Department of Applied Mechanics

STANFORD UNIVERSITY

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ELASTIC-PLASTIC BOUNDARIES IN PLANE AND CYLINDRICAL WAVE PROPAGATION OF COMBINED STRESSES

By

T.C.T. Ting

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ABSTRACT

A general study is given of plane and cylindrical wave propagation of combined stresses in an elastic-plastic medium. The coefficients of the governing differential equations, when written in matrix notation, are symmetric matrices and can be divided into sub-matrices each of which has a special form. The relations between the stresses on both sides of an elastic-plastic boundary are derived. Also presented are the restrictions on the speed of an elastic-plastic boundary.

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1. Introduction.

The equations of motion for a continuum body are

$$\frac{\partial \sigma_{ij}}{\partial x_j} = \rho \frac{\partial v_i}{\partial t} , \quad (i,j=1,2,3) \tag{1}$$

where σ_{ij} is the stress, v_i the velocity and ρ is the mass density of the body, and summation is implied by repeated indices. The relation between the strain ϵ_{ij} and the velocity v_i is

$$\frac{\partial \epsilon_{ij}}{\partial t} = \frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \tag{2}$$

while the stress-strain relation for an elastic, isotropic work-hardening material is (see [1])

$$\frac{\partial \epsilon_{ij}}{\partial t} = \frac{1+\nu}{E} \frac{\partial \sigma_{ij}}{\partial t} - \frac{\nu}{E} \delta_{ij} \frac{\partial \sigma_{kk}}{\partial t} + G(k) \frac{\partial f}{\partial \sigma_{ij}} \frac{\partial f}{\partial \sigma_{k\ell}} \frac{\partial \sigma_{k\ell}}{\partial t} \tag{3}$$

E is Young's modulus, ν is Poisson's ratio. k is the yield stress and the yield condition can be written as

$$f(\sigma_{ij}) = k^2 \tag{4}$$

$G(k)$ in Eq. (3) is a given function of k which characterizes the work-hardening property. Equations (1) ~ (4) give a complete description of wave propagation in three-dimensional elastic-plastic media.

In this paper, we will restrict our attention to special cases in which the governing equations depend on only one space variable.

Plane wave propagation and cylindrical wave propagation are such special cases. The most general plane wave propagation is the one in which v_1, v_2, and v_3 are functions of x_1 and t only. Using x instead of x_1 for simplicity, σ_1, τ_2, τ_3 for σ_{11}, σ_{21}, σ_{31} respectively, Eq. (1) gives

$$\sigma_{1,x} = \rho\, v_{1,t} \tag{5}$$

$$\tau_{2,x} = \rho\, v_{2,t} \tag{6}$$

$$\tau_{3,x} = \rho\, v_{3,t} \tag{7}$$

where the subscripts x and t denote partial differentiation with respect to these variables. If we use von Mises yield condition:

$$f = \frac{1}{2} s_{ij}\, s_{ij} = k^2$$

where

$$s_{ij} = \sigma_{ij} - \frac{1}{3} \delta_{ij}\, \sigma_{kk} ,$$

Eq. (3) gives, making use of Eq. (2) and the fact that $\epsilon_{22} = \epsilon_{33} = 0$ leads to $\sigma_{22} = \sigma_{33}$ in the present case,

$$v_{1,x} = \frac{1}{E} \sigma_{1,t} - \frac{2\nu}{E} \sigma_{2,t} + s_1 GQ \tag{8}$$

$$0 = - \frac{2\nu}{E} \sigma_{1,t} + \frac{2(1-\nu)}{E} \sigma_{2,t} + 2s_2 GQ \tag{9}$$

$$v_{2,x} = \frac{1}{\mu} \tau_{2,t} + 2\tau_2 GQ \tag{10}$$

$$v_{3,x} = \frac{1}{\mu} \tau_{3,t} + 2\tau_3 GQ \tag{11}$$

where s_1, s_2 and σ_2 stand for s_{11}, s_{22}, and σ_{22} respectively. μ is the shear modulus and

$$Q = s_1\sigma_{1,t} + 2s_2\sigma_{2,t} + 2\tau_2\tau_{2,t} + 2\tau_3\tau_{3,t} \tag{12}$$

Equations (5 ~ 11) give a complete description of a general plane wave propagation in an elastic-plastic medium.

