

ELASTIC WAVES IN ROTATING MEDIA*

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Abstract. Plane harmonic waves in a rotating elastic medium are considered. The inclusion of centripetal and Coriolis accelerations in the equations of motion with respect to a rotating frame of reference leads to the result that the medium behaves as if it were dispersive and anisotropic. The general techniques of treating anistropic media are used with some necessary modifications. Results concerning slowness surfaces, energy flux and mode shapes are derived. These concepts are applied in a discussion of the behavior of harmonic waves at a free surface.

Introduction. In this paper plane wave propagation in a linear, homogeneous, isotropic elastic medium will be considered, with the assumption that the entire elastic medium is rotating with a uniform angular velocity. If the coordinate system is taken as fixed in the rotating medium, this introduces additional terms in the equations of motion: a centripetal and a Coriolis acceleration. We consider small-amplitude waves propagating in the medium and exclude any discussion of the time-independent stresses and displacements that are caused by centrifugal forces and other possible body forces.

In the following section the equations governing plane-wave solutions in an infinite rotating medium are formulated and it is shown that there are three real slowness surfaces, each corresponding to the root of a cubic characteristic equation. It is shown that the phase speed in all cases depends on the ratio of the wave frequency to the rotational frequency, thus making it clear that the rotation causes the material to be dispersive. The actual slowness surfaces are given for various values of Poisson's ratio and the frequency ratio.

In the next section the energy flux for plane waves is discussed and it is proved, for any admissible plane wave, to be perpendicular to the slowness surface at the point indicated by the slowness vector (essentially the wave number vector) of the wave.

Actual displacements that occur are discussed qualitatively in the subsequent section. It is seen that, in general, the various modes are neither shear nor compressional, but combinations of both. All exceptional cases of pure shear or pure compressional modes are discussed.

In the last section free surface phenomena are discussed. To describe the reflection of plane waves from a plane free surface, use is made of the slowness surfaces. Much qualitative information on types of reflected waves can be brought out even without the use of the explicit expressions for the slowness vector, such as under what circumstances one or two of the reflected waves will be surface waves, i.e. have a complex

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slowness vector. As an extension of this, a solution which consists only of surface waves is discussed. This can be thought of as a generalized Rayleigh wave.

The analysis involved in the derivations is roughly similar to that used for wave propagation in anisotropic media by Synge [1], [2] and Musgrave [3], among others. A rotating medium can be thought of as a type of transverse isotropic medium; i.e., all directions orthogonal to some direction, in this case the axis of rotation, are equivalent. However, there are basic differences between rotational and material anisotropy. For a rotating medium, substitution of a plane wave solution into the equations of motion leads to a homogeneous set of linear equations for which the matrix of the coefficients is Hermitian, instead of symmetric, as is the case for non-rotating media. Hence the eigenvectors, i.e. the displacement vectors, even for real slowness vectors, are complex instead of real, implying that the particle trajectories are elliptical. Further, as mentioned above, the solutions for a rotating medium are frequency-dependent. In addition, there is no easily perceived eigenvector that can be used to find one root of the cubic characteristic equation, thus leaving only a quadratic equation to be solved, as for a non-rotating transverse isotropic medium (see [1, p. 331]).

The standard index notation is used throughout, e.g. for the position vector

$$\mathbf{x} = x_i\hat{\mathbf{e}}_i \equiv x_1\hat{\mathbf{e}}_1 + x_2\hat{\mathbf{e}}_2 + x_3\hat{\mathbf{e}}_3 \,.$$

The cap will always denote unit vectors. Use is made both of vector and indicial notation.

The rotating elastic medium. Consider an infinite homogeneous, isotropic, linear elastic medium characterized by a density ρ, a shear modulus and a bulk modulus. These quantities determine, in the usual fashion, a shear wave speed C_s and a pressure wave speed C_p. The medium is rotating uniformly with respect to an inertial frame, and the constant rotation vector in an x_1, x_2, x_3 rectangular Cartesian frame rotating with the medium is $\mathbf{\Omega} = \Omega\hat{\mathbf{w}}$. The unit vector $\hat{\mathbf{w}}$ will denote the direction of the axis of rotation (according to the right-hand rule) throughout.

The displacement equation of motion in such a rotating frame has two terms that do not appear in the non-rotating situation. As we are looking for time-varying dynamic solutions, the time-independent part of the centripetal acceleration $\mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{x})$, as well as all body forces, will be neglected. Thus our dynamic displacement $\mathbf{u}$ is actually measured from a steady-state deformed position, the deformation of which, however, is assumed small. The equation which governs the dynamic displacement $\mathbf{u}$ is

$$(C_p^2 - C_s^2)\nabla(\nabla\cdot\mathbf{u}) + C_s^2\nabla^2\mathbf{u} = \ddot{\mathbf{u}} + \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{u}) + 2\mathbf{\Omega} \times \dot{\mathbf{u}}, \tag{1}$$

where the dot denotes time differentiation. The term $\mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{u})$ is the additional centripetal acceleration due to the time-varying motion only, and the term $2\mathbf{\Omega} \times \dot{\mathbf{u}}$ is the Coriolis acceleration. All other terms are as usual for a linear elastic medium under the assumptions of small strains and displacements.

