Electroconvective Instability with a

Stabilizing Temperature Gradient

I. Theory

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Abstract

A uniform vertical electric field produces an instability in a poorly-conducting fluid subject to a vertical temperature gradient. A gradient in conductivity resulting from the temperature gradient causes free charge to accumulate in the fluid when an electric field is applied. For the cases considered the gradient in dielectric constant can be neglected with the significant electric force that due to the free charge. The threshold conditions for the instability are predicted using linear perturbation theory. Approximations are made which allow the equations with space-varying coefficients to be solved. The analysis shows that, for fluids with short or moderate electrical

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relaxation times, the electric field causes the gravity wave propagating downward to become unstable.

I. INTRODUCTION

A fluid that is uniformly heated from above is stabilized by gravity because its density increases with depth. If the fluid is a poor electrical conductor, the conductivity can be a very strong function of temperature and a vertical electric field will produce an instability. A theory to predict the threshold conditions for this instability is developed in this paper.

Figure 1 illustrates the problem of interest. An incompressible fluid with a small electrical conductivity is placed between two horizontal highly conducting plates. Each of these plates is maintained at a constant temperature with the upper one being warmer. A vertical d-c electric field is applied to produce convection. In the absence of an electric field the fluid is stable since it is most dense on the bottom.

The electrical force on a fluid is[1]

$$\overline{f}_e = \rho_f \overline{E} - \frac{1}{2} \overline{E}\cdot\overline{E}\nabla\varepsilon + \frac{1}{2}\nabla \left(\rho \frac{\partial \varepsilon}{\partial \rho} \overline{E}\cdot\overline{E}\right) \tag{1}$$

The last term in the force expression is the gradient of a scaler and thus it has no effect on an incompressible fluid. The first term involves the free charge, ρ_f, which is non-zero only when the electrical

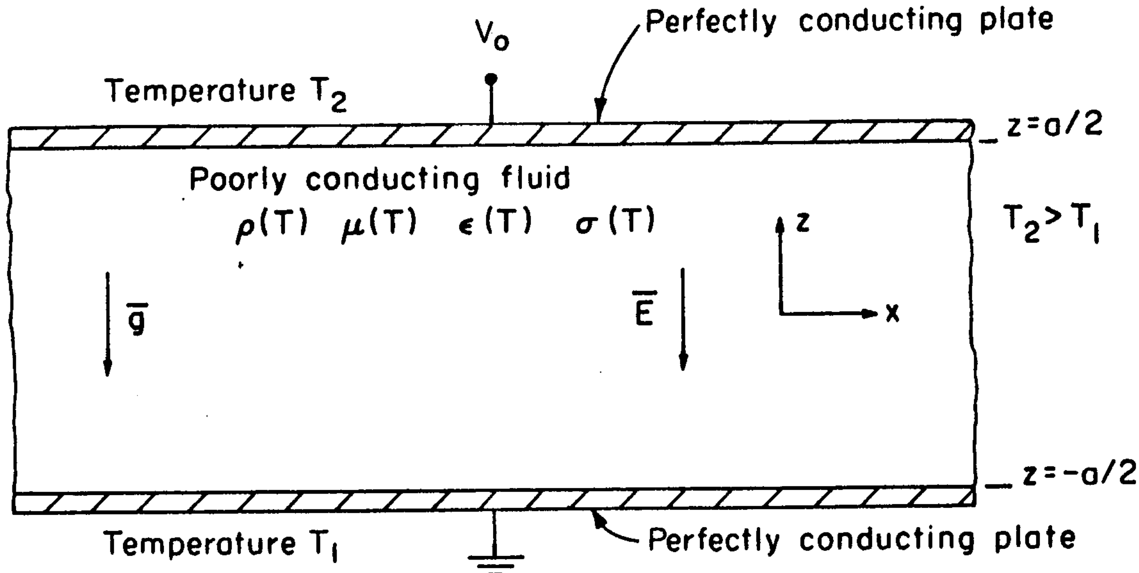


Fig. 1 Cross-section view of fluid with vertical temperature gradient and electric field

properties of the fluid are space-varying. The second term depends on the gradient in dielectric constant. Therefore the electrical forces can have no effect in the bulk of the fluid unless there is an inhomogeneity in the fluid.

The electrical properties of the fluid are functions of temperature and therefore vary in the vertical direction because of the temperature gradient. The resulting electrical force densities are essentially determined by the gradient in conductivity since it is a much stronger function of temperature than the dielectric constant. This is shown in reference 2.

An inviscid fluid with a stabilizing temperature gradient supports a class of waves known as internal gravity waves.[3] These waves are damped by viscosity. It will be shown later that the electric field acting through the gradient in conductivity decreases the damping of the gravity waves. If the electric field exceeds some threshold value, the waves become unstable and convection results.

When a strong, uniform electric field is imposed on a poorly-conducting fluid, convection is observed even in the absence of a temperature gradient. In the 1930's Avsec and Luntz[4] observed steady cellular motion in light oils in the presence of an electric field but with no temperature gradient. Observations of bulk convections were made by Ostromov[5] in a variety of slightly-conducting fluids. He used a temperature gradient and a Schlieren apparatus to detect the motions, but the purpose of the temperature gradient was only to detect the motions, not to produce them. The causes of this electroconvection are not understood, but these effects can be neglected in this work because in the class of fluids of interest the voltages needed for instability with a temperature gradient are much less than those needed without a temperature gradient.

The hydrodynamic problem of a fluid heated from below (Bénard problem) is one where the viscosity, boundaries and thermal conductivity team up to retard the free convection and produce a threshold for instability.[6] The Bénard problem appears similar to the electrohydrodynamic situation analyzed here.

II. CONDITION FOR THE ONSET OF CONVECTION

A. Equations

The equations needed to solve for the motions are given below. Conservation of mass may be expressed as

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \bar{v}) = 0 \tag{2}$$

The fluids considered are incompressible, i.e., the density is independent of the pressure. However ρ is still a variable since it depends on the temperature.

