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ELECTIONS, COALITIONS, AND LEGISLATIVE OUTCOMES

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ABSTRACT

This paper develops a multi-stage game-theoretic model of three-party competition under proportional representation. The final policy outcome of the game is generated by a non-cooperative bargaining game between the parties in the elected legislature. This game is essentially defined by the vote shares each party receives in the general election, and the parties' electoral policy positions. At the electoral stage parties and voters are strategic in that they take account of the legislative implications of any electoral outcome. We solve for equilibrium electoral positions by the parties and final policy outcomes.

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1. INTRODUCTION

Spatial theories of elections and legislatures are now well-established, if not thoroughly worked out [for recent reviews, see Shepsle (1986), Calvert (1986), Austen-Smith (1983)]. For the most part, theories of elections and theories of legislatures have developed independent of one-another. This is unfortunate because, inter alia, voters are interested in policy outcomes, not policy promises. And policy outcomes are determined within an elected legislature which typically comprises representatives of several districts or political parties. Rational voters, therefore, will take into account the subsequent legislative game in making their decisions at the electoral stage of the process. In turn, rational candidates will take account of such deliberations in selecting their electoral strategy and subsequent legislative behavior conditional on electoral success. So, to understand more fully both electoral and legislative behavior - in the sense of being able to explain and predict policy positions, policy outcomes, and coalition structures - it is necessary to develop a theory of both political arenas simultaneously.

This paper makes a modest attempt at such a goal. We describe a

multi-stage game-theoretic model of electoral and legislative behavior where three parties are competing for votes in a proportional representation (PR) system. We initially solve for the equilibria of a non-cooperative bargaining game among the parties at the legislative stage, and then analyze the equilibria at the electoral stage, where the payoffs to the participants are those induced from the equilibrium behavior at the legislative stage.

The rationale for starting with the assumption of PR is two-fold: first, it allows us to examine coalition formation in legislatures using the party as the unit of analysis; second, the discreteness problem induced by any plurality system is absent - i.e., under PR, the number of legislative seats won by a party may be treated as essentially proportional to the number of votes that party attracts. And in addition to PR being an important electoral mechanism in most of western continental Europe, there is renewed analytical interest in PR and other alternative rules to simple majority or plurality voting [e.g. Sugden (1984), Greenberg and Weber (1985)]. But, to our knowledge, in all of this work the voters are assumed to vote over <u>candidates</u> and not over <u>final policies</u>. As the remarks above suggest, this is a misspecification of the choice set. Although candidate characteristics other than policy positions surely matter in elections [Enelow and Hinich (1984)], it is inappropriate to ignore the <u>legislative</u> implications of electing one candidate over another, as is explicitly done in the papers cited above. Likewise, while there are strategic models of legislative coalition formation and bargaining [e.g. Riker

(1962), McKelvey, Ordeshook, and Winer (1978), Schofield (1985)], none of these explicitly consider the electoral implications of the legislative behavior studied.

The notion that a political model should involve both electoral and legislative stage is of course not new. Downs (1957) examines a fairly informal model in which a legislature is formed via PR, and then simple majority voting within the legislature determines the government. His conclusions are imprecise and vague, and his focus is more on showing how any voter's decision calculus is made more difficult when, at the electoral stage, the voter is ignorant of the eventual coalition structure in the legislature. Robertson (1977) analyzes a multi-district model in which the party whose candidates win the most districts controls the legislature. Thus, it is evidently not sensible for a party to maximize votes; what matters is obtaining a controlling number of seats in the legislature. With only two parties and simple majority voting in the legislature, there is no room for post-election coalition-building. Recognizing this, rational voters vote on the basis of party, rather than candidate, policy. Still, Robertson does not exploit a rigorous strategic model and his conclusions are correspondingly "broad-brush."

Austen-Smith (1981),(1984),(1986) develops a sequence of multi-district models in which simple plurality voting in districts generates a legislature, and voters vote on the basis on legislative policy outcomes. In Austen-Smith (1981), there are several "Downsian" parties, where all party candidates coordinate their policy positions so

as to win control of the legislature, but the issue of coalition formation at the legislative stage is ignored. In Austen-Smith (1984) there are only two parties, all candidates belong to one or other party, and candidates are free to adopt any policy they wish. Thus the analysis focuses on the mechanism which aggregates candidates' electoral positions into party positions at the legislative stage. Finally, in Austen-Smith (1986) it is assumed that all candidates in a multi-district, simple plurality election are completely independent. In this case, rational voters at the electoral stage will, given any list of candidate positions, form estimates of (1) which legislature will form, (2) given a legislature, which coalition will form, and (3) given a coalition, what will be the policy that is implemented. In this paper, the legislative stage is not formulated explicitly; rather, each component is treated in an essentially probabilistic fashion.

In contrast, the model developed here provides a structure for solving for the policy outcome from the formation of a given legislature, by positing an institution in which the parties attempt to form a government. The typical approach to predicting the formation of coalitions and policy outcomes has been with the theory on cooperative games [cf. McKelvey, Ordeshook, and Winer (1978), Schofield (1985), inter alia]. This approach avoids identifying which of the possible winning coalitions form, and instead generates families of coalition/payoff combinations which satisfy certain stability properties. Since our goal is to allow the parties and voters to "look ahead" to the future consequences of current actions, we prefer instead to adopt an approach which might give unique behavioral predictions at the legislative stage as a function the results from the electoral stage. Hence we adopt a non-cooperative approach to coalition formation which, given the generic non-existence of the core, necessitates the imposition of some exogenous institutional structure to the problem. Thus we are trading off generality in favor of analytic tractability; this is in the spirit of recent work which examines outcomes as a function of institutions as well as preferences [e.g. Shepsle (1979), Ferejohn and Krehbiel (1985)].

The particular institutional feature we have in mind is the widespread convention of first asking the party with the largest number of votes to attempt to form a winning coalition; i.e. a government (Parliaments of the World, vol.II, 2nd ed., 1986, Table 39). If this party is unsuccessful, the party with the next largest number of votes is allowed to try a form a government, and so on. In the event that no government is able to form, a "caretaker" government forms which is assumed to make the choice of legislative outcomes "equitably".

Once the general election results are determined, the mechanism described generates a noncooperative bargaining game in the legislature. Given parties' electoral platforms, this game has a unique equilibrium outcome for any distribution of electoral votes across parties. An outcome in the model is a winning coalition, a legislative policy position implemented by that coalition, and a distribution of portfolios across the coalition. Since only one policy can be implemented and

parties have different "preferences over what it should be, coalition governments are sustained partly through sharing the benefits of being a member of the government. These benefits are modelled here as portfolios, and a party can be induced to join a government and compromise over the policy choice by offering it some portfolios. Thus parties have an incentive to join a government other than policy-implementation alone. And, as we shall see, this incentive is important in supporting equilibrium policy positions.

Voters in the model care about final policy outcomes and not about party platforms per se, or about the distribution of portfolios in any resulting government. Voters are also presumed to be rational. Given a list of electoral platforms of parties and given the structure of the legislative bargaining game, voters can compute the final legislative policy decision as a function of the distribution of electoral votes. Therefore, given everyone else's voting behavior, each individual will cast his or her vote to promote the final policy outcome he or she most prefers. In a two-party, simple plurality election, this amounts to voting sincerely over the party platforms. In a multi-party election with proportional representation, in which individuals cast at most one vote, sincere voting is typically not rational [Austen-Smith (1987)]. Furthermore, such voting behavior effectively eliminates any stable set of party policy positions; strategic (rational) voting here is necessary to support electoral equilibria.

In the next section we present the model of electoral and legislative behavior formally. We then provide a characterization of a class of equilibria generated by the induced multi-stage game. There are three features of the equilibria identified worth anticipating:

(1) Given the electoral policy positions, only the rank-order of the electoral vote-shares matters in determining the winning coalition (government) which emerges from the legislative bargaining process. Assuming no one party receives an overall majority of votes, the government will comprise the largest and smallest legislative parties; the middle-ranked party in terms of votes will be excluded. Thus a party's influence in the legislature is not monotonic in its vote share, and the winning coalition may not be of minimum size in the sense of Riker (1962) or connected in the sense of Axelrod (1970).