We look for plane wave solutions of the form

$$\mathbf{u} = \Re\left[\mathbf{U} \exp i\omega\left(\frac{\hat{\mathbf{n}}\cdot\mathbf{x}}{C} - t\right)\right] = \Re\left[U \exp i\omega\left(\frac{\mathbf{S}\cdot\mathbf{x}}{C_p} - t\right)\right] \tag{2}$$

where $\Re$ denotes the real part, $\mathbf{U}$ is a constant vector, in general complex, ω is the angular frequency, C is the phase speed, $\hat{\mathbf{n}}$ is a unit vector in the direction of propagation, and $\mathbf{S}$ is the non-dimensional slowness vector with amplitude $S = C_p/C$ in the direction of $\hat{\mathbf{n}}$.

Substitution of (2) into (1) gives

$$(1-\beta)(\mathbf{S}\cdot\mathbf{U})\mathbf{S}+\beta S^2\mathbf{U}=\mathbf{U}-(1/\omega^2)\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{U})+(2i/\omega)\boldsymbol{\Omega}\times\mathbf{U},\qquad \beta=(C_s/C_p)^2, \tag{3}$$

or, by making use of $\boldsymbol{\Omega}=\Omega\hat{\mathbf{w}}$ and the vector identity $\hat{\mathbf{w}}\times(\hat{\mathbf{w}}\times\mathbf{U})=(\hat{\mathbf{w}}\cdot\mathbf{U})\hat{\mathbf{w}}-\mathbf{U}$

$$(1-\beta)(\mathbf{S}\cdot\mathbf{U})\mathbf{S}+\beta S^2\mathbf{U}=(1+\Gamma^2)\mathbf{U}-\Gamma^2\hat{\mathbf{w}}\cdot\mathbf{U}\hat{\mathbf{w}}+2i\Gamma\hat{\mathbf{w}}\times\mathbf{U},\qquad \Gamma=\Omega/\omega. \tag{4}$$

This vector equation stands for three scalar equations on the three components U_i, and they can be written as

$$M_{ij}U_j\equiv[(1+\Gamma^2)\,\delta_{ij}-\Gamma^2w_iw_j-2i\Gamma e_{ijk}w_k-(1-\beta)s_is_j-\beta s_ks_k\,\delta_{ij}]U_j=0, \tag{5}$$

where δ_{ij} is the Kronecker delta and e_{ijk} is the permutation tensor. Note that, because of the Coriolis term, the matrix M_{ij} is no longer symmetric but Hermitian. The necessary and sufficient condition for the existence of non-trivial eigenvectors, U_j, is det $M_{ij}=0$. In general, there may be real and complex vectors, S_i, which satisfy the condition det $M_{ij}=0$. However, complex S_i substituted in (2) give waves of exponentially varying amplitude along the real plane of constant phase and hence cannot exist in an infinite medium. Thus for now we shall confine ourselves to slowness vectors given by $S\hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is the real direction of propagation and S is the magnitude. To that end, substitute $\mathbf{S}=S\hat{\mathbf{n}}$ into (5), giving

$$A_{ij}U_j=S^2B_{ij}U_j\,,\qquad A_{ij}=(1+\Gamma^2)\,\delta_{ij}-\Gamma^2w_iw_j-2i\Gamma e_{ijk}w_k\,,$$
$$B_{ij}=(1-\beta)n_in_j+\beta\delta_{ij}\,. \tag{6}$$

Note that B_{ij} is symmetric, a special case of Hermitian, and that A_{ij} is Hermitian. The condition for non-zero U_j is that det $(A_{ij}-S^2B_{ij})=0$. This is the equation for the eigenvalues of Eq. (6). If $S^2=\lambda$ is a particular eigenvalue for an arbitrary propagation direction $\hat{\mathbf{n}}$, and $\mathbf{V}$ is its associated eigenvector, then from (6) we can write

$$V_i^*A_{ij}V_j=\lambda V_i^*B_{ij}V_j\,, \tag{7}$$

where the asterisk denotes the complex conjugate. As the value of a Hermitian form is always real (see, for example, [4]), $V_i^*A_{ij}V_j$ and $V_i^*B_{ij}V_j$ are both real and hence λ is real. In addition, both A and B are positive definite (the eigenvalues of A are 1, $(1+\Gamma)^2$, $(1-\Gamma)^2$ and those of B are 1, β, β) for all $\hat{\mathbf{w}}$ and $\hat{\mathbf{n}}$ except when $\Gamma=1$, in which case one of the eigenvalues of A is zero. Hence, from (7) λ must be positive except when $\Gamma=1$ and zero is one eigenvalue of (6). Further, it is easy to show that two eigenvectors corresponding to distinct eigenvalues are orthogonal in the Hermitian sense relative to the matrix B and to the matrix A.

