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Electrodynamics in curved spacetime: 3 + 1 formulation*

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Summary. This paper develops the mathematical foundations for a companion paper on 'Black-Hole Electrodynamics'. More specifically, it re-expresses the equations of curved-spacetime electrodynamics in terms of a 3 + 1 (space + time) split, in which the key quantities are three-dimensional vectors (electric field $\utilde{E}$, magnetic field $\utilde{B}$, etc.) that lie in hypersurfaces of constant time t. Three-dimensional vector analysis is used to express Maxwell's equations, the Gauss, Faraday and Ampere laws, the Lorentz force law, and the laws of energy and momentum conservation in forms closely resembling their flat-spacetime counterparts.

After developing the 3 + 1 formalism for general spacetimes, this paper specializes to the spacetime outside a stationary but rotating black hole. The Znajek–Damour boundary conditions at the hole's horizon are re-expressed in 3 + 1 language. Because the black hole's hypersurfaces of constant time all have identical three-dimensional geometries, one can abandon entirely Einstein's view of spacetime and return to Galileo's: The electric and magnetic fields $\utilde{E}$ and $\utilde{B}$ can be regarded as living in an absolute (but curved) three-dimensional space, and as evolving in this space with the passage of universal time t. This viewpoint and associated mathematics are the foundation for the companion paper.

1 Introduction

There is a close relationship between the theory of axisymmetric pulsar magnetospheres (e.g. Goldreich & Julian 1969; Mestel, Phillips & Wang 1979), and the theory of black-hole and accretion-disc magnetospheres (Blandford & Znajek 1977). For this reason it is curious that astrophysicists have spent enormous effort on the axisymmetric pulsar problem, an idealized problem somewhat far from the structure of real (non-axisymmetric) pulsars, but have put little effort into the theory of black-hole magnetospheres, for which the assumption of axisymmetry is probably justified in Nature.

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We think that this may be due to the fact that general relativity plays crucial roles in the black-hole problem, but not in the pulsar problem, and that the language and mathematical formalism of black-hole electrodynamic theory (Blandford & Znajek 1977) are therefore somewhat different from those of pulsar electrodynamics, and somewhat alien to pulsar theorists. For example, the black-hole theory of Blandford and Znajek uses as its fundamental electrodynamic variables the components A_0, A_ϕ, $F_{r\theta}$, J^0, J^r, J^θ, J^ϕ of the four-vector potential $\mathfrak{A}$, the electromagnetic field tensor $\mathbf{F}$, and the charge-current four-vector $\mathbf{J}$ – and those components are taken in the Boyer–Lindquist coordinate basis of Kerr spacetime. It is not easy for an astrophysicist to get an intuitive, physical feeling for these variables or for their relationship to the electric vector $\underset{\sim}{E}$, magnetic vector $\underset{\sim}{B}$, current vector $\underset{\sim}{j}$, and charge density ρ_e of his flat-space pulsar theory.

Fortunately, it is possible – indeed straightforward – to rewrite curved-spacetime black-hole electrodynamic theory in terms of the physically measured $\underset{\sim}{E}$, $\underset{\sim}{B}$, $\underset{\sim}{j}$, and ρ_e and thereby to obtain a formalism that is very similar to the theory of pulsar electrodynamics and that therefore might be a powerful tool in future black-hole research. The prescription for this rewrite of the curved-spacetime theory is as follows: (i) Choose at each event in spacetime a fiducial reference frame; i.e. split spacetime up into three space directions and one uniquely chosen time direction ('3 + 1 split'). (ii) In this fiducial reference frame, split the electromagnetic field tensor $\mathbf{F}$ into electric and magnetic fields $\underset{\sim}{E}$ and $\underset{\sim}{B}$ in the usual manner of flat spacetime ($\underset{\sim}{E}$ is the time-space part of $\mathbf{F}$; $\underset{\sim}{B}$ is the space-space part). (iii) Similarly, in the fiducial frame, split the four-current vector $\mathbf{J}$ into a time part $J^0 \equiv \rho_e$ = (charge density) and a three-space part $\underset{\sim}{j}$ = (current density). (iv) Rewrite in terms of $\underset{\sim}{E}$, $\underset{\sim}{B}$, ρ_e, and $\underset{\sim}{j}$ the curved-spacetime Maxwell equations, the Lorentz force law, and the law of charge conservation.

Many relativity theorists dislike such a 3 + 1 split because of the arbitrariness of the choice of fiducial reference frame. However, in the case of stationary black-hole electrodynamics there is one set of fiducial frames preferred over all others: the frames of observers who are at rest in the hole's stationary gravitational field, and who see neighbouring fiducial observers inertially fixed with respect to the gyroscopes of their inertial guidance systems ('ZAMO' or 'zero angular momentum observers'). When one uses these ZAMO frames one finds that the '3 + 1' equations of black-hole electrodynamics are nearly identical to the flat-space equations of pulsar electrodynamics. Moreover, when using these frames one can mentally adopt a new viewpoint on the 3 + 1 formalism: one can regard electrodynamics and all other physics as occurring in a fixed, unchanging, absolute three-dimensional space and one can regard time as merely a parameter which demarks the evolution of the matter and fields. In other words, one can return to the absolute-space and universal-time viewpoint of Galileo, which underlies most modern-day astrophysical intuition.

Previous research on black-hole electrodynamics has not used either the 3 + 1 viewpoint, or the absolute-space/universal-time viewpoint. The purpose of this paper and its companion is to introduce those viewpoints and thereby, we hope, to make it easier for astrophysicists to carry their pulsar-based intuition over to the black-hole problem.

We have split our presentation into two papers, so as to make the 3 + 1 formalism more accessible to astrophysicists. Paper I (this paper) derives 3 + 1 electrodynamics from the relativist's more usual four-dimensional formalism – and in doing so it makes free use of the mathematical tools of general relativity theory. Paper II (Macdonald & Thorne 1982) reformulates the Blandford–Znajek theory of black-hole magnetospheres in 3 + 1 language, using the absolute-space/universal-time viewpoint – and in doing so it avoids the mathematics of general relativity.

Paper II can be read separately from Paper I if one is willing to accept the equations of 3 + 1 electrodynamics on faith.

The rest of this paper is on *Microfiche* MN 198/1 and is organized as follows. Section 2 introduces the mathematics of the 3 + 1 split, including: a brief historical survey of the subject (Section 2.1); the fiducial observers and their hypersurfaces of simultaneity $\mathscr{S}_t$ with respect to which the 3 + 1 split is made (Section 2.2); the dot product, cross product, gradient, divergence and curl of spatial vectors (three-vectors) lying in the fiducial hypersurfaces (Section 2.3); three different types of time derivative (Section 2.4); identities for transforming volume, surface and line integrals and their time derivatives into each other (Section 2.5).

Section 3 presents the 3 + 1 formulation of electrodynamics in terms of differential equations, including: the relationship between 3 + 1 electrodynamic quantities and four-dimensional quantities (Section 3.1); the Maxwell equations (Section 3.2); expressions for $\underset{\sim}{E}$ and $\underset{\sim}{B}$ in terms of the scalar potential ϕ and vector potential $\underset{\sim}{A}$ (Section 3.3); the law of charge conservation (Section 3.4); the Lorentz force law and equation of motion of a charged test particle (Section 3.5); and the differential laws of energy and momentum conservation for the electromagnetic field and a continuous medium (Section 3.6).

Section 4 presents the integral formulation of 3 + 1 electrodynamics: Gauss's law, Ampere's law, Faraday's law and the law of charge conservation.

Section 5 specializes the 3 + 1 formalism to the spacetime of a stationary, axisymmetric black hole, including: the selection of the ZAMO observers as our fiducial observers and the resulting simplifications of various 3 + 1 kinematic equations (Section 5.1); the 3 + 1 electrodynamic equations specialized to our black-hole spacetime (Section 5.2); the pathological behaviour of the hypersurfaces $\mathscr{S}_t$ near the hole's horizon, and the resulting delicate definition of 'the limit of a physical quantity as one approaches the horizon' (Section 5.3); and the Znajek–Damour theory of electromagnetic boundary conditions at the horizon, rewritten in 3 + 1 language (Section 5.4).

Section 6 illustrates the 3 + 1 formalism by rewriting in 3 + 1 language two known solutions to the vacuum Maxwell equations: the electric field of a point charge outside a Schwarzschild hole (Section 6.1), and a uniform magnetic field surrounding and deformed by a Kerr hole (Section 6.2).

Throughout this paper we use the mathematical notation and conventions of Misner, Thorne & Wheeler (1973; cited henceforth as MTW), including units in which the speed of light c is unity. (Nowhere, except in the examples of Sections 5.3 and 6, do we need to set Newton's gravitation constant G to unity.) Electromagnetic quantities are expressed in Gaussian units (electric fields in statvolts per centimetre, magnetic fields in Gauss). We denote four-vectors and four-tensors by bold-face letters, e.g. **U** and **F**, and their components by Greek indices, e.g. U^α and $F_{\alpha\beta}$. We denote spatial vectors (three-vectors) and spatial tensors (three-tensors) by underscored letters, e.g. $\underset{\sim}{E}$ and $\underset{\sim}{\gamma}$, and their components by Latin indices, e.g. E^j and γ_{jk}.

[See *Microfiche* MN 198/1 for continuation of this paper.]

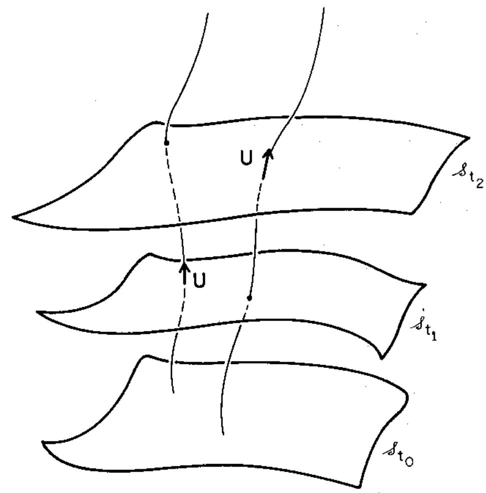

Figure 1. The world lines of fiducial observers with four-velocities U, and the space-like hypersurfaces of simultaneity $\mathscr{S}_t$ which are orthogonal to the fiducial world lines.

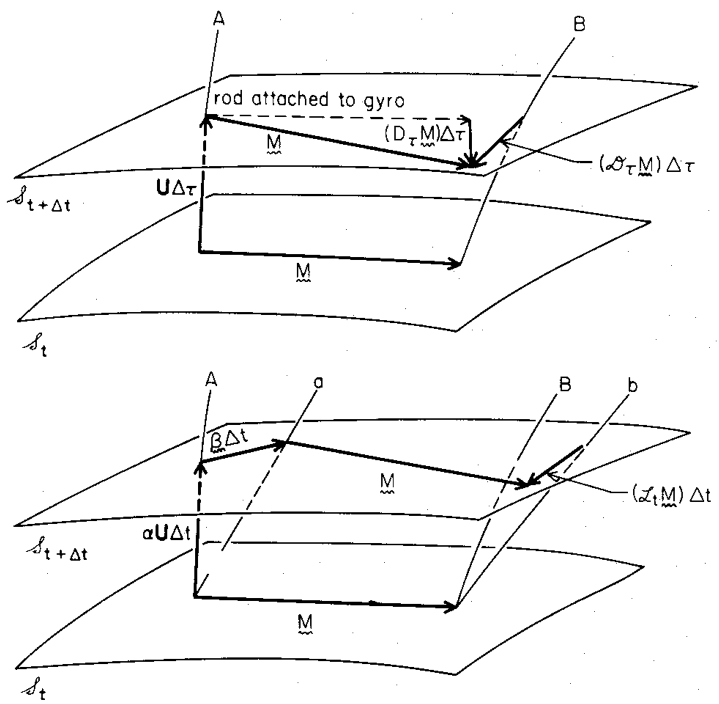

Figure 2. The Fermi–Walker time derivative $D_\tau \underset{\sim}{M}$, Lie time derivative $\mathscr{D}_\tau \underset{\sim}{M}$, and shifting time derivative $\mathscr{L}_t \underset{\sim}{M}$ of a spatial vector $\underset{\sim}{M}$. The two hypersurfaces are separated by global time Δt, and fiducial observer A sees them separated by proper time $\Delta\tau = \alpha\Delta t$.