(2) The equilibrium electoral policy positions of the parties are symmetrically distributed around the median voter's most-preferred policy, with one party adopting this position to contest the election. This party, however, receives the fewest votes. As a result, the <u>expected</u> legislative policy outcome (i.e. prior to the legislative process being completed) is the median position, but the actual outcome in any election period will be skewed away from this point.

(3) Not all individuals vote sincerely relative to the party positions in equilibrium. Therefore, the equilibrium party vote-shares will not necessarily reflect the distribution of preferences of the electorate. Furthermore, as the minimum number of votes necessary for a party to be elected to the legislature goes down, the number of individuals not voting sincerely goes up.

Advocates of proportional representation often predicate their

arguments upon two premises. First, the composition of the legislature will mirror the preference distribution of the population at large; second, legislative outcomes will reflect the relative weights of the elected parties in the legislature [see Sugden (1984), pp. 31-33 for a discussion). The results reported here suggest that neither of these premises may be well-founded. In the concluding section of this paper, we take up these issues at greater length.

2. THE MODEL

There are three parties, α, β, and γ, where $\Omega=\{\alpha,\beta,\gamma\}$, competing on a one-dimensional policy space $P \subset \mathbb{R}$, for the votes from a finite set N of individuals. Assume $|P| < \infty$, and $|N| \equiv n$ is sufficiently large (≥ 15) and odd. Let $S(\Omega)$ denote the set of all subsets of Ω. At time $t=-2$, the parties simultaneously announce policy positions p_k in P, where $\mathbf{p} = (p_\alpha, p_\beta, p_\gamma)$, and at $t=-1$ the voters each cast a single ballot for one of the three parties. The method of determining a legislature is by proportional representation, where a party needs at least s votes to gain entrance to the legislature. We assume that s is odd and $s \in [3, 1/3 \cdot n)$. Let w_k be the proportion of votes party k receives in the election at $t=-1$, and let $\mathbf{w} = (w_\alpha, w_\beta, w_\gamma)$. If one or more parties receives less than s votes, we normalize the weights of the remaining parties so that they sum to 1. For example, if only party γ gets less than s votes, then $w_\alpha' = w_\alpha/(w_\alpha+w_\beta)$. For the following we assume that all parties receive at least s votes, so that we use the vector $\mathbf{w}$ rather than $\mathbf{w}'$ in establishing legislative influence.

For all coalitions $C \in S(\Omega)$ let $w_C = \sum_{k\in C} w_k$, and define

$$D(\mathbf{w}) = \{C\in S(\Omega) : w_C > 1/2\}.$$

We assume that $D(\mathbf{w})$ identifies the set of winning coalitions in the legislature given the vector of seats $\mathbf{w}$. Also, for all $k\in\Omega$ let

$$D_k(\mathbf{w}) = \{C\in D(\mathbf{w}) : k\in C\}$$

be the set of winning coalitions of which party k is a member.

From t=1 on the parties attempt to form a <u>government</u>, or a winning coalition, which will collectively choose, i) a policy $y\in P$ and ii) a distribution of portfolios among the parties, which we characterize as choosing a distribution of a fixed amount G of transferable benefits across the parties; let

$$\Delta(G)=\{(g_\alpha,g_\beta,g_\gamma) : g_k\geq 0,\ \forall\ k\in\Omega \text{ and } \sum_{k\in\Omega} g_k = G\}$$

be the set of such distributions. The process by which a government is formed is as follows: at t=1, the party with the largest number of seats proposes a coalition $C_1\in D(\mathbf{w})$, a policy $y_1\in P$, and a distribution of benefits $g_1\in\Delta(G)$, where $g_1=(g_{1\alpha},g_{1\beta},g_{1\gamma})$. The members of the coalition either accept or reject the proposal. If the parties which accept the proposal constitute a winning coalition, then they form a government, implement y_1, and distribute g_1. If not enough parties accept the proposal, then at time t=2 the party with the second highest number of seats proposes a coalition, a policy, and a distribution of benefits, and again the members of the proposed coalition either accept or reject. If a government has not formed after the t=3 proposal, then a "caretaker" government is implemented which "equitably" makes the policy and benefits decisions.

Given this description, then, a <u>strategy</u> for party k consists of three elements: an <u>electoral position</u> $p_k\in P$, a <u>proposal</u> $\Gamma_k \in D(\mathbf{w})\times P\times\Delta(G)$, and a <u>response strategy</u>

$$r_k: D_k(\mathbf{w})\times P\times\Delta(G)\times T \rightarrow \{0,1\},$$

specifying whether or not party k accepts (1) or rejects (0) a proposal which includes k in the coalition, where this response may be a function of the time [t=1,2,3] at which it is offered. Note that our definition of a proposal is <u>ahistorical</u>; while a complete description of a strategy would imply the proposal being a function of past electoral positions, proposals, and responses, the nature of the model eliminates the necessity of carrying around this extra notation. Let $\Gamma = (\Gamma_\alpha,\Gamma_\beta,\Gamma_\gamma)$, and $r = (r_\alpha,r_\beta,r_\gamma)$. A <u>strategy</u> for voter i is a function

$$\sigma_i : P\times P\times P \rightarrow \Delta(\Omega)$$

specifying the probability i votes for each party given their electoral positions. Let $\sigma_i(p) = (\sigma_i(\alpha),\sigma_i(\beta),\sigma_i(\gamma))$, where $\sigma_i(k)$ is the probability that voter i votes for party k, and

$$\sigma(p) = (\sigma_1(p),\ldots,\sigma_n(p)).$$

Voters are assumed to be purely policy-oriented, with preferences characterized by quadratic utility functions $u_i(\cdot) = u(\cdot;x_i)$ over the policy space P, where x_i is voter i's ideal point in P. It is assumed that $x = (x_1,\ldots,x_n)$ is common knowledge and ordered so that $\forall\ i < n$, $x_i < x_{i+1}$. Further, assume that voter ideal points are distributed symmetrically about the median voter's ideal point. Let $\mu = (n+1)/2$ be the median voter. The assumption of quadratic preferences implies that, if the policy outcome from the legislative stage is uncertain, but there exists a probability distribution $\rho(\cdot)$ over P, then the expected utility for voter i is

$$E_\rho[u_i(\cdot)] = -(y^\rho - x_i)^2 - s^\rho,$$

where y^ρ is the mean and s^ρ the variance of the distribution ρ.

Parties will have utility functions defined over $\Delta(G)$ as well as, at the legislative stage, over P. Ex ante, however, their policy preferences will be a function only of the difference between their electoral positions and the final policy outcome. The motivation for this is as follows: voters and parties are actually engaged in a continuing relationship which spans a number of elections. Voters therefore have the ability to condition future decisions on the past performance of the parties; in particular, they can condition their votes on the degree to which party promises (i.e. electoral positions) differ from the actual policy outcomes as a way of imposing costs on the parties at the legislative stage for deviating from their announced positions. Even if the parties are only concerned with winning elections and collecting the transferable benefits, future benefits will be a function of the current difference between the electoral position of a party and the final policy outcome from the legislature if the voters adopt these "retrospective" strategies. Hence rational parties will take this difference into account when choosing electoral positions and legislative proposals. Since it is in the interest of the voters to adopt these strategies, it seems consistent, in a "single-election" model, to characterize party preferences in the manner that we do.

This assumption also implies that, if a party receives less than s

votes and hence is not represented in the legislature, their payoff would not be a function of the final policy outcome. Thus we assume that party preferences are represented by a utility function taking on the values $U_k(y,g;p)$ if elected and $-c$ otherwise, where later we assume that the "cost" c is sufficiently large. The function U_k is assumed to be quasi-linear, i.e. additively separable and linear in g_k, and quadratic in y:

$$U_k(y,g;\mathbf{p}) = g_k - (y - p_k)^2,$$

where p_k is the electoral position of party k. Again, use of quadratic utility functions implies that the expected utility for party k generated by the distribution $\rho(\cdot)$ over P and $f(\cdot)$ over $\Delta(G)$ is

$$E_{f,\rho}[U_k(\cdot,\cdot;p)] = g_k^f - (y^\rho - p_k)^2 - s^\rho,$$

where y^ρ and s^ρ are defined as above, and g_k^f is the mean value of g_k with respect to the distribution f.