The condition det $(A_{ij}-S^2B_{ij})=0$ is a bicubic equation, given by

$$-\beta^2S^6+\beta[2+\beta+\Gamma^2(1+f)]S^4$$
$$-[(1+2\beta)(1+\Gamma^2)-f\Gamma^2(3-\Gamma^2)]S^2+(1-\Gamma^2)^2=0, \tag{8}$$
$$f=\beta+(1-\beta)\cos^2\theta,\qquad \cos\theta=\hat{\mathbf{n}}\cdot\hat{\mathbf{w}}.$$

We have seen above that the roots S^2 of (8) must always be positive except for $\Gamma=1$ in which case there is one zero root. Note that this zero root with its associated eigenvector found from (5) represents a rigid body motion which is a circular translational motion of the entire medium about the axis of rotation in the opposite direction as the rotation and of course at the same frequency. In a laboratory system, it is easy to show that this is only a rigid-body time-independent displacement of the entire medium and henceforth this mode will be ignored.

In the special case $\hat{\mathbf{n}} \cdot \hat{\mathbf{w}} = 1$ the three roots of (8) are $S^2 = 1$, $(1 \pm \Gamma)^2/\beta$ and in the special case $\hat{\mathbf{n}} \cdot \hat{\mathbf{w}} = 0$, one of the three roots of (8) is $1/\beta$. These cases will be discussed in more detail below.

In the general case obtaining the roots of (8) involves solving a cubic equation. We see that all the coefficients and thus the roots of (8) depend on three parameters, i.e. Γ, the frequency ratio which must be positive, β, limited to values between 0 and $\frac{3}{4}$ (β depends only on Poisson's ratio and for positive Poisson's ratio, β is further restricted to be less than $\frac{1}{2}$) and $\cos^2 \theta$, where θ is the angle between the axis of rotation and the direction of propagation.

We now consider the possibility of repeated roots of (8). Clearly a triple root is impossible. For $\Gamma = 0$, i.e. no rotation, we have a double root, $S^2 = 1/\beta$, corresponding to the two independent shear waves that can propagate in any direction. If, for non-zero Γ, there is a root $S^2 = \lambda$ which is a double root, then, as A and B are Hermitian, there are two linearly independent eigenvectors associated with λ and hence the matrix $A - \lambda B$ must be of rank 1. This means that all 2×2 minors of $A - \lambda B$ must have determinants equal to zero, and setting all these minor determinants to zero implies that, for $\Gamma > 0$, either 1) $\sin \theta = 0$, $\lambda = 1$, and $\Gamma = 1 \pm \beta^{1/2}$ or 2) $\cos \theta = 0$, $\lambda = 1/\beta$ and $\Gamma = (3 + \beta^{-1})^{1/2}$. Under no other circumstances can there be a multiple root.

Thus, we have found that in a given rotating elastic medium, for any direction $\hat{\mathbf{n}}$, there are three distinct slowness vectors (except as noted above) $\mathbf{S}_{(N)} = S_{(N)}(\cos^2 \theta)\hat{\mathbf{n}}$, $N = 1, 2, 3$. Each of these vector functions of direction defines a real slowness surface that is axisymmetric about the axis of rotation and symmetric about the plane specified by $\theta = \pi/2$ (see Fig. 1). These non-intersecting surfaces are such that the inner surface surrounds the origin, the next surface surrounds the inner one, and the outer one surrounds this second surface. Contact is possible only along the axis of rotation under condition 1 above and at $\theta = \pi/2$ under condition 2 above. Fig. 2 shows quadrants of the actual slowness surfaces for $\beta = \frac{1}{3}$ (Poisson's ratio $= \frac{1}{4}$) for several different values of Γ.

With the three real slowness surfaces we have a very powerful tool. It will be proved in the next section that the energy flux vector associated with a given wave is normal to the slowness surface, as is the case with anisotropic non-rotating elastic solids. Further, the slowness surfaces provide a means of visualizing the application of Snell's law in the case of reflection from a plane surface. This will be discussed at length in the last section.

The energy flux. The energy flux at any time, at any point, is the vector

$$F_i^\dagger = -\sigma_{ij}\dot{u}_j \,, \tag{9}$$

where σ_{ij} is the stress tensor. Expressing the stress in terms of the displacement gradients, and recalling the expressions for the Lamé constants in terms of ρ, C_s, and C_p, and that $\beta = C_s^2/C_p^2$, gives

$$F_i^\dagger = -\rho C_p^2[\beta(u_{j,i} + u_{i,j}) + (1 - 2\beta)u_{k,k}\,\delta_{ij}]\dot{u}_j \,. \tag{10}$$

To find the energy flux associated with a given plane wave, substitute (2) into (10), noting that since we no longer have a linear expression in $\mathbf{u}$ we must keep track of the real part. This gives an expression that oscillates in time with a frequency 2ω. We

