\qquad = \int_V dV\,[s_i^* f_i'] - \int_{\Sigma_0} dA_0\,n_j^0\left[s_i^*\,T_{ji}' + \phi_1^*\left(\frac{1}{4\pi G}\partial_j\phi_1' + \rho_0 s_j'\right)\right]_-^+\Big\}. \quad (75)
\end{aligned}$$

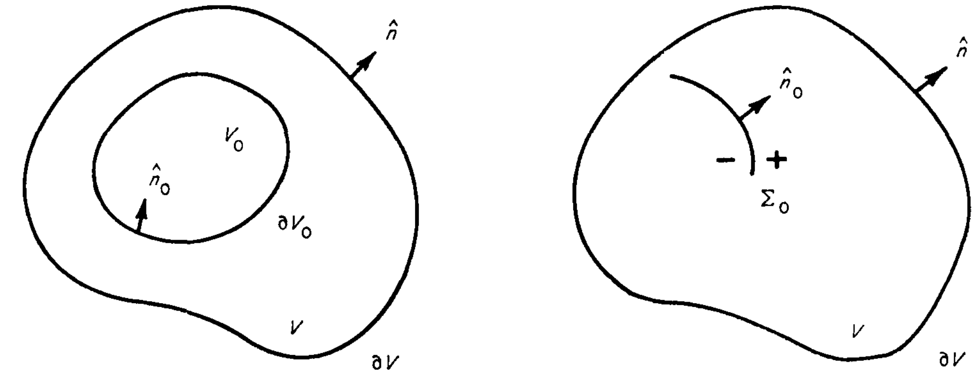


FIG. 3. Schematic diagram of an imaginary interior volume element $V_0$ in $V$ which is allowed to shrink to zero in such a way that $\partial V_0$ collapses on to the fault plane $\Sigma_0$.

Equation (75) is a relation between any two arbitrary combinations $\mathbf{s}(\mathbf{r}, \omega)$, $\phi_1(\mathbf{r}, \omega)$, $\mathbf{f}(\mathbf{r}, \omega)$, and $\mathbf{s}'(\mathbf{r}, \omega)$, $\phi_1'(\mathbf{r}, \omega)$, $\mathbf{f}'(\mathbf{r}, \omega)$ of Fourier-transformed variables in the Earth model V, provided both combinations satisfy the free surface boundary conditions (32) on the external surface $\partial V$.

In order to derive the desired representation theorem, take the primed variables $\mathbf{s}'(\mathbf{r}, t)$, $\phi_1'(\mathbf{r}, t)$, $\mathbf{T}'(\mathbf{r}, t)$ to describe the response of the Earth model V to the faulting on $\Sigma_0$, and take the unprimed variables $\mathbf{s}^p(\mathbf{r}, t; \mathbf{x}, s)$, $\phi_1^p(\mathbf{r}, t; \mathbf{x}, s)$, $\mathbf{T}^p(\mathbf{r}, t; \mathbf{x}, s)$ to be the response at the point $(\mathbf{r}, t)$ to a unit impulsive point force $\mathbf{f}(\mathbf{r}, t)$ acting in an unfaulted Earth model in the direction $\hat{\mathbf{x}}_p$ and at the point $(\mathbf{x}, s)$. Since $\mathbf{f}(\mathbf{r}, t)$ is assumed to be acting in an unfaulted medium, the various discontinuities $[\mathbf{s}(\mathbf{r}_0, t)]_-^+$, $[\phi_1(\mathbf{r}_0, t)]_-^+$, $[\hat{\mathbf{n}}_0(\mathbf{r}_0) \cdot \nabla\phi_1(\mathbf{r}_0, t) + 4\pi G\rho_0(\mathbf{r}_0)\, \hat{\mathbf{n}}_0(\mathbf{r}_0) \cdot \mathbf{s}(\mathbf{r}_0, t)]_-^+$, and $[\hat{\mathbf{n}}_0(\mathbf{r}_0) \cdot \mathring{\mathbf{T}}(\mathbf{r}_0, t)]_-^+$ are all zero across $\Sigma_0$; the continuity conditions across $\Sigma_0$ satisfied by the primed variables are equations (73). Making these substitutions, and once again utilizing the convolution theorem to transform to the time domain, equation (75) for this special case takes the form

$$s_p'(\mathbf{x}, s) = \int_{-\infty}^{\infty} dt \int_V dV\, [\mathring{s}_i^p(\mathbf{r}, -t; \mathbf{x}, -s) f_i'(\mathbf{r}, t)]$$

$$+ \int_{-\infty}^{\infty} dt \int_{\Sigma_0} dA_0 \{[n_j^0(\mathbf{r}_0)\, T_{ji}^p(\mathbf{r}_0, -t; \mathbf{x}, -s)]^* \,[s_i'(\mathbf{r}_0, t)]_-^+$$

$$- \mathring{s}_i^p(\mathbf{r}_0, -t; \mathbf{x}, -s)[n_j^0(\mathbf{r}_0)\, \mathring{T}_{ji}'(\mathbf{r}_0, t)]_-^+\}. \tag{76}$$

Equation (76) is a time domain representation theorem which allows one to determine the response $\mathbf{s}'(\mathbf{x}, t)$ of the Earth model V in terms of a prescribed applied body force $\mathbf{f}'(\mathbf{r}, t)$ throughout the volume V, and in terms of prescribed discontinuities in the displacement $[\mathbf{s}'(\mathbf{r}_0, t)]_-^+$ and the normal pseudo-stress $[\hat{\mathbf{n}}_0(\mathbf{r}_0) \cdot \mathring{\mathbf{T}}_0'(\mathbf{r}_0, t)]_-^+$ across a fault plane $\Sigma_0$.