In the upper diagram observer A carries with himself a gyroscope, applying an acceleration $\underset{\sim}{a}$ at its centre of mass to keep it moving with him. He orients the gyroscope along the direction of $\underset{\sim}{M}$ at time t, and he attaches a rod to the gyroscope with precisely the same length as $\underset{\sim}{M}$. After proper time lapse $\Delta\tau = \alpha\Delta t$ the rod is located along the dashed arrow. The difference between $\underset{\sim}{M}$ and this dashed arrow is $(D_\tau \underset{\sim}{M})\Delta\tau$.

At time t in the upper diagram the tail of $\underset{\sim}{M}$ sits on fiducial observer A and the tip on fiducial observer B. After proper time lapse $\Delta\tau = \alpha\Delta t$ the tail is still on A but the tip has been displaced away from B. Its vector displacement is $(\mathscr{D}_\tau \underset{\sim}{M})\Delta\tau$.

The lower diagram shows trajectories a and b of the shifting congruence. The velocity of a trajectory relative to fiducial observer A is $d(\text{proper distance})/d\tau = \beta/\alpha$. At time t the tail of $\underset{\sim}{M}$ sits on trajectory a and the tip on trajectory b. After global time lapse Δt the tail is still on a but the tip has been displaced away from b. Its vector displacement is $(\mathscr{L}_t \underset{\sim}{M})\Delta t$.

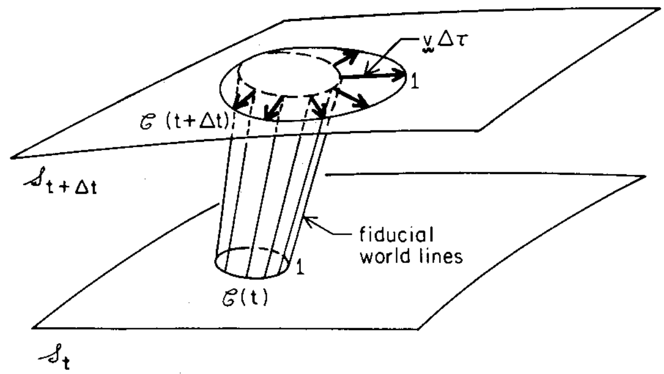

Figure 3. A curve $\mathscr{C}(t)$, lying in the hypersurface $\mathscr{S}_t$, changes in some arbitrary manner as time t passes. A point labelled 1 moves with velocity $\underset{\sim}{\mathrm{v}}$ as measured by a fiducial observer near it. During proper time $\Delta\tau = \alpha\Delta t$ point 1 gets displaced by $\underset{\sim}{\mathrm{v}}\Delta\tau$ relative to the fiducial observer.

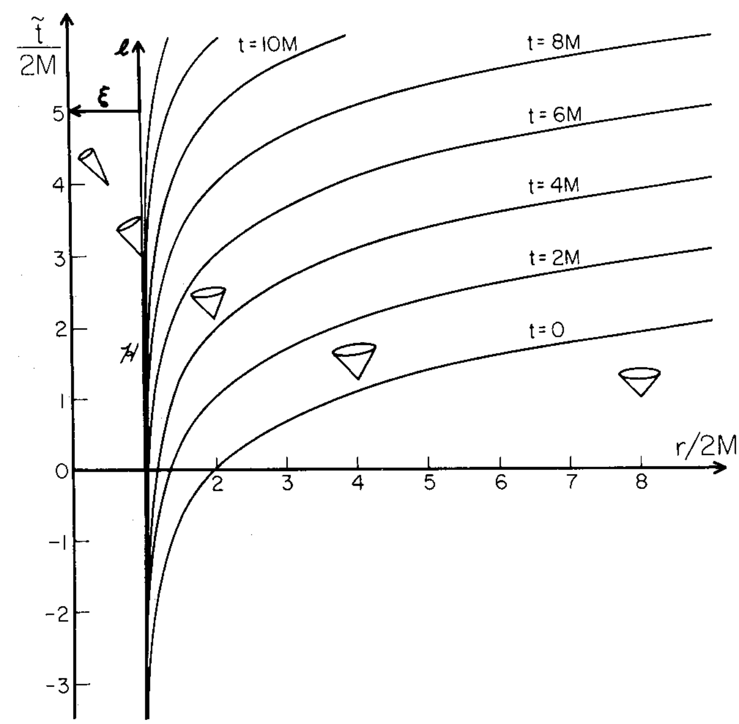

Figure 4. The hypersurfaces of simultaneity $\mathscr{S}_t$ around a Schwarzschild black hole, as viewed in Eddington–Finkelstein coordinates (MTW, Box 31.2). Plotted upward is the Eddington–Finkelstein time coordinate $\tilde{t}$, which is related to the Schwarzschild time and radial coordinates by $\tilde{t} = t + 2M\ln(r/2M - 1)$. Plotted horizontally is the Eddington–Finkelstein radial coordinate r, which is identical to the Schwarzschild radial coordinate. The curves shown are our fiducial hypersurfaces $\mathscr{S}_t$, and the cones are the radial light cones as given by the metric (5.19).

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2 3+1 Mathematical Formalism

2.1 HISTORICAL REMARKS

There are two rather different ways to make a 3+1 split of the laws of physics in curved spacetime. The first way selects a fiducial congruence of timelike world lines, and at each event identifies "time" as the direction along the fiducial world line and "space" as the three directions orthogonal to it. In this congruence approach the space directions at neighboring events will not mesh to form global spacelike hypersurfaces (3-spaces of constant time), unless the congruence is constrained to be "rotation-free".

The second approach selects a foliation of fiducial 3-dimensional hypersurfaces (3-spaces of constant time), and at each event identifies "space" as the directions lying in the hypersurface. In this hypersurface approach one can identify as "time" the direction orthogonal to the hypersurface — in which case the formalism is identical to the rotation-free limit of the congruence approach. Alternatively, one can identify as "time" a nonorthogonal direction (nonzero "shift vector").

The congruence approach to 3+1 splits was developed in brief form by Landau and Lifshitz (1941) and in greater detail by Zel'manov (1956, 1959), who refers to spatial vectors and tensors as "chronometric invariants". Today this congruence approach is much used by Russian relativistic astrophysicists, in no small measure because of the influence of Igor Novikov, who was a student of Zel'manov (see, e.g., §1.6 of Zel'dovich and Novikov, 1971). In the West the congruence approach was developed in brief form by Cattaneo (1959) and in great detail by Estabrook and Wahlquist (1964), who called it the "dyadic formalism". None of these workers, Russian or Western, wrote down Maxwell's equations in 3+1 congruence language; that was done later, by Ellis (1973). And as far as we know, nobody has ever used the Ellis equations in astrophysics or relativity research, except for our present study of black-hole electrodynamics.

The hypersurface approach to 3+1 splits was developed by Lichnerowicz (1944) in his pioneering studies of the dynamical evolution of spacetime geometry; and it was further developed in the 1950's by Bergmann, Dirac, Wheeler, Arnowitt, Deser, Misner, and others as part of their efforts to create a Hamiltonian formulation of general relativity and thereby to lay the foundations for canonical quantization of the gravitational field; see Arnowitt, Deser, and Misner (1962) ("ADM"). As part of this program, Misner and Wheeler (1957) wrote down the curved-space, vacuum Maxwell equations in 3+1 hypersurface form, but using the language of exterior calculus rather than vector analysis; Arnowitt, Deser, and Misner (1960a,b) wrote down the 3+1 Maxwell equations with point charges, but using the language of vector densities rather than vectors; and Stachel (1969) wrote down those portions of the 3+1 Maxwell equations which are metric-independent, using the language of vectors and tensors. These "3+1 Maxwell equations" have been much used since 1960 as a guide to formal mathematical studies of the dynamics of geometry (see, e.g., chapter 21 of MTW). However, they seem never to have been used in astrophysics research. In recent years the full ADM 3+1 hypersurface formalism has been adopted as the canonical foundation for numerical solutions of the Einstein field equations and of hydrodynamical equations in curved spacetime ("numerical relativity"); see Smarr and York (1978), York (1979), Smarr, Taubes, and Wilson (1980). In the Soviet Union the hypersurface approach to 3+1 splits has been formulated by Zel'manov (1973); he refers to spatial vectors and tensors in this formalism as "kinematic invariants".

In the present work we shall use the ADM hypersurface approach to 3+1 splits. Initially we shall choose our time direction orthogonal to the hypersurfaces, thereby making our formalism identical to the rotation-free limit of the congruence approach. This will permit us to use Ellis's beautiful 3+1 formulation of Maxwell's equations. Later, when specializing to black-hole spacetimes, we shall introduce a "shift" into our time direction, so that instead of being orthogonal to the fiducial hypersurfaces, it is along the Killing direction $\mathbf{k} = \partial/\partial t$ of the stationary spacetime geometry.

2.2 FIDUCIAL HYPERSURFACES AND CONGRUENCE

Our mathematical formalism for the fiducial hypersurfaces and congruence and for the 3+1 split is essentially the same as that used in current research on numerical relativity (e.g., York 1979), with these exceptions: (i) Our notation is slightly different; for example, we use index-free expressions $\underset{\sim}{A}\cdot\underset{\sim}{B}$, $\underset{\sim}{\nabla}\times\underset{\sim}{E}$, etc. and we "think" in coordinate-free language, whereas numerical relativists always have a coordinate mesh and use index notation $A^j B^k \gamma_{jk}$ with components taken on that mesh. (ii) In the early part of this paper we use different kinds of time derivatives than they — our D_τ and $\mathcal{D}_\tau$. (iii) We develop and make extensive use of 3+1 integral identities, which are not part of present-day numerical relativity.

Consider a region $\mathcal{E}$ of 4-dimensional spacetime in which electrodynamic phenomena are to be studied. Introduce into $\mathcal{E}$ a family of spacetime-filling, 3-dimensional spacelike hypersurfaces; and introduce a parameter t which (i) labels the hypersurfaces, and (ii) increases smoothly as one moves forward in time from hypersurface to hypersurface, but (iii) is otherwise arbitrary. Denote by $\mathcal{S}_t$ the hypersurface which has label t. Give t the name "global time parameter" or simply "global time". See Figure 1.

There will exist a congruence of timelike curves which are orthogonal to the hypersurfaces. These curves can be regarded as the world lines of a family of "fiducial observers" [numerical relativists call them "Eulerian observers"] who think of the hypersurfaces $\mathcal{S}_t$ as "slices of simultaneity". Parametrize each fiducial world line by the proper time τ of its observer. Then the observer's 4-velocity (unit tangent to the fiducial world line) is $\mathbf{U} = d/d\tau$. [In numerical relativity (e.g., York 1979) the notation $\mathbf{n}$ is used rather than $\mathbf{U}$. In this paper $\mathbf{n}$ is reserved for the normal to the horizon of a black hole (Eq. 5.23 below).] Proper time τ and global time t typically will not march forward at the same rate along a fiducial world line; the ratio of their rates is called the "lapse function" α

$$\alpha = (d\tau/dt)_{\text{along fiducial world line}} \quad . \tag{2.1a}$$

Since the fiducial 4-velocity $\mathbf{U}$ is orthogonal to hypersurfaces of constant t, it must be parallel to the 4-gradient of t with a proportionality constant determined by $\mathbf{U}^2 = -1$ and by $\mathbf{U}\cdot{}^{(4)}\boldsymbol{\nabla} t = dt/d\tau = \alpha^{-1}$:

$$\mathbf{U} = -\alpha\, {}^{(4)}\boldsymbol{\nabla} t \quad , \qquad \alpha = [-({}^{(4)}\boldsymbol{\nabla} t)^2]^{-1/2} \quad . \tag{2.1b}$$

Here and below we use a prefix (4) on the spacetime gradient ${}^{(4)}\boldsymbol{\nabla}$ to distinguish it clearly from the spatial gradient $\underset{\sim}{\nabla}$.

2.3 THREE-DIMENSIONAL VECTOR ANALYSIS

Any 4-vector $\mathbf{M}$ or 4-tensor $\mathbf{T}$ which is orthogonal to the fiducial 4-velocity, $M^\alpha U_\alpha = 0$ or $U_\alpha T^{\alpha\beta} = 0 = T^{\alpha\beta}U_\beta$, can be regarded as a purely spatial vector $\underset{\sim}{M}$ or tensor $\underset{\sim}{T}$— i.e., it can be regarded as living in a fiducial hypersurface $\mathcal{S}_t$. When adopting the 3+1 viewpoint we shall denote it $\underset{\sim}{M}$ or $\underset{\sim}{T}$ and its components M_j or T_{jk}. When adopting the 4-dimensional spacetime viewpoint we shall use the notation $\mathbf{M}$, $\mathbf{T}$, M_α, $T_{\alpha\beta}$.