A sequential equilibrium to this game will consist of voter and party strategies which are optimal for each participant at every point in time, given the assumed equilibrium behavior of the others. To characterize these equilibria we first determine the equilibrium behavior at the legislative stage for any vectors **p** of policy positions and **w** of party weights. By the sequential nature of the decision-making at the legislative stage, we initially solve for the optimal proposals and responses at time t=3. This then allows us to solve for the optimal behavior at t=2 as a function of the optimal behavior to occur at t=3, and so on. As we shall see, the equilibrium prediction for the legislative stage will in general be unique for any (p,**w**). Therefore, for any p and set of voting strategies $\sigma(p)$, voters can deduce the final legislative outcome. This allows us to analyze equilibrium behavior at the voting stage, for all party positions p, by solving for the optimal behavior of the voters. In equilibrium, the vector of party weights $\mathbf{w}$ will be a known function of the party positions p, where this functional relationship will be determined by the voting strategies σ. This then constitutes the basis for analyzing the competition among the parties at the electoral stage as well, since now the legislative outcome is a function only of the positions the parties choose.

In the next section we initially describe the equilibrium behavior at the legislative stage, and then work back to the voting and electoral stages.

3. EQUILIBRIUM BEHAVIOR

Equilibrium legislative outcomes

As described above, in this section the vector of party policies $\mathbf{p} = (p_\alpha, p_\beta, p_\gamma)$ and weights $\mathbf{w} = (w_\alpha, w_\beta, w_\gamma)$ are treated parametrically. It will be convenient to relabel the parties according to their relative positions on the policy space P. Let $p_L = \min \{p_\alpha, p_\beta, p_\gamma\}$, $p_M = \text{mid} \{p_\alpha, p_\beta, p_\gamma\}$, $p_R = \max \{p_\alpha, p_\beta, p_\gamma\}$; similarly define w_L, w_M, and w_R as the weights of the left, middle, and right parties, respectively, and let $\Omega = \{L,M,R\}$. If the weights of any two parties are equal, while the remaining party has less weight, then it is assumed that prior to t=1 a fair coin is flipped to decide which party will make the t=1 proposal; a similar assumption holds when the two parties have the lowest weight or

when all parties have equal weight. Most of the following analysis will focus on the case where each party has a distinct electoral position; however, the outcomes when some of the positions coincide are easily derived from Proposition 1.

If only two parties receive the necessary s votes, then as discussed in Section 2 the weights of the parties in the legislature are normalized to reflect this fact. Thus the party with the higher vote share will hold a majority of the seats in the legislature. If the parties have the same vote shares, then it is assumed that the coin flip determines who will hold the majority in the legislature. In what follows we assume that all three parties receive at least s votes; given the above description of events the subsequent analysis is easily extended to the case where only two parties are represented.

We assume that G is sufficiently large ($\geq |P|^2$) so that it is always possible for a coalition to form at any time, and further that any caretaker government has the ability to, and in fact would, choose a policy $y \in P$ and a distribution $g \in \Delta(G)$ such that the utilities for the parties are all equal to 0 in the event of no agreement at t=1,2, or 3. Note that we could have equivalently assumed that there existed a positive coefficient on the linear term in the parties' utility functions and, rather than assuming that G was sufficiently large, assume that this coefficient were sufficiently large.

If any party has a majority of the seats, say for example party L, it is clear that the only equilibrium is for that party to choose $y_1 = p_L$ and $g_{1L} = G$, since it needs no other party to form a government.

Furthermore, by the assumption of complete information with regard to the payoffs of the parties, we need only consider minimum winning coalitions, so that all members of a proposed coalition must agree to the proposal.

Suppose party k is attempting to form a government. In order to induce party j to agree to a proposal, party k must offer j at least as much as j could get by rejecting k's proposal; i.e., party j's <u>opportunity cost</u> of joining the coalition. Let u_j^t be party j's opportunity cost at time t. It is clear from the above assumptions that $u_L^3 = u_M^3 = u_R^3 = 0$, since by rejecting a proposal at t=3 each party guarantees a payoff of 0 from the ensuing caretaker government. If a government would implement (y_3,g_3) at t=3, then the opportunity cost for party k at t=2 will be determined by k's utility from the outcome (y_3,g_3); i.e. $u_k^2 = U_k(y_3,g_3;\mathbf{p})$, since this is the utility k would receive from rejecting a proposal at t=2. Similarly, if (y_2,g_2) would be implemented at t=2, then $u_k^1 = U_k(y_2,g_2;\mathbf{p})$.

In general the parties' opportunity costs will depend on the responses of the parties to subsequent proposals. Let

$$\delta(C,y,g,t) = \prod_{k\in C} r_k(C,y,g,t)$$

be the product of party responses to a proposal of (C,y,g) at time t; thus $\delta(C,y,g,t)$ will be 1 if all parties agree to the proposal and 0 otherwise. Since only minimum winning coalitions will be proposed $\delta(C,y,g,t)$ is sufficient to deduce whether a government will form at t.

<u>Definition</u> Given proposals Γ and responses r, the opportunity cost of party k at time t, $u_k^t(\Gamma,r)$, is

$$u_k^3(\Gamma,r) = 0$$
$$u_k^2(\Gamma,r) = \delta(C_3,y_3,g_3,3)\cdot U_k(y_3,g_3;\mathbf{p})$$
$$u_k^1(\Gamma,r) = \delta(C_2,y_2,g_2,2)\cdot U_k(y_2,g_2;\mathbf{p}) + (1 - \delta(C_2,y_2,g_2,2))\cdot u_k^2.$$

Given a list Γ of proposals, then, we can inductively define equilibrium responses for the parties. To determine the equilibrium proposals for the parties, define

$$\tilde{U}_k(\Gamma_k,r) = U_k(y,g;\mathbf{p})\cdot\delta(C,y,g,k)$$

as the utility for party k generated by the proposal $\Gamma_k = (C,y,g)$ if Γ is accepted.

<u>Definition</u> A <u>legislative equilibrium</u> consists of response strategies $r^*(\cdot) = (r_L^*(\cdot),r_M^*(\cdot),r_R^*(\cdot))$ and proposals $\Gamma^* = (\Gamma_L^*,\Gamma_M^*,\Gamma_R^*)$ such that $\forall$ t, $\forall$ k $\in \Omega$,

i) $\forall\, \Gamma,\ r_k^*(C,y,g,t) = \begin{cases} 1 & \text{if } U_k(y,g;p) \geq u_k^t(\Gamma,r^*) \\ 0 & \text{else} \end{cases}.$

ii) Γ_k^* maximizes $\tilde{U}_k(\Gamma_k,r^*)$.

The logic of this definition follows from the sequential nature of the actions: since $r_k^*(\cdot,3)$ is known for all $k\in\Omega$, the optimal proposal from the party with the lowest weight can be explicitly solved. This then generates $r^{*}_{k}(\cdot,2)$, so that the optimal proposal at t=2 can be solved, and so on.

The presence of perfect information guarantees that, in equilibrium, if there exists a proposal at t=3 which gives the party with the lowest weight and some other party non-negative utility, then the equilibrium proposal at t=3 will be accepted; similar logic holds for t=2 and t=1. Furthermore, if parties k and j agree to form a coalition at t=k, it must be that the proposal (y_k, g_k) is such that y_k lies between p_k and p_j and $g_{kk}+ g_{kj}= G$; in words, the proposal must be Pareto-efficient for the coalition C={k,j}. Also, if j accepts k's proposal, then it must be that either j is receiving exactly his opportunity cost, or $y_k = p_k$ and $g_{kk}=G$, since otherwise k could offer j less than g_{kj} or a policy closer to p_k and still gain j's acceptance. The assumption that G is sufficiently large relative to $|P|$ implies that in equilibrium the proposal by the party with the highest weight will be accepted at an outcome which is either "first-best" for that party or makes the joining party indifferent between accepting and rejecting the proposal. [This is typical of bargaining models with perfect information, Rubinstein (1982)]. To determine who will join this party and at what outcome, then, we need to analyze the equilibrium proposals at t=3 and t=2 decisions to generate the opportunity costs of the parties at t=1.