The discontinuity $[\hat{\mathbf{n}}_0(\mathbf{r}_0) \cdot \mathbf{T}_0'(\mathbf{r}_0, t)]_-^+$ may be written, using (70) or (72), in terms of the prescribed discontinuity $[\mathbf{s}(\mathbf{r}_0, t)]_-^+$ ,and its contribution to the surface integral in equation (76) may be put in a more convenient form using the extension of Gauss's theorem to curved surfaces (Spivak 1965). Let $\hat{\mathbf{b}}(\mathbf{r}_0)$ denote the unit vector tangent to $\Sigma_0$ and normal to $\partial\Sigma_0$ at $\mathbf{r}_0$ and pointing out of $\Sigma_0$. Then

$$\int_{\Sigma_0} dA_0\{\overset{*}{s}_i{}^p(\mathbf{r}_0, -t; \mathbf{x}, -s)[n_j{}^0(\mathbf{r}_0)\, \overset{*}{T}_{ji}'(\mathbf{r}_0, t)]_-^+\}$$

$$= -\int_{\Sigma_0} dA_0\{n_j{}^0(\mathbf{r}_0)\, T_{ij}{}^0(\mathbf{r}_0)\,\partial_k{}^0\, \overset{*}{s}_i{}^p(\mathbf{r}_0, -t; \mathbf{x}, -s)[s_k'(\mathbf{r}_0, t)]_-^+\}$$

$$+ \int_{\partial\Sigma_0} dL_0\{b_k(\mathbf{r}_0)[s_k'(\mathbf{r}_0, t)]_-^+\, n_j{}^0(\mathbf{r}_0)\, T_{ji}{}^0(\mathbf{r}_0)\, \overset{*}{s}_i{}^p(\mathbf{r}_0, -t; \mathbf{x}, -s)\} \qquad (77)$$

Now $[\mathbf{s}'(\mathbf{r}_0, t)]_-^+ = 0$ on $\partial\Sigma_0 - \partial\Sigma_0 \cap \partial V$ and $\hat{\mathbf{n}}_0(\mathbf{r}_0)\cdot\mathbf{T}_0(\mathbf{r}_0) = 0$ on $\partial V$, so the line integral over $\partial\Sigma_0$ vanishes, even if $\Sigma_0$ and $\partial V$ intersect. Substituting equation (77) into (76) one obtains the representation theorem in the final form.

Define a new incremental stress tensor $G_{ij}{}^p(\mathbf{r}_0, -t; \mathbf{x}, -s)$ and a new fourth order isentropic elastic tensor $\Sigma_{ijkl}(\mathbf{r}_0)$ by

$$G_{ij}{}^p(\mathbf{r}_0, -t; \mathbf{x}, -s) = \Sigma_{ijkl}(\mathbf{r}_0)\,\partial_k{}^0\, s_l{}^p(\mathbf{r}_0, -t; \mathbf{x}, -s) \qquad (78)$$

where

$$\Sigma_{ijkl} = \Lambda_{jikl} + T_{jl}{}^0\,\delta_{ik}$$
$$= C_{ijkl} + \tfrac{1}{2}(T_{ij}{}^0\,\delta_{kl} + T_{kl}{}^0\,\delta_{ij} - T_{ik}{}^0\,\delta_{jl} + T_{jk}{}^0\,\delta_{il} - T_{il}{}^0\,\delta_{jk} + T_{jl}{}^0\,\delta_{ik}). \qquad (79)$$

The final form of the desired representation may then be written in the form

$$s_p'(\mathbf{x}, s) = \int_{-\infty}^{\infty} dt \int_V dV\, [\overset{*}{s}_i{}^p(\mathbf{r}, -t; \mathbf{x}, -s)\, f_i'(\mathbf{r}, t)]$$

$$+ \int_{-\infty}^{\infty} dt \int_{\Sigma_0} dA_0\, n_j{}^0(\mathbf{r}_0)\{\overset{*}{G}_{ij}{}^p(\mathbf{r}_0, -t; \mathbf{x}, -s)[s_i'(\mathbf{r}_0, t)]_-^+\}. \qquad (80)$$

Equation (80) provides an explicit representation of the elastic-gravitational displacement response $\mathbf{s}'(\mathbf{x}, t)$ of the Earth model V to an applied body force $\mathbf{f}'(\mathbf{r}, t)$ and to a prescribed tangential displacement discontinuity $[\mathbf{s}'(\mathbf{r}_0, t)]_-^+$ on a fault surface $\Sigma_0$. For the case where there is no applied body force $\mathbf{f}'(\mathbf{r}, t)$, equation (80), reduces to

$$s_p'(\mathbf{x}, s) = \int_{-\infty}^{\infty} dt \int_{\Sigma_0} dA_0\, n_j{}^0(\mathbf{r}_0)\{\overset{*}{G}_{ij}{}^p(\mathbf{r}_0, -t; \mathbf{x}, -s)[s_i'(\mathbf{r}_0, t)]_-^+\}. \qquad (81)$$

Equation (81) is the appropriate extension to the case under consideration here of the fundamental relation of elastic dislocation theory, commonly called the Volterra dislocation relation.