The most important spatial tensor we shall deal with is the metric $\underset{\sim}{\gamma}$ of the fiducial hypersurfaces $\mathcal{S}_t$. In 4-dimensional notation, $\gamma^{\alpha\beta}$ is the tensor

$$\gamma^{\alpha\beta} = g^{\alpha\beta} + U^{\alpha}U^{\beta} \tag{2.2}$$

which projects 4-vectors into the fiducial hypersurfaces. Here $g^{\alpha\beta}$ is the metric of 4-dimensional spacetime.

The 3+1 equations of physics will involve the kinematic properties of the fiducial world lines — their expansion θ, 4-acceleration $\underset{\sim}{a}$, and shear $\underset{\sim}{\sigma}$. (Their rotation $\underset{\sim}{\omega}$ vanishes because the fiducial world lines are hypersurface orthogonal.) Viewed as 4-dimensional quantities, θ, a^{α}, and $\sigma_{\alpha\beta}$ are defined by

$$\theta = U^{\alpha}{}_{;\alpha} \quad , \qquad a^{\alpha} = U^{\alpha}{}_{;\beta}U^{\beta} \quad ,$$
$$\sigma_{\alpha\beta} = \frac{1}{2}\gamma_{\alpha}{}^{\mu}\,\gamma_{\beta}{}^{\nu}(U_{\mu;\nu} + U_{\nu;\mu}) - \frac{1}{3}\theta\,\gamma_{\alpha\beta} \quad , \tag{2.3a}$$

which can be inverted to give

$$U_{\alpha;\beta} = -a_{\alpha}U_{\beta} + \sigma_{\alpha\beta} + \frac{1}{3}\theta\,\gamma_{\alpha\beta} \tag{2.3b}$$

(cf. Exercise 22.6 of MTW). Here the semicolon denotes the covariant derivative ${}^{(4)}\nabla$ with respect to the spacetime geometry. One can easily verify that a^{α} and $\sigma_{\alpha\beta}$ are orthogonal to U^{α} and are therefore a spatial vector and spatial tensor, respectively. A fiducial observer interprets θ as the fractional time rate of change $V^{-1}dV/d\tau$ of the volume V of a "fluid element" whose walls are attached to the world lines of nearby fiducial observers — i.e., $\theta = 3\times$ (Hubble expansion rate of fiducial observers averaged over all directions in space). If a fiducial observer carries an accelerometer, he interprets $\underset{\sim}{a}$ as the vector acceleration which it reads. If a fiducial observer studies the motions of other nearby fiducial observers, he interprets $\underset{\sim}{\sigma}$ as the rate of shear of those motions, as defined in nonrelativistic fluid mechanics. For further detail see, e.g., Ellis (1971,1973). One can show, using equation (2.1b) and the definition of the spatial gradient given below, that the acceleration and the lapse function are related by

$$\underset{\sim}{a} = \underset{\sim}{\nabla} \ln \alpha \quad ; \tag{2.4}$$

and one can show using equations (2.3a) and (2.2), that the shear $\underset{\sim}{\sigma}$ is symmetric and trace-free

$$\sigma_{jk} = \sigma_{kj} \quad , \quad \sigma_{jk}\, \gamma^{jk} = \sigma_{\alpha\beta}\, \gamma^{\alpha\beta} = 0 \quad . \tag{2.5}$$

The "extrinsic curvature" $\underset{\sim}{K}$ of the hypersurface $\mathcal{S}_t$ is related to the shear and expansion of the fiducial congruence by

$$\underset{\sim}{K} = -(\underset{\sim}{\sigma} + \frac{1}{3}\,\theta\, \underset{\sim}{\gamma}) \quad . \tag{2.6}$$

Spatial vectors and tensors living in $\mathcal{S}_t$ can be manipulated in much the same manner as in flat space: If $\underset{\sim}{L}$ and $\underset{\sim}{M}$ are spatial vectors, their inner product and cross product are

$$\underset{\sim}{L} \cdot \underset{\sim}{M} = L_j M_k \gamma^{jk} \quad , \quad (\underset{\sim}{L} \times \underset{\sim}{M})^j = \varepsilon^{jk\ell}\, L_k M_\ell \tag{2.7}$$

where $\varepsilon^{jk\ell}$ is the spatial Levi-Civita tensor [equal to $(\det \|\gamma_{ij}\|)^{-1/2}$ (antisymmetric symbol)]. The spatial gradient operator (denoted $\underset{\sim}{\nabla}$ in abstract and $|_j$ in component notation) can be defined in either of two equivalent ways: as the 4-dimensional covariant derivative projected into $\mathcal{S}_t$, or as the spatial covariant derivative associated with the spatial metric γ_{jk}. From the former viewpoint, if $\underset{\sim}{M}$ is a spatial vector then $\underset{\sim}{\nabla}\underset{\sim}{M}$ is a spatial tensor with spacetime components

$$M^{\alpha}{}_{|\beta} = \gamma^{\alpha}{}_{\mu}\, \gamma_{\beta}{}^{\nu}\, M^{\mu}{}_{;\nu} \quad . \tag{2.8a}$$

From the latter viewpoint, $\underset{\sim}{\nabla} \underset{\sim}{M}$ has spatial components

$$M^{j}{}_{|k} = M^{j}{}_{,k} + \Gamma^{j}{}_{\ell k}\, M^{\ell} \tag{2.8b}$$

where $\Gamma^{j}{}_{\ell k}$ are connection coefficients computed in the usual way from the spatial metric γ_{jk}. The divergence and curl are defined in terms of $\underset{\sim}{\nabla}$ by

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{M} = M^{j}{}_{|j} \quad , \qquad (\underset{\sim}{\nabla} \times \underset{\sim}{M})^{j} = \varepsilon^{jk\ell}\, M_{\ell|k} \quad . \tag{2.9}$$

Note that because the geometry of $\mathcal{S}_t$ is not flat, spatial gradients of vectors do not commute:

$$M^{j}{}_{|k\ell} - M^{j}{}_{|\ell k} = R^{j}{}_{i\ell k}\, M^{i} \tag{2.10}$$

where $R^{j}{}_{i\ell k}$ is the Riemann tensor of $\mathcal{S}_t$; cf. Exercise 16.3 of MTW. Despite this noncommutation, the following identities are valid for any vector field $\underset{\sim}{M}$ and scalar field ψ

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{\nabla} \times \underset{\sim}{M} = 0 \quad , \qquad \underset{\sim}{\nabla} \times \underset{\sim}{\nabla}\psi = 0 \quad . \tag{2.11}$$

2.4 TIME DERIVATIVES OF SPATIAL VECTORS

Three different time derivatives are useful in studying the evolution of spatial vectors and tensors.

When focussing attention on physical measurements made by a fiducial observer, one may prefer a time derivative defined by Fermi transport ("gyroscope transport") along the fiducial world lines:

$$D_\tau M^\beta \equiv \gamma^{\beta\mu} M_{\mu;\nu} U^\nu$$

$$= M^\beta{}_{;\mu} U^\mu - U^\beta a_\mu M^\mu \quad . \tag{2.12}$$

Here M^α and $D_\tau M^\alpha$ are the spacetime components of a spatial vector $\underset{\sim}{M}$ and its time derivative $D_\tau \underset{\sim}{M}$. This is the type of time derivative used in the congruence version of the 3+1 formalism (e.g., Estabrook and Wahlquist 1964).

In deriving integral identities (§2.5 below) we shall find it easiest to make geometric constructions involving the Lie derivative of $\underset{\sim}{M}$ along fiducial world lines

$$\mathcal{D}_\tau M^\beta \equiv \alpha^{-1} \mathcal{L}_{\alpha \boldsymbol{U}} M^\beta$$

$$= \alpha^{-1}[M^\beta{}_{;\mu}\, \alpha U^\mu - (\alpha U^\beta)_{;\mu} M^\mu] \quad . \tag{2.13}$$

Note that all of the 4-vectors $\alpha \boldsymbol{U}$ whose tails sit on the same hypersurface $\mathcal{S}_t$ have their tips on the same hypersurface $\mathcal{S}_{t+1\,\mathrm{second}}$. This together with the pictorial interpretation of the Lie derivative (Box 9.2 of MTW) guarantees that, just as $\underset{\sim}{M}$ is a 4-vector lying in $\mathcal{S}_t$, so $\mathcal{D}_\tau \underset{\sim}{M}$ is a 4-vector lying in $\mathcal{S}_t$—i.e., it is a spatial vector. This type of time derivative is occasionally used in the numerical relativity 3+1 formalism (e.g., York 1979, where $\alpha \mathcal{D}_\tau$ is denoted $\mathcal{L}_N$).

One can also define a third kind of time derivative: the Lie derivative along a "shifting congruence" with tangent vector $\alpha U^\mu + \beta^\mu$. Here $\underset{\sim}{\beta}$, the "shift vector", is a spatial vector field lying in $\mathcal{S}_t$. As measured by fiducial observers [called "Eulerian observers" by numerical relativists], the shifting congruence [called "Lagrangian congruence" by numerical relativists] has ordinary velocity

$$\boldsymbol{\beta}/\alpha = d(\text{proper distance})/d\tau \tag{2.14}$$

The shift vector field $\boldsymbol{\beta}$ could be specified arbitrarily, but when we consider the case of stationary, axially symmetric spacetimes, a natural choice of $\boldsymbol{\beta}$ presents itself, namely the one for which $\alpha\mathbf{U}+\boldsymbol{\beta}$ is equal to the Killing vector $\mathbf{k}$ associated with the time isometry of the spacetime. The time derivative along the shifting congruence will be defined by

$$\mathcal{L}_t M^\mu \equiv \mathcal{L}_{\alpha\mathbf{U}+\boldsymbol{\beta}} M^\mu$$

$$= M^\mu{}_{;\nu}(\alpha U^\nu + \beta^\nu) - (\alpha U^\mu + \beta^\mu)_{;\nu} M^\nu \quad . \tag{2.15}$$

Just as $\mathbf{M}$ is a spatial vector, so $\mathcal{L}_t\mathbf{M}$ is a spatial vector. The mixed Eulerian-Lagrangian equations of numerical relativity are formulated in terms of this shifting time derivative $\mathcal{L}_t$ (York 1979; Smarr and York 1978; Smarr, Taubes, and Wilson 1980).

Physically the Fermi time derivative $D_\tau\mathbf{M}$ describes the rate of change of $\mathbf{M}$ with respect to proper time τ along a fiducial world line — the change being measured relative to an inertial guidance system of physical rods and gyroscopes carried by the fiducial observer; see §13.6 of MTW and Figure 2 of this paper. The Lie time derivative $\mathfrak{D}_\tau\mathbf{M}$ also describes the rate of change of $\mathbf{M}$ with respect to proper time τ along a fiducial world line — but now the change is measured relative to the (changing) spatial locations of other fiducial observers; see Schild (1967) and Figure 2 of this paper. The Lie time derivative $\mathcal{L}_t\mathbf{M}$ describes the rate of change of $\mathbf{M}$ with respect to global time t along a trajectory of the shifting congruence — the change being measured relative to the spatial locations of other trajectories in the shifting congruence; see Figure 2.