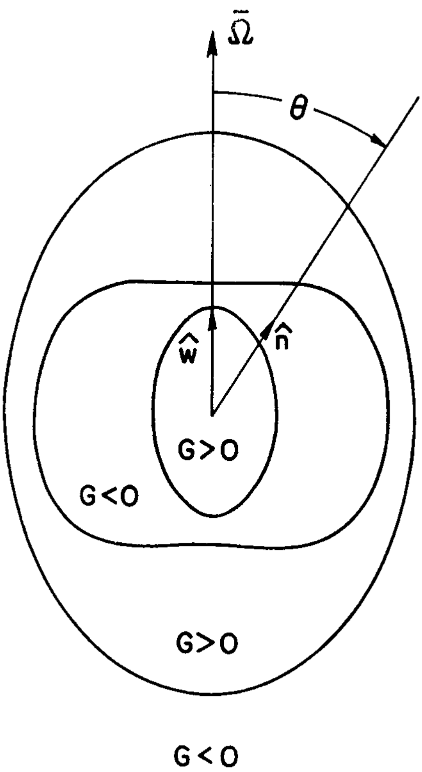

FIG. 1. An example of the three slowness surfaces in the infinite medium.

are interested in the time-averaged value of the energy flux and thus letting

$$F_i \equiv \lim_{t_1 - t_0 \to \infty} \frac{1}{t_1 - t_0} \int_{t_0}^{t_1} F_i^\dagger \, dt = \frac{\omega}{\pi} \int_0^{\pi/\omega} F_i^\dagger \, dt \tag{11}$$

gives

$$\mathbf{F} = \frac{\rho\omega^2 C_p}{2} \left[\beta(\mathbf{U}^* \cdot \mathbf{U})\mathbf{S} + \frac{1-\beta}{2} (\mathbf{S} \cdot \mathbf{U}\mathbf{U}^* + \mathbf{S} \cdot \mathbf{U}^*\mathbf{U}) \right], \tag{12}$$

where $\mathbf{S}$ is one of the three real slowness vectors, for a given direction $\hat{\mathbf{n}}$, of (5), and $\mathbf{U}$ is its corresponding eigenvector. $\mathbf{U}^*$ is the complex conjugate of $\mathbf{U}$.

Now we wish to prove that the energy flux for a particular wave is parallel to the normal of the slowness surface at the particular point $\mathbf{S}$ in S_1, S_2, S_3 space. We shall adapt the proof of Synge [2] for anisotropic elastic media to our present situation, the difference being that our determining matrix for the slowness and the eigenvectors is Hermitian instead of symmetric.

The condition for non-zero $\mathbf{U}$, from (5), gives the real slowness surfaces

$$G(S_i) \equiv \det M_{ij} = 0. \tag{13}$$

For any real vector $\mathbf{S} = S_i \hat{\mathbf{e}}_i$, G has some value, in general not equal to zero. In particular, when $\mathbf{S} = 0$, $G = (1 - \Gamma^2)^2 > 0$ and for $|\mathbf{S}|$ larger than some sufficiently large