For the special case where the initial static stress is purely hydrostatic, $\mathbf{T}_0(\mathbf{r}_0) = -p_0(\mathbf{r}_0)\,\mathbf{I}$, everywhere on the fault surface $\Sigma_0$, the above relations can be reduced considerably. For that case $\Sigma_{ijkl}(\mathbf{r}_0)$ reduces to $C_{ijkl}(\mathbf{r}_0)$ and thus $G_{ij}{}^p(\mathbf{r}_0, -t; \mathbf{x}, -s)$ reduces to the elastic stress $E_{ij}{}^p(\mathbf{r}_0, -t; \mathbf{x}, -s)$; the Volterra relation thus takes the simpler form

$$s_p'(\mathbf{x}, s) = \int_{-\infty}^{\infty} dt \int_{\Sigma_0} dA_0\, n_j{}^0(\mathbf{r}_0)\{\overset{*}{E}_{ij}{}^p(\mathbf{r}_0, -t; \mathbf{x}, -s)[s_i'(\mathbf{r}_0, t)]_-^+\}. \qquad (82)$$

This is the form of the Volterra relation previously deduced (for this case only) by a faulty argument and subsequently utilized by Dahlen (1971) to study the excitation of the Chandler wobble by earthquakes.

## 11. Body force equivalents

The argument used by Burridge & Knopoff (1964) may be used to derive body force equivalents to the prescribed discontinuity $[\mathbf{s}'(\mathbf{r}_0, t)]_-^+$ across the fault surface $\Sigma_0$. Note that for the case where $\Sigma_0$ and $\partial V$ have no points in common, one may write $\partial_k{}^0 \overset{*}{s}_l{}^p(\mathbf{r}_0, -t; \mathbf{x}, -s)$ in the form

$$\partial_k{}^0 \overset{*}{s}_l{}^p(\mathbf{r}_0, -t; \mathbf{x}, -s) = -\int_V dV\,[\partial_k \delta(\mathbf{r}-\mathbf{r}_0)\, \overset{*}{s}_l{}^p(\mathbf{r}_0, -t; \mathbf{x}, s)]. \tag{83}$$

Substituting (78) and (83) into (80), one obtains the result

$$s_p'(\mathbf{x}, s) = \int_{-\infty}^{\infty} dt \int_V dV\,[\overset{*}{s}_l{}^p(\mathbf{r}, -t; \mathbf{x}, -s)][f_l'(\mathbf{r}, t) + e_l'(\mathbf{r}, t)] \tag{84}$$

where

$$e_l'(\mathbf{r}, t) = -\int_{\Sigma_0} dA_0\, n_j{}^0(\mathbf{r}_0)\{\partial_k \delta(\mathbf{r}-\mathbf{r}_0)\, \Sigma_{ijkl}(\mathbf{r}_0)[s_i'(\mathbf{r}_0, t)]_-^+\}. \tag{85}$$

It is clear from equation (84) that the dynamic elastic-gravitational effect of the prescribed tangential displacement discontinuity $[\mathbf{s}'(\mathbf{r}_0, t)]_-^+$ across the fault surface $\Sigma_0$ is exactly the same as the effect of the purely hypothetical introduction of an extra applied body force $\mathbf{e}'(\mathbf{r}, t)$ given by equation (85) into an unfaulted medium. It can be shown that equation (85) giving the equivalent body forces is valid also for the case where the fault surface $\Sigma_0$ intersects the external surface $\partial V$ of the Earth model. The expression (85) for the equivalent body forces is similar to that given by Burridge & Knopoff (1964) for the case of zero initial stress; the only difference is that in this case the fourth order elastic tensor $\Sigma_{ijkl}(\mathbf{r}_0)$ given in (79) appears instead of $C_{ijkl}(\mathbf{r}_0)$. Note in particular that as in the case discussed by Burridge & Knopoff (1964), the body force equivalents depend only upon the kinematical prescription of the source and upon the elastic properties (including the initial stress $\mathbf{T}_0(\mathbf{r}_0)$) on the fault surface itself. The expression (85) for the equivalent body forces $\mathbf{e}'(\mathbf{r}, t)$ reduces to the equivalent expression of Burridge & Knopoff (1964) not only for the case of zero initial stress, but also for the case where the initial stress is purely hydrostatic $\mathbf{T}_0(\mathbf{r}_0) = -p_0(\mathbf{r}_0)\,\mathbf{I}$, since in that case $\Sigma_{ijkl}(\mathbf{r}_0) = C_{ijkl}(\mathbf{r}_0)$.

It is a simple matter to show that the equivalent body forces $\mathbf{e}'(\mathbf{r}, t)$ defined by equation (85) exert no net force or torque on the Earth model V. The total force acting on the entire Earth model V is obtained by integration over $V$.

$$\int_V dV[e_l'(\mathbf{r}, t)] = -\int_{\Sigma_0} dA_0\, n_j{}^0(\mathbf{r}_0)\left\{\Sigma_{ijkl}(\mathbf{r})[s_i'(\mathbf{r}_0, t)]_-^+ \int_V dV\,[\partial_k \delta(\mathbf{r}-\mathbf{r}_0)]\right\}. \tag{86}$$

Now

$$\int_V dV[\partial_k \delta(\mathbf{r}-\mathbf{r}_0)] = \int_{\partial V} dA\, n_k\, \delta(\mathbf{r}-\mathbf{r}_0) = 0$$

if $\Sigma_0$ and $\partial V$ have no points in common. Hence the total equivalent force acting on $V$ is at all times zero; this result is also valid for the case where $\Sigma_0$ and $\partial V$ intersect.