The derivatives D_τ, $\mathfrak{D}_\tau$, and $\mathcal{L}_t$ can act on scalar fields and 3-tensor fields as well as on vector fields. The action of D_τ is always defined by parallel transport $U^{\alpha}\,{}^{(4)}\nabla_{\alpha}$, followed by projection with $\gamma^{\mu}{}_{\nu}$ on all indices; $\mathcal{D}_{\tau}$ and $\mathcal{L}_t$ are always defined by $\mathcal{D}_{\tau} \equiv \alpha^{-1}\pounds_{\alpha \boldsymbol{U}}$ and $\mathcal{L}_t \equiv \pounds_{\alpha \boldsymbol{U}+\boldsymbol{\beta}}$ where $\pounds$ is the Lie derivative which acts on scalars and tensors in the usual fashion (Schild 1967). When acting on a scalar field these three derivatives are related by

$$D_{\tau}\psi = \mathcal{D}_{\tau}\psi = \alpha^{-1}(\mathcal{L}_t\psi - \underset{\sim}{\beta}\cdot\underset{\sim}{\nabla}\psi) \quad . \tag{2.16a}$$

When acting on a vector field they are related by

$$\mathcal{D}_{\tau}\underset{\sim}{M} = D_{\tau}\underset{\sim}{M} - \underset{\sim}{\sigma}\cdot\underset{\sim}{M} - \frac{1}{3}\,\theta\,\underset{\sim}{M} \quad , \tag{2.16b}$$

$$\mathcal{L}_t\underset{\sim}{M} = \alpha\,\mathcal{D}_{\tau}\underset{\sim}{M} + \pounds_{\underset{\sim}{\beta}}\underset{\sim}{M} \tag{2.16c}$$

where $\pounds_{\underset{\sim}{\beta}}\underset{\sim}{M}$ is the Lie derivative of the spatial vector $\underset{\sim}{M}$ along the spatial vector $\underset{\sim}{\beta}$

$$\pounds_{\underset{\sim}{\beta}}\underset{\sim}{M} = (\underset{\sim}{\beta}\cdot\underset{\sim}{\nabla})\underset{\sim}{M} - (\underset{\sim}{M}\cdot\underset{\sim}{\nabla})\underset{\sim}{\beta} \quad . \tag{2.17}$$

The 3-metric $\underset{\sim}{\gamma}$ is unchanging as measured by the Fermi time derivative D_{τ}, but it changes as measured by the Lie derivatives

$$D_{\tau}\underset{\sim}{\gamma} = 0 \tag{2.18a}$$

$$\mathcal{D}_{\tau}\gamma_{jk} = 2(\sigma_{jk} + \frac{1}{3}\theta\gamma_{jk}) \tag{2.18b}$$

$$\mathcal{L}_t\gamma_{jk} = 2\alpha(\sigma_{jk} + \frac{1}{3}\,\theta\,\gamma_{jk}) + \beta_{j|k} + \beta_{k|j} \quad . \tag{2.18c}$$

Because of this, one must be careful about scalar products when using $\mathcal{D}_{\tau}$ and $\mathcal{L}_t$; for example,

$$\mathcal{L}_t(\underset{\sim}{E}\cdot\underset{\sim}{B}) = \underset{\sim}{E}\cdot\mathcal{L}_t\underset{\sim}{B} + \underset{\sim}{B}\cdot\mathcal{L}_t\underset{\sim}{E} + \underset{\sim}{E}\cdot(\mathcal{L}_t\underset{\sim}{\gamma})\cdot\underset{\sim}{B} \quad . \tag{2.19}$$

[Relations (2.16) and (2.18) are derived from the 4-dimensional definitions (2.12), (2.13), (2.15) of D_{τ}, $\mathcal{D}_{\tau}$, and $\mathcal{L}_t$. In (2.18b,c) note that $\sigma_{jk} + \frac{1}{3}\,\theta\,\gamma_{jk} =$

$-K_{jk}$ (extrinsic curvature; equation 2.6).]

Time derivatives do not commute with spatial gradients. From definitions (2.12) and (2.8a) one can show that for any scalar field ψ and spatial vector field $\underset{\sim}{M}$

$$D_{\tau}\underset{\sim}{\nabla}\psi = \frac{1}{\alpha}\underset{\sim}{\nabla}(\alpha D_{\tau}\psi) - \frac{1}{3}\theta\,\underset{\sim}{\nabla}\psi - \underset{\sim}{\sigma}\cdot\underset{\sim}{\nabla}\psi \quad , \tag{2.20a}$$

$$D_{\tau}(M_{j|k}) = \frac{1}{\alpha}(\alpha D_{\tau}M_{j})_{|k} - \frac{1}{3}\theta\, M_{j|k} - \sigma_{k}{}^{i}\, M_{j|i} + \mathcal{R}^{i}{}_{jk}\, M_{i}$$
$$+ (\underset{\sim}{a}\cdot\underset{\sim}{M})(\sigma_{jk} + \frac{1}{3}\theta\,\gamma_{jk}) - M_{i}a_{j}(\sigma_{k}{}^{i} + \frac{1}{3}\theta\,\gamma_{k}{}^{i}) \quad . \tag{2.20b}$$

Here $\mathcal{R}^{i}{}_{jk}$ is a spatial tensor related to the Riemann curvature of 4-dimensional spacetime by

$$\mathcal{R}^{\alpha}{}_{\beta\gamma} = \gamma^{\alpha}{}_{\mu}\,\gamma_{\beta}{}^{\nu}\,\gamma_{\gamma}{}^{\sigma}\,{}^{(4)}R^{\mu}{}_{\nu\sigma\rho}\,U^{\rho} \quad . \tag{2.21a}$$

Using the Gauss-Codazzi equations along with (2.6), one can rewrite $\mathcal{R}^{i}{}_{jk}$ in terms of the kinematic quantities of the fiducial congruence:

$$\mathcal{R}_{ijk} = \sigma_{jk|i} - \sigma_{ik|j} + \frac{1}{3}(\gamma_{jk}\,\theta_{,i} - \gamma_{ik}\,\theta_{,j}) \quad . \tag{2.21b}$$

2.5 THREE-DIMENSIONAL INTEGRAL THEOREMS

In passing from the differential formulation of Maxwell's equations to the integral formulation, we shall use various integral identities. If $\mathcal{V}$ is a region of 3-dimensional space lying in $\mathcal{S}_t$ and $\partial\mathcal{V}$ is its closed 2-dimensional boundary, then Gauss's theorem says that for any vector field $\underset{\sim}{M}$

$$\int_{\mathcal{V}} \underset{\sim}{\nabla}\cdot\underset{\sim}{M}\, dV = \int_{\partial\mathcal{V}} \underset{\sim}{M}\cdot d\underset{\sim}{\Sigma} \quad . \tag{2.22}$$

Here dV is an element of spatial proper volume in $\mathcal{V}$, and $d\underset{\sim}{\Sigma}$ is an element of area in $\partial\mathcal{V}$ ($d\underset{\sim}{\Sigma}$ points orthogonally out of $\mathcal{V}$, and $|d\underset{\sim}{\Sigma}| = (\gamma^{jk}\, d\Sigma_{j}\, d\Sigma_{k})^{1/2}$ is proper area). If $\mathcal{A}$ is a 2-dimensional region lying in the 3-space $\mathcal{S}_t$ and $\partial\mathcal{A}$ is its closed 1-dimensional boundary, then Stokes's theorem says that for any vector field $\underset{\sim}{M}$

$$\int_{\mathcal{A}} (\underset{\sim}{\nabla} \times \underset{\sim}{M}) \cdot d\underset{\sim}{\Sigma} = \int_{\partial\mathcal{A}} \underset{\sim}{M} \cdot d\underset{\sim}{\ell} \quad . \tag{2.23}$$

Here $d\underset{\sim}{\Sigma}$ is an element of proper area in $\mathcal{A}$, $d\underset{\sim}{\ell}$ is an element of proper length along $\partial\mathcal{A}$, and the directions of $d\underset{\sim}{\ell}$ and $d\underset{\sim}{\Sigma}$ must be chosen in accord with the standard right-hand rule.

Gauss's theorem and Stokes's theorem involve spatial vector analysis in a single hypersurface $\mathcal{S}_t$, chosen once-and-for-all. We shall also need identities which relate integrals on one hypersurface $\mathcal{S}_t$ to integrals on an adjacent hypersurface $\mathcal{S}_{t+\Delta t}$. In these identities we must pay attention to the motion, relative to fiducial observers, of the regions of integration. For this purpose we give each point on a region of integration a label, and we define by

$$\underset{\sim}{v} = d(\underset{\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim}{\text{proper spatial distance}})/d\tau \tag{2.24}$$

the velocity of that labeled point as measured by a fiducial observer who sits beside it; see Figure 3.

Let ϕ be a smoothly varying scalar field in spacetime; let $\mathcal{V}(t)$ be a spatial volume in $\mathcal{S}_t$ which changes in some arbitrary but smooth manner as time passes; and let $\partial\mathcal{V}(t)$ be the 2-dimensional closed boundary of $\mathcal{V}(t)$ in $\mathcal{S}_t$. Then between global time t and time $t+\Delta t$ the integral of ϕ over $\mathcal{V}$ changes by

$$\Delta \int_{\mathcal{V}} \phi \, dV = \int_{\mathcal{V}} (D_\tau \phi)\alpha\Delta t \, dV + \int_{\mathcal{V}} \phi(\theta\alpha\Delta t) \, dV + \int_{\partial\mathcal{V}} \phi(\underset{\sim}{v}\alpha\Delta t) \cdot d\underset{\sim}{\Sigma} \quad .$$

The first term accounts for the change $(D_\tau\phi)\Delta\tau = (D_\tau\phi)\alpha\Delta t$ in ϕ. The second term accounts for the change $\Delta dV = (\theta dV)\Delta\tau$ in a physical volume element dV which is attached to fiducial observers. The third term accounts for the opening up of new volume (or closing off of old volume) at the moving boundary of $\mathcal{V}$, $\Delta dV = (\underset{\sim}{v}\Delta\tau) \cdot d\underset{\sim}{\Sigma}$. Dividing this equation by Δt and taking the limit $\Delta t \to 0$ we obtain the integral identity

$$\frac{d}{dt}\int_{\mathcal{V}(t)} \phi\, dV = \int_{\mathcal{V}(t)} \alpha(D_\tau\phi + \theta\phi)dV + \int_{\partial\mathcal{V}(t)} \alpha\phi\underset{\sim}{v} \cdot d\underset{\sim}{\Sigma} \quad . \tag{2.25}$$

Let $\underset{\sim}{M}$ be a smoothly varying vector field in spacetime; let $\mathcal{A}(t)$ be a 2-dimensional surface in $\mathcal{S}_t$ which changes in some arbitrary but smooth manner as time passes; and let $\partial\mathcal{A}(t)$ be the 1-dimensional closed boundary of $\mathcal{A}(t)$ with line element $d\underset{\sim}{\ell}$ related to the area element $d\underset{\sim}{\Sigma}$ of $\mathcal{A}$ by the right-hand rule. Then between global time t and $t+\Delta t$ the integral of $\underset{\sim}{M}$ over $\mathcal{A}(t)$ changes by

$$\Delta \int_{\mathcal{A}(t)} \underset{\sim}{M} \cdot d\underset{\sim}{\Sigma} = \int_{\mathcal{A}(t)} (\alpha\Delta t)(\mathcal{L}_\tau \underset{\sim}{M}) \cdot d\underset{\sim}{\Sigma} + \int_{\mathcal{A}(t)} (\theta\alpha\Delta t)\underset{\sim}{M} \cdot d\underset{\sim}{\Sigma}$$
$$+ \int_{\mathcal{A}(t)} (\underset{\sim}{\nabla} \cdot \underset{\sim}{M})(\underset{\sim}{v}\alpha\Delta t) \cdot d\underset{\sim}{\Sigma} + \int_{\partial\mathcal{A}(t)} \underset{\sim}{M} \cdot (\underset{\sim}{v}\alpha\Delta t) \times d\underset{\sim}{\ell} \quad .$$