The momentum equation for the problem is

$$\rho \frac{D\bar{v}}{Dt} = - \nabla p + \rho\bar{g} + \bar{f}_e + \nabla \cdot \bar{\bar{T}}^{(v)} \tag{3}$$

where $\frac{D}{Dt} = \frac{\partial}{\partial t} + \overline{v}\cdot\nabla$, $\overline{f}_e$ is the force of electrical origin given by equation (1), and $\overline{\overline{T}}^{(v)}$ is the viscous stress tensor. If the pressure is replaced by

$$p_1 = p - \frac{1}{2}\rho\frac{\partial\varepsilon}{\partial\rho}\overline{E}\cdot\overline{E} \tag{4}$$

the electrostriction term disappears from the equation. The coefficients of the viscous stress tensor for an incompressible fluid are

$$T_{ij}^{(v)} = \mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \tag{5}$$

The energy equation, neglecting the work of compression, is

$$\rho c_p \frac{DT}{Dt} = \nabla\cdot(k\nabla T) + P_v + P_e \tag{6}$$

where P_v is the viscous power dissipated and P_e the electrical loss. Because we are dealing only with poorly conducting fluids, the magnetic fields are negligible and the electrical equations become[8]

$$\nabla \times \overline{E} = 0 \tag{7}$$

$$\nabla\cdot\varepsilon\overline{E} = \rho_f \tag{8}$$

$$\frac{\partial\rho_f}{\partial t} + \nabla\cdot\overline{J} = 0 \tag{9}$$

An ohmic conduction law is assumed for the current throughout this paper. Although not always obeyed by fluids, Ohm's law is valid for many fluids and has been used to successfully interpret a variety of experiments.[9-12] If the fluid is in motion an additional term is added to the current to account for convection of charge.

$$\bar{J} = \sigma \bar{E} + \rho_f \bar{v} \tag{10}$$

Finally, the fluid properties are assumed to be functions of temperature alone.

$$\rho = \rho(T) \qquad \sigma = \sigma(T)$$
$$\varepsilon = \varepsilon(T) \qquad \mu = \mu(T) \tag{11}$$

B. Equilibrium Point

To decide on the stability of a system, one finds the equilibrium point, considers small perturbations from the equilibrium, and then determines whether the perturbations grow or decay in time. In the equilibrium state the fluid is stationary with a steady-state temperature distribution.

At equilibrium

$$\bar{v} = 0 \tag{12}$$

and

$$\nabla \cdot (k \nabla T) + P_e = 0 \tag{13}$$

Thermal conductivities of liquids are not very strong functions of temperature at moderate temperature.[13] Therefore, the thermal conductivity may, as a first approximation, be made a constant and equation (13) becomes

$$k \nabla^2 T + P_e = 0 \qquad (14)$$

For poorly conducting fluids ($\sigma \leq 10^{-9}$ mho/m) P_e is negligible for field strengths of the order of 10^5 volts/m and the temperature distribution is approximately linear.

As a consequence of the linear temperature distribution, the fluid properties are assumed to vary as follows

$$\rho(z) = \rho_o + \rho_1 \, z/a \quad ; \quad (\rho_1 < 0) \qquad (15)$$

$$\mu(z) = \mu + \mu_1 \, z/a \quad ; \quad (\mu_1 < 0) \qquad (16)$$

$$\sigma(z) = \sigma_o \, (1 + \alpha \, z/a + \beta \, z^2/a^2) \; : \; (\alpha, \beta > 0) \qquad (17)$$

It can be shown that gradients in dielectric constant may be neglected since the conductivity is a much stronger function of temperature than the dielectric constant.[2]

The equilibrium electric field is vertical and is given by

$$E_z(z) = \frac{E_o}{1 + \alpha z/a + \beta z^2/a^2} . \tag{18}$$

where E_o is determined by the applied voltage. The forces are curl-free and are therefore balanced by the pressure.

C. Linear Theory

To test for stability, the fluid is now slightly displaced from equilibrium. The perturbation variables are the velocity v, the electric field $\tilde{e}$, pressure p', density ρ', conductivity σ', viscosity μ', and temperature T'. Since three-dimensional wave motion may be obtained by the superposition of two-dimensional waves, the perturbations are assumed to be two-dimensional and of the form:

$$\rho = \mathrm{Re}[\hat{\rho}'(z)\, e^{j(\omega t - kx)}] \tag{19}$$

(The circumflex is used to denote complex amplitudes.) Since the perturbations are small, the equations will be linearized with respect to the perturbation variables.

Considering the heat conduction equation (6), we find that P_v is negligible since it is proportional to the velocity squared. Since we have already neglected P_e, the equation reduces to the diffusion equation. For a poorly conducting fluid with a characteristic length of 1 cm, the time constant for diffusion of heat is about 20 minutes.

Because the gravity waves have periods of less than 1 minute heat diffusion will be neglected and equation (6) reduces to

$$\frac{DT}{Dt} = 0 \qquad (20)$$

Equation (20) says that each fluid particle retains it original temperature while it moves. Because the fluid properties are functions of temperature only, they also do not change for each particle as the fluid moves. Thus:

$$\frac{D(\mu + \mu')}{Dt} = 0 \qquad (21)$$

$$\frac{D(\sigma + \sigma')}{Dt} = 0 \qquad (22)$$

$$\frac{D(\rho + \rho')}{Dt} = 0 \qquad (23)$$

Equations (2) and (22) yield

$$\nabla \cdot \bar{v} = 0 \qquad (24)$$

We may now assume a velocity stream function $\bar{\Psi}$ defined by

$$\bar{v} = - \nabla \times \bar{\Psi} \qquad (25)$$

For the two-dimensional motion, only the y-component of $\bar{\Psi}$ is non-zero. This component will be called ψ .

Since the electric field is curl-free, we may introduce a potential function

$$\bar{e} = - \nabla \phi \qquad (26)$$

The curl of the momentum equation (3) combined with equation (22) and linearized results in

$$j\omega\rho(D^2 - k^2)\hat{\psi} + j\omega(D\rho)(D\hat{\psi}) = j\frac{k^2}{\omega} g(D\rho)\hat{\psi} + jk\varepsilon(D^2E)\hat{\phi} - jk\varepsilon E(D^2 - k^2)\hat{\phi}$$

$$+ \mu(D^2 - k^2)^2\hat{\psi} + 2(D\mu)(D^2 - k^2)D\hat{\psi} + (D^2\mu)(D^2 + k^2)\hat{\psi} \qquad (27)$$

The linearized electrical equations (7-10,23) combined to form one equation yield

$$jk\varepsilon(D^2E)\hat{\psi} - (\sigma + j\omega\varepsilon)(D^2 - k^2)\hat{\phi} - \frac{k}{\omega} D[(D\sigma)E]\ \hat{\psi}$$

$$- \frac{k}{\omega} E(D\sigma)(D\hat{\psi}) - D\sigma D\hat{\phi} = 0 \qquad (28)$$

These equations, (27) and (28) are linear but have space-varying coefficients. In the next section approximations are developed which permit solutions to be found.

D. Electrohydrodynamic "Boussinesq Approximations"

In ordinary hydrodynamics, the equations have space-varying coefficients whenever the fluid has a temperature gradient. An approximation given by Boussinesq[14] and applied to the Bènard problem by Rayleigh[15] yields constant coefficient equations by assuming that the density is constant except in the gravitational term where it varies linearly with height. Also, all the other fluid properties are assumed to be constant.