If $\hat{\mathbf{x}}_1$, $\hat{\mathbf{x}}_2$, $\hat{\mathbf{x}}_3$, is an arbitrary Cartesian axis system with origin 0, then the net moment about the point 0 of the equivalent body force distribution $\mathbf{e}'(\mathbf{r}, t)$ is

$$\mathbf{M}_0(t) = \int_V dV[\mathbf{r}\times\mathbf{e}'(\mathbf{r}, t)]\, dV. \tag{87}$$

Denoting the component of $M_0(t)$ about the axis $\hat{\mathbf{x}}_m \times \hat{\mathbf{x}}_l$ by $M_{ml}{}^0(t)$, one has

$$M_{ml}{}^0(t) = \int_V dV[r_m e_l'(\mathbf{r}, t) - r_l e_m'(\mathbf{r}, t)]. \tag{88}$$

This is

$$M_{ml}{}^o(t) = -\int_{\Sigma_0} dA_0\, n_j{}^0(\mathbf{r}_0)\left\{[s_i'(\mathbf{r}_0, t)]_-^+ \left[\Sigma_{ijkl}(\mathbf{r}_0)\int_V dV\mathbf{r}_m\partial_k\delta(\mathbf{r}-\mathbf{r}_0) - \Sigma_{ijkm}(\mathbf{r}_0)\int_V dV r_l\partial_k\delta(\mathbf{r}-\mathbf{r}_0)\right]\right\}. \tag{89}$$

But for the case where $\Sigma_0$ and $\partial V$ do not intersect

$$\Sigma_{ijkl}(\mathbf{r}_0)\int_V dV r_m\partial_k\delta(\mathbf{r}-\mathbf{r}_0) - \Sigma_{ijkm}(\mathbf{r}_0)\int_V dV\mathbf{r}_l\partial_k\delta(\mathbf{r}-\mathbf{r}_0) = -\Sigma_{ijml}(\mathbf{r}_0) + \Sigma_{ijlm}(\mathbf{r}_0) \tag{90}$$

and from (79), this is zero. Hence the net torque exerted on $V$ by the equivalent forces $\mathbf{e}'(\mathbf{r}, t)$ is at all times zero; this result as well may also be demonstrated for the case where $\Sigma_0$ and $\partial V$ intersect.

## 12. An infinitesimal fault surface

Consider the special case of an infinitesimal, non-propagating fault surface $\Sigma_0$ whose total area $A_0$ is very small compared to the dimensions of any inhomogeneities in the Earth model V. Such a model is likely to be a sufficiently good approximation for many applications, e.g. studies of the excitation of the normal modes of the Earth model V (Gilbert 1971) or of the teleseismic radiation field. For this case the expression (85) for the equivalent body forces $\mathbf{e}'(\mathbf{r}, t)$ may be written

$$e_l'(\mathbf{r}, t) = -A_0\, n_j{}^0(\mathbf{r}_0)\{\partial_k\delta(\mathbf{r}-\mathbf{r}_0)\,\Sigma_{ijkl}(\mathbf{r}_0)[s_i'(\mathbf{r}_0, t)]_-^+\} \tag{91}$$

where $\mathbf{r}_0$ is the hypocentre or location of the infinitesimal fault surface $\Sigma_0$ and where $[\mathbf{s}'(\mathbf{r}_0, t)]_-^+$ now represents the averaged tangential displacement discontinuity across that surface. Specializing further, let the elastic properties of the medium in the vicinity of the source be isotropic so that the elastic coefficients $C_{ijkl}(\mathbf{r}_0)$ may be expressed in terms of the compressibility $\kappa_0 = \kappa(\mathbf{r}_0)$ and the rigidity $\mu_0 = \mu(\mathbf{r}_0)$ by equation (27). Up until now the Cartesian axis system $\hat{\mathbf{x}}_1$, $\hat{\mathbf{x}}_2$, $\hat{\mathbf{x}}_3$ utilized for computation has been completely arbitrary. Now for simplicity take $\hat{\mathbf{x}}_1$, $\hat{\mathbf{x}}_2$, $\hat{\mathbf{x}}_3$ to be a hypocentral co-ordinate system with its origin 0 at the point $\mathbf{r}_0$ and with $\hat{\mathbf{x}}_3$ along $\hat{\mathbf{n}}_0(\mathbf{r}_0)$ and $\hat{\mathbf{x}}_1$ along the direction of slip on the fault.

$$\left.\begin{aligned} n_j{}^0(\mathbf{r}_0) &= \delta_{j3} \\ [s_i'(\mathbf{r}_0, t)]_-^+ &= \Delta s_0(t)\,\delta(\mathbf{r}_0)\,\delta_{i1}. \end{aligned}\right\} \tag{92}$$

In this case the only non-zero $\Sigma_{ijkl}(\mathbf{r}_0)$ are $\Sigma_{13kl}(\mathbf{r}_0)$.

Consider first the case where the initial static stress at the hypocentre is perfectly hydrostatic, $\mathbf{T}_0(\mathbf{r}_0) = -p_0(\mathbf{r}_0)\,\mathbf{I}$. In this case equation (91) reduces to

$$\left.\begin{aligned} e_1'(\mathbf{r},t) &= -[\mu_0 A_0\,\Delta s_0(t)]\,\delta(r_1)\,\delta(r_2)\,\delta'(r_3) \\ e_2'(\mathbf{r},t) &= 0 \\ e_3'(\mathbf{r},t) &= -[\mu_0 A_0\,\Delta s_0(t)]\,\delta'(r_1)\,\delta(r_2)\,\delta(r_3). \end{aligned}\right\} \tag{93}$$

The equivalent force system for this case, equation (93), is thus the familiar double couple with each couple having a net moment $\mu_0 A_0\,\Delta s_0$ (see Fig. 4). This result was first obtained by Burridge and Knopoff (1964) for the case of zero initial stress, and has since received a great deal of application. The earthquake moment, defined as as $\mu_0 A_0\,\Delta s_0$, has become an almost routinely measured parameter used to describe and characterize individual earthquakes. The above argument establishes that the familiar double couple equivalent force (93) is dynamically equivalent to an infinitesimal tangential displacement dislocation even in the presence of an initial static stress field, so long as the initial stress at the hypocentre is purely hydrostatic, $\mathbf{T}_0(\mathbf{r}_0) = -p_0(\mathbf{r}_0)\,\mathbf{I}$.