The first term accounts for changes $\Delta\underset{\sim}{M} = (\mathcal{L}_\tau \underset{\sim}{M})\Delta\tau$ of $\underset{\sim}{M}$ relative to Lie transport by fiducial observers; cf. Figure 2. If $\underset{\sim}{M}$ and $\mathcal{A}$ were both attached to (i.e., Lie transported by) fiducial observers, then $\underset{\sim}{M} \cdot d\underset{\sim}{\Sigma}$ would be a 3-volume attached to them, and $\theta\underset{\sim}{M} \cdot d\underset{\sim}{\Sigma}$ would be the time rate of change of this 3-volume due to the fiducial expansion θ; the second term accounts for this change. The third term accounts for the displacement $\underset{\sim}{v}\Delta\tau$ of points on the interior of $\mathcal{A}$ relative to fiducial observers — the integral of $\underset{\sim}{\nabla} \cdot \underset{\sim}{M}$ over $(\underset{\sim}{v}\Delta\tau) \cdot d\underset{\sim}{\Sigma}$ = (volume through which $\mathcal{A}$ was displaced) can be converted by Gauss's theorem (2.22) to the difference in surface integrals between the displaced $\mathcal{A}$ and the fiducially transported $\mathcal{A}$. The fourth term accounts for the displacement $\underset{\sim}{v}\Delta\tau$ of the boundary of $\mathcal{A}$ relative to fiducial observers, which opens up a new area element $(\underset{\sim}{v}\Delta\tau) \times d\underset{\sim}{\ell}$. Dividing the above equation by Δt, taking the limit $\Delta t \to 0$, expressing the Lie derivative in terms of the Fermi time derivative by (2.16b), and using the vector identity $\underset{\sim}{A} \cdot \underset{\sim}{B} \times \underset{\sim}{C} = \underset{\sim}{A} \times \underset{\sim}{B} \cdot \underset{\sim}{C}$, we obtain the integral identity

$$\frac{d}{dt} \int_{\mathcal{A}(t)} \underset{\sim}{M} \cdot d\underset{\sim}{\Sigma} = \int_{\mathcal{A}(t)} \alpha [D_\tau \underset{\sim}{M} + \frac{2}{3}\theta \underset{\sim}{M} - \underset{\sim}{\sigma} \cdot \underset{\sim}{M} + (\underset{\sim}{\nabla} \cdot \underset{\sim}{M})\underset{\sim}{v}] \cdot d\underset{\sim}{\Sigma}$$

$$+ \int_{\partial\mathcal{A}(t)} \alpha \underset{\sim}{M} \times \underset{\sim}{v} \cdot d\underset{\sim}{\ell} \quad . \qquad (2.26)$$

Let M be a smoothly varying vector field in spacetime; and let $\mathcal{C}(t)$ be a closed curve in $\mathcal{S}_t$ which changes in some arbitrary but smooth manner as time passes. Then between time t and t+Δt the integral of $\underset{\sim}{M}$ over $\mathcal{C}(t)$ changes by

$$\Delta \int_{\mathcal{C}(t)} \underset{\sim}{M} \cdot d\underset{\sim}{\ell} = \int_{\mathcal{C}(t)} (\alpha\Delta t)(\mathcal{L}_\tau \underset{\sim}{M}) \cdot d\underset{\sim}{\ell}$$

$$+ \int_{\mathcal{C}(t)} 2\alpha\Delta t \, \underset{\sim}{M} \cdot (\frac{1}{3}\theta \underset{\sim}{\gamma} + \underset{\sim}{\sigma}) \cdot d\underset{\sim}{\ell} + \int_{\mathcal{C}(t)} (\underset{\sim}{\nabla} \times \underset{\sim}{M}) \cdot (\alpha\Delta t \, \underset{\sim}{v} \times d\underset{\sim}{\ell}) \; .$$

The first term accounts for changes of $\underset{\sim}{M}$ relative to Lie transport by fiducial observers;cf. Figure 2. If $\underset{\sim}{M}$ and $d\underset{\sim}{\ell}$ were both Lie transported by fiducial observers, then in time $\Delta\tau$ $\underset{\sim}{M} \cdot d\underset{\sim}{\ell}$ would change by $\Delta(\underset{\sim}{M} \cdot d\underset{\sim}{\ell}) = \underset{\sim}{M} \cdot (\Delta\tau \, \mathcal{L}_\tau \underset{\sim}{\gamma}) \cdot d\underset{\sim}{\ell} = 2\alpha\Delta t \, \underset{\sim}{M} \cdot (\underset{\sim}{\sigma} + \frac{1}{3}\theta \underset{\sim}{\gamma}) \cdot d\underset{\sim}{\ell}$ (Eq. 2.18b); the second term accounts for this. The third term accounts for the displacement of $\mathcal{C}(t)$ relative to fiducial observers, i.e., for the failure of $d\underset{\sim}{\ell}$ to be Lie transported; the integral of $\underset{\sim}{\nabla} \times \underset{\sim}{M}$ over the area $(\underset{\sim}{v}\Delta\tau) \times d\underset{\sim}{\ell}$ can be transformed by Stokes's theorem (2.23) into the integral of $\underset{\sim}{M}$ along the displaced $\mathcal{C}(t)$ minus the integral along the fiducially transported $\mathcal{C}(t)$. Dividing the above equation by Δt, taking the limit $\Delta t \to 0$, expressing the Lie derivative in terms of the Fermi time derivative by (2.16b), and using $\underset{\sim}{A} \cdot \underset{\sim}{B} \times \underset{\sim}{C} = \underset{\sim}{A} \times \underset{\sim}{B} \cdot \underset{\sim}{C}$, we obtain the integral identity

$$\frac{d}{dt} \int_{\mathcal{C}(t)} \underset{\sim}{M} \cdot d\underset{\sim}{\ell} = \int_{\mathcal{C}(t)} \alpha[D_\tau \underset{\sim}{M} + \frac{1}{3}\theta \underset{\sim}{M} + \underset{\sim}{\sigma} \cdot \underset{\sim}{M} + (\nabla \times \underset{\sim}{M}) \times \underset{\sim}{v}] \cdot d\underset{\sim}{\ell} \quad . \tag{2.27}$$

3 3+1 Electrodynamics in Differential Form

3.1 ELECTROMAGNETIC QUANTITIES

The 3+1 formulation of electrodynamics involves the following quantities, which are measured by the fiducial observers in the usual manner of flat spacetime, and which therefore have the usual physical interpretation:

$$\begin{aligned}
\rho_e &= \text{charge density} && (\text{esu/cm}^3)\\
\underset{\sim}{j} &= \text{current density} && (\text{esu/cm}^3)\\
\underset{\sim}{E} &= \text{electric field} && (\text{statvolts/cm})\\
\underset{\sim}{B} &= \text{magnetic field} && (\text{gauss})\\
\phi &= \text{scalar potential} && (\text{statvolts})\\
\underset{\sim}{A} &= \text{vector potential} && (\text{gauss cm, or statvolts}) \quad .
\end{aligned} \tag{3.1}$$

One can reconstruct the charge-current 4-vector J^α, the electromagnetic field tensor $F^{\alpha\beta}$, and the 4-vector potential $\mathfrak{A}^\alpha$ from these 3+1 quantities, the fiducial 4-velocity U^α, and the 4-dimensional Levi-Civita tensor $\varepsilon_{\alpha\beta\gamma\delta}$ by regarding $\underset{\sim}{j}$, $\underset{\sim}{E}$, $\underset{\sim}{B}$, and $\underset{\sim}{A}$ as 4-vectors orthogonal to U^α and then computing

$$\begin{aligned}
J^\alpha &= \rho_e U^\alpha + j^\alpha \quad ,\\
F^{\alpha\beta} &= U^\alpha E^\beta - E^\alpha U^\beta + \varepsilon^{\alpha\beta\gamma\delta} U_\gamma B_\delta \quad ,\\
\mathfrak{A}^\alpha &= \phi U^\alpha + A^\alpha \quad .
\end{aligned} \tag{3.2}$$

One can invert these relations to get

$$\rho_e = -J^{\alpha}U_{\alpha} \quad , \qquad j^{\alpha} = \gamma^{\alpha\beta} J_{\beta} \quad ,$$
$$E^{\alpha} = F^{\alpha\beta}U_{\beta} \quad , \qquad B^{\alpha} = -\frac{1}{2}\,\varepsilon^{\alpha\beta\gamma\delta}\, U_{\beta}\, F_{\gamma\delta} \quad , \tag{3.3}$$
$$\phi = -\mathfrak{A}^{\alpha}U_{\alpha} \quad , \qquad A^{\alpha} = \gamma^{\alpha\beta}\,\mathfrak{A}_{\beta} \quad .$$

3.2 MAXWELL'S EQUATIONS

Ellis (1973) has derived Maxwell's equations in the 3+1 congruence formalism from their 4-dimensional formulations $F^{\alpha\beta}{}_{;\beta} = 4\pi J^{\alpha}$ and $F_{[\alpha\beta;\gamma]} = 0$. We can take his 3+1 equations over into our hypersurface formalism by simply setting the fiducial rotation $\underset{\sim}{\omega}$ to zero. The result is

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{E} = 4\pi\rho_e \quad , \tag{3.4a}$$

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{B} = 0 \quad , \tag{3.4b}$$

$$D_{\tau}\underset{\sim}{E} + \frac{2}{3}\theta\,\underset{\sim}{E} - \underset{\sim}{\sigma} \cdot \underset{\sim}{E} = \alpha^{-1}\,\underset{\sim}{\nabla} \times (\alpha\underset{\sim}{B}) - 4\pi\underset{\sim}{j} \quad , \tag{3.4c}$$

$$D_{\tau}\underset{\sim}{B} + \frac{2}{3}\theta\,\underset{\sim}{B} - \underset{\sim}{\sigma} \cdot \underset{\sim}{B} = -\alpha^{-1}\,\underset{\sim}{\nabla} \times (\alpha\underset{\sim}{E}) \quad . \tag{3.4d}$$

Equations (3.4a,b) have the form familiar from flat-spacetime, Lorentz-frame electrodynamics. They permit one (following Hanni and Ruffini 1973; Christodoulou and Ruffini 1973; and King et al. 1975) to characterize $\underset{\sim}{E}$ and $\underset{\sim}{B}$ by electric and magnetic field lines which lie in the hypersurfaces $\mathcal{S}_t$. The magnetic field lines never end ($\underset{\sim}{\nabla} \cdot \underset{\sim}{B} = 0$); the electric field lines terminate on electric charge ($\underset{\sim}{\nabla} \cdot \underset{\sim}{E} = 4\pi\rho_e$).

Equations (3.4c,d) have a slightly different form from the corresponding flat-spacetime, Lorentz-frame equations. The differences are due to the peculiar motion of the fiducial observers (expansion θ, shear $\underset{\sim}{\sigma}$, and acceleration $\underset{\sim}{a} = \underset{\sim}{\nabla} \ln \alpha$).

Consider first equation (3.4d). If the fiducial observers were to carry a perfectly conducting medium with them, then they would never see an electric field, and equation (3.4d) would become $D_\tau \mathbf{B} + \frac{2}{3}\theta \mathbf{B} - \boldsymbol{\sigma} \cdot \mathbf{B} = 0$. This is precisely the equation for the evolution of a magnetic field that is "frozen into" the conducting medium (cf. Cowling 1957 or Lichnerowicz 1967). The expansion θ of the fiducial observers moves the field lines apart with a "Hubble-type expansion rate" $\dot{\ell}/\ell = \frac{1}{3}\theta$, thereby reducing the field strength (conservation of flux), $D_\tau \mathbf{B} = -\frac{2}{3}\theta \mathbf{B}$. The shear $\boldsymbol{\sigma}$ rotates the frozen-in field lines relative to parallel transport (relative to directions defined by gyroscopes), $D_\tau \mathbf{B} = \boldsymbol{\sigma} \cdot \mathbf{B}$; this shearing also changes the distance between field lines and thereby changes the field strength, $D_\tau |\mathbf{B}| = \sigma_{jk} B_j B_k / |\mathbf{B}|$.

If the fiducial observers do not carry a perfectly conducting medium, then they can see an electric field whose curl produces a time-changing magnetic field $D_\tau \mathbf{B} = -\alpha^{-1} \boldsymbol{\nabla} \times (\alpha \mathbf{E}) = -\boldsymbol{\nabla} \times \mathbf{E} - \mathbf{a} \times \mathbf{E}$ (right side of equation 3.4d). The lapse function α gives rise to the unfamiliar term $D_\tau \mathbf{B} = -\mathbf{a} \times \mathbf{E}$, which has the following physical interpretation: Because the fiducial observers accelerate, they acquire in time $\Delta\tau$ a velocity $\mathbf{v} = \mathbf{a}\Delta\tau$ relative to their initial inertial frame. This motion, together with the electric field $\mathbf{E}$ in the initial inertial frame, causes the fiducial observers to see a changed magnetic field, $\Delta \mathbf{B} = -\mathbf{v} \times \mathbf{E} = -(\mathbf{a} \times \mathbf{E})\Delta\tau$.

The unfamiliar terms in equation (3.4c) have the same origin as those in equation (3.4d).