From equation (27) we see that neglecting the density gradient in the inertial term requires that

$$\left(\frac{D\rho}{\rho}\right)\left(\frac{D\hat{\psi}}{(D^2 - k^2)\hat{\psi}}\right) << 1 \tag{29}$$

But

$$\left(\frac{D\rho}{\rho}\right)\left(\frac{D\hat{\psi}}{(D^2 - k^2)\hat{\psi}}\right) \sim a\left(\frac{D\rho}{\rho}\right) \tag{30}$$

where $\sim$ means the two sides are of the same order of magnitude and a is the tank height. In order for the ratio of equation (30) to be equal to .1 for a liquid, a temperature difference across the tank of greater than 100° C would be necessary. If we restrict ourselves to smaller temperature gradients, we can neglect the density gradients in the inertial term.

Applying the Boussinesq approximation reduces all the mechanical terms in the equations to constant coefficient form. The electrical conductivity cannot be made constant because it is a very strong function of temperature. In addition, the instability under investigation requires a gradient in conductivity. An approximation that could be made without neglecting any important effects, is to assume that the conductivity is constant in any term where it appears without being differentiated. In terms where a derivative of the conductivity appears, that derivative will be approximated as a constant. This process will be repeated for the electric field and the viscosity, even though for the usual Boussinesq approximation $\nabla \mu = 0$. The approximations were used successfully in analyzing a steady flow problem consisting of a traveling-wave electric field pumping a fluid with a temperature gradient.[9] We can see that these approximations are in the same spirit as those of Boussinesq by considering equation (27) without the electric field terms. The density multiplies the velocity in the inertial term so ρ is taken to be constant there. However, in the gravity term, only $D\rho$ appears so it is considered to be constant there yielding the same results as Boussinesq obtained.

E. Dispersion Relation

The Boussinesq approximations require the following substitutions into equations (27) and (28)

$$\begin{aligned} &\rho = \rho_o \quad ; \quad D\rho = \rho_1/a \\ &\mu = \mu \quad ; \quad D\mu = \mu_1/a \quad ; \quad D^2\mu = 0 \\ &\sigma = \sigma_o \quad ; \quad D\sigma = \sigma_o \frac{\alpha}{a} \quad ; \quad D^2\sigma = \sigma_o \frac{2\beta}{a^2} \\ &E = E_o \quad ; \quad DE = -E_o \frac{\alpha}{a} \quad ; \quad D^2E = 2E_o \left(\frac{\alpha^2 - \beta}{a^2}\right) \end{aligned} \tag{31}$$

If equations (27) and (28) are reduced to constant coefficient form, combined and non-dimensionalized the result is

$$\begin{aligned} &\{[(1 + j\omega^* R)(D^{*2} - k^{*2}) + \alpha D^*] \; [M\omega^{*2}(D^{*2} - k^{*2}) \\ &+ k^{*2}M + j\omega^*(D^{*2} - k^{*2})^2 - j\omega^* A(D^{*2} - k^{*2})D^*] \\ &- k^{*2}H[D^* - (\frac{\alpha^2 - 2\beta}{\alpha}) - j\omega^* 2R(\frac{\alpha^2 - \beta}{\alpha})][D^{*2} - k^{*2} - 2(\alpha^2 - \beta)]\}\hat{\psi} = 0 \end{aligned} \tag{32}$$

where we have introduced the following quantities (an asterisk indicates a dimensionless number)

$$k^* = ka \qquad M = \frac{a\sqrt{-\rho_1 \rho_o g a}}{\mu} \tag{33}$$

$$D^* = Da$$

The relaxation time $\tau_o = \frac{\varepsilon}{\sigma_o}$

$$H = \frac{\varepsilon E^2 \alpha}{\mu\sqrt{\frac{-\rho_1 g}{\rho_o a}}}$$

$$\tau_1 = \sqrt{\frac{-\rho_o a}{\rho_1 g}} \quad , \quad (\rho_1 < 0)$$

$$A = -\frac{2\mu_1}{\mu}$$

$$R = \frac{\tau_o}{\tau_1}$$

The problem depends on six parameters R, H, M, A, α, and β. The length scale is a and the time scale τ_1.

Equation (32) is a sixth order ordinary differential equation and therefore requires six boundary conditions to specify a solution. These conditions are: the normal and tangential velocities and the tangential electric field must vanish at each electrode. An iterative numerical method must be used to find the solutions. The difficulties of this approach force one to look for an approximate solution. One possible approximation is to assume that the boundaries allow an integer number of half wavelengths in the vertical direction. In the absence of an electric field and viscosity this solution satisfies the boundary conditions. The real justification for this approximation is that it gives results that agree qualitatively and quantitatively with experiments.[16]

We now have

$$\hat{\psi} = \hat{\psi}_o e^{jqz}$$
$$q = \frac{n\pi}{a} \tag{34}$$

The resulting dispersion relation in non-dimensional form is

$$\begin{aligned} & j\omega^3 MR(q^2+k^2)^2 + \omega^2[(q^2+k^2)^2(q^2+k^2+jqA)R + M(q^2+k^2-j\alpha q) \\ & (q^2+k^2)] - j\omega[R(q^2+k^2)\,k^2M + (q^2+k^2+jqA)(q^2+k^2)(q^2+k^2-j\alpha q) \\ & + k^2H2R\left(\frac{\alpha^2-\beta}{\alpha}\right)\left(q^2+k^2+2(\alpha^2-\beta)\right)] - k^2M(q^2+k^2-j\alpha q) \\ & + k^2H[q^2+k^2+2(\alpha^2-\beta)]\left(jq - \frac{\alpha^2-2\beta}{\alpha}\right) = 0 \end{aligned} \tag{35}$$

To find the conditions for incipient instability we range over all allowed values of q and k and find the smallest voltage (or H) which has a solution for ω with a negative imaginary part. Since we are allowing only real values of q and k, and since at the threshold ω is purely real, the threshold conditions may be calculated directly.

This is done by dividing equation (35) into real and imaginary parts and then solving the two resulting equations for ω and H. H can then be minimized over the allowed values of q and k.

F. Small Temperature Gradient Approximations

If the temperature gradient is very small, further approximations may be made. These approximations enable us to explain physically the basic mechanisms that cause the instability. Also, it is hoped that a theory which is valid for small temperature gradients may still be a good approximation for moderate ones.

For small temperature gradients $\left|\frac{\mu_1}{\mu}\right| << 1$. The ratio of the viscous terms in the mechanical equation (27), using the substitution (30), is

$$\left|\frac{\frac{2\mu_1}{a}(D^2 - k^2)D\hat{\psi}}{\mu(D^2 - k^2)^2\,\hat{\psi}}\right| \sim \left|\frac{\frac{2\mu_1}{a}}{\mu/a^2}\right| = \left|\frac{2\mu_1}{\mu}\right| << 1 \qquad (36)$$

The electrical force terms in the same equation have the ratio

$$\left|\frac{jk\varepsilon 2E_o\left(\frac{\alpha^2 - \beta}{a^2}\right)\hat{\phi}}{jk\varepsilon E_o(D^2 - k^2)\hat{\phi}}\right| \sim |\alpha^2 - \beta| \qquad (37)$$

But for small temperature gradients α^2 and β are much less than one. To linear terms, the electrical force consists of the perturbation charge times the equilibrium electric field plus the equilibrium free charge times the perturbation electric field. We have just neglected the latter.