Suppose that $w_L > w_M > w_R$, so that party R has the lowest weight, and hence makes a proposal at t=3, if no government has yet formed. Given the form of the parties' utility functions, it is clear that party

R will attempt to form a coalition with the party whose electoral position p_j is closest to p_R, since the opportunity costs of the other parties are equal. Thus, in this case, R would attempt to form the coalition {M,R}. Since R cannot implement $y = p_R$ and $g_{RR} = G$, R chooses (y_3, g_3) to solve

$$\max_{y,g} \; g_R - (y - p_R)^2 + \lambda(G - g_R - (y - p_M)^2), \; y \in P, \; g_R \in [0,G].$$

Since the utility functions are separable and quadratic in y, the solution to the above optimization will be

$$y^{*}_{3} = \frac{p_R + p_M}{2} \equiv p_{RM},$$

$$g^{*}_{3R} = G - (p_M - p_{RM})^2,$$

$$g^{*}_{3M} = (p_M - p_{RM})^2.$$

Thus, if a government has not formed at t=1,2, then at t=3 the policy outcome of the government will be the midpoint between the electoral positions of R and M, while the benefits G will be distributed in such a way as to give M a utility of exactly 0; i.e. M's opportunity cost. [Note: this solution holds if we weaken the assumption on preferences from quadratic to symmetric concave utilites over policy.]

Notice that this logic is quite general; for any $(\mathbf{p},\mathbf{w})$, the equilibrium proposal at t=3 will be such that the policy y is the midpoint between the electoral positions of the party with the lowest weight and the party with the nearest electoral position.

At t=2, then, the opportunity costs of the parties will be

$u_L^2 = -(p_{RM} - p_L)^2$, $u_M^2 = 0$, and $u_R^2 = G - 2(p_R - p_{RM})^2$. Thus, party L will accept a proposal which is "first-best" for party M, $y = p_M$ and $g_M = G$, since this gives L utility of $-(p_M - p_L)^2$, which is greater than $-(p_{RM} - p_L)^2$.

At t=1, then, $u_L^1 = -(p_M - p_L)^2$, $u_M^1 = G$, and $u_R^1 = -(p_R - p_M)^2$, implying that at t=1 the coalition {L,R} will form, since party M's opportunity cost at t=1 implies that L could never make a proposal which would keep M indifferent while making L better off, but there do exist proposals which make both L and R better off. If $d_L \equiv (p_M - p_L) \geq (p_R - p_M) \equiv d_R$, then, as at t=3, the optimal proposal from party L at t=1 will be such that $y = p_{RL}$, and R receives sufficient transferable benefits to meet its opportunity cost. If $d_L \leq d_R$, then the optimal proposal would be to choose $y = p_M$ and $g_L = G$, since this gives R precisely his opportunity cost and no other proposal would make L better off without making R worse off. Thus, if $w_L > w_M > w_R$, the equilibrium policy will either be p_M or p_{LR}, depending on the distances between the electoral positions.

Suppose instead that $w_M > w_L > w_R$. By the same logic as above, at t=3 the coalition {M,R} would form, with policy $y_3 = p_{MR}$, and benefits $g_{3M} = (p_{MR} - p_M)^2$, $g_{3R} = G - g_{3M}$. At t=2, then, the coalition {L,M} would form with some policy $y_2 \in (p_L, p_M)$, where again the exact policy will be a function of the distances between electoral positions. At t=1, then, the opportunity cost of party R, u_R^1, will be less than $-(p_R - p_M)^2$; hence R will accept a proposal by party M of $y_1 = p_M$ and $g_{1M} = G$. Thus the equilibrium policy outcome when $w_M > w_L > w_R$ will be $y = p_M$,

regardless of the distances between the electoral positions.

Note that the above analysis applies directly to the symmetric cases where $w_R > w_M > w_L$ and $w_M > w_R > w_L$. The remaining cases, where party M has the lowest weight, can be analyzed similarly, although in these cases the algebra is somewhat trickier.

The following proposition summarizes the equilibrium coalitions C^* and outcomes y^*, g^* from the legislative stage. The (lengthy) formal statement of the proposition can be found in the Appendix.

<u>Proposition 1</u> Let party k offer the proposal at t=1, party h at t=2, and party j at t=3.

(a) If k has a majority in the legislature, then $y^* = p_k$, $g_k^* = G$;

(b) If k does not have a majority, then $C^* = \{k,j\}$, y^* lies between p_k and p_{kj}, and

$$\max g_j^* = \begin{cases} (p_k - p_{kj})^2 & \text{if } y^* = p_{kj} \\ 0 & \text{else} \end{cases}$$

$$g_k^* = G - g_j^*, \quad g_h^* = 0.$$

Thus in equilibrium it will always be the parties with the highest and lowest weights which form the governing coalition. The logic of this follows directly from the recursive nature of the analysis: the party with the middle weight is excluded precisely because it would make the t=2 proposal, thus implying a high opportunity cost at t=1 and hence a degree of bargaining power vis-a-vis the party with the highest weight which exeeds that of the party with the lowest weight. The non-cooperative bargaining model of coalition formation developed here generates a unique coalition prediction, where this coalition is minimum winning, but is not of minimum size [Riker (1962)] and is not necessarily connected [Axelrod (1970)]. It is worth remarking that these features are not peculiar to quasi-linear preferences.

The cases in which the party with the lowest weight is in the middle and $d_L = d_R$ yield a sequential equilibrium prediction which is non-unique. The reason for this is that, at t=3, party M is indifferent between forming a government with L or with R, since either will give M the same payoff. However, the equilibrium payoffs to M depend on exactly this choice at t=3. In particular, M receives a higher payoff if it would form with the party with the highest weight than it would from forming with the party with the middle weight. The selection we make is the equilibrium with the higher payoff for party M, since ex ante M could credibly threaten the party with the highest weight that it would take such an (equilibrium) action at t=3 if it were called upon to do so.

Equilibrium voting strategies

Let $y(\mathbf{w},p)$ be the equilibrium policy outcome from the legislative stage given the vector of weights $\mathbf{w}$ and positions p. Define

$$\Lambda(p) = \{y \in P:\ y = y(\mathbf{w},p) \text{ for some } \mathbf{w}\}$$

to be the set of possible equilibrium policy outcomes given p. The vector of weights $\mathbf{w}$ will be determined by the individual voting

behavior; in particular, assuming that all voters adopt pure strategies, for any $k \in \Omega$,

$$w_k = |\{i \in N:\ \sigma_i(k)=1\}|/N \equiv v_k(\sigma(p))/N,$$

where $\sigma(p)=(\sigma_1(p),\ldots,\sigma_n(p))$. Thus the probability of any specific policy in $y \in \Lambda(p)$ being the final outcome is a function of voter strategies; let $\pi(\cdot|\sigma,p):\Lambda(p) \to [0,1]$ denote this probability.

Definition For any $C \in S(\Omega)$, with $|C| \geq 2$ and any $p \in P^3$, voter i's sincere strategy relative to C, $\sigma_i^C(\cdot)$, is defined as

$$\sigma_i^C(k') = 1 \text{ if and only if } u_i(p_{k'}) > \max_{C \setminus \{k'\}} u_i(p_k).$$

Notice that if $|C|=2$, an individual who votes sincerely relative to C at p does not necessarily vote for the party offering his most preferred policy in $\{p_L,p_M,p_R\}$.

Definition A voting equilibrium is an n-tuple $\sigma^*(p)$ such that $\forall$ p, $\forall$ $i \in N$, $\forall\ \sigma_i(p)$:

$$E_{\pi(\sigma^*,p)}[u_i(y)] \geq E_{\pi(\sigma_i,\sigma^*_{-i},p)}[u_i(y)].$$

Thus, given p, a voting equilibrium is simply a Nash equilibrium to the game with players N and payoffs induced by the equilibrium behavior in the legislative game generated by p. For simple-plurality, two-candidate electoral competition, Nash equilibrium is too weak a concept; it admits equilibria which are supported only by weakly dominated strategies. Consequently, in such games voter strategies are additionally required to be undominated. In the three-party proportional representation game developed here, however, no strategy is weakly dominated. The reason for this is that, in contrast to the two-party case, the final outcome from the legislative game for any **p** is not monotonic in vote shares: given that no party has an overall majority, it is always the largest and smallest parties which form the government. Thus requiring voter strategies to be undominated is vacuous here, even when two of the three parties adopt identical platforms.