3.3 $\mathbf{E}$ AND $\mathbf{B}$ IN TERMS OF POTENTIALS

From the 4-vector relationship $F_{\alpha\beta} = \mathfrak{A}_{\beta;\alpha} - \mathfrak{A}_{\alpha;\beta}$ and from equations (3.2) and (3.3) one can derive the following expressions for $\mathbf{E}$ and $\mathbf{B}$ in terms of the scalar and vector potentials:

$$\mathbf{E} = -\alpha^{-1} \boldsymbol{\nabla}(\alpha\phi) - \left(D_\tau \mathbf{A} + \frac{1}{3}\theta \mathbf{A} + \boldsymbol{\sigma} \cdot \mathbf{A}\right) , \tag{3.5a}$$

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A} . \tag{3.5b}$$

When $\underset{\sim}{E}$ and $\underset{\sim}{B}$ are expressed in this manner, the two Maxwell equations (3.4b) and (3.4d) are automatically satisfied. [To verify (3.4b) is trivial; to verify (3.4d) is a somewhat lengthy calculation, making use of the identities (2.11), (2.20b), and (2.21b).]

3.4 CHARGE CONSERVATION

The 3+1 equation of charge conservation

$$D_\tau \rho_e + \rho_e \theta + \alpha^{-1} \underset{\sim}{\nabla} \cdot (\alpha \underset{\sim}{j}) = 0 \tag{3.6}$$

can be derived by a nontrivial calculation from the Maxwell equations (3.4a,c). The $\rho_e \theta$ term is the rate of decrease of ρ_e due to expansion of the fiducial congruence (volume element carried by observers gets bigger, so charge density decreases). The lapse function α gives rise to an unfamiliar term $(\alpha^{-1} \underset{\sim}{\nabla} \alpha) \cdot \underset{\sim}{j} = \underset{\sim}{a} \cdot \underset{\sim}{j}$, which is the rate at which current density $\underset{\sim}{j}$ gets Lorentz transformed into charge density ρ_e by the changing velocity of the fiducial observer.

3.5 EQUATION OF MOTION OF A CHARGED PARTICLE

Consider a particle with rest mass μ and charge q. Denote by $\underset{\sim}{v}$ its ordinary velocity, as measured in the local rest frame of the fiducial observer whom the particle is passing. Then $\underset{\sim}{v}$ and the particle's 3-momentum

$$\underset{\sim}{p} \equiv \mu \Gamma \underset{\sim}{v} \quad , \qquad \Gamma \equiv (1 - \underset{\sim}{v}^2)^{-1/2} \tag{3.7}$$

are 3-vectors lying in $\mathcal{S}_t$. The 4-dimensional equation of motion for a charged particle, when rewritten in 3+1 form, says

$$(D_\tau + \underset{\sim}{v} \cdot \underset{\sim}{\nabla}) \underset{\sim}{p} = -(\mu \Gamma \underset{\sim}{a} + \underset{\sim}{\sigma} \cdot \underset{\sim}{p} + \tfrac{1}{3} \theta \underset{\sim}{p}) + q(\underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B}) \quad . \tag{3.8}$$

Here $(D_\tau + \underset{\sim}{v} \cdot \underset{\sim}{\nabla}) \underset{\sim}{p}$ is the "convective derivative" of $\underset{\sim}{p}$ along the particle's world line — it is the rate of change of $\underset{\sim}{p}$ with respect to (i) Fermi transport from the particle's initial position in $\mathcal{S}_t$, along the observer's world line to $\mathcal{S}_{t+\Delta\tau/\alpha}$ [the D_τ part of (3.8)], followed by (ii) spatial parallel transport, in $\mathcal{S}_{t+\Delta\tau/\alpha}$, along $\underset{\sim}{v}\Delta\tau$ to the particle's new position.

The term $-(\mu\Gamma\underset{\sim}{a} + \underset{\sim}{\sigma}\cdot\underset{\sim}{p} + \frac{1}{3}\theta\underset{\sim}{p})$ on the right-hand side of equation (3.8) is an "inertial force" to compensate for the fact that the fiducial observers at the old and new positions of the particle have a relative velocity $\Delta\underset{\sim}{v} = (\underset{\sim}{a} + \underset{\sim}{\sigma}\cdot\underset{\sim}{v} + \frac{1}{3}\theta\underset{\sim}{v})\Delta\tau$ as seen by inertial observers. The term $q(\underset{\sim}{E} + \underset{\sim}{v}\times\underset{\sim}{B})$ is the usual Lorentz force in 3+1 notation.

3.6 CONSERVATION OF ENERGY AND MOMENTUM FOR ELECTROMAGNETIC FIELD AND A CONTINUOUS MEDIUM

Let $T^{\alpha\beta}$ be the stress-energy tensor of the electromagnetic field and/or of a continuous medium with which it interacts. Denote by ε the mass-energy density, by $\underset{\sim}{S}$ the energy flux, and by $\underset{\sim}{W}$ the stress tensor — all as measured in the fiducial reference frame:

$$\varepsilon = T^{\mu\nu} U_\mu U_\nu \quad , \qquad S^\alpha = -\gamma^\alpha{}_\mu T^{\mu\nu} U_\nu \quad , \qquad W^{\alpha\beta} = \gamma^\alpha{}_\mu T^{\mu\nu} \gamma^\beta{}_\nu \, . \tag{3.9}$$

For the electromagnetic field

$$\varepsilon = \frac{1}{8\pi}(\underset{\sim}{E}^2 + \underset{\sim}{B}^2) \quad , \qquad \underset{\sim}{S} = \frac{1}{4\pi}\underset{\sim}{E}\times\underset{\sim}{B} \, ,$$
$$\underset{\sim}{W} = \frac{1}{4\pi}\left[-(\underset{\sim}{E}\otimes\underset{\sim}{E} + \underset{\sim}{B}\otimes\underset{\sim}{B}) + \frac{1}{2}(\underset{\sim}{E}^2 + \underset{\sim}{B}^2)\underset{\sim}{\gamma}\right] . \tag{3.10}$$

For a perfect fluid with rest-frame density of mass-energy ρ and pressure p and with velocity $\underset{\sim}{v}$ as measured by fiducial observers

$$\varepsilon = \Gamma^2(\rho + p\underset{\sim}{v}^2) \quad , \qquad \underset{\sim}{S} = (\rho + p)\Gamma^2\underset{\sim}{v} \quad ,$$
$$\underset{\sim}{W} = (\rho + p)\Gamma^2\underset{\sim}{v}\otimes\underset{\sim}{v} + p\underset{\sim}{\gamma} \, , \qquad \Gamma \equiv (1 - \underset{\sim}{v}^2)^{-1/2} \, . \tag{3.11}$$

The 3+1 split of the law of energy-momentum conservation $T^{\alpha\beta}{}_{;\beta} = 0$ has been worked out and applied in a variety of contexts by workers in numerical relativity; see, e.g., York (1979); Smarr, Taubes, and Wilson (1980); Wilson (1977). We record here in our notation York's (1979) general form of the law of energy conservation $U_\mu T^{\mu\nu}{}_{;\nu} = 0$:

$$D_\tau \varepsilon + \theta\varepsilon + \alpha^{-2}\underset{\sim}{\nabla}\cdot(\alpha^2 \underset{\sim}{S}) + W^{jk}(\sigma_{jk} + \tfrac{1}{3}\theta\,\gamma_{jk}) = 0 \quad , \tag{3.12}$$

and his general form of the law of momentum conservation (force balance) $\gamma^\alpha{}_\mu T^{\mu\nu}{}_{;\nu} = 0$:

$$D_\tau \underset{\sim}{S} + \tfrac{4}{3}\theta\underset{\sim}{S} + \underset{\sim}{\sigma}\cdot\underset{\sim}{S} + \varepsilon\underset{\sim}{a} + \alpha^{-1}\underset{\sim}{\nabla}\cdot(\alpha\underset{\sim}{W}) = 0 \quad , \tag{3.13}$$

cf., York's equations (40) and (41). Here ε, $\underset{\sim}{S}$, and $\underset{\sim}{W}$ contain all forms of energy, momentum, and stress.

The analogous equations describing the energy and momentum transfer from matter to electromagnetic fields, $U_\mu T^{\mu\nu}_{EM;\nu} = -U_\mu F^{\mu\nu} J_\nu$ and $\gamma^\alpha{}_\mu T^{\mu\nu}_{EM;\nu} = -\gamma^\alpha{}_\mu F^{\mu\nu} J_\nu$, have the 3+1 form

$$D_\tau \varepsilon + \theta\varepsilon + \alpha^{-2}\underset{\sim}{\nabla}\cdot(\alpha^2\underset{\sim}{S}) + W^{jk}(\sigma_{jk} + \tfrac{1}{3}\theta\,\gamma_{jk}) = -\underset{\sim}{j}\cdot\underset{\sim}{E} \quad , \tag{3.14}$$

$$D_\tau \underset{\sim}{S} + \tfrac{4}{3}\theta\underset{\sim}{S} + \underset{\sim}{\sigma}\cdot\underset{\sim}{S} + \varepsilon\underset{\sim}{a} + \alpha^{-1}\underset{\sim}{\nabla}\cdot(\alpha\underset{\sim}{W}) = -(\rho_e\underset{\sim}{E} + \underset{\sim}{j}\times\underset{\sim}{B}) \quad . \tag{3.15}$$

Here ε, $\underset{\sim}{S}$, and $\underset{\sim}{W}$ are the electromagnetic energy density, momentum density and stress (equations 3.10).

4 3+1 Electrodynamics in Integral Form

As in flat spacetime, so also in curved spacetime, one can use integral identities to rewrite in integral form the differential Maxwell equations (3.4) and the law of charge conservation (3.6).

Gauss's law for electric flux follows from $\underset{\sim}{\nabla}\cdot\underset{\sim}{E} = 4\pi\rho_e$ and Gauss's integral identity (2.22). It says that the total electric flux through a closed

2-surface $\partial\mathcal{V}$ lying in a fiducial hypersurface $\mathcal{S}$ is equal to 4π times the total charge enclosed

$$\int_{\partial\mathcal{V}} \underset{\sim}{E} \cdot d\underset{\sim}{\Sigma} = 4\pi \int_{\mathcal{V}} \rho_e \, dV \quad . \tag{4.1}$$

Similarly, Gauss's law for magnetic flux, which is equivalent to $\underset{\sim}{\nabla} \cdot \underset{\sim}{B} = 0$, says that the total flux through any closed 2-surface in $\mathcal{S}$ must vanish,

$$\int_{\partial\mathcal{V}} \underset{\sim}{B} \cdot d\underset{\sim}{\Sigma} = 0 \quad . \tag{4.2}$$

Faraday's law of magnetic induction can be derived by applying the integral identity (2.26) to the Maxwell equation (3.4d), and by then replacing $\underset{\sim}{\nabla} \cdot \underset{\sim}{B}$ by zero and using Stokes's law (2.23) to rewrite the surface integral of $\underset{\sim}{\nabla} \times (\alpha \underset{\sim}{E})$. The result is

$$\int_{\partial\mathcal{A}(t)} \alpha(\underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B}) \cdot d\underset{\sim}{\ell} = -\frac{d}{dt} \int_{\mathcal{A}(t)} \underset{\sim}{B} \cdot d\underset{\sim}{\Sigma} \quad . \tag{4.3}$$

Here $\mathcal{A}(t)$ is a 2-surface lying in $\mathcal{S}_t$; $\partial\mathcal{A}(t)$ is the closed boundary curve of $\mathcal{A}(t)$, and $\underset{\sim}{v}$ is the velocity of a point on the boundary curve as measured by the fiducial observer whom it is passing. As in flat spacetime, so also here, Faraday's law says that the time changing magnetic flux through a curve $\partial\mathcal{A}$ generates on EMF around the curve. The derivative of the flux in this case is with respect to global time t (the only universally defined time parameter, and therefore the only kind of time with respect to which one can differentiate outside the flux integral). The EMF is the integral around the curve $\partial\mathcal{A}$ of the electromagnetic force $\underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B}$ acting on a unit charge which moves with the curve, multiplied by $\alpha = d\tau/dt$ to convert the force into a "rate of change of momentum $\underset{\sim}{p}$ with respect to global time t" instead of "with respect to fiducial proper time τ".