To find similar approximations in the electric field equation (28) we use the same reasoning and get

$$\left| \frac{\sigma_o \frac{\alpha}{a} D\hat{\phi}}{\sigma_o (D^2 - k^2)\hat{\phi}} \right| \sim \alpha << 1 \tag{38}$$

$$\left| \frac{\frac{\alpha^2 - 2\beta}{a^2} \hat{\psi}}{\frac{\alpha}{a} D \hat{\psi}} \right| \sim \frac{\alpha^2 - 2\beta}{\alpha} << 1 \tag{39}$$

The small termperature gradient approximation reduces equation (32) to

$$M\omega^{*2}(D^{*2} - k^{*2})\hat{\psi}^* + k^{*2}M\hat{\psi}^* + \frac{k^{*2} H}{1 + j\omega^* R}\left[j\omega^* R2(\frac{\alpha^2 - \beta}{\alpha}) - D^*\right]\hat{\psi}^*$$
$$+ j\omega^*(D^{*2} - k^{*2})^2\hat{\psi}^* = 0 \tag{40}$$

The above equation is only fourth order so two boundary conditions have been eliminated. Since the force due to the perturbation electric field has been neglected, the boundary conditions on the tangential electric field must have been eliminated. The number of non-dimensional parameters is now also four, M, H, R and $(\alpha^2 - \beta)/\alpha$.

Satisfying the boundary conditions presents the same problems as before, so again sinusoidal variation in the z-direction is assumed.

The dispersion relation is, with the asterisks omitted:

$$- j\omega^3 MR(q^2+ k^2) - \omega^2 [R(q^2+ k^2)^2 + M(q^2+ k^2)]$$
$$+ j\omega[MRk^2 + k^2 H2R(\frac{\alpha^2 - \beta}{\alpha}) + (q^2+ k^2)^2] + k^2 M - jqk^2 H = 0 \qquad (41)$$

With q and k real the threshold conditions (ω real) may be found by separating equations (41) into real and imaginary parts. This yields

$$\omega^2 = \frac{k^2}{(q^2+ k^2) + \frac{R}{M}(q^2+ k^2)^2} \qquad (42)$$

$$H = \frac{R^2 k^2 (q^2+ k^2) + (q^2+ k^2)^2 + \frac{R}{M}(q^2+ k^2)^3}{k^2 \left[\frac{q}{\omega} - 2R(\frac{\alpha^2 - \beta}{\alpha})\right]\left[1 + \frac{R}{M}(q^2+ k^2)\right]} \qquad (43)$$

The voltage for instability is found by using equation (42) to calculate ω and then calculating H from equation (43). H is then minimized over allowable values of q and k to give the threshold voltage.

III. SPECIAL CASES

A. Instantaneous Charge Relaxation

If charge relaxation occurs instantaneously, R = 0 and equations

(42) and (43) become

$$\omega^2 = \frac{k^2}{q^2 + k^2} \tag{44}$$

$$H = \frac{\omega(q^2 + k^2)^2}{qk^2} \tag{45}$$

Eliminating ω gives

$$H = \frac{(q^2 + k^2)^{3/2}}{qk} \tag{46}$$

The stability of the system now depends on only one number, H. To find the voltage at which instability first occurs, we now minimize H over the allowed values of q and k. If $q = n\pi$, H has a minimum at $k = n\pi/\sqrt{2}$. The minimum H is then

$$H = \frac{n\pi \; 3\sqrt{3}}{2} \tag{47}$$

If $n = 1$, the fluid is unstable for $H > 8.15$. Recalling the definition of H, equation (33), we see that the instability is caused by the electrical forces acting through the gradient in conductivity overcoming the viscous damping of the internal gravity waves.

B. Inviscid

The small temperature gradient approximation applied to an inviscid fluid yields an equation which requires only two boundary conditions. Thus the solutions satisfying the boundary conditions may be found analytically. If the viscosity is set to zero in equation (40), $M \to \infty$ and $H \to \infty$ but H/M remains finite. The inviscid limit for that equation is

$$\omega^2(D^2 - k^2)\hat{\psi} + k^2\hat{\psi} + \frac{k^2 F}{1 + j\omega R}\left[j\omega R2\left(\frac{\alpha^2 - \beta}{\alpha}\right) - D\right]\hat{\psi} = 0 \quad (48)$$

where

$$F = (H/M) = \frac{\varepsilon E_o^2 \alpha}{a(-\rho_1 g)} \quad (49)$$

The remaining boundary conditions are that the normal velocities must vanish at each electrode giving

$$\hat{v}_z = jk\hat{\psi} = 0 \quad \text{at } z = \pm \frac{1}{2} \quad (50)$$

The solutions are

$$\hat{\psi} = A_1 e^{p_1 z} + A_2 e^{p_2 z} \quad (51)$$

where p_1 and p_2 are the solutions to

$$p^2\omega^2 - p\frac{k^2 F}{1 + j\omega R} + k^2\left(1 - \omega^2 + \frac{j\omega 2RF\left(\frac{\alpha^2 - \beta}{\alpha}\right)}{1 + j\omega R}\right) = 0 \quad (52)$$

Equations (50), (51) and (52) result in

$$4R^2(k^2 + n^2\pi^2)\omega^6 - 8(k^2 + n^2\pi^2)Rj\omega^5 - 4[k^2(1 + R^2) + n^2\pi^2 + R^2k^2 2F\left(\frac{\alpha^2 - \beta}{\alpha}\right)]\omega^4 + 8k^2R[1 + F\left(\frac{\alpha^2 - \beta}{\alpha}\right)]j\omega^3 + 4k^2\omega^2 - k^4F^2 = 0 \quad (53)$$

Two limits to equation (53) which give simple results are the limit of zero conductivity and the limit of zero relaxation time.

1) zero conductivity ($R \rightarrow \infty$)

The solution to equation (53) is now

$$\omega^2 = \frac{k^2\left(1 + 2F\left(\frac{\alpha^2 - \beta}{\alpha}\right)\right)}{q^2 + k^2} \tag{54}$$

For this case, the instability is a static one and the necessary and sufficient condition for stability is

$$1 + 2F\left(\frac{\alpha^2 - \beta}{\alpha}\right) > 0 \tag{55}$$

This is equivalent to

$$gD\rho - \epsilon ED^2E < 0 \tag{56}$$

This solution is identical to that which is obtained using sinusoidal variation in the vertical direction. The reason for this is that disturbances of all wavelengths go unstable at the same time and boundaries cannot affect the threshold.