If the initial stress at the hypocentre is not purely hydrostatic, then there are additional equivalent body forces $\mathbf{e}'(\mathbf{r},t)$ which must be added to the system (93) in order to properly model a point tangential displacement dislocation. Denote the deviatoric initial stress at the hypocentre $\boldsymbol{\tau}_0(\mathbf{r}_0)$ simply by $\boldsymbol{\tau}_0$. The additional equivalent body forces $\mathbf{e}'(\mathbf{r},t)$ in the presence of such a deviatoric initial hypocentral stress are

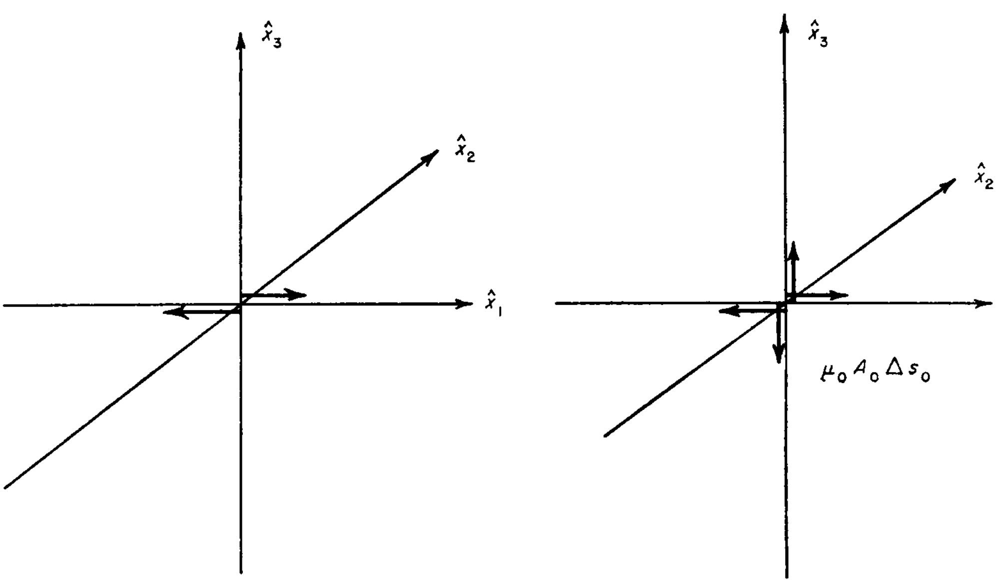


FIG. 4. Left: point tangential displacement dislocation in the presence of an initial hydrostatic pressure $p_0$. Right: equivalent double couple of individual moment $\mu_0 A_0\,\Delta s_0$.

easily catalogued, using (91) and (79)

$$\left.\begin{aligned} e_1'(\mathbf{r},t) &= -\tfrac{1}{2}A_0\,\Delta s_0(t)[3\tau_{13}{}^0\,\delta'(r_1)\delta(r_2)\delta(r_3)+\tau_{23}{}^0\,\delta(r_1)\delta'(r_2)\delta(r_3) \\ &\qquad +(\tau_{33}{}^0-\tau_{11}{}^0)\,\delta(r_1)\delta(r_2)\delta'(r_3)] \\ e_2'(\mathbf{r},t) &= -\tfrac{1}{2}A_0\,\Delta s_0(t)[\tau_{23}{}^0\,\delta'(r_1)\delta(r_2)\delta(r_3)+\tau_{13}{}^0\,\delta(r_1)\delta'(r_2)\delta(r_3) \\ &\qquad -\tau_{12}{}^0\,\delta(r_1)\delta(r_2)\delta'(r_3)] \\ e_3'(\mathbf{r},t) &= -\tfrac{1}{2}A_0\,\Delta s_0(t)[(\tau_{33}{}^0-\tau_{11}{}^0)\,\delta'(r_1)\delta(r_2)\delta(r_3)-\tau_{12}{}^0\,\delta(r_1)\delta'(r_2)\delta(r_3) \\ &\qquad +\tau_{13}{}^0\,\delta(r_1)\delta(r_2)\delta'(r_3)]. \end{aligned}\right\} \tag{94}$$

The equations (94) are an explicit list of the additional $\mathbf{e}'(\mathbf{r}, t)$ expressed in terms of the hypocentral deviatoric stress components $\tau_{ij}{}^0$, and resolved along the Cartesian axis vectors $\hat{\mathbf{x}}_1$, $\hat{\mathbf{x}}_2$, $\hat{\mathbf{x}}_3$. As written, these additional equivalent body forces appear to consist of a combination of three double couples and three linear vector dipoles (see Fig. 5). In the next section, it will be shown how a more suitable choice of a hypocentral Cartesian axis system allows one to simplify this representation.