An example will illustrate this fact. Suppose n=15, s=3, and $x_i=i$ for all i=1,...,15. Suppose also that $p_L=p_M=11<p_R=12$. Evidently, $u_1(p_M)>u_1(p_R)$. Is voting for R a dominated strategy for 1? The answer is No. To see this, suppose individuals i=2,...,5 vote for L, individuals i=6,...,11 vote for M, and i=12,...,15 vote for R. All these voters are voting sincerely relative to the positions **p** = (11,11,12). Since s=3, all parties get elected to the legislature in the absence of 1's vote. If 1 votes sincerely for either L or M, then no party has an overall majority, and the legislative weights of the parties are $w_M>w_L>w_R$ if 1 votes for L, and $w_M>w_L=w_R$ if 1 votes M. By Proposition 1, the final policy outcome from the legislative bargaining process will be $(p_M+p_R)/2$ = 11.5 if 1 votes L, and will be (in expectation) $1/2\cdot(p_M+p_R)/2 + 1/2\cdot p_M$ = 11.25 if 1 votes M. Now assume 1 votes for party R. Then $w_M>w_R>w_L$, in which case the final policy outcome is surely p_M=11. Hence, individual 1 is better off voting for R in these circumstances than he is by voting sincerely.

So eliminating voting equilibria involving dominated strategies buys us nothing. Consequently, we make a selection from the set of voting equilibria which is simple, supports an intuitively reasonable class of equilibrium party positions (see below), and has two desirable properties. First, at any equilibrium set of party positions, every voter is decisive between at least two parties. Thus, although we cannot apply the "weak dominance" argument for all electoral positions **p**, in equilibrium each voter will have a non-trivial decision problem in that the final policy outcome will be a function of how he votes. Second, at any out-of-equilibrium party positions, the voting equilibrium strategies provide incentives for the parties to "move toward" the equilibrium positions.

The voting equilibrium is described formally and in detail in the Appendix. For current purposes, and to provide a reference for later on, it is sufficient to identify the key features of the equilibrium informally.

<u>Proposition 2</u> A voting equilibrium $\sigma^*(p)$ is well-defined for all **p** $\in$ P×P×P. It is such that at least one party is penalized (in terms of votes) if, relative to the distribution of voter preferences:

(a) any two parties are "too close", or

(b) no party is centrally located, or

(c) parties are "too dispersed".

Together, (a),(b), and (c) insure that, in equilibrium, parties will adopt distinct positions symmetrically distributed about the median of the voter distribution. Exactly what constitutes being "too close" or "too dispersed" will become clear once we analyze the parties' strategic choice of electoral policy platforms.

It is worth noting that, in the voting equilibrium we select, no individual votes sincerely relative to Ω for all $p \in P\times P\times P$. Moreover, as we remarked in the Introduction, if individuals are constrained always to vote sincerely, then there is no set of party positions which could be an equilibrium; given any set of party platforms p, there is invariably one party which can unilaterally improve its payoff by deviating from p. From a theoretical perspective, therefore, strategic behavior on the part of the voters is required to generate stable electoral outcomes. And there exists considerable empirical evidence for strategic voting in legislative elections [Riker (1982)].

Equilibrium party positions

We are now in a position to define the equilibrium path of the entire multi-stage game by analyzing the electoral game among the parties, where the payoffs are those induced by the equilibrium behavior of the voters at t=-1 and the subsequent equilibrium behavior of the parties at the legislative stage. Let

$$\psi_k(p) = E_{\pi(\sigma^*,p)}[U_k(y^*(w(\sigma^*(p)),p),g^*(w(\sigma^*(p)),p);p)]$$

be the (expected) indirect utility for party k from the electoral positions p given the equilibrium behavior at the voting stage $\sigma^*(\cdot)$ and at the subsequent legislative stage $y^*(\cdot)$, $g^*(\cdot)$.

Definition An electoral equilibrium is a triple $p^*=(p^*_\alpha,p^*_\beta,p^*_\gamma)$ such that $\forall\, k\in\Omega$, $\forall\, p_k\in P$,

$$\psi_k(p^*) \geq \psi_k(p_k,p^*_{-k}).$$

For any $a\in\mathbb{R}$ define int[a] as the smallest integer greater than or equal to a.

Proposition 3 Relative to the voting strategies of Proposition 2, $p^* \in P\times P\times P$ is an electoral equilibrium for any $s \in [3,n/3)$, s odd, if and only if

(1) $p^*_M = x_\mu$,

(2) $(p^*_M - p^*_L) = (p^*_R - p^*_M) \in [8/3\cdot(x_{i^*} - x_\mu), 4\cdot(x_{j^*} - x_\mu))$,

where $i^* \equiv \mu + (s-1)/2$, $j^* \equiv \mu + \text{int}[(n-1)/4]$.

Under the assumption of a symmetric distribution of voter ideal points, at any p^* in this class, the equilibrium vote shares $w(\sigma^*(\cdot))$ are

$$w_L = w_R > w_M,$$

where party M receives exactly s votes. Consequently, from the analysis above, the equilibrium policy outcome from the legislative stage will be either p^*_{LM} or p^*_{MR}, with each of these occuring with probability 1/2 due to the equal weights of the extreme parties. Note that, by the assumption of quadratic utilities, all of the equilibria in this class are Pareto-inefficient; that is, ex ante everyone would prefer the outcome $y^* = p_M$, $g^*_L = (1/2)\cdot G = g^*_R$, since this gives the same mean utility as all the equilibria but at zero variance. The equilibrium which is Pareto-efficient among the class of equilibria is where the extreme parties adopt the innermost positions defined in (2) above.

Figure 1 gives an example of equilibrium party positions along with the associated voting behavior of the electorate. Define x(d) as the point in P such that, in an equilibrium where $(p_R-p_M) = (p_M-p_L) = d$, an individual with x(d) as an ideal point would be indifferent between voting for M and giving M precisely s votes, and voting for R and giving R a subsequent majority in the legislature. Thus x(d) solves

$$-(x - x_\mu)^2 - d^2/4 = -(d - (x - x_\mu))^2.$$

Solving this, we get that

$$x(d) = x_\mu + 3/8 \cdot d.$$

Condition (2) then implies that M receives at least s votes, and $x(d) \geq x_{i^*}$; otherwise voter i^* would prefer to vote for R, thereby upsetting the equilibrium. To see that M receives exactly s votes, suppose that some voter $i > i^*$ were voting for M in an equilibrium described in Proposition 3. Then, by switching his vote to R, party M still receives at least s votes, but now party R will surely make the first proposal in the legislature, thus implying that the policy outcome will be p_{RM} with probability 1. It is easy to see that this outcome would be preferred by i to the proposed equilibrium outcome.

4. DISCUSSION

From the perspective of positive political theory, little is known about the comparative properties of proportional representation and

simple-plurality decision-making schemes. What is known is largely confined to the abstratc structures of various aggregate preference relations. For example, we can say how Simple Majority Preference and the Single Transferable Vote match up on desiderata such as "anonymity", "neutrality", or "independence of irrelevant alternatives", but we have little idea about how strategic agents - candidates for office, voters, etc. - would behave differentially under these mechanisms, or what the difference in the final policy outcomes might be, in an otherwise fixed environment.

This paper develops a multi-stage game-theoretic model of three party competition under proportional representation. The particular PR mechanism assumed has two parts. First, at the election stage, a fixed-standard, or quota, rule determines the composition of the legislature. And second, in the elected legislature, a noncooperative bargaining process determines the membership of the government, the distribution of portfolios across this membership, and the final policy outcome. The legislative bargaining process is defined both by the relative electoral vote shares of parties - the "weights" of the parties in the legislature - and by the policy platforms they adopt to contest the election. As claimed in the Introduction and established in the subsequent sections of the paper, the identified equilibria to this game have three main substantive features:

(1) The government consists of the parties with the largest and the smallest weights. Hence, the legislative influence of an elected party is not monotonic in its vote share;

(2) Parties' electoral platforms are symmetrically distributed about the median voter's ideal point in the one-dimensional issue space. The party adopting this position to contest the election receives the smallest number of votes, and the remaining parties have an equal likelihood of being first-ranked in the legislature. Therefore, by conclusion (1), the expected final policy outcome will be at the median voter's ideal point, but the realized final outcome will lie between the median and either the right-most, or the left-most, party's position;

(3) Not all individuals vote sincerely for the party platforms they most prefer. So, even with equilibrium party platforms, vote-shares will not reflect the true distribution of preferecnes of the electorate.