2) Zero relaxation time ($R \rightarrow 0$)

The solutions to equation (53) with $R = 0$ are

$$\omega^2 = \frac{k^2 \pm \sqrt{k^4 - k^4(k^2 + n^2\pi^2)F^2}}{2(k^2 + n^2\pi^2)} \tag{57}$$

The system is stable if and only if ω^2 is real and positive, which occurs when

$$F < \frac{1}{\sqrt{k^2 + n^2\pi^2}} \tag{58}$$

For any given voltage, the fluid is unstable for short enough wavelengths. However, the viscosity, which was neglected, has its greatest effect at short wavelengths. This indicates that the inviscid limit is unrealistic unless the conductivity is zero.

IV. CHARACTERISTICS OF THE INCIPIENT ELECTROCONVECTION

A. Properties of the Solutions to the Dispersion Relation

In the absence of an electric field, an inviscid stratified fluid supports internal gravity waves[3] with real frequencies. The addition of viscosity adds a damping to the gravity waves and also decreases the real part of the frequency. The effect of an electric field can be most easily seen for the case of instantaneous charge relaxation. The solutions to the dispersion relation (40) are, for this case

$$\omega = \frac{j(n^2\pi^2 + k^2)}{2M} \pm \sqrt{\frac{k^2}{n^2\pi^2 + k^2}\left(1 - jn\pi\frac{H}{M}\right) - \frac{(n^2\pi^2 + k^2)^2}{4M^2}} \tag{59}$$

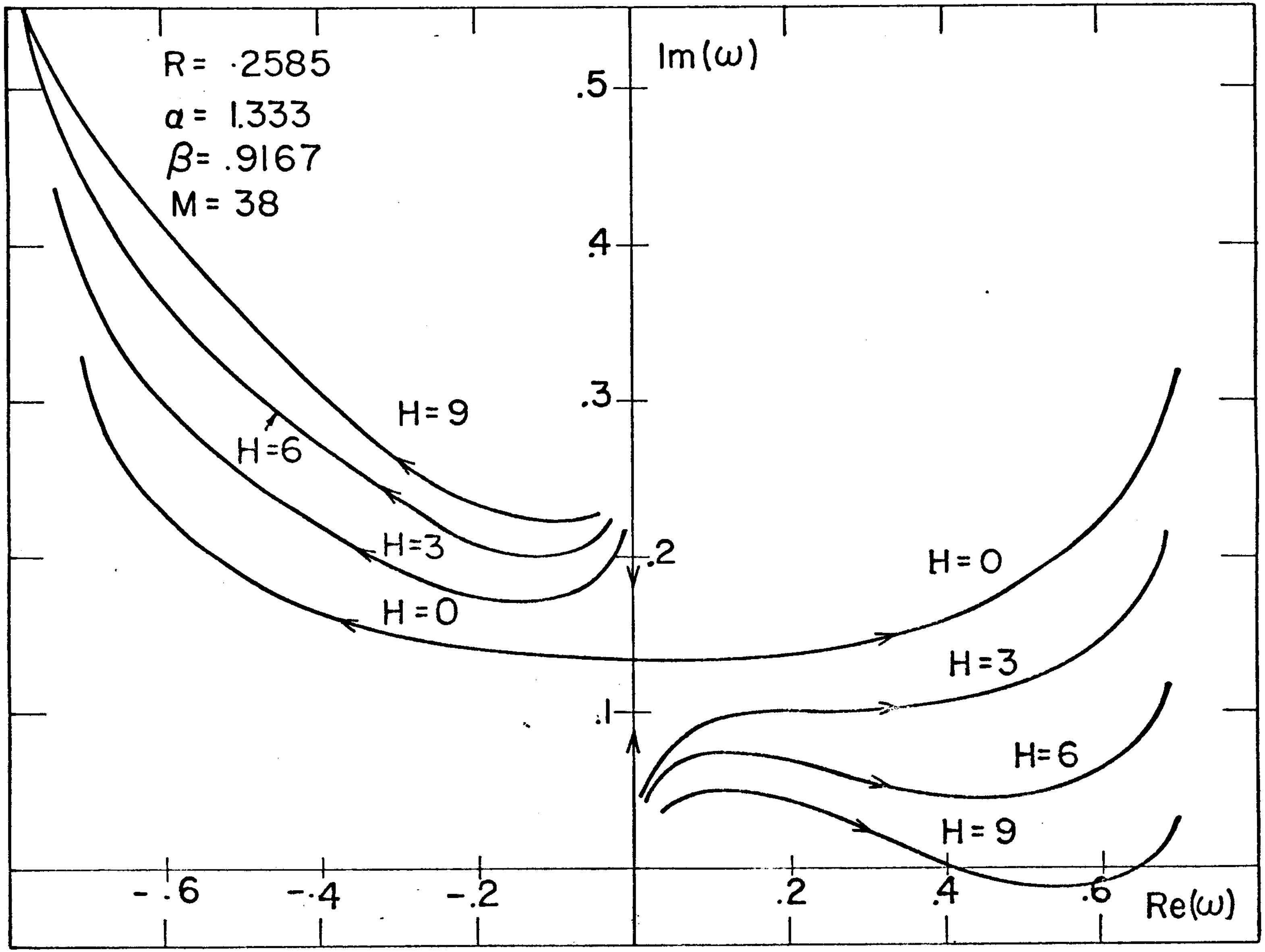
For n positive, the electric field reduces the damping for the root with $\mathrm{Re}(\omega) > 0$ and increases it for the other root.

Figures 2-6 are plots of solutions to the dispersion relation for different fluid properties. The properties which are varied are the electrical relaxation time (R) and the viscosity. All the graphs are plots of complex ω for real k with $q = \pi$. Each line represents a particular voltage. The arrowheads indicate the direction of increasing k.

The parameters chosen for Figure 2 are those of corn oil in a one inch high tank with a temperature range of 25-45° C. The H = 0 curve represents the zero voltage case. Increasing the voltage causes one root to go unstable at $H \approx 8$, $k \approx .6\pi$, and increases the damping for the other gravity wave root. There is a third root, basically due to charge relaxation, which does not appear in Figure 2 since it is too heavily damped.

Figure 3 is plotted for the same fluid as Figure 2 except that the relaxation time is increased by a factor of 10. In this case the charge relaxation root interacts strongly with one of the gravity wave roots. The threshold voltage has increased by a factor of four from that of Figure 2. Increasing the relaxation time by another factor of 10 produces Figure 4. In this case it is the charge relaxation root which goes unstable and the threshold voltage has again been increased by a factor of 4.