Burridge & Knopoff (1964) go on to list the body force equivalents in the case of zero initial stress for a number of other simple dislocation models, e.g. for various propagating ruptures. This could also be done now for the more general case of propagating ruptures in an elastic-gravitational medium with an initial static stress. The results of such an exercise are exactly what one would expect. No new features

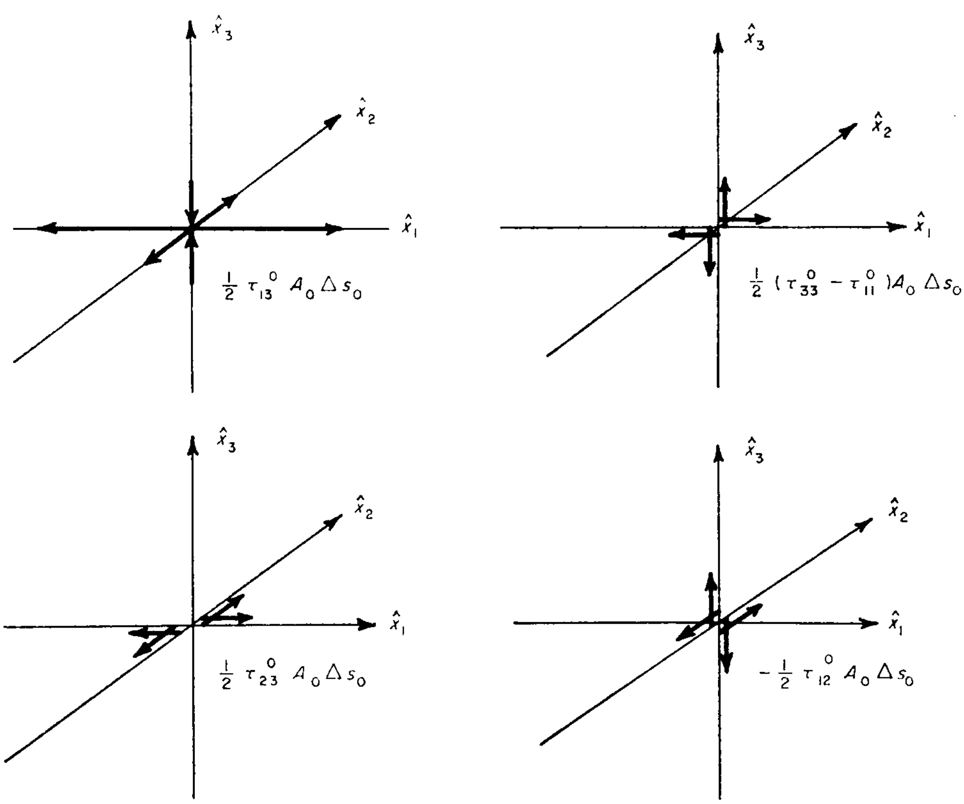


FIG. 5. The additional equivalent forces in the presence of a deviatoric initial hypocentric stress with Cartesian components $\tau_{ij}{}^0$.

appear; propagating tangential displacement dislocations are dynamically equivalent to propagating equivalent force systems. Such examples will not be discussed explicitly here.

## 13. The dislocation moment tensor

Consider once again the expression (91) giving the body forces $\mathbf{e}'(\mathbf{r}, t)$ equivalent to a point tangential displacement dislocation in an arbitrary Cartesian axis system $\hat{\mathbf{x}}_1, \hat{\mathbf{x}}_2, \hat{\mathbf{x}}_3$. Let $\mathbf{e}_0(\mathbf{r}_0)$ be a unit vector in the direction of slip on the fault so that

$$[\mathbf{s}'(\mathbf{r}_0, t)]_-^+ = \Delta s_0(t)\,\hat{\mathbf{e}}(\mathbf{r}_0). \tag{95}$$

Equation (91) may be rewritten in the form

$$e_l'(\mathbf{r}, t) = -M_{kl}(\mathbf{r}_0, t)\,\partial_k\,\delta(\mathbf{r}-\mathbf{r}_0) \tag{96}$$

where

$$M_{kl}(\mathbf{r}_0, t) = A_0\,\Delta s_0(t)\,e_i{}^0(\mathbf{r}_0)\,n_j{}^0(\mathbf{r}_0)\,\Sigma_{ijkl}(\mathbf{r}_0). \tag{97}$$

The coefficients $M_{kl}(\mathbf{r}_0, t)$ are the nine Cartesian components of a second order tensor $\mathbf{M}(\mathbf{r}_0, t)$ which will be called the dislocation moment tensor. Because of the symmetry relation $\Sigma_{ijkl} = \Sigma_{ijlk}$, the dislocation moment tensor is symmetric

$$M_{kl}(\mathbf{r}_0, t) = M_{lk}(\mathbf{r}_0, t). \tag{98}$$

Rewritten in invariant notation, equation (96) takes the form

$$\mathbf{e}'(\mathbf{r}, t) = -\mathbf{M}(\mathbf{r}_0, t)\cdot\nabla\delta(\mathbf{r}-\mathbf{r}_0). \tag{99}$$

The dislocation moment tensor, whose Cartesian components are given in equation (97), is the extension to this more general case of an initial static stress of the tensor previously called by Kostrov (1970) the tensor of moments and by Gilbert (1971) the action tensor. The symmetry (98) of the dislocation moment tensor is a direct consequence of the fact that a point tangential displacement dislocation does not exert a net torque on the Earth model V.

It can be seen from (79) that in general $\Sigma_{ijkl}(\mathbf{r}_0) \neq \Sigma_{jikl}(\mathbf{r}_0)$, but that this symmetry relation is satisfied for the case where the initial static stress is purely isotropic, $\mathbf{T}_0(\mathbf{r}_0) = -p_0(\mathbf{r}_0)\mathbf{I}$. Inspection of equation (91) or (97) thus reveals that in the latter case, there exists for a point tangential displacement dislocation an inherent ambiguity between the unit normal $\hat{\mathbf{n}}_0$ and the unit vector $\hat{\mathbf{e}}_0$ in the direction of slip, but that there is no such ambiguity in the general case $\Sigma_{ijkl}(\mathbf{r}_0) \neq \Sigma_{jikl}(\mathbf{r}_0)$. This ambiguity is the well-known fault plane-auxiliary plane ambiguity encountered in first motion focal mechanism studies, and it is of some interest to observe that in general the existence of a non-isotropic initial stress $\boldsymbol{\tau}_0(\mathbf{r}_0)$ serves to remove the ambiguity.