When $v_2 \equiv 0$, and hence $\tau_2 \equiv 0$, Eqs. (5), (7), (8), (9), and (11) reduce to the case of pressure-shear wave propagation considered in [2,3,4]. If v_2, v_3 are the only non-zero velocity components, then τ_2 and τ_3 are the only non-zero stresses and Eqs. (6), (7), (10) and (11) reduce to the case of two-shear waves studied in [3,5,6]. The equations derived in [7] for combined longitudinal and torsinal waves in a thin-walled tube can also be reduced from Eqs. (5), (6), (8) and (10) by letting $\sigma_2 \equiv 0$.

If v_r, v_θ, v_z are the velocity components of a particle in cylindrical coordinates (r,θ,z), the most general cylindrical wave propagation is the one in which v_r, v_θ and v_z are functions of r and t only. Then, the only non-zero strains are ϵ_r, ϵ_θ, γ_θ, γ_z where γ_θ, γ_z stand for $\epsilon_{r\theta}$, ϵ_{rz} respectively. Consequently σ_r, σ_θ, σ_z, τ_θ, τ_z are the non-zero stresses where $\tau_\theta = \sigma_{r\theta}$, $\tau_z = \sigma_{rz}$. Now, instead of writing the governing equations for cylindrical wave propagation in the form shown in Eqs. (5 ~ 11), we will use matrix notation and write the governing equations, Eqs. (1~3) in a matrix differential equation which can be applied to both plane and cylindrical wave propagation.

2. The Matrix Differential Equation.

For wave propagation which involves only one space variable x or r, the equations of motion (1) can be written in matrix notation as

$$\underset{\sim}{M}\,\underset{\sim}{\sigma}_x + \underset{\sim}{b}_1 = \rho\,\underset{\sim}{v}_t \tag{13}$$

where $\underset{\sim}{\sigma}$ and $\underset{\sim}{v}$ are column vectors whose elements are stress and velocity components respectively. $\underset{\sim}{b}_1$ is also a column vector whose elements are functions of stress $\underset{\sim}{\sigma}$ and space variable x only. (In cylindrical waves, x becomes r). If $\underset{\sim}{\sigma}$ has m elements and $\underset{\sim}{v}$ has n elements, then $\underset{\sim}{M}$ is an $n \times m$ matrix whose elements are constants. The continuity condition Eq. (2) can be written as

$$\underset{\sim}{\epsilon}_t = \underset{\sim}{N}\,\underset{\sim}{v}_x + \underset{\sim}{b}_2 \tag{14}$$

where $\underset{\sim}{\epsilon}$ is a column vector with strains as its elements while $\underset{\sim}{b}_2$ is a column vector whose elements are functions of velocity $\underset{\sim}{v}$ and space variable x only. $\underset{\sim}{N}$ is an $m \times n$ matrix whose elements are constants. Finally, the stress-strain relation, Eq. (3) can be written as

$$\underset{\sim}{\epsilon}_t = \underset{\sim}{S}\,\underset{\sim}{\sigma}_t \tag{15}$$

where $\underset{\sim}{S}$ is an $m \times m$ square matrix. In the plastic region, $\underset{\sim}{S}$ can be written more precisely as

$$\underset{\sim}{S}^p = \underset{\sim}{S}^e + G(k)(\underset{\sim}{\nabla} f)(\underset{\sim}{\nabla} f)^T \tag{16}$$

$\underset{\sim}{S}^e$ is also an $m \times m$ square matrix whose elements are functions of elastic constants only. Thus

$$\underset{\sim}{\epsilon}_t = \underset{\sim}{S}^e \underset{\sim}{\sigma}_t \tag{17}$$

gives the elastic stress-strain relation. $\underset{\sim}{\nabla} f$ is the gradient of $f(\underset{\sim}{\sigma})$ with respect to the components of $\underset{\sim}{\sigma}$. Hence, by (4)

$$(\underset{\sim}{\nabla} f)^T \underset{\sim}{\sigma}_t = 2kk_t \tag{18}$$

Now, by eliminating $\underset{\sim}{\epsilon}_t$ between Eqs. (14) and (15), we can write Eqs. (13)~(15) in one matrix equation:

$$\underset{\sim}{A}\, \underset{\sim}{w}_t + \underset{\sim}{B}\, \underset{\sim}{w}_x = \underset{\sim}{b} \tag{19}$$