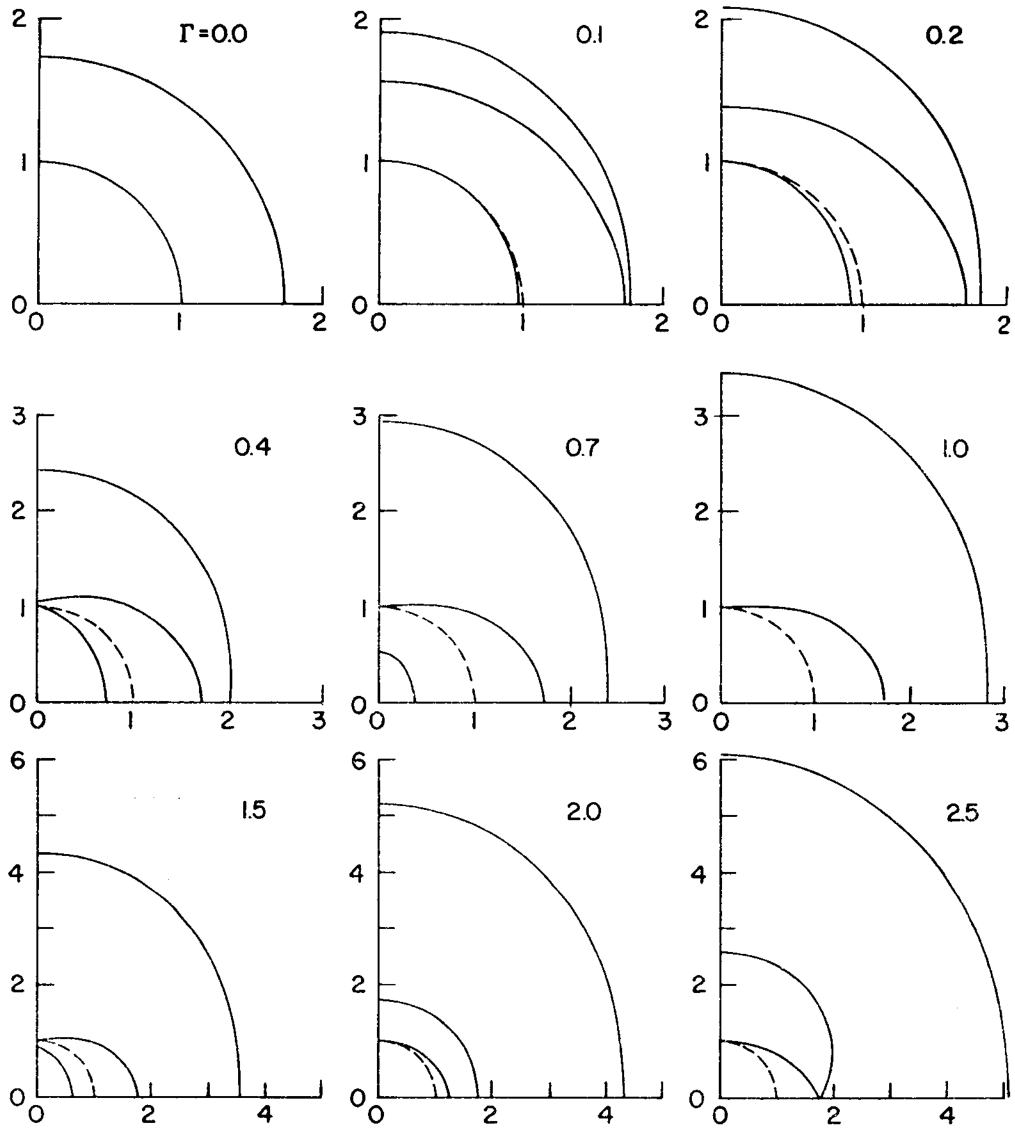

FIG. 2. Slowness surface quadrants for various values of Γ with $\beta = \frac{1}{3}$. The dashed line denotes the unit circle and θ is measured from the vertical axis.

number, G is negative. In Fig. 1 the regions of positive and negative G are shown. They are separated by the slowness surfaces. The gradient of G at any point is

$$\nabla G = (\partial G/\partial S_i)\hat{\mathbf{e}}_i = N_{ik}(\partial M_{ik}/\partial S_i)\hat{\mathbf{e}}_i = N_{jk}[-(1-\beta)(S_k\hat{\mathbf{e}}_j + S_j\hat{\mathbf{e}}_k) - 2\beta\mathbf{S}\delta_{jk}], \quad (14)$$

where N_{jk} is the cofactor of M_{jk}. At any point on a slowness surface, i.e., a point where $G = 0$,

$$M_{ij}N_{kj} = \delta_{ik}\, G = 0, \quad (15)$$

$$N_{ik}M_{ij} = \delta_{kj}\, G = 0. \quad (16)$$

But Eq. (5), at a point where $G = 0$, is

$$M_{ij}U_j = 0, \tag{17}$$

and this defines the eigenvector U_j uniquely to within a constant. Thus, comparing with (15), we see that

$$N_{kj} = A_k U_j\,. \tag{18}$$

The complex conjugate of (17), due to the fact that M_{ij} is Hermitian, is

$$U_i^* M_{ij} = 0, \tag{19}$$

and comparison with (16) implies that

$$N_{ik} = U_i^* A_k\,. \tag{20}$$

Eqs. (18) and (20) can both be valid if and only if

$$N_{jk} = A U_j^* U_k\,. \tag{21}$$

Substitution of (21) into (14) gives

$$\nabla G = -2A[\beta \mathbf{U}^*\cdot\mathbf{U}\mathbf{S} + ((1-\beta)/2)(\mathbf{S}\cdot\mathbf{U}\mathbf{U}^* + \mathbf{S}\cdot\mathbf{U}^*\mathbf{U})], \tag{22}$$

which, by comparing with (12), proves the result.

As each slowness surface is a closed surface about the origin with only one value of S for any $\hat{\mathbf{n}}$, the outward normal from any point on the slowness surface must form an angle less than $\pi/2$ with $\mathbf{S}$, i.e. $\mathbf{S}\cdot\mathbf{e}_n > 0$. From (12)

$$\mathbf{S}\cdot\mathbf{F} = \frac{\rho\omega^2 C_p}{2}[\beta(\mathbf{U}^*\cdot\mathbf{U})S^2 + (1-\beta)(\mathbf{S}\cdot\mathbf{U})(\mathbf{S}\cdot\mathbf{U}^*)] > 0; \tag{23}$$

thus $\mathbf{F}$ is directed along the outward normal of the slowness surface. From Fig. 1 we can see that on the inner and outer surfaces ∇G is directed inwards, i.e. opposite to $\mathbf{F}$, and on the middle surface ∇G is directed outwards, i.e. in the direction of $\mathbf{F}$. At the singular points where two of the slowness surfaces are in contact, $G = \nabla G = 0$ and these points are saddle points. At such points the slowness surfaces are kinked; however the energy flux for each of the two modes is parallel to the slowness vector.