Since the dislocation moment tensor $\mathbf{M}(\mathbf{r}_0, t)$ is symmetric, it may always be reduced to its principal axes. If one chooses the principal axes or eigenvectors of $\mathbf{M}(\mathbf{r}_0, t)$ as Cartesian co-ordinate axes in (91) or (96), it is clear that the equivalent body forces $\mathbf{e}'(\mathbf{r}, t)$ resolved in that axis system will appear to consist of three unequal mutually perpendicular vector dipoles. The algebraic determination of this canonical equivalent force representation in terms of the hypocentral deviatoric stress components $\tau_{ij}{}^0$ is straightforward and will not be discussed here.

## 14. Some remarks on the derivation

The main result of this paper is equation (85), giving the equivalent body forces $\mathbf{e}'(\mathbf{r}, t)$ which must be applied to an Earth model V in order to produce an elastic-gravitational response $\mathbf{s}'(\mathbf{r}, t)$, $\phi_1'(\mathbf{r}, t)$ equivalent to that produced by a tangential displacement dislocation across a fault surface $\Sigma_0$. This result could not unfortunately be obtained without laying considerable preliminary groundwork, namely the linearized theory of elastic-gravitational deformation in the presence of an initial static stress $\mathbf{T}_0(\mathbf{r})$. In the interest of brevity, and in an attempt to keep the derivation as direct and uncluttered as possible, two difficulties which arise in the course of the argument have been overlooked. The existence and the resolution of these two difficulties will be very briefly discussed here.

The first difficulty is in fact an apparent flaw in the argument as given; it occurs in the application of the Betti reciprocal theorem (66) to obtain the Volterra dislocation relation (80) and (81), and in particular in the specification of the body force $\mathbf{f}(\mathbf{r}, t)$ to be a unit impulsive point force. Such a unit impulsive point force will of course in general exert a net force and torque on the Earth model V giving rise to rigid body translations and rotations, and for that reason the response $\mathbf{s}^p(\mathbf{r}, -t; \mathbf{x}, -s)$ to this applied force cannot be uniquely determined (Love 1927). Since Volterra's relation (81) is supposed to be a specific representation of the response $\mathbf{s}'(\mathbf{x}, s)$ to a tangential displacement dislocation in terms of $\mathbf{s}^p(\mathbf{r}, -t; \mathbf{x}, -s)$, it is clear that Volterra's relation in the simple form (81) cannot be entirely correct. The resolution of this apparent difficulty is straightforward. A slightly more careful examination of the argument leading from (66) to (81) reveals that in general Volterra's relation (81) can be used to determine explicitly *almost all* of the response $\mathbf{s}'(\mathbf{x}, s)$, but that a certain correction must be applied by requiring that a tangential displacement dislocation cannot give rise to a net rigid body translation or rotation of the earth model V. For example, for the case of a spherically symmetric earth model V, Volterra's relation may be used to determine all of $\mathbf{s}'(\mathbf{x}, s)$ except for the $l = 1$ vector spherical harmonic contribution; the $l = 1$ terms constitute a special case which must be treated separately by requiring that there be no net rigid body motion. This type of situation has been amply discussed in the past in at least three different contexts (Ben-Menahem & Singh 1968; Dahlen 1971; Cathles 1971; Farrell 1971), and it does not seem worthwhile here to go into any detail regarding the precise nature of the necessary considerations. The important thing is that equation (85) giving the equivalent body forces $\mathbf{e}'(\mathbf{r}, t)$ will not be affected by any such considerations. This is clear because of the fact, demonstrated above, that the equivalent body forces $\mathbf{e}'(\mathbf{r}, t)$, in contrast to the unit impulsive point force $\mathbf{f}(\mathbf{r}, t)$, do not exert a net force or torque on the earth model V.

The second difficulty is not an apparent flaw in the present argument, but is rather a problem which arises in any attempt to extend the argument to an Earth model which is not everywhere solid (e.g. to an Earth model which has a fluid core and/or which is partially covered by fluid oceans). The extension to this more general case is not entirely trivial for the following reason. The continuity conditions (32) apply only to a *welded* boundary (in particular they do not apply to a fluid-solid boundary, where there can be tangential slip), and in the course of the argument, particularly when using Gauss's theorem, the conditions (32) were assumed to apply throughout $V$ except on $\Sigma_0$. If one wishes to extend the argument to an Earth model V with a fluid core, then additional surface integral terms over the undeformed core–mantle boundary surface arise in expressions like equations (51) and (53), Rayleigh's principle. It may however be shown that the existence of these additional terms in Rayleigh's principle does not give rise to any extra terms in the Volterra dislocation relation (80) or (81), and thus the equivalent body forces $\mathbf{e}'(\mathbf{r}, t)$ in (85) are also unaffected. This is clear since for this purely theoretical (i.e. non-numerical) application, it is perfectly reasonable to take as a model V of the Earth a completely solid elastic continuum, but with a very low rigidity core, in which case the derivation as given is applicable. The principal situation in which the consideration of these additional core–mantle boundary terms will be necessary for a complete treatment is in any future attempt to utilize normal mode perturbation theory to discuss the effect of lateral inhomogeneities (especially non-hydrostatic undulations of the core–mantle interface) on the Earth's free oscillations. Since it may be shown that there are in fact no additional terms in the case of a purely hydrostatic initial stress, Dahlen's (1968) results for the normal modes of a rotating, hydrostatic ellipsoidal Earth model are unaffected. Madariaga (1971) has given a thorough and systematic discussion of the free oscillations of laterally heterogeneous Earth models, but he at the outset simply neglected (probably justifiably) all terms involving the initial deviatoric static stress $\boldsymbol{\tau}_0(\mathbf{r})$, including the extra core–mantle boundary terms mentioned here.