where

$$\underset{\sim}{A} = \begin{bmatrix} \rho \underset{\sim}{I} & \underset{\sim}{O} \\ \underset{\sim}{O} & \underset{\sim}{S} \end{bmatrix}, \qquad \underset{\sim}{w} = \begin{bmatrix} \underset{\sim}{v} \\ \underset{\sim}{\sigma} \end{bmatrix}$$

$$\underset{\sim}{B} = \begin{bmatrix} \underset{\sim}{O} & -\underset{\sim}{M} \\ -\underset{\sim}{N} & \underset{\sim}{O} \end{bmatrix}, \qquad \underset{\sim}{b} = \begin{bmatrix} \underset{\sim}{b}_1 \\ \underset{\sim}{b}_2 \end{bmatrix} \tag{20}$$

and $\underset{\sim}{I}$ is a unit square matrix. It can be checked easily that the governing equations for general plane wave propagation derived in Eqs. (5 ~ 11) as well as the equations for other plane wave propagation reduced from Eqs. (5 ~ 11) can be written in the form of Eq. (19). Moreover, $\underset{\sim}{A}$ and $\underset{\sim}{B}$ are both symmetric in all cases. In particular,

$\underset{\sim}{N} = \underset{\sim}{M}^T$ and the elements of $\underset{\sim}{N}$ and $\underset{\sim}{M}$ are either one or zero. The same is true for all cylindrical waves. For the most general cylindrical wave propagation, it can be shown that, using von Mises yield condition:

$$\underset{\sim}{v} = \begin{bmatrix} v_r \\ v_\theta \\ v_z \end{bmatrix}, \quad \underset{\sim}{b}_1 = \begin{bmatrix} (\sigma_1 - \sigma_2)/r \\ 2\tau_\theta/r \\ \tau_z/r \end{bmatrix}, \quad \underset{\sim}{M} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\underset{\sim}{\sigma} = \begin{bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \\ \tau_\theta \\ \tau_z \end{bmatrix}, \quad \underset{\sim}{b}_2 = \begin{bmatrix} 0 \\ v_r/r \\ 0 \\ -v_\theta/r \\ 0 \end{bmatrix}, \quad \underset{\sim}{N} = \underset{\sim}{M}^T \tag{21}$$

$$\underset{\sim}{S}^e = \begin{bmatrix} 1/E & -\nu/E & -\nu/E & 0 & 0 \\ -\nu/E & 1/E & -\nu/E & 0 & 0 \\ -\nu/E & -\nu/E & 1/E & 0 & 0 \\ 0 & 0 & 0 & 1/\mu & 0 \\ 0 & 0 & 0 & 0 & 1/\mu \end{bmatrix}, \quad \underset{\sim}{\nabla} f = \begin{bmatrix} s_r \\ s_\theta \\ s_z \\ 2\tau_\theta \\ 2\tau_z \end{bmatrix}$$

Cristescu [8] has studied an axilly symmetric wave propagation in which v_r and v_θ are the only non-vanishing velocities. The governing equations can be obtained from Eq. (21) by letting $v_z \equiv \tau_z \equiv 0$. The particular cases of two-shear cylindrical waves and pressure-shear cylindrical waves have also been derived earlier in [5] but the resulting coefficient matrices were not symmetric. The formulation presented here yields symmetric coefficient matrices for all cylindrical waves.

The analyses in the rest of this paper will be based on the matrix differential equation (19). Although for all plane waves and cylindrical waves the matrices $\underset{\sim}{A}$ and $\underset{\sim}{B}$ are symmetric, the analyses presented in the following do not require the symmetry property of $\underset{\sim}{A}$ and $\underset{\sim}{B}$. The yield condition f of Eq. (4) is not restricted to von Mises yield condition.

3. An Identity.

In this section, we will derive an identity which is useful in the analyses of the present problem.