The displacement vectors. In order to simplify the expressions and increase our understanding of the actual types of displacements that occur for these various waves, we select an x_i' coordinate system in which $\hat{\mathbf{w}}$ is equal to $\hat{\mathbf{e}}_3'$ and $\hat{\mathbf{n}}$ lies in the $x_1'x_2'$-plane, and is expressed by $\cos\theta\hat{\mathbf{e}}_1' + \sin\theta\hat{\mathbf{e}}_2'$. In this coordinate system, (6) becomes

$$\begin{bmatrix} 1-(\cos^2\theta+\beta\sin^2\theta)S^2 & -(1-\beta)\sin\theta\cos\theta S^2 & 0 \\ -(1-\beta)\sin\theta\cos\theta S^2 & 1+\Gamma^2-(\sin^2\theta+\beta\cos^2\theta)S^2 & -2i\Gamma \\ 0 & +2i\Gamma & 1+\Gamma^2-\beta S^2 \end{bmatrix}\begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{24}$$

which yields three eigenvectors, one for each of the eigenvalues $S_{(N)}$, $N = 1, 2, 3$, given to within an arbitrary constant by

$$\begin{aligned} \mathbf{U}^{(N)} = {} & (1-\beta)\sin\theta\cos\theta S_{(N)}^2(1+\Gamma^2-\beta S_{(N)}^2)\hat{\mathbf{e}}_1' \\ & + (1+\Gamma^2-\beta S_{(N)}^2)[1-(\cos^2\theta+\beta\sin^2\theta)S_{(N)}^2]\hat{\mathbf{e}}_2' \\ & - 2i\Gamma[1-(\cos^2\theta+\beta\sin^2\theta)S_{(N)}^2]\hat{\mathbf{e}}_3' \end{aligned} \tag{25}$$

for $\theta \neq 0, \pi/2$.

We see that for all modes, the component of the displacement normal to the $\hat{\mathbf{n}}$, $\hat{\mathbf{w}}$-plane is out of phase with the component of the displacement that lies in the $\hat{\mathbf{n}}$, $\hat{\mathbf{w}}$-plane. Thus the particle trajectory is an ellipse with one of its axes in the x_3' direction and the other in the direction of $\Re[\mathbf{U}^{(N)}]$. In general, $\mathbf{U}^{(N)}$ is not perpendicular to $\hat{\mathbf{n}}$ and not parallel to $\hat{\mathbf{n}}$; thus none of the three modes are pure shear or pure dilational modes. Exceptions occur for $\theta = 0$ and $\theta = \pi/2$. For $\theta = \pi/2$, there is one pure shear mode with $S = \beta^{-1/2}$, which from (24) has a displacement parallel to $\hat{\mathbf{w}}$. This shear mode does not 'see' the rotation of the medium as there is no displacement or velocity perpendicular to the axis of rotation. The other two modes that occur at $\theta = \pi/2$ are both combined shear and dilational modes. The reflection and scattering of these combined waves that propagate perpendicular to the axis of rotation will be discussed in detail in a subsequent paper.

For $\theta = 0$, there is a pure dilational wave with $S^2 = 1$ for which the displacement is in the direction of propagation. As this is parallel to the axis of rotation, this mode too does not 'see' the rotation of the medium. The other two modes are pure shear modes with slownesses given by $S_{1,2} = \beta^{-1/2}\,|1 \pm \Gamma|$. From (24), we find that

$$\mathbf{U}^{(1)} = \hat{\mathbf{e}}_2' + i\hat{\mathbf{e}}_3' , \qquad \mathbf{U}^{(2)} = \hat{\mathbf{e}}_2' - i\hat{\mathbf{e}}_3' \tag{26}$$

and hence the particle trajectories for both of these waves are circles about the vector $\hat{\mathbf{w}}$, the motion for the first (slower) wave being right-handed about $\hat{\mathbf{w}}$, and the motion of the second (faster) being left-handed. If we take a wave made up of $\mathbf{U}^{(1)}$ and $\mathbf{U}^{(2)}$ with equal amplitudes, the total displacement, from (2), noting that $\beta^{1/2}C_p = C_s$, is

$$\begin{aligned} \mathbf{u} &= 2\cos\omega\left(\frac{x_1'}{C_s} - t\right)\left[\hat{\mathbf{e}}_2'\cos\frac{\omega\Gamma x_1'}{C_s} - \hat{\mathbf{e}}_3'\sin\frac{\omega\Gamma x_1'}{C_s}\right], \qquad \Gamma \le 1, \\ \mathbf{u} &= 2\cos\omega\left(\frac{\Gamma x_1'}{C_s} - t\right)\left[\hat{\mathbf{e}}_2'\cos\frac{\omega x_1'}{C_s} - \hat{\mathbf{e}}_3'\sin\frac{\omega x_1'}{C_s}\right], \qquad \Gamma \ge 1. \end{aligned} \tag{27}$$

Therefore the combined wave is linearly polarized, i.e. at fixed x_1' the tip of the displacement vector moves along a fixed line perpendicular to $\hat{\mathbf{w}}$. However, as x_1' varies, the orientation line of this changes. In particular, the orientation rotates in a right-handed fashion as x_1' increases. This is analogous to the Faraday rotation of electromagnetic waves in a magnetized plasma; however, for electromagnetic waves propagating in the direction of an externally applied magnetic vector, $\mathbf{B}_0$, the rotation of the orientation is opposite to that in our case. This is due to the fact that the Lorentz force is given by $\mathbf{u} \times \mathbf{B}_0$ while the Coriolis force is given by $2\mathbf{\Omega} \times \mathbf{u}$.