If the viscosity is increased by a factor of 10 from Figure 2 with all other parameters unchanged, Figure 5 shows the solutions to the dispersion relation. When there is no electric field, the perturbations decay in time with no sinusoidal component. The threshold H



Mapping of the Real k – Axis into the ω – Plane for Corn Oil

Fig. 2 Solutions to the dispersion relation in the complex ω– plane for corn oil in a tank of height one inch with a temperature range of 25° – 45° C. The vertical wavenumber is π and the horizontal wavenumber increases from 0 to π along each of the lines in the direction indicated by the arrowheads. Instability occurs when $Im(\omega) < 0$.

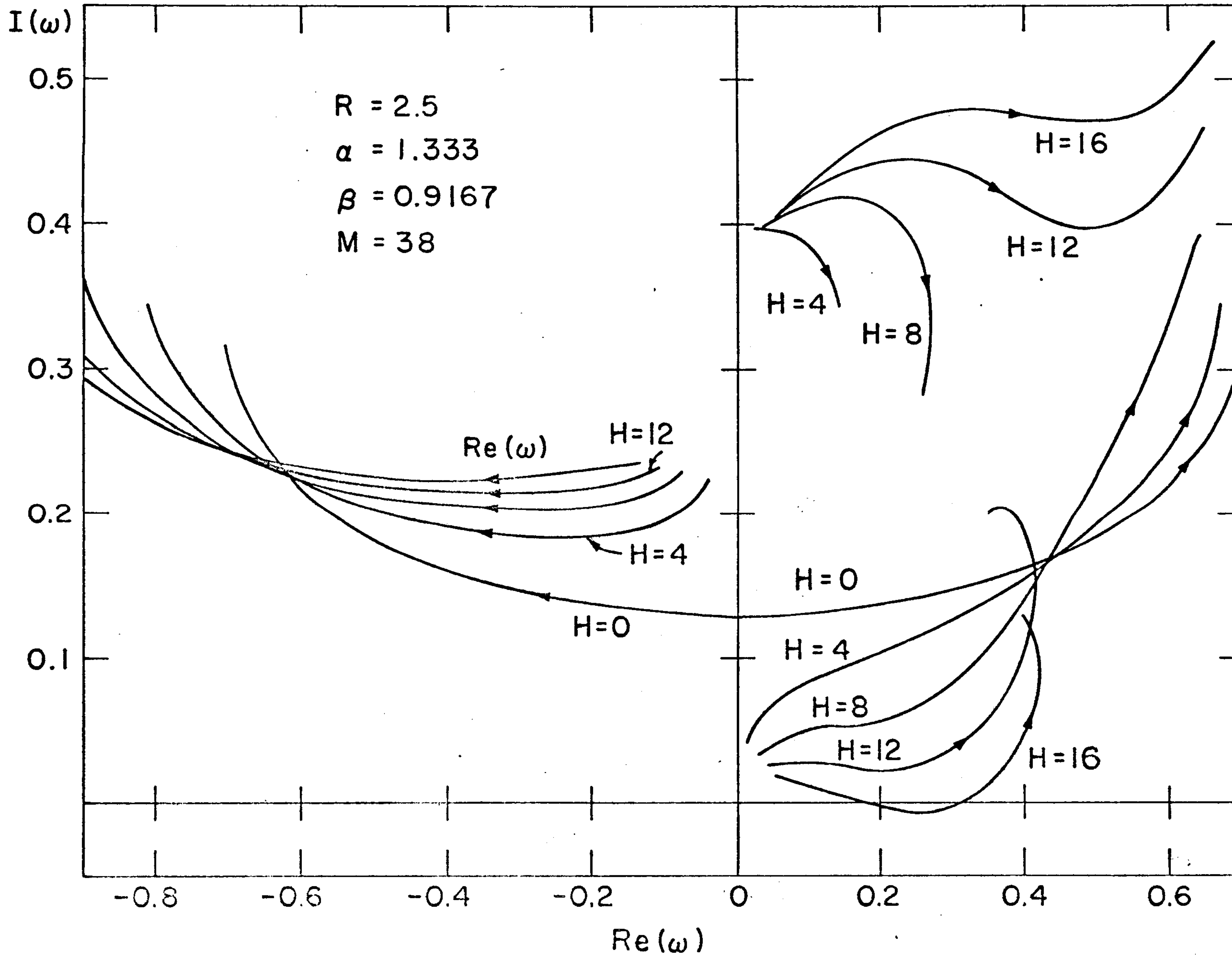


Fig. 3 Same as Fig. 2 except that the electrical relaxation time has been increased by a factor of 10.