Let $\underset{\sim}{P}$ be a $r \times r$ matrix and $\underset{\sim}{g}$ and $\underset{\sim}{h}$ are column vectors with r components. Then, if a is a scalar,

$$\|\underset{\sim}{P} + a\, \underset{\sim}{h}\, \underset{\sim}{g}^T\| = \|\underset{\sim}{P}\| + a\, \underset{\sim}{h}^T \underset{\sim}{P}^* \underset{\sim}{g} \tag{22}$$

where $\|\underset{\sim}{P}\|$ denote the determinant of $\underset{\sim}{P}$. $\underset{\sim}{P}^*$ is the adjoint of $\underset{\sim}{P}$, i.e., the element P^*_{ij} in $\underset{\sim}{P}^*$ is the cofactor of the element P_{ij} in $\underset{\sim}{P}$. Hence $\underset{\sim}{P}^*$ has the property:

$$(\underset{\sim}{P}^*)^T \underset{\sim}{P} = \|\underset{\sim}{P}\|\, \underset{\sim}{I} \tag{23}$$

To prove Eq. (22), we write $\underset{\sim}{P}$ in terms of its columns as

$$\underset{\sim}{P} = [\underset{\sim}{p}_1, \underset{\sim}{p}_2, \ldots, \underset{\sim}{p}_r] \tag{24}$$

where $\underset{\sim}{p}_1, \underset{\sim}{p}_2, \ldots$ are column vectors. If $g_1, g_2, \ldots, g_r$ denote the components of $\underset{\sim}{g}$, the left hand side of Eq. (22) can be written as

$$\|\underset{\sim}{P} + a\, \underset{\sim}{h}\, \underset{\sim}{g}^T\| = \|\underset{\sim}{p}_1 + a\, g_1 \underset{\sim}{h},\ \underset{\sim}{p}_2 + a\, g_2 \underset{\sim}{h}, \ldots, \underset{\sim}{p}_r + a\, g_r \underset{\sim}{h}\| \tag{25}$$

Now, by the theory of determinants, it is known that

$$\|\mathbf{p}_1 + a\,\mathbf{h}\ ,\ \mathbf{p}_2,\ \mathbf{p}_3, \ldots, \mathbf{p}_r\| = \|\mathbf{P}\| + a\|\mathbf{h},\ \mathbf{p}_2,\ \mathbf{p}_3, \ldots, \mathbf{p}_r\|\ , \tag{26}$$

$$\|\mathbf{h},\ a\,\mathbf{h},\ \mathbf{p}_3,\ \mathbf{p}_4, \ldots, \mathbf{p}_r\| = 0 \tag{27}$$

By repeatedly applying Eqs. (26) and (27) to the right hand side of Eq. (25), we obtain:

$$\begin{aligned} \|\mathbf{P} + a\,\mathbf{h}\,\mathbf{g}^T\| &= \|\mathbf{p}_1,\ \mathbf{p}_2, \ldots, \mathbf{p}_r\| + a\ g_1\|\mathbf{h},\ \mathbf{p}_2,\ \mathbf{p}_3, \ldots, \mathbf{p}_r\| \\ &\quad + a\ g_2\|\mathbf{p}_1,\ \mathbf{h},\ \mathbf{p}_3, \ldots, \mathbf{p}_r\| + \cdots + a\ g_r\|\mathbf{p}_1,\ \mathbf{p}_2, \ldots, \mathbf{h}\| \end{aligned} \tag{28}$$

With this, it is not difficult to see that Eq. (28) can be written in the form of Eq. (22). This completes the proof.

For the particular case in which $\mathbf{P}$ is a unit matrix, Eq. (22) reduces to

$$\|\mathbf{I} + a\ \mathbf{h}\ \mathbf{g}^T\| = 1 + a\ \mathbf{h}^T\mathbf{g} \tag{29}$$

3. <u>The Characteristic Equation</u>.

The characteristics c of Eq. (19) are the roots of the equation (see [9]):

$$\|c\mathbf{A} - \mathbf{B}\| = 0\ . \tag{30}$$

Since

$$c\mathbf{A}-\mathbf{B} = \begin{bmatrix} \rho c\mathbf{I} & \mathbf{M} \\ \mathbf{N} & c\mathbf{S} \end{bmatrix} = \begin{bmatrix} \rho c\mathbf{I} & \mathbf{0} \\ \mathbf{N} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \frac{1}{\rho c}\mathbf{M} \\ \mathbf{0} & \frac{1}{\rho c}\mathbf{I} \end{bmatrix}\ ,$$

where

$$\underset{\sim}{D} = \rho c^2 \underset{\sim}{S} - \underset{\sim}{N}\,\underset{\sim}{M} \;, \tag{31}$$

we have

$$\| cA - B \| = (\rho c)^{n-m} \| \underset{\sim}{D} \| \;. \tag{32}$$

Thus, instead of expanding the determinant $\| c\underset{\sim}{A} - \underset{\sim}{B} \|$ which is of order $m+n$, we can expand the determinant $\| \underset{\sim}{D} \|$ which is of order m.