The comparison with the two-party, winner-take-all, electoral mechanism in this environment is straightforward. In this case:

(1') The party with the most votes has monopolistic control of the legislature. Legislative influence, therefore, is monotonic in votes;

(2') In equilibrium, both parties adopt the median voter's position, and this is surely the final policy outcome;

(3') All voters vote sincerely, whether or not the parties adopt the equilibrium policy platforms.

In sum, the popular conception that, in contrast with simple-plurality schemes, proportional representation leads to legislatures, and hence to final policy outcomes, which reflect the variety of interests in the electorate, seems mistaken. Such a conception rest on the more-or-less implicit assumption of non-strategic behavior by the voters and parties which, on both theoretical and

empirical grounds, is unwarranted. Having said this, two caveats should be noted in regard to the model here.

First, the question of entry into the electoral competition is ignored. This is clearly important, since the number of candidates or parties contesting the election will be functionally dependent on the particular electoral and legislative schemes in place. However, allowing free entry, say, is not going to remove the incentives for strategic behavior by the voters in the election, and the logic of the legislative bargaining process studied in this paper is invariant to the number of parties in the legislature (although the location of the final policy outcome is, of course sensitive to this number).

Second, the equilibrium location of the three parties' policy platforms depends on the specification of the (equilibrium) voting behavior for any set of platforms. If voting strategies are altered, then the associated equilibrium party policies will be altered as well. Unlike in two-party competition, sincere voting by everyone is not capable of supporting any equilibrium in party positions; strategic voting is essential to generate stable outcomes. Moreover, as we argued earlier in the text, no voting strategy is weakly dominated - i.e. suppose we fix party positions, and fix some individual's (j's) vote arbitrarily; then there exists a distribution of votes by others such that j voting otherwise makes him strictly worse off. Consequently, for any distribution of party platforms there is a multiplicity of undominated voting equilibria, and the selection of exactly which one to adopt in order to solve for the electoral equilibrium in party platforms is somewhat arbitrary. The criterion used here was to insist~~s~~ that, in <u>any</u> electoral equilibrium, <u>every</u> voter must be pivotal, i.e. capable of unilaterally altering the final policy outcome from the legislative bargaining process by affecting the rank-order of parties' electoral vote-shares. This is a non-trivial prerequisite which refines the set of admissable equilibria considerably. But it is clear that work needs to be done on this problem.

Appendix

<u>Proposition 1</u> The following constitute the legislative equilibrium coalitions and outcomes:

1) if $w_M = \max\{w_L, w_M, w_R\}$, then $C^* = \{M,k\}$, where $w_k = \min\{w_L, w_M, w_R\}$, $y^* = p_M$, $g_M^* = G$;

2) if $w_L > w_M > w_R$, then $C^* = \{L,R\}$ and if

   (a) $d_L \leq d_R$, then $y^* = p_M$, $g_L^* = G$;

   (b) $d_L > d_R$, then $y^* = p_{LR}$, $g_R^* = (p_{LR} - p_R)^2$, and $g_L^* = G - g_R^*$;

3) if $w_R > w_M > w_L$, then $C^* = \{R,L\}$ and if

   (a) $d_L \leq d_R$, then $y^* = p_{LR}$, $g_L^* = (p_{LR} - p_R)^2$, and $g_R^* = G - g_L^*$;

   (b) $d_L > d_R$, then $y^* = p_M$, $g_R^* = G$;

4) if $w_L > w_R > w_M$ and $d_L \leq d_R$, then $C^* = \{L,M\}$ and

   a) if $2d_L \geq d_R$, $y^* = p_{LM}$, $g_M^* = (p_{LM} - p_M)^2 - (p_{RL} - p_M)^2$, and $g_L^* = G - g_M^*$;

   b) if $2d_L < d_R \leq 3d_L$, $y^* = 2p_M - p_{RL}$, $g_L^* = G$;

   c) if $3d_L < d_R$, $y^* = p_L$, $g_L^* = G$;

5) if $w_L > w_R > w_M$ and $d_L > d_R$, then $C^* = \{L,M\}$, $y^* = p_{LM}$, $g_M^* = (p_M - p_{LM})^2 - (p_{RM} - p_M)^2$, and $g_L^* = G - g_M^*$;

6) if $w_R > w_L > w_M$ and $d_L < d_R$, then $C^* = \{R,M\}$, $y^* = p_{MR}$, $g_M^* = (p_{MR} - p_M)^2 - (p_M - p_{ML})^2$, and $g_R^* = G - g_M^*$;

7) if $w_R > w_L > w_M$ and $d_L \geq d_R$, then $C^* = \{R,M\}$ and

   a) if $d_L \leq 2d_R$, $y^* = p_{MR}$, $g_M^* = (p_M - p_{RM})^2 - (p_M - p_{RL})^2$, and $g_R^* = G - g_M^*$;

b) if $2d_R < d_L \le 3d_R$, $y^* = 2p_M - p_{RL}$, $g_R^* = G$;

c) if $d_L > 3d_R$, $y^* = p_R$, $g_R^* = G$.

Next, we state and prove the formal version of Proposition 2. For any $p \in P\times P\times P$, $k \in \Omega$, let $B_k(p) = \{i \in N: u_i(p_k) > u_i(y), \forall y \in \Lambda(p)\}$.

Proposition 2 The following n-tuple σ^* of voter strategies is a voting equilibrium for any $s \in [3,n/3)$, s odd:

1a) $p_L = p_M = p_R \Rightarrow \sigma_i^*(k) = 1/3$, $\forall i \in N$, $\forall k \in \Omega$.

b) $p_L = p_M < p_R \Rightarrow$

$\sigma_i^*(R) = 1$, $i = \mu-1,\dots,n$; $\sigma_i^*(k) = 1/2$, $i = 1,\dots,\mu-2$, $k=L,M$.

c) $p_L < p_M = p_R \Rightarrow$

$\sigma_i^*(L) = 1$, $i = 1,\dots,\mu+1$; $\sigma_i^*(k) = 1/2$, $i = \mu+2,\dots,n$, $k=L,M$.

2) $p_L < p_M < p_R$ and $|B_k| \ge (n+1)/2$, some $k \in \Omega \Rightarrow$

$\sigma_i^* = \sigma_i^\Omega$, $\forall i \in N$.

Now suppose that $p_L < p_M < p_R$ and $|B_k| < (n+1)/2$, $\forall k \in \Omega$. Then $x_\mu \in (p_L,p_R)$, and

3a) $x_\mu > p_M \Rightarrow \sigma_i^*(M) = 1$, $i = 1,\dots,s$; $\sigma_i^*(R) = 1$, $i = s+1,\dots,n$.

b) $x_\mu < p_M \Rightarrow \sigma_i^*(L) = 1$, $i = 1,\dots,n-s$; $\sigma_i^*(M) = 1$, $i = n-s+1,\dots,n$.

4) $x_\mu = p_M$ and $d_L < (>)\ d_R \Rightarrow \sigma_i^* = \sigma_i^{\{L,M\}}\ (\sigma_i^{\{R,M\}})$, $\forall i \in N$.

5) $x_\mu = p_M$ and $d_L = d_R = d < (x_{(2\mu+s-1)/2} - x_\mu)\cdot 8/3 \Rightarrow$

$\sigma_\mu^*(L) = \sigma_\mu^*(R) = 1/2$; $\sigma_i^* = \sigma_i^{\{L,R\}}$, $\forall i \ne \mu$.

6) $x_\mu = p_M$ and $d_L = d_R = d \ge (x_{(2\mu+s-1)/2} - x_\mu)\cdot 8/3 \Rightarrow$

$\sigma_i^*(L) = 1$, $i = 1,\dots,(2\mu-s-3)/2$,

$\sigma_i^*(M) = 1$, $i = (2\mu-s-1)/2,\dots,(2\mu+s-1)/2$,

$\sigma_i^*(R) = 1$, $i = (2\mu+s+1)/2,\dots,n$.

Proof.

1a) Suppose $p = (y,y,y)$. Then $\Lambda(p) = \{y\}$, in which case all voters are indifferent over voting strategies. Hence $\sigma^*(p)$ as specified is an equilibrium.