Ampere's law can be derived by applying the integral identity (2.26) to the Maxwell equation (3.4c), and by then replacing $\underset{\sim}{\nabla} \cdot \underset{\sim}{E}$ with $4\pi\rho_e$ and using Stokes's law (2.23) to rewrite the surface integral of $\utilde{\nabla} \times (\alpha \utilde{B})$. The result is

$$\int_{\partial \mathcal{A}(t)} \alpha(\utilde{B} - \utilde{v} \times \utilde{E}) \cdot d\utilde{\ell} = \frac{d}{dt} \int_{\mathcal{A}(t)} \utilde{E} \cdot d\utilde{\Sigma} + 4\pi \int_{\mathcal{A}(t)} \alpha(\utilde{j} - \rho_e \utilde{v}) \cdot d\utilde{\Sigma} \quad . \tag{4.4}$$

The left side and the first term on the right are identical to Faraday's law (4.3) plus a "duality transformation" $\utilde{E} \to \utilde{B}$, $\utilde{B} \to -\utilde{E}$. The last term is 4π times the rate per unit global time that charge crosses the moving area $\mathcal{A}(t)$. [Note: $\utilde{v}$ is the velocity of a point on $\mathcal{A}(t)$ as measured by fiducial observers.]

The integral law of charge conservation can be derived by integrating the differential conservation law (3.6) over $\mathcal{V}(t)$ and by then using the integral identities (2.25) and (2.22). The result is

$$\frac{d}{dt} \int_{\mathcal{V}(t)} \rho_e dV = - \int_{\partial \mathcal{V}(t)} \alpha(\utilde{j} - \rho_e \utilde{v}) \cdot d\utilde{\Sigma} \quad . \tag{4.5}$$

Here $\mathcal{V}(t)$ is a 3-volume lying in $\mathcal{S}_t$; $\partial\mathcal{V}(t)$ is the closed 2-surface boundary of $\mathcal{V}(t)$, and $\utilde{v}$ is the velocity of a point on the boundary 2-surface as measured by the fiducial observer whom it is passing. The left side of this conservation law is the rate of increase, per unit global time t, of the charge in $\mathcal{V}(t)$. The right side is the rate, per unit global time t, at which charge flows in through the moving boundary of $\mathcal{V}(t)$.

5 3+1 Electrodynamics Outside a Stationary Black Hole

5.1 THE ZAMO REFERENCE FRAMES AND THE CHOICE OF GLOBAL TIME

We now specialize to the spacetime region $\mathcal{E}$ outside a stationary, axisymmetric black hole: $\mathcal{E}$ extends from the hole's absolute event horizon $\mathcal{H}$ out to spatial and null infinity. We require that the spacetime geometry of $\mathcal{E}$ be stationary and axisymmetric. It will be the Kerr geometry if the hole's gravity is far stronger than the gravity due to external matter. Otherwise, the external matter will deform the geometry away from that of Kerr.

The theory of stationary, axisymmetric black holes is reviewed by Carter (1979). We adopt his notation $\mathbf{k}$ and $\mathbf{m}$ for the mutually commuting Killing vector fields which generate invariant "time translations" and invariant "rotations about the axis of symmetry". Far from the hole $\mathbf{k}^2 \to -1$ and $\mathbf{m}^2 \to (\text{distance from axis of symmetry})^2$. Inside the hole's ergosphere $\mathbf{k}$ is spacelike, but the Killing vector

$$\boldsymbol{\ell} \equiv \mathbf{k} + \Omega^H \mathbf{m} \tag{5.1}$$

(where Ω^H is the hole's angular velocity) is timelike. As one approaches the horizon, $\boldsymbol{\ell}$ becomes tangent to the horizon's null-geodesic generators.

We shall require that our fiducial congruence, hypersurfaces, and global time parameter mesh with the hole's stationary exterior geometry in the following senses: (i) The fiducial congruence completely covers the exterior region $\mathcal{E}$. (ii) Each fiducial observer moves along a Killing direction, so that he sees a forever unchanging spacetime geometry in his neighborhood. (iii) The hypersurfaces of simultaneity all have identical spatial geometries. (iv) Far from the hole the global time parameter t becomes equal to proper time τ as measured by the fiducial observers.

These four demands fix the fiducial congruence, hypersurfaces, and global time parameter uniquely (up to the addition of a constant to t): The fiducial observers are the "zero-angular-momentum observers" (ZAMOs) of Bardeen, Press, and Teukolsky (1973). Their 4-velocities can be expressed in terms of the Killing vectors $\mathbf{k}$ and $\mathbf{m}$ as

$$\mathbf{U} = \alpha^{-1}(\mathbf{k} + \omega\mathbf{m}) \quad , \tag{5.2}$$

where ω is the ZAMO angular velocity

$$\omega = -\mathbf{k}\cdot\mathbf{m}/\mathbf{m}^2 \quad , \tag{5.3}$$

and where α, the lapse function, can be expressed as

$$\alpha = [-\boldsymbol{k}^2 + (\boldsymbol{k}\cdot\boldsymbol{m})^2/\boldsymbol{m}^2]^{1/2} \quad . \tag{5.4}$$

The rotational Killing vector $\boldsymbol{m}$ is orthogonal to $\boldsymbol{U}$ and thus lies in the fiducial hypersurfaces $\mathcal{S}_t$ and can be regarded as a spatial vector $\underset{\sim}{m}$.

For the Schwarzschild geometry of a nonrotating black hole the global time parameter t is equal to the standard Schwarzschild time coordinate t (MTW chapter 31). For the Kerr geometry, t is the Boyer-Lindquist time coordinate (MTW chapter 33).

The congruence of Killing trajectories generated by $\boldsymbol{k}$ threads its way from one hypersurface $\mathcal{S}_t$ to the next and the next in a non-orthogonal manner. The equations of black-hole electrodynamics will take on a particularly simple form if we express them in terms of the time derivative $\mathcal{L}_t$ along this "shifting Killing congruence", rather than in terms of the Fermi time derivative D_τ along the fiducial world lines. To make $\mathcal{L}_t$ differentiate along $\boldsymbol{k} = \alpha\boldsymbol{U} - \omega\boldsymbol{m}$ we must choose as our shift vector

$$\underset{\sim}{\beta} = -\omega\underset{\sim}{m} \quad . \tag{5.5}$$

The magnitude, $|\underset{\sim}{\beta}|/\alpha$, of the ordinary velocity associated with this shift vector will be greater than the speed of light near the black hole, and less far from the hole. With this choice of shift vector, the 3+1 splits of the Killing equations $k_{(\alpha;\beta)} = 0$ and $m_{(\alpha;\beta)} = 0$ along with the mutual commutivity of $\boldsymbol{k}$ and $\boldsymbol{m}$, imply the following

$$\mathcal{L}_t\alpha = \mathcal{L}_t\omega = 0 \quad , \tag{5.6a}$$

$$\underset{\sim}{m}\cdot\underset{\sim}{\nabla}\alpha = \underset{\sim}{m}\cdot\underset{\sim}{\nabla}\omega = 0 \quad , \tag{5.6b}$$

$$\mathcal{L}_t\underset{\sim}{m} = 0 \quad , \tag{5.6c}$$

$$m_{j|k} + m_{k|j} = 0 \quad , \tag{5.6d}$$

$$\mathcal{L}_t \gamma_{jk} = 0 \quad , \tag{5.6e}$$

$$\theta = 0 \quad , \tag{5.6f}$$

$$\underset{\sim}{\sigma} = \frac{1}{2} \alpha^{-1} [\underset{\sim}{m} \otimes (\underset{\sim}{\nabla\omega}) + (\underset{\sim}{\nabla\omega}) \otimes \underset{\sim}{m}] \quad . \tag{5.6g}$$

Equations (5.5) and (5.6b,f,g), together with (2.16b,c), imply the following relationship between the fiducial observers' Fermi time derivative and the Lie time derivative $\mathcal{L}_t$ along $\mathbf{k}$:

$$D_\tau \underset{\sim}{M} = \alpha^{-1}[\mathcal{L}_t \underset{\sim}{M} + \omega \overset{\vee}{\mathcal{L}}_{\underset{\sim}{m}} \underset{\sim}{M} + \frac{1}{2} (\underset{\sim}{m} \times \underset{\sim}{\nabla\omega}) \times \underset{\sim}{M}] \quad . \tag{5.7}$$

Because our hypersurfaces $\mathcal{S}_t$ all have the same spatial geometry, and because our time derivatives $\mathcal{L}_t$ act along Killing trajectories, we can now abandon the spacetime viewpoint of relativity and return to the Galilean-Newtonian viewpoint that physics occurs in an absolute 3-dimensional space $\mathcal{S}$. As in Galilean-Newtonian physics there is a universal time parameter t which marks the evolution of fields and particles in $\mathcal{S}$; but t is no longer united with $\mathcal{S}$ in a 4-dimensional spacetime structure. We shall adopt this Galilean-Newtonian viewpoint in Paper II; but for the remainder of Paper I we shall retain the spacetime viewpoint, using it as a tool in deriving further features of the 3+1 formalism.

5.2 ELECTRODYNAMIC EQUATIONS

The Maxwell equations (3.4) can be brought into the following form by use of equations (5.6f,g) and (5.7)

$$\nabla \cdot \mathbf{E} = 4\pi\rho_e \quad , \tag{5.8a}$$

$$\nabla \cdot \mathbf{B} = 0 \quad , \tag{5.8b}$$

$$\mathcal{L}_t \mathbf{E} + \omega \mathcal{L}_{\mathbf{m}} \mathbf{E} - (\mathbf{E} \cdot \nabla\omega)\mathbf{m} = \nabla \times (\alpha \mathbf{B}) - 4\pi\alpha \mathbf{j} \quad , \tag{5.8c}$$

$$\mathcal{L}_t \mathbf{B} + \omega \mathcal{L}_{\mathbf{m}} \mathbf{B} - (\mathbf{B} \cdot \nabla\omega)\mathbf{m} = -\nabla \times (\alpha \mathbf{E}) \quad . \tag{5.8d}$$

Expressions (3.5) for $\mathbf{E}$ and $\mathbf{B}$ in terms of the scalar and vector potentials similarly can be brought into the form

$$\mathbf{E} = \alpha^{-1}(\nabla A_0 + \omega \nabla A_\varphi) - \alpha^{-1}(\mathcal{L}_t \mathbf{A} + \omega \mathcal{L}_{\mathbf{m}} \mathbf{A}) \quad , \tag{5.9a}$$

$$\mathbf{B} = \nabla \times \mathbf{A} \quad , \tag{5.9b}$$

where

$$A_\varphi \equiv \mathbf{A} \cdot \mathbf{m} \quad , \tag{5.10a}$$

$$A_0 \equiv -\alpha\phi - \omega A_\varphi = (\text{4-vector potential } \mathfrak{A}) \cdot \mathfrak{k} \quad . \tag{5.10b}$$

(The notation A_φ and A_0 is motivated by the fact that one will often use co-ordinate systems in which $\mathbf{m} = \partial/\partial\varphi$ and $\mathfrak{k} = \partial/\partial t$.)

The law of charge conservation (3.6) can be rewritten, using expressions (2.16a), (5.5), and (5.6f), as

$$\mathcal{L}_t \rho_e + \omega \mathbf{m} \cdot \nabla\rho_e + \nabla \cdot (\alpha \mathbf{j}) = 0 \quad . \tag{5.11}$$

Equations (5.8) - (5.11) will simplify considerably if the electromagnetic field is stationary and axisymmetric. Then all terms involving $\mathcal{L}_t$, $\mathcal{L}_{\mathbf{m}}$, and $\mathbf{m} \cdot \nabla$ will vanish.

The equation of motion (3.8) for a test particle acted on by the electromagnetic field becomes, upon using expressions (2.4), (5.6f,g), and (5.7),

$$[\mathcal{L}_t + (\alpha \underset{\sim}{v} + \omega \underset{\sim}{m}) \cdot \underset{\sim}{\nabla}]\underset{\sim}{p} = \omega(\underset{\sim}{p} \cdot \underset{\sim}{\nabla})\underset{\sim}{m} - (\underset{\sim}{p} \cdot \underset{\sim}{m})\underset{\sim}{\nabla}\omega - \mu\Gamma\underset{\sim}{\nabla}\alpha + \alpha q(\underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B}) \quad . \tag{5.12}$$

Here μ and q are the particle's rest mass and charge; $\underset{\sim}{v}$ is its velocity as measured by the ZAMOs; $\alpha\underset{\sim}{v} + \omega\underset{\sim}{m}$ is its velocity, on a per unit global time basis d/dt, with respect to the Killing trajectories of $\mathbb{k}$; and $\underset{\sim}{p} = \mu\Gamma\underset{\sim}{v} = \mu\underset{\sim}{v}(1 - \underset{\sim}{v}^2)^{-1/2}$ is its momentum as measured by the ZAMOs.