## 15. Concluding remarks

The body forces $\mathbf{e}'(\mathbf{r}, t)$ equivalent to a given arbitrary tangential seismic dislocation are thus given by equation (85), even for a fault surface $\Sigma_0$ located in the solid portion of a partially fluid earth, i.e. one with a fluid core and/or oceans. The equivalent body forces are independent of the initial hydrostatic pressure $p_0(\mathbf{r}_0)$ at points $\mathbf{r}_0$ on the fault surface, but they do depend on the magnitude and orientation of the initial deviatoric stress $\boldsymbol{\tau}_0(\mathbf{r}_0)$ on the fault surface. For the case of an infinitesimal tangential dislocation, the precise nature of the dependence on the components $\tau_{ij}(\mathbf{r}_0)$ is indicated in equation (94). Theoretically, the results presented here would allow one to deduce information about the magnitude and orientation of the initial deviatoric stress $\boldsymbol{\tau}_0(\mathbf{r}_0)$ at the hypocentre of an earthquake directly from a consideration of the geometry of the resulting elastic-gravitational displacement field. The possibility of using data about the geometric or spatial variation of the seismic displacement field to directly measure hypocentral stresses is presently being investigated, but it is clear that although it is a theoretical possibility, it will be very difficulty to implement. The reason is that the relative contribution of the terms which depend directly on the hypocentral stress $\boldsymbol{\tau}_0(\mathbf{r}_0)$ is of the order $\tau_0/\mu_0$, and for reasonable values of the deviatoric stresses (say a few hundred bars), this ratio is very small (about $10^{-3}$). In this sense, the results presented here may almost be viewed as negative results. Until the various phenomena which act to degrade the earthquake-radiated elastic-gravitational displacement field (e.g. crustal and upper mantle inhomogeneities) are better understood, it will be difficult to utilize purely geometrical measurements to deduce any information about hypocentral deviatoric stresses. The estimates of hypocentral stresses which have been obtained in the past have been deduced by a study of the time or spectral behaviour of the displacement field at a single seismographic location. For example, Wyss (1970) and others have shown how a comparison of the spectral amplitudes of seismic waves at high frequencies with those at low frequencies can be used to deduce the so-called apparent stress (defined as the seismic efficiency times the average stress) at an earthquake focus. If the difficulties associated with measuring the required data with sufficient accuracy can be overcome, then the techniques described here could be used to determine the actual deviatoric hypocentral stresses, not merely the apparent stresses.

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## References


Backus, G. E., 1967. Converting vector and tensor equations to scalar equations in spherical coordinates, *Geophys. J. R. astr. Soc.*, **13**, 71–101.

Backus, G. E. & Gilbert, J. F., 1967. Numerical applications of a formalism for geophysical inverse problems, *Geophys. J. R. astr. Soc.*, **13**, 247–276.

Ben-Menahem, A & Singh, S. J., 1968. Eigenvector expansions of Green's dyads with applications to geophysical theory, *Geophys. J. R. astr. Soc.*, **16**, 417–452.

Biot, M. A., 1965. *Mechanics of incremental deformations*, John Wiley, New York, 504 pp.

Burridge, R. & Knopoff, L., 1964. Body force equivalents for seismic dislocations, *Bull. seism. Soc. Am.*, **54**, 1875–1888.

Cathles, L. M., 1971. *The viscosity of the Earth's mantle*, Ph.D. thesis, Princeton University.

Dahlen, F. A., 1968. The normal modes of a rotating, elliptical Earth, *Geophys. J. R. astr. Soc.*, **16**, 329–367.

Dahlen, F. A., 1971. The excitation of the Chandler wobble by earthquakes, *Geophys. J. R. astr. Soc.*, **25**, 157.

Eringen, A. C., 1967. *Mechanics of continua*, John Wiley and Sons, New York, 502 pp.

Farrell, W. E., 1972. Deformation of the Earth by surface loads, *Rev. Geophys.*, in press.

Gilbert, J. F., 1971. Excitation of the normal modes of the Earth by earthquake sources, *Geophys. J. R. astr. Soc.*, **22**, 223–226.

Kostrov, B. V., 1970. The theory of the focus for tectonic earthquakes, *Izv., Earth Physics*, **4**, 84–101.

Love, A. E. H., 1927. *A treatise on the mathematical theory of elasticity*, New York, Dover Publications, 634 pp.

Madariaga, R., 1971. *Free oscillations of the laterally heterogeneous Earth*, Ph.D. thesis, Massachusetts Institute of Technology.

Malvern, L. E., 1969, *Introduction to the mechanics of a continuous medium*, Prentice-Hall, Englewood Cliffs, New Jersey, 713 pp.

Spivak, M., 1965, *Calculus on manifolds*, W. A. Benjamin, Inc., New York, 146 pp.

Whittaker, E. T., 1936. *A treatise on the analytical dynamics of particles and rigid bodies*, Cambridge University Press, Cambridge, England, 456 pp.

Wyss, M., 1970. Stress estimates for South American shallow and deep earthquakes, *J. geophys. Res.*, **75**, 1529–1544.