For propagation in the $-x_1'$ ($\theta = \pi$) direction, the same arguments apply and it is easy to see that the Faraday rotation will be in the same sense relative to $\hat{\mathbf{w}}$, i.e. in the opposite sense relative to the new direction of propagation.

One other case of pure shear wave occurs. For $\Gamma = 2$, one root of (9) is $S^2 = 1/\beta$ for all θ; thus this slowness surface is a sphere. From (25), the displacement vector associated with this mode is

$$\mathbf{U} = -\sin\theta\hat{\mathbf{e}}_1' + \cos\theta\hat{\mathbf{e}}_2' - \mathrm{i}\cos\theta\hat{\mathbf{e}}_3' , \tag{28}$$

and clearly this displacement vector is perpendicular to the direction of propagation, so this is a pure shear mode.

Free surface phenomena. Consider the case of a rotating elastic half-space. This is the most elementary geometry that can be introduced into the problem but it enables us to consider reflection phenomena and surface waves. Let the medium occupy the region $x_2 \leq 0$. The vector $\hat{\mathbf{w}}$ is specified by w_i, $i = 1, 2, 3$, and the three slowness surfaces are figures of revolution about the direction given by $\hat{\mathbf{w}}$. Consider a body wave in the medium, of any of the three modes, incident to the surface (incident meaning a wave for which $\hat{\mathbf{e}}_2 \cdot \mathbf{F}$ is positive i.e. the energy flux is towards the surface. A reflected wave, similarly, is one for which $\hat{\mathbf{e}}_2 \cdot \mathbf{F}$ is negative.) Thus, we can divide each slowness surface into two regions, the top containing all incident slowness vectors, and the bottom containing all reflected slowness vectors.

Let the incident wave on the surface $x_2 = 0$, have a phase given by

$$i\omega[(S_{1(I)}x_1 + S_{3(I)}x_3)/C_p - t] \tag{29}$$

The reflected waves must have the same phase on the surface $x_2 = 0$. Using the slowness surfaces, we can visualize where the real slowness vectors of the reflected waves must lie. To do this, drop a perpendicular to the surface through the tip of $\mathbf{S}_{(I)}$. The points of intersection with the reflecting regions of the slowness surfaces are the endpoints of the real slowness vectors of the reflected waves. There is no loss of generality in selecting our axes so that $\mathbf{S}_{(I)}$ lies in the x_1x_2-plane and $S_{3(I)} = 0$ and it is clear that all the reflected waves lie in the x_1x_2-plane. Fig. 3 shows the intersection of the three slowness surfaces with the x_1x_2-plane and depicts the graphical representation of the slowness vectors of the reflected waves for the cases a) of three real reflected waves, b) of two real reflected waves, and c) of one real reflected wave. To do this analytically, the condition for the existence of non-zero eigenvectors, Eq. (13) must be used. For the case where we picked a real $\hat{\mathbf{n}}$ and wanted to find S, this became the bicubic Eq. (8). Here S_1 is given (equal to $S_{1(I)}$) and we have set $S_3 = 0$. Eq. (13) becomes

$$\begin{aligned}-\beta^2(S^2)^3 + \beta\{[2 + \beta + \Gamma^2(1 + \beta)]S^2 + \Gamma^2(1 - \beta)(\hat{\mathbf{w}}\cdot\mathbf{S})^2\}S^2 \\ - \{[(1 + 2\beta)(1 + \Gamma^2) - \beta\Gamma^2(3 - \Gamma^2)]S^2 - (1 - \beta)\Gamma^2(3 - \Gamma^2)(\hat{\mathbf{w}}\cdot\mathbf{S})^2\} \\ + (1 - \Gamma^2)^2 = 0, \qquad S^2 = S_1^2 + S_2^2, \qquad \hat{\mathbf{w}}\cdot\mathbf{S} = w_1S_1 + w_2S_2\end{aligned} \tag{30}$$

Eq. (30) has been kept in the same form as (8) but this equation is in general a full sextic equation for S_2. Note, however, that one root is known to be $S_{2(I)}$, as can be seen from Fig. 3, leaving a quintic equation. Case a) corresponds to the situation in which there are no complex roots and the desired three values of S_2 are the three smallest roots of Eq. (27). Case b) corresponds to (27) having four real roots, the smallest two being the real values of S_2, and a complex conjugate pair of roots, the one with the negative imaginary part being the complex value of S_2 so as to have attenuation into the medium in the $-x_2$ direction. This cannot occur for the incident slowness vector $\mathbf{S}_{(I)}$ on the inner slowness surface. Fig. 3 shows the limits of $S_{1(I)}$ for this case to apply. Case c) corresponds to two real roots, the smaller of which is the real value of S_2. The other two values are the complex roots with negative imaginary parts. This case can occur only when $\mathbf{S}_{(I)}$ is on the outer slowness surface. Fig. 3 also shows the limits of $S_{1(I)}$ for this case to occur. The case of repeated real roots corresponds to the normal to the free surface through $\mathbf{S}_{(I)}$ being tangent to one of the slowness surfaces. In such a case, the energy flux for this wave is parallel to the free surface.