If we define

$$\underset{\sim}{D}^e = \rho c^2 \underset{\sim}{S}^e - \underset{\sim}{N}\,\underset{\sim}{M} \;, \tag{33}$$

then by Eqs. (16) and (31), we have

$$\underset{\sim}{D}^p = \underset{\sim}{D}^e + \rho c^2 G(k)(\underset{\sim}{\nabla} f)(\underset{\sim}{\nabla} f)^T \;. \tag{34}$$

Using the identity derived in the previous section, we obtain

$$\| \underset{\sim}{D}^p \| = \| \underset{\sim}{D}^e \| + \rho c^2 G(k)(\underset{\sim}{\nabla} f)^T (\underset{\sim}{D}^e)^* (\underset{\sim}{\nabla} f) \;. \tag{35}$$

Equation (35) can be used to study the relative positions of the roots of $\| \underset{\sim}{D}^p \| = 0$ and $\| \underset{\sim}{D}^e \| = 0$. (see [4]).

4. The Elastic-Plastic Boundary.

If c is the speed of an elastic-plastic boundary, then

$$\frac{d\underset{\sim}{w}}{dt} = \underset{\sim}{w}_x c + \underset{\sim}{w}_t \tag{36}$$

is the total derivative of $\underset{\sim}{w}$ along the boundary. Elimination of $\underset{\sim}{w}_x$ between Eqs. (19) and (36) yields

$$(c\underset{\sim}{A} - \underset{\sim}{B})\underset{\sim}{w}_t = c\underset{\sim}{b} - \underset{\sim}{B}\frac{d\underset{\sim}{w}}{dt} \ . \tag{37}$$

Since $\underset{\sim}{w}$ is continuous across an elastic-plastic boundary, Eq. (37) gives

$$(c\underset{\sim}{A}^p - \underset{\sim}{B})\underset{\sim}{w}_t^p = (c\underset{\sim}{A}^e - \underset{\sim}{B})\underset{\sim}{w}_t^e \tag{38}$$

where the superscripts e and p denote the values in elastic and plastic regions respectively. With Eq. (20), Eq. (38) is equivalent to:

$$\rho\, c\, \underset{\sim}{v}_t^p + \underset{\sim}{M}\, \underset{\sim}{\sigma}_t^p = \rho\, c\, \underset{\sim}{v}_t^e + \underset{\sim}{M}\, \underset{\sim}{\sigma}_t^e \tag{39}$$

$$\underset{\sim}{N}\, \underset{\sim}{v}_t^p + c\, \underset{\sim}{S}^p \underset{\sim}{\sigma}_t^p = \underset{\sim}{N}\, \underset{\sim}{v}_t^e + c\, \underset{\sim}{S}^e \underset{\sim}{\sigma}_t^e \ . \tag{40}$$

It is more convenient to write Eqs. (39) and (40) in the following forms, using Eqs. (16) and (18):

$$\rho c(\underset{\sim}{v}_t^e - \underset{\sim}{v}_t^p) + \underset{\sim}{M}(\underset{\sim}{\sigma}_t^e - \underset{\sim}{\sigma}_t^p) = 0 \tag{41}$$

$$\underset{\sim}{N}(\underset{\sim}{v}_t^e - \underset{\sim}{v}_t^p) + c\underset{\sim}{S}^e(\underset{\sim}{\sigma}_t^e - \underset{\sim}{\sigma}_t^p) = 2cG(k)k\ k_t^p(\underset{\sim}{\nabla} f) \tag{42a}$$

or

$$\underset{\sim}{N}(\underset{\sim}{v}_t^e - \underset{\sim}{v}_t^p) + c\underset{\sim}{S}^p(\underset{\sim}{\sigma}_t^e - \underset{\sim}{\sigma}_t^p) = 2cG(k)k\ k_t^e(\underset{\sim}{\nabla} f) \tag{42b}$$

Equations (41) and (42a) yield

$$\underset{\sim}{D}^e(\underset{\sim}{\sigma}_t^e - \underset{\sim}{\sigma}_t^p) = 2\rho c^2 G(k)k\ k_t^p(\underset{\sim}{\nabla} f) \tag{43a}$$

while Eqs. (41) and (42b) yield

$$\underset{\sim}{D}^p(\underset{\sim}{\sigma}_t^e - \underset{\sim}{\sigma}_t^p) = 2\rho c^2 G(k)k\ k_t^e(\underset{\sim}{\nabla} f) \tag{43b}$$