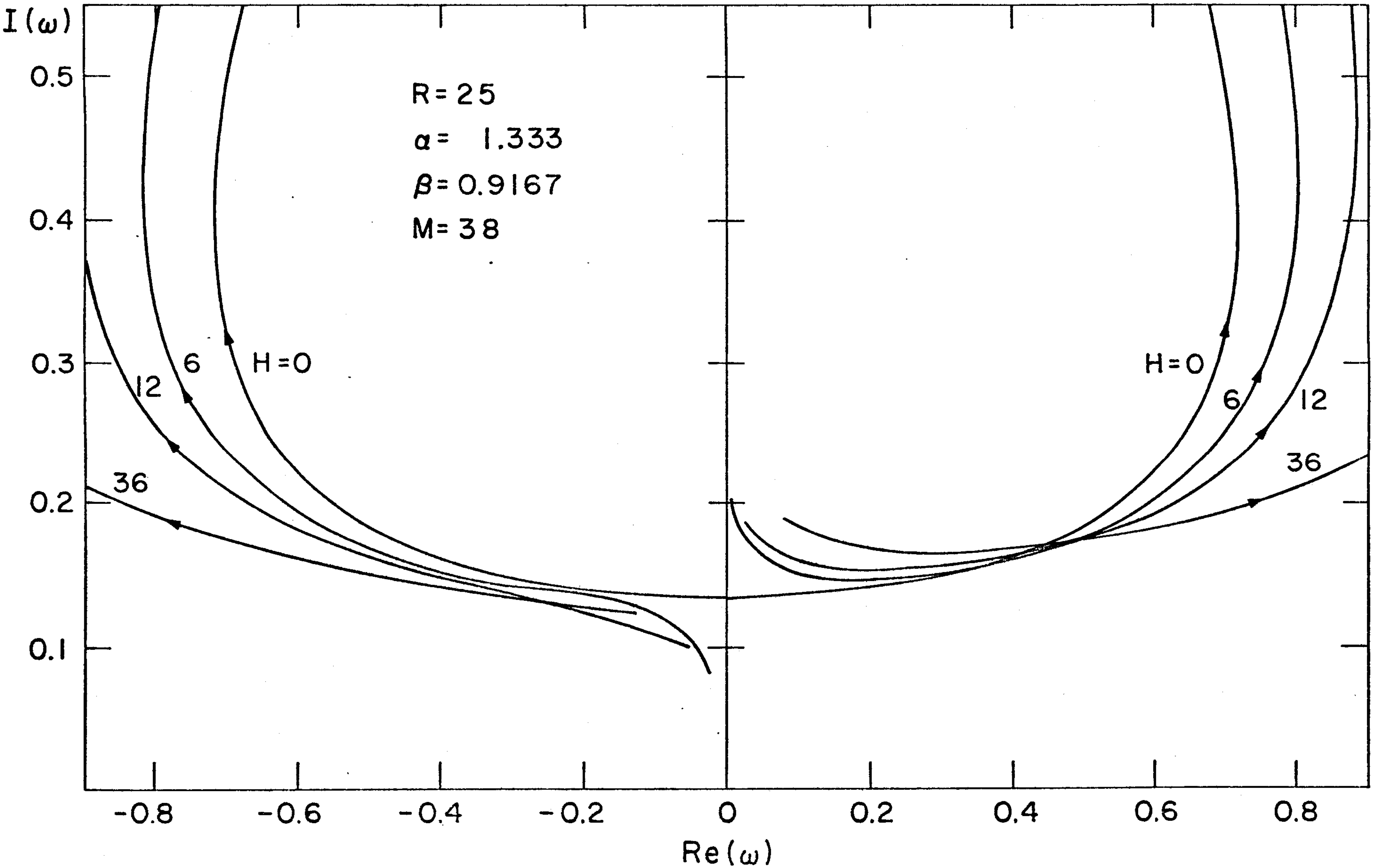


Fig. 4 Same as Fig. 2 except that the electrical relaxation time has been increased by a factor of 100. a) Gravity wave roots