1b) Suppose $p = (y',y',p_R)$, where $p_R > y' = p_L = p_M$. Then $\Lambda(p)=\{y',(y'+p_R)/2,p_R\}$, $v_R(\sigma^*(p)) \ge (n+3)/2$, and $y(\varpi(\sigma^*(p)),p) = p_R$. Clearly all $i \in B_R$ are using maximizing strategies. And given $\sigma^*(p)$, no $i \in N\setminus B_R$ is pivotal between $\{p_L,p_M\}$ and p_R. Hence $\sigma^*(p)$ is an equilibrium.

1c) An argument symmetric to that used in 1b) applies.

2) If $|B_k(p)| \ge (n+1)/2$, for some $k\in\Omega$, then $\sigma_i^*(k) = 1$ is clearly a best response for any $i \in B_k(p)$, since $y(\varpi(\sigma^*),p) = p_k$. And given σ_{-i}^*, no $i \in N\setminus B_k$ is pivotal between p_k and any other possible outcome. Hence, $\sigma^*(p)$ is an equilibrium.

3a) In this instance, $v_R(\sigma^*(p)) = n-s > 2n/3$, so that $y(\varpi(\sigma^*(p)),p) = p_R$. Since no individual is pivotal, $\sigma^*(p)$ is an equilibrium. Mutatis mutandis, the same is true for case 3b).

4) Suppose $x_\mu = p_M$ and $d_L < d_R$. Then $v_M(\sigma^*(p)) \ge (n+1)/2$ and $y(\varpi(\sigma^*(p)),p) = p_M$. If any individual i is pivotal, then $\sigma_i^*(M) = 1$ and $v_M(\sigma^*(p)) = v_L(\sigma^*(p))+1 = (n+1)/2$. Since $s \ge 3$, $\sigma_i \ne \sigma_i^*$ implies $y(\varpi(\sigma_i,\sigma_{-i}^*),p) \le p_M$, with strict inequality if σ_i is a pure strategy. But $[x_\mu=p_M$, $v_M(\sigma^*(p))=(n+1)/2$, and $\sigma_i^{\{L,M\}}(M)=1]$ implies $x_i \ge p_M$.

Hence, $\sigma^*(p)$ is an equilibrium. A symmetric argument holds when $d_L > d_R$.

5) By definition of p_M and d, the argument for this case follows immediately from that of case 6), below.

6) Given $\sigma^*(p)$, $v_L(\sigma^*(p)) = v_R(\sigma^*(p)) = (n-s)/2 > v_M(\sigma^*(p)) = s$. Hence, $y(w(\sigma^*(p)),p) \in \{p_{LM},p_{MR}\}$, where each occurs with probability 1/2. By supposition, $d_L = d_R = d$. Thus, for all $j \in N$,

$$Eu_j(y(w(\sigma^*(p)),p)) = -(x_j - p_M)^2 - d^2/4.$$

Consider any i such that $\sigma_i^*(M) = 1$, and suppose i switches to $\sigma_i(R) = 1$. Then, $v_R(\sigma_i,\sigma_{-i}^*) = (n-s+2)/2 > v_L(\sigma_i,\sigma_{-i}^*) = (n-s)/2 > v_M(\sigma_i,\sigma_{-i}^*) = s-1$. Hence M is not elected to the legislature, and $y(w(\sigma_i,\sigma_{-i}^*),p) = p_R$. In this event, i's utility is

$$Eu_i(y(w(\sigma_i,\sigma_{-i}^*),p)) = -(x_i - p_R)^2.$$

Therefore,

$$Eu_i(y(w(\sigma^*(p)),p) - Eu_i(y(w(\sigma_i,\sigma_{-i}^*),p)) < 0 \iff$$
$$(x_i-p_R)^2 - (x_i-p_M)^2 - d^2/4 < 0 \iff$$
$$[(x_i-p_R) - (x_i-p_M)]\cdot[(x_i-p_R) + (x_i-p_M)] - d^2/4 < 0 \iff$$
$$x_i > p_M + 3d/8.$$

Similarly, if $\sigma_i(L) = 1$ for any i such that $\sigma_i^*(M) = 1$,

$$Eu_i(y(w(\sigma^*(p)),p)) - Eu_i(y(w(\sigma_i,\sigma_{-i}^*),p)) < 0 \iff x_i < p_M - 3d/8.$$

By definition of this case, $d \geq (x_{(2\mu+s-1)/2}-p_M)\cdot 8/3$. Hence, all i such that $\sigma_i^*(M) = 1$ are using best responses. Now consider any i such that $\sigma_i^*(L) = 1$. If $\sigma_i(k) = 1$, $k\in\{M,R\}$, then, because $3 \leq s < n/3$, all parties get elected and $y(w(\sigma_i,\sigma_{-i}^*),p) = p_{MR}$ with probability 1. Hence,

$$Eu_i(y(w(\sigma_i,\sigma_{-i}^*),p)) = -(x_i - p_{MR})^2, \text{ and}$$
$$Eu_i(y(w(\sigma^*(p),p)) - Eu_i(y(w(\sigma_i,\sigma_{-i}^*),p)) < 0 \iff x_i > p_M.$$

But $\sigma_i^*(L) = 1$ only if $x_i < p_M$. Hence, $\sigma_i^*(L) = 1$ is a best response for such individuals. By symmetry, $\sigma_i^*(R) = 1$ is a best response for $i > (2\mu+s+1)/2$. This completes the proof of case 6) [and case 5)]. □

<u>Proposition 3</u> Relative to the voting strategies of Proposition 2, $p^* \in P\times P\times P$ is an electoral equilibrium for any $s \in [3,n/3)$, s odd, if and only if:

(1) $p_M^* = x_\mu$,

(2) $(p_M^* - p_L^*) = (p_R^* - p_M^*) \in [8/3\cdot(x_{i^*} - x_\mu),4\cdot(x_{j^*} - x_\mu))$,

where $i^* = \mu + (s-1)/2$, $j^* \equiv \mu + \text{int}[(n-1)/4]$.

<u>Proof</u>.

(suff.) Suppose p^* satisfies (1) and (2). Then $\sigma^*(p^*)$ is described by the voter strategies of case 6) of the voting equilibrium. Hence, $v_L(\sigma^*(p^*)) = v_R(\sigma^*(p^*)) = (n-s)/2 > s = v_M(\sigma^*(p^*))$, and $y(w(\sigma^*(p^*)),p^*) \in \{p_{LM}^*,p_{MR}^*\}$. Let $|p_M^* - p_k^*| \equiv d^*$, k=L,R. Since each outcome occurs with probability 1/2, Proposition 1 yields:

$$\psi_L(p^*) = \psi_R(p^*) = 1/2\cdot(G - d^{*2}/4) - d^{*2} - d^{*2}/4 = G/2 - 11/8\cdot d^{*2},$$
$$\psi_M(p^*) = 0.$$

By the assumption on the size of G, $\psi_k(p^*) > 0$, k=L,R.

Consider $p = (p_L^*,p,p_R^*)$ and suppose that $p \in (p_L^*,p_M^*)$. Since $p_M^* = x_\mu$, $|B_k(p)| < (n+1)/2$, $\forall\ k\in\Omega$. Hence case 3a) of the voting equilibrium obtains at p, in which case $v_M(\sigma^*(p)) = s$, $v_R(\sigma^*(p)) = n-s$, and

$y(w(\sigma^*(p),p) = p_R^*$. Therefore, $\psi_M(p) = -(p-p_R^*)^2 < 0$. Now suppose $p = p_L^*$. Then case 1b) of the voting equilibrium obtains, and $v_R(\sigma^*(p) > (n+1)/2$, so that $y(w(\sigma^*(p)),p) = p_R^*$. Therefore $\psi_M(p) < 0$. If $p < p_L^*$, then again case 3a) of the voting equilibrium obtains, but here $v_M(\sigma^*(p)) = 0$ so that $\psi_M(p) = -c < 0$. A symmetric argument holds for $p > p_M^*$. Thus, p_M^* is a best response to (p_L^*,p_R^*).