Equation (43a) or (43b) gives a relation between $\underset{\sim}{\sigma}_t^e$ and $\underset{\sim}{\sigma}_t^p$ on both sides of an elastic-plastic boundary. For the case of combined longitudinal and torsional stresses in a tube,

$$\underset{\sim}{D}^e = \begin{bmatrix} \frac{\rho c^2}{E} - 1 & 0 \\ 0 & \frac{\rho c^2}{\mu} - 1 \end{bmatrix}$$

and Eq. (43a) reduces to the result obtained in [10].

Equation (43a) can be solved for $(\underset{\sim}{\sigma}_t^e - \underset{\sim}{\sigma}_t^p)$ by pre-multiplying both sides of equation by $(\underset{\sim}{D}^e)^{*T}$ and making use of Eq. (23). Hence

$$\underset{\sim}{\sigma}_t^e - \underset{\sim}{\sigma}_t^p = \frac{2\rho c^2 G(k)}{\|\underset{\sim}{D}^e\|} \, k \, k_t^p (\underset{\sim}{D}^e)^{*T} (\underset{\sim}{\nabla} f) \,. \qquad (44)$$

Pre-multiplying once more both sides of Eq. (44) by $(\underset{\sim}{\nabla} f)^T$ and making use of Eq. (18), we obtain

$$k_t^e - k_t^p = \frac{\rho c^2 G(k)}{\|D^e\|} \, k_t^p \, (\underset{\sim}{\nabla} f)^T (\underset{\sim}{D}^e)^{*T} (\underset{\sim}{\nabla} f) \qquad (45)$$

Since the right hand side of the equation is a scalar,

$$(\underset{\sim}{\nabla} f)^T (\underset{\sim}{D}^e)^{*T} (\underset{\sim}{\nabla} f) = (\underset{\sim}{\nabla} f)^T (\underset{\sim}{D}^e)^{*} (\underset{\sim}{\nabla} f),$$

and by virtue of Eq. (35) we have finally

$$\frac{k_t^e}{k_t^p} = \frac{\|\underset{\sim}{D}^p\|}{\|\underset{\sim}{D}^e\|} \qquad (46)$$

A simplest particular case of Eq. (46) is the case of longitudinal wave propagation in a thin rod studied in [13]. In the case of combined longitudinal and torsional stresses in a thin-walled tube,

$$\|\underset{\sim}{D}^e\| = (\frac{c^2}{c_o^2} - 1)(\frac{c^2}{c_2^2} - 1)$$

$$\|\underset{\sim}{D}^p\| = (\frac{c^2}{c_f^2} - 1)(\frac{c^2}{c_s^2} - 1)$$

where $c_o^2 = E/\rho$, $c_2^2 = \mu/\rho$, and c_f, c_s are the roots of $\|\underset{\sim}{D}^p\| = 0$, Eq. (46) reduces to the result obtained in [10].

If $k_t^p = 0$, then by Eq. (43a) either $\underset{\sim}{\sigma}_t^e = \underset{\sim}{\sigma}_t^p$ or $\|\underset{\sim}{D}^e\| = 0$. Similarly, if $k_t^e = 0$, then by Eq. (43b) either $\underset{\sim}{\sigma}_t^e = \underset{\sim}{\sigma}_t^p$ or $\|\underset{\sim}{D}^p\| = 0$. Consequently, if $k_t^e = k_t^p = 0$, $\underset{\sim}{\sigma}_t^e = \underset{\sim}{\sigma}_t^p$. Moreover, $\underset{\sim}{v}_t^e = \underset{\sim}{v}_t^p$ by Eq. (41). Thus when $k_t^e = k_t^p = 0$, $\underset{\sim}{w}_t^e = \underset{\sim}{w}_t^p$. To obtain relations between $\underset{\sim}{\sigma}_{tt}^e$ and $\underset{\sim}{\sigma}_{tt}^p$ on both sides of an elastic-plastic boundary when $k_t^e = k_t^p = 0$, we differentiate both sides of Eqs. (43) and noticing that $\underset{\sim}{\sigma}_t^e = \underset{\sim}{\sigma}_t^p$, $k_t^e = k_t^p = 0$,

$$\underset{\sim}{D}^e(\underset{\sim}{\sigma}_{tt}^e - \underset{\sim}{\sigma}_{tt}^p) = 2\rho c^2 G(k)k\ k_{tt}^p(\underset{\sim}{\nabla} f) \tag{47a}$$

$$\underset{\sim}{D}^p(\underset{\sim}{\sigma}_{tt}^e - \underset{\sim}{\sigma}_{tt}^p) = 2\rho c^2 G(k)k\ k_{tt}^e(\underset{\sim}{\nabla} f) \ . \tag{47b}$$