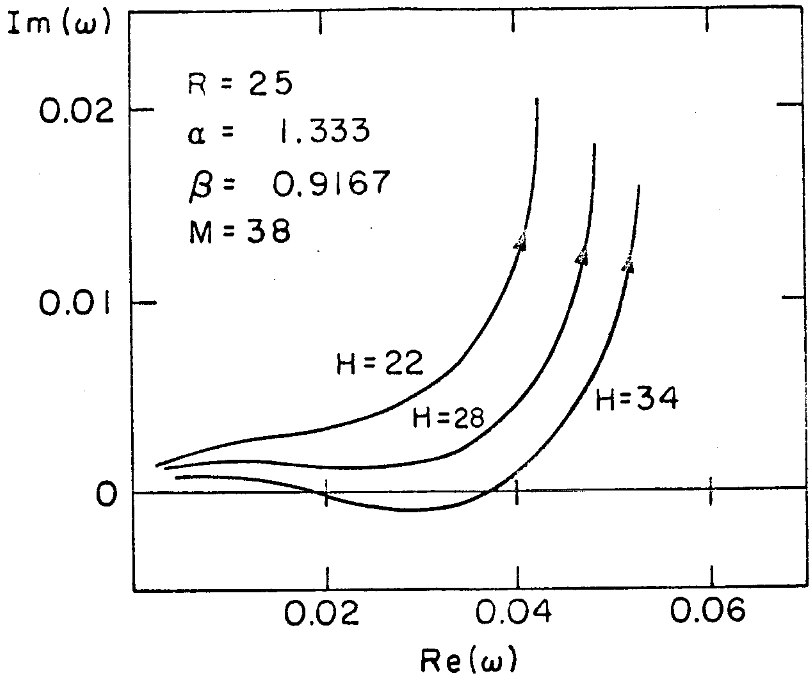


Fig. 4 b) charge relaxation root

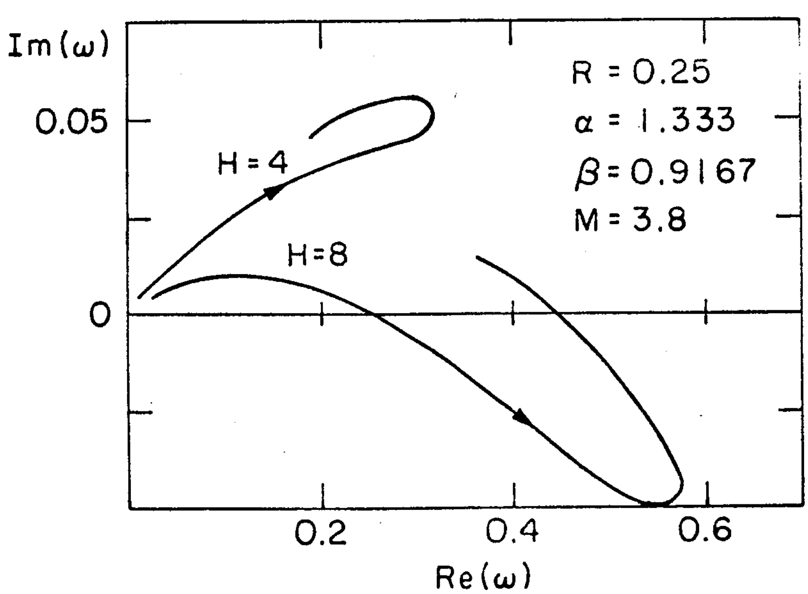


Fig. 5 Same as Fig. 2, but with the viscosity increased by a factor of 10

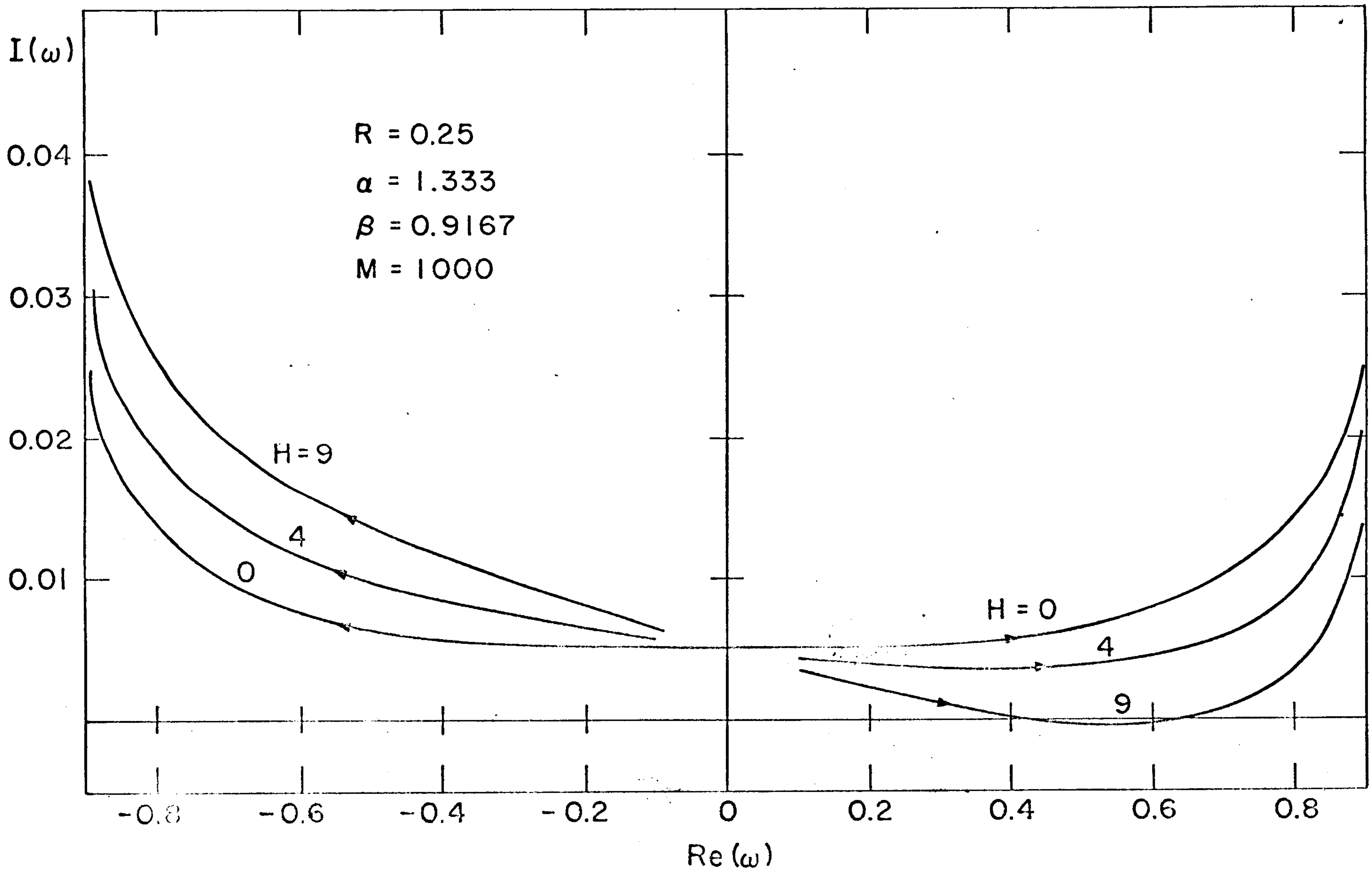


Fig. 6 Same as Fig. 2 with the viscosity decreased by a factor of 25

remains about the same with the voltage increasing because of the increased viscosity. The other two roots are not shown since they are heavily damped. Reducing the viscosity by a factor of 25 from Figure 2 does not change the features of the solutions much. Figure 6, where the low viscosity solutions are shown, differs from Figure 2 mainly in the magnitude of the damping of the gravity waves.

B. Effect of Electrical Conduction and Charge Convection on the Threshold

The perturbation forces are produced by free charges which occur either because of convection of the equilibrium free charge or from electrical conduction or from both. In this section the effects of these two phenomena will be compared. First we will consider each separately.

Case 1 If the perturbation free charge results solely from electrical conduction then relaxation is instantaneous. For this case the stability criterion is given by equation (47) and is

$$\frac{\varepsilon E_o^2 \alpha}{\mu\sqrt{\frac{\rho_1 g}{\rho_o a}}} < 8.15 \qquad (60)$$

Case 2 If only charge convection is present, the electrical conductivity must be zero and the stability criterion is equation (55) which is

$$g\rho_{1/a} - \epsilon 2E_o^2 \left(\frac{\alpha^2 - \beta}{\alpha}\right) < 0 \tag{61}$$

If the temperature gradient is uniform, $\alpha^2 - \beta$ is generally a positive number and charge convection can never produce an instability. Rather, it tends to stabilize the fluid.

In Section II both charge conduction and convection were present and the threshold H was given by equation (43). In this expression charge conduction and convection effects appear in the first bracket in the denominator. The conduction produces the q/ω while convection produces the $-2R(\alpha^2 - \beta)/\alpha$. Since H must be positive to be physically realizable, it may be easily seen that charge convection raises the threshold voltage.

C. Properties of the Unstable Mode

The properties of the instability will be shown by examining the part played by the different forces. For this we use the small temperature gradient approximation. The curl of the force equation is

$$\rho_o \frac{\partial \overline{\Omega}}{\partial t} = \nabla\rho \times \overline{g} + \nabla\rho_f \times \overline{E} + \mu\nabla^2 \overline{\Omega} \tag{62}$$

The stream function is

$$\psi = \psi_o \cos(\omega t - kx + qz) \tag{63}$$

and the resulting vorticity is

$$\bar{\Omega} = -\bar{i}_y\ (q^2+k^2)\psi_o \cos(\omega t - kx + qz) \tag{64}$$

Assuming instantaneous relaxation the y- component of equation (62) is

$$\rho_o\omega(q^2+k^2)\psi_o \sin(\omega t - kx + qz) = -\frac{k^2\rho_1\psi_o g}{\omega\ a} \sin(\omega t - kx + qz)$$
$$-\frac{k^2 q\alpha\epsilon E_o^2\psi_o}{\omega\ a} \cos(\omega t - kx + qz) + \mu(q^2+k^2)^2\psi_o \cos(\omega t - kx + qz) \tag{65}$$

The curl of the electrical force, $\nabla\rho_f \times E$, is in phase with the vorticity if (q/ω) is positive and exactly out of phase if (q/ω) is negative. Therefore, instability can only occur for (q/ω) positive. The phases of the unstable mode can only propagate downward. For this reason the instability can never appear to be a standing wave.

Equation (65) also shows that the instability will occur at the longest possible wavelength, since the electric field forces depends on the wave number cubed and the viscous force on the wave number to the fourth power. This means that the boundaries play an important role in determining the stability of the fluid. The threshold conditions can be determined by equating the curl of the electrical force with the curl of the viscous force.

Another result which may be obtained from equation (65) is the frequency at which the instability occurs. This is obtained by equating the gravitational and inertial terms. A consequence of equation (63) is that the fluid velocities are always tangential to the planes of constant phase.

The theory developed in this paper is verified both qualitatively and quantitatively by experiments described in a separate paper.[16]

ACKNOWLEDGMENTS:

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FIGURE CAPTIONS

1) Cross section view of fluid with vertical temperature gradient and electric field.

2) Solutions to the dispersion relation in the complex ω-plane for corn oil in a tank of height 1 inch with a temperature range of 25^{o} - 45^{o} C. The vertical wavenumber is π and the horizontal wavenumber increases from 0 to π along each of the lines in the direction indicated by the arrowheads. Instability occurs when $Im(\omega) < 0$.

3) Same as Figure 2 except that the electrical relaxation time has been increased by a factor of 10.

4) Same as Figure 2 except that the electrical relaxation time has been increased by a factor of 100.

   a) Gravity wave roots

   b) Charge relaxation root

5) Same as Figure 2 but with the viscosity increased by a factor of 10.

6) Same as Figure 2 with the viscosity decreased by a factor of 25.