In the special case that w_1 or w_2 is zero, (30) becomes a bicubic equation on S_2. This is seen graphically as in this case the intersections of the slowness surfaces with

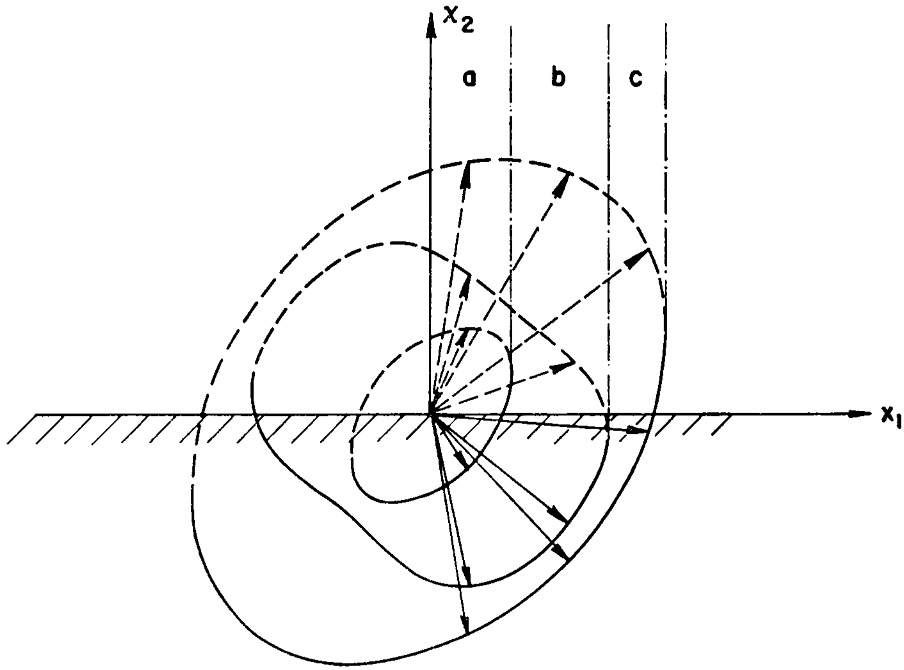

FIG. 3. Section of the three real slowness surfaces by the x_1, x_2-plane. The dashed lines refer to incident waves, the solid lines to reflected waves. Region a) contains those incident slowness vectors associated with three real reflected waves; region b), two real reflected waves; and region c) one real reflected wave.

the x_1x_2-plane are symmetric about the x_1-axis. Then in case b) the one complex value of S_2 must be pure imaginary.

Knowing the three slowness vectors of the reflected waves enables us to compute each of the displacement vectors to within an arbitrary constant, according to (5) or by transformation of Eq. (25) from the primed to the unprimed coordinate system. Assuming the displacement vectors are normalized, i.e. $\mathbf{U}^* \cdot \mathbf{U} = 1$, and the incident wave is of normal amplitude, the full displacement field can be written

$$\mathbf{u} = \mathbf{U}^{(I)} \exp i\omega\left(\frac{\mathbf{S}_{(I)} \cdot \mathbf{x}}{C_p} - t\right) + \sum_{r=1}^{3} C_r \mathbf{U}^{(r)} \exp i\omega\left(\frac{\mathbf{S}_{(r)} \cdot \mathbf{x}}{C_p} - t\right). \tag{31}$$

There are three boundary conditions to satisfy on the surface which determine the constants C_r, and the reflection problem is complete. A check on the solution is that the total energy flux across the free surface must be zero.

We are now led to consider a solution which consists only of surface waves, a generalized Rayleigh wave. This would consist of displacements of the form

$$\mathbf{U} \exp i\omega\left(\frac{S_1x_1 + S_2x_2 + S_3x_3}{C_p} - t\right) \tag{32}$$

where S_2 is complex with a negative real part. Thus we must have S_1 and S_3 such that a line perpendicular to the free surface through S_1 and S_3 does not touch any of the real slowness surfaces. Then S_2 assumes the values of the three roots of (13) with negative real parts. The procedure for determining possible combinations of S_1 and S_3 for Rayleigh waves to exist is outlined by Synge [1] for an anisotropic material. To summarize, a linear combination of the three waves is formed and the three boundary conditions are applied. This leads to three homogeneous equations on the three amplitudes. For a non-trivial solution, the complex determinant must be zero, yielding two real equations on S_1 and S_3. If these equations can be satisfied simultaneously then a solution S_1, S_3 represents a possible direction and speed of Rayleigh wave propagation on the rotating half-space at the given frequency.

References

[1] J. L. Synge, *Elastic waves in anisotropic media*, J. Math. Physics **35**, 323–334 (1957)

[2] J. L. Synge, *Flux of energy for elastic waves in anisotropic media*, Proc. Roy. Irish Acad. **A58**, 13–21 (1956)

[3] M. J. P. Musgrave, *Elastic waves in anisotropic media*, Prog. Solid Mech. **2**, 63–85 (1960)

[4] R. Courant and D. Hilbert, *Methods of mathematical physics*, Volume I, Interscience Publishers, Inc., New York, 1953