Similarly, if we differentiate both sides of Eq. (45), we obtain, after using Eq. (35),

$$\frac{k_{tt}^e}{k_{tt}^p} = \frac{\|\underset{\sim}{D}^p\|}{\|\underset{\sim}{D}^e\|} \ . \tag{48}$$

Equations (47) and (48) are identical to Eqs. (43) and (46) respectively with the exception that the order of derivatives is changed.

From the above derivations we can generalize the results and state as in the following:

If $\partial^n k^e/\partial t^n = \partial^n k^p/\partial t^n = 0$ for $n=1,2,\ldots,\alpha-1$ but $\partial^\alpha k^e/\partial t^\alpha$ and $\partial^\alpha k^p/\partial t^\alpha$ are not both zero, we have

$$\frac{\partial^n \underset{\sim}{w}^e}{\partial t^n} = \frac{\partial^n \underset{\sim}{w}^p}{\partial t^n} , \quad n=1,2,\ldots,\alpha-1 \tag{49}$$

$$\underset{\sim}{D}^e\left(\frac{\partial^\alpha \underset{\sim}{\sigma}^e}{\partial t^\alpha} - \frac{\partial^\alpha \underset{\sim}{\sigma}^p}{\partial t^\alpha}\right) = 2\rho c^2 G(k)k \, \frac{\partial^\alpha k^p}{\partial t^\alpha} \, (\underset{\sim}{\nabla} f) \tag{50a}$$

or

$$\underset{\sim}{D}^p\left(\frac{\partial^\alpha \underset{\sim}{\sigma}^e}{\partial t^\alpha} - \frac{\partial^\alpha \underset{\sim}{\sigma}^p}{\partial t^\alpha}\right) = 2\rho c^2 G(k)k \, \frac{\partial^\alpha k^e}{\partial t^\alpha} \, (\underset{\sim}{\nabla} f) \tag{50b}$$

and

$$\frac{\partial^\alpha k^e/\partial t^\alpha}{\partial^\alpha k^p/\partial t^\alpha} = \frac{\|\underset{\sim}{D}^p\|}{\|\underset{\sim}{D}^e\|} , \tag{51}$$

For a loading wave,

$$\frac{\partial^\alpha k^e/\partial t^\alpha}{\partial^\alpha k^p/\partial t^\alpha} \begin{cases} > 0 & \text{if } \alpha \text{ is an odd integer} \\ < 0 & \text{if } \alpha \text{ is an even integer} \end{cases} \tag{52a}$$

while for an unloading wave,

$$\frac{\partial^\alpha k^e/\partial t^\alpha}{\partial^\alpha k^p/\partial t^\alpha} \begin{cases} < 0 & \text{if } \alpha \text{ is an odd integer} \\ > 0 & \text{if } \alpha \text{ is an even integer} \end{cases} \tag{52b}$$

With this, Eq. (51) furnishes a restriction on the speed of an elastic-plastic boundary as in the particular cases considered in [10,11,13]. It whould be noticed that Eq. (51) applies if $\partial^e k/\partial t^n$ $(n=1,2,\ldots,\alpha-1)$ are zero on both sides of an elastic-plastic boundary as well as along the boundary. If $\partial^n k/\partial t^n$ $(n=1,2,\ldots,\alpha-1)$ are zero on both sides of the boundary only at the point concerned but not at other points on the boundary, the right hand side of Eq. (51) should be modified. An example of this modification for the particular problem of wave propagation in a rod was given in [12].

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A general study is given of plane and cylindrical wave propagation of combined stresses in an elastic-plastic medium. The coefficients of the governing differential equations, when written in matrix notation, are symmetric matrices and can be divided into sub-matrices each of which has a special form. The relations between the stresses on both sides of an elastic-plastic boundary are derived. Also presented are the restrictions on the speed of an elastic-plastic boundary.

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