Consider $p = (p,p_M^*,p_R^*)$ and suppose that $p < p_L^*$. Then case 4) of the voting equilibrium obtains with $d_L > d_R$, in which case $v_L(\sigma^*(p)) = 0$, and $\psi_L(p) = -c < 0$. Suppose $p \in (p_L^*,p_M^*)$. Then case 4) again obtains, with $d_L < d_R$, in which case $v_M(\sigma^*(p)) \geq (n+1)/2$. Therefore $y(w(\sigma^*(p)),p) = p_M^*$ and $\psi_L(p) = -(p - p_M^*)^2 < 0$. Suppose $p = p_M^*$. Then case 1b) of the voting equilibrium obtains; $Ev_L(\sigma^*(p)) = (n-3)/4 < v_R(\sigma^*(p)) = (n+3)/2$, and $\psi_L(p) \in \{-c,-(p - p_R^*)^2\} < 0$. If $p > p_M^*$, then case 2) of the voting equilibrium obtains, so that $v_M(\sigma^*(p)) \geq (n+1)/2$ and $\psi_L(p) \in \{-c,-(p - p_M^*)^2\} < 0$. Therefore, p_L^* is a best response to (p_M^*,p_R^*). By symmetry, p_R^* is a best response to (p_L^*,p_M^*).

(nec.) We prove necessity by first showing that if $p \in P{\times}P{\times}P$ is such that one of cases 1) - 5) of the voting equilibrium obtains, then p cannot be an electoral equilibrium. Let $\Sigma(\sigma^*)$ be the set of electoral equilibria relative to the voter strategies σ^*.

Let $p = (y,y,y)$. Then $\psi_k(p) = G/3$, $\forall\, k \in \Omega$. Consider party α. If $y \neq x_\mu$, choosing $p_\alpha' = x_\mu$ implies $|B_\alpha(x_\mu,y,y)| \geq (n+1)/2$, so that $\psi_\alpha(x_\mu,y,y)) = G$. If $y = x_\mu$, choose $p_\alpha' > y$. Then case 1b) of the voting equilibrium obtains, in which case $v_\alpha(\sigma^*(p)) = (n+3)/2 > Ev_k(\sigma^*(p))$, $k \neq \alpha$. Thus $\psi_\alpha(p_\alpha',y,y) = G$. Therfore, $p \notin \Sigma(\sigma^*)$.

Let $p = (p_\alpha,y,y)$ and suppose $p_\alpha > y$. Consider party β. At p, case 1b) of the voting equilibrium obtains. Therefore, $\psi_\beta(p) \in \{-c,-(y - p_\alpha)^2\}$. If $p_\alpha \neq x_\mu$, choose $p_\beta' = x_\mu$. Then either case 2) or 4) obtains. In either case, $v_\beta(p_\alpha,x_\mu,y) \geq (n+1)/2$ and $\psi_\beta(p_\alpha,x_\mu,y) = G$. If $p_\alpha = x_\mu$, choose $p_\beta' > x_\mu$ such that $(p_\beta' - x_\mu) < (x_\mu - y)/t$, $t \geq 2$. Then case 4) of the voting equilibrium obtains with $d_L > d_R$, so that $v_\gamma(p_\alpha p_\beta',y) = 0 < v_\beta(p_\alpha,p_\beta',y) < v_\alpha(p_\alpha,p_\beta',y)$. Hence for sufficiently large t,

$$\psi_\beta(p_\alpha,p_\beta',y) = -(p_\beta' - p_\alpha)^2 > \max \psi_\beta(p).$$

Therefore, $p \notin \Sigma(\sigma^*)$. By symmetry, the same is true for $p_\alpha < y$.

Let $p = (p_\alpha,p_\beta,p_\gamma)$ and suppose all parties adopt disticnt positions. Then we can write $p = (p_L,p_M,p_R)$. Let p be such that case 2) of the voting equilibrium occurs. Suppose $|B_L(p)| \geq (n+1)/2$, and consider $p' = (p_L,p_M,p)$. By the assumption of symmetric utilities, $x_\mu < (p_L+p_M)/2$. If $p_L \neq x_\mu$, choose $p = x_\mu$. Then either case 2) occurs or case 4). In both cases, $v_R(p') \geq (n+1)/2$, so that $\psi_R(p') = G > \psi_R(p)$. If $p_L = x_\mu$, choose $p < x_\mu$ so that $(x_\mu - p) = (p_M - x_\mu)/t$, $t \geq 2$. Then case 4) occurs at p' with $d_L < d_R$. Thus we have $\psi_R(p') > \max \psi_R(p)$ for sufficiently large t. Therefore, if $p \in \Sigma(\sigma^*)$ and case 2) of the voting equilibrium occurs, $|B_L(p)| < (n+1)/2$; by symmetry, $|B_R(p)| < (n+1)/2$. And if $|B_M(p)| \geq (n+1)/2$, the same arguments, mutatis mutandis, apply. Therefore, $p \notin \Sigma(\sigma^*)$

Let $p = (p_L,p_M,p_R)$ and assume hereafter that $|B_k(p)| < (n+1)/2$, for $k = L,M,R$.

Suppose that $x_\mu > p_M$. Then $v_L(\sigma^*(p)) = 0$ and $\psi_L(p) = -c$. Assume first that $(p_M+p_R)/2 \neq x_\mu$ and consider $p' = (x_\mu,p_M,p_R)$, where the party offering p_L in p now offers x_μ. Then case 4) occurs and $\psi_L(\mathbf{p}') = G > \psi_L(\mathbf{p})$. Now assume $(p_M+p_R)/2 = x_\mu$ and consider $\mathbf{p}' = (p_L, x_\mu, p_R)$, where the party offering p_M in $\mathbf{p}$ now offers x_μ. Then again case 4) occurs, and $\psi_M(\mathbf{p}') = G > \psi_M(\mathbf{p}) = -(p_M-p_R)^2$. So $\mathbf{p} \notin \Sigma(\sigma^*)$; by symmetry the same is true when $x_\mu < p_M$.

Let $\mathbf{p} = (p_L, p_M, p_R) = (p_L, x_\mu, p_R)$ and suppose $d_L \neq d_R$. Then case 4) of the voting equilibrium occurs and $v_k(\sigma^*(\mathbf{p})) = 0$ for some $k = L,R$. Let $k = L$ so that $d_L > d_R$, and $\psi_L(\mathbf{p}) = -c$. Consider $\mathbf{p}' = (p, p_M, p_R)$, $p < p_M$ and $(p_M-p) = (p_R-p_M)/t$, $t \geq 2$. Then case 4) obtains at $\mathbf{p}'$ and $d_L' < d_R' = d_R$. Hence, $v_L(\sigma^*(\mathbf{p})) \geq s$ and, for sufficiently large t, $\psi_L(\mathbf{p}') > \psi_L(\mathbf{p})$. Therefore, $\mathbf{p} \notin \Sigma(\sigma^*)$; by symmetry, the same is true when $k = R$.

Let $\mathbf{p} = (p_L, p_M, p_R) = (p_L, x_\mu, p_R)$ and suppose $d_L = d_R = d < 8/3 \cdot (x_{(2\mu+s-1)/2} - x_\mu)$. Then case 5) of the voting equilibrium occurs. Hence, $v_M(\sigma^*(\mathbf{p})) = 0$ and $\psi_M(\mathbf{p}) = -c$. Consider $\mathbf{p}' = (p_L, p_M-\epsilon, p_R)$, $\epsilon>0$. Then case 3a) of the voting equilibrium occurs with $x_\mu > p_M-\epsilon$, in which case $v_M(\sigma^*(\mathbf{p}')) = s$ and $\psi_M(\mathbf{p}') = -(d+\epsilon)^2$. Therefore,

$$\psi_M(\mathbf{p}') - \psi_M(\mathbf{p}) = c - (d+\epsilon)^2 > 0 \iff$$
$$c - d^2 > \epsilon \cdot (2d+\epsilon).$$

By assumption, $c \geq 8/3 \cdot [(x_{(2\mu+s-1)/2} - x_\mu)]^2 > d^2$, so for sufficiently small ϵ, $\psi_M(\mathbf{p}') > \psi_M(\mathbf{p})$. Therefore, $\mathbf{p} \notin \Sigma(\sigma^*)$.

Putting the previous arguments together, we have that $\mathbf{p} \in \Sigma(\sigma^*)$ implies $\sigma^*(\mathbf{p})$ is such that case 6) of the voting equilibrium occurs. To complete the argument for necessity, note that, by the symmetry of voter preferences and the distribution of ideal points,

$$[p_M = x_\mu,\ d_L = d_R = d, \text{ and } d \geq 4 \cdot (x_{j^*} - x_\mu)] \Rightarrow$$
$$|B_M(\mathbf{p})| \geq (n+1)/2. \qquad \square$$

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FIGURE 1