The differential laws of energy conservation (3.12) and (3.14) for an electromagnetic field and/or a continuous medium become, upon using expressions (2.16a), (5.5), and (5.6f,g),

$$\begin{aligned} \mathcal{L}_t\varepsilon + \omega\underset{\sim}{m} \cdot \underset{\sim}{\nabla}\varepsilon + \alpha^{-1}\underset{\sim}{\nabla} \cdot (\alpha^2\underset{\sim}{S}) + \underset{\sim}{m} \cdot \underset{\sim}{W} \cdot \underset{\sim}{\nabla}\omega &= 0 \quad \text{if all stress-energy is included in } \varepsilon, \underset{\sim}{S}, \underset{\sim}{W} \\ &= -\alpha\underset{\sim}{j} \cdot \underset{\sim}{E} \quad \text{if only electromagnetic stress-energy is included;} \end{aligned} \tag{5.13}$$

and the laws of momentum conservation (3.13) and (3.15) become, upon using (2.4), (5.6f,g), and (5.7),

$$\begin{aligned} \mathcal{L}_t\underset{\sim}{S} + \omega\mathcal{L}_{\underset{\sim}{m}}\underset{\sim}{S} + (\underset{\sim}{S} \cdot \underset{\sim}{m})\underset{\sim}{\nabla}\omega + \varepsilon\underset{\sim}{\nabla}\alpha + \underset{\sim}{\nabla} \cdot (\alpha\underset{\sim}{W}) &= 0 \quad \text{all included} \\ &= -\alpha(\rho_e\underset{\sim}{E} + \underset{\sim}{j} \times \underset{\sim}{B}) \quad \text{only electromagnetism included .} \end{aligned} \tag{5.14}$$

These differential equations cannot be converted into integral conservation laws. However, there do exist integral conservation laws associated with two special combinations of these equations — combinations associated with the two Killing vector fields $\mathbb{k}$ and $\mathbb{m}$. Associated with $\mathbb{k}$ is a conserved "redshifted energy" or "energy at infinity" with energy density $\varepsilon_E = T^{\mu\nu}k_\mu U_\nu$, i.e. [cf. equations (5.2), (3.9), (3.10)]

$$\begin{aligned} \varepsilon_E &= \alpha\varepsilon + \omega\underset{\sim}{S} \cdot \underset{\sim}{m} \quad \text{in general} \\ &= \frac{\alpha}{8\pi}(\underset{\sim}{E}^2 + \underset{\sim}{B}^2) + \frac{\omega}{4\pi}(\underset{\sim}{E} \times \underset{\sim}{B}) \cdot \underset{\sim}{m} \quad \text{for electromagnetism ,} \end{aligned} \tag{5.15a}$$

and with energy flux $S^{\alpha}_{E} = -\gamma^{\alpha}{}_{\mu} T^{\mu\nu} k_{\nu}$, i.e.

$$\underset{\sim}{S}_E = \alpha \underset{\sim}{S} + \omega \underset{\sim}{W} \cdot \underset{\sim}{m} \qquad \text{in general}$$

$$= \frac{1}{4\pi} [\alpha \underset{\sim}{E} \times \underset{\sim}{B} - \omega(\underset{\sim}{E} \cdot \underset{\sim}{m})\underset{\sim}{E} - \omega(\underset{\sim}{B} \cdot \underset{\sim}{m})\underset{\sim}{B} + \frac{1}{2}\omega(\underset{\sim}{E}^2 + \underset{\sim}{B}^2)\underset{\sim}{m}] \quad \text{for electromagnetism.} \tag{5.15b}$$

Associated with m is a conserved "angular momentum about the hole's symmetry axis" with density $\varepsilon_L = -T^{\mu\nu} m_{\mu} U_{\nu}$, i.e. [cf. equations (3.9) and (3.10)]

$$\varepsilon_L = \underset{\sim}{S} \cdot \underset{\sim}{m} \qquad \text{in general}$$

$$= \frac{1}{4\pi} (\underset{\sim}{E} \times \underset{\sim}{B}) \cdot \underset{\sim}{m} \qquad \text{for electromagnetism,} \tag{5.16a}$$

and with flux $S^{\alpha}_{L} = +\gamma^{\alpha}{}_{\mu} T^{\mu\nu} m_{\nu}$, i.e.

$$\underset{\sim}{S}_L = \underset{\sim}{W} \cdot \underset{\sim}{m} \qquad \text{in general}$$

$$= \frac{1}{4\pi} [-(\underset{\sim}{E} \cdot \underset{\sim}{m})\underset{\sim}{E} - (\underset{\sim}{B} \cdot \underset{\sim}{m})\underset{\sim}{B} + \frac{1}{2}(\underset{\sim}{E}^2 + \underset{\sim}{B}^2)\underset{\sim}{m}] \qquad \text{for electromagnetism.} \tag{5.16b}$$

The differential and integral conservation laws for redshifted energy and for angular momentum have the same form as those for electric charge [equations (5.11) and (4.5)]:

$$\mathcal{L}_t \varepsilon_E + \omega \underset{\sim}{m} \cdot \underset{\sim}{\nabla} \varepsilon_E + \underset{\sim}{\nabla} \cdot (\alpha \underset{\sim}{S}_E) = 0 \quad \text{if all stress-energy is included in } \varepsilon_E, \underset{\sim}{S}_E$$

$$= -\alpha^2 \underset{\sim}{j} \cdot \underset{\sim}{E} - \alpha\omega(\rho_e \underset{\sim}{E} + \underset{\sim}{j} \times \underset{\sim}{B}) \cdot \underset{\sim}{m} \quad \text{if only electromagnetic stress-energy is included,} \tag{5.17a}$$

$$\mathcal{L}_t \varepsilon_L + \omega \underset{\sim}{m} \cdot \underset{\sim}{\nabla} \varepsilon_L + \underset{\sim}{\nabla} \cdot (\alpha \underset{\sim}{S}_L) = 0 \quad \text{all included}$$

$$= -\alpha(\rho_e \underset{\sim}{E} + \underset{\sim}{j} \times \underset{\sim}{B}) \cdot \underset{\sim}{m} \quad \text{only electromagnetism included;} \tag{5.17b}$$

$$\frac{d}{dt} \int_{\mathcal{V}(t)} \varepsilon_E dV + \int_{\partial\mathcal{V}(t)} \alpha(\underset{\sim}{S}_E - \varepsilon_E \underset{\sim}{v}) \cdot d\underset{\sim}{\Sigma} = 0 \quad \text{all included}$$

$$= -\int_{\mathcal{V}(t)} [\alpha^2 \underset{\sim}{j} \cdot \underset{\sim}{E} + \alpha\omega(\rho_e \underset{\sim}{E} + \underset{\sim}{j} \times \underset{\sim}{B}) \cdot \underset{\sim}{m}] dV \quad \text{only electromagnetism included,} \tag{5.18a}$$

$$\frac{d}{dt}\int_{\mathscr{V}(t)} \varepsilon_L dV + \int_{\partial\mathscr{V}(t)} \alpha(\underset{\sim}{S}_L - \varepsilon_L \underset{\sim}{v}) \cdot d\underset{\sim}{\Sigma} = 0 \quad \text{all included}$$

$$= - \int_{\mathscr{V}(t)} \alpha(\rho_e \underset{\sim}{E} + \underset{\sim}{j} \times \underset{\sim}{B}) \cdot \underset{\sim}{m}\, dV \quad \text{only electromagnetism included} . \tag{5.18b}$$

The integral formulations of Maxwell's equations [Gauss's laws (4.1) and (4.2), Faraday's law (4.3), Ampere's law (4.4), and charge conservation (4.5)] do not simplify when we specialize to stationary, axially symmetric spacetimes.

5.3 SPACETIME STRUCTURE NEAR THE HORIZON

Our foliation of hypersurfaces becomes pathological near the horizon of the black hole. The pathology can be understood most clearly in the simple case of a nonrotating Schwarzschild black hole with gravitational radius 2M (MTW chapter 31). Figure 4 is a spacetime diagram for the hole's exterior $\mathcal{E}$ $(r/2M > 1)$ and interior $(r/2M < 1)$ in ingoing Eddington-Finkelstein coordinates $\tilde{t}$ and r (Box 31.2 of MTW). The key feature of this coordinate system, for our purposes, is the fact that it is well behaved everywhere except at the $r = 0$ singularity; all the metric coefficients in the line element

$$ds^2 = -d\tilde{t}^2 + dr^2 + (2M/r)(d\tilde{t} + dr)^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2) \tag{5.19}$$

are of order unity outside, on, and near the horizon $\mathscr{H}$ $(r = 2M)$. The light cones tilt near the horizon (light trapping), but do not squash down to slivers.

Our global time parameter t in Eddington-Finkelstein coordinates is given by

$$t = \tilde{t} - 2M \ln (r/2M - 1) \quad . \tag{5.20}$$

Curves of constant t (our hypersurfaces of simultaneity $\mathcal{S}_t$) are plotted in Figure 4. Note that our hypersurfaces sink deep into the past as they approach the horizon $\mathcal{H}$ $(r = 2M)$. This is a manifestation of the very slow rate at which our fiducial proper time τ marches forward near the horizon

$$d\tau/dt = \alpha = (1 - 2M/r)^{1/2} \to 0 \quad \text{at} \quad \mathcal{H} \quad , \tag{5.21}$$

and it is characteristic of all black-hole spacetimes, not just Schwarzschild.

Suppose that one (mathematically) approaches the horizon $r = 2M$ by moving inward along a fixed hypersurface of simultaneity $\mathcal{S}_t$. In principle, one will then explore the entire past history of the spacetime region just above the horizon. For a 10^8 solar mass hole ($2M = 3 \times 10^8$ km) one will see, plastered into the region between $r - 2M = 100$ microns and $r - 2M = 2$ microns, the near-horizon electromagnetic field structure laid down there $\Delta\tilde{t} = 10$ to 11 hours ago. Far beneath this, at $r - 2M = 2 \times 10^{-18}$ cm to 4×10^{-20} cm, one will see the structure characteristic of $\Delta\tilde{t} = 20$ to 21 hours ago. These structures will be layered down one after another like ancient sediment deposits on the bottom of the sea.

In view of this multilayered structure, how can we define the "limits of $\underset{\sim}{E}$ and $\underset{\sim}{B}$ as one approaches the horizon"? In principle, we can choose any layer we wish as the horizon limit. We need only note which part of the horizon (which $\tilde{t}$) is near the layer chosen, and announce our results as the limiting horizon fields at that specific moment of $\tilde{t}$ time.

In practice, the 3+1 formalism of this paper will probably be useful only when the external electromagnetic field evolves very slowly compared to $2M \simeq (17 \text{ minutes}) \cdot (M/10^8 M_\odot)$ ["quasi-stationary" evolution]. In this case the horizon structure being laid down now (in $\tilde{t}$ time) will extend so deep [to $r - 2M \lesssim 10^{-18}$ cm if the evolution timescale is $\Delta\tilde{t} \gtrsim (20 \text{ hours}) \cdot (M/10^8 M_\odot)$] that previous structures can be totally ignored. One can pretend that the present structure extends all the way in to $r = 2M$.

Although the above discussion is couched in the language, formulae, and numbers of the Schwarzschild geometry, its qualitative features will be the same for any stationary, axially symmetric hole.

In the generic case, as one approaches the horizon $\mathcal{H}$, one sees the lapse function go to zero and the fiducial (ZAMO) congruence become null

$$\left.\begin{aligned} \alpha &= (d\tau/dt)_{\text{along congruence}} \to 0 \\ \omega &\to \Omega^H = (\text{angular velocity of hole}) \\ \alpha\mathbf{U} &\to \boldsymbol{\ell} = \mathbf{k} + \Omega^H \mathbf{m} = \begin{pmatrix}\text{tangent to null} \\ \text{generator of } \mathcal{H}\end{pmatrix} \end{aligned}\right\} \quad \text{at } \mathcal{H} \quad . \tag{5.22a}$$

One can show, using the formulas on pages 251 and 252 of Bardeen (1973), that near the horizon the magnitude $a \equiv |\underset{\sim}{a}|$ of the acceleration of the fiducial ZAMO congruence behaves as

$$\alpha a \equiv \alpha|\underset{\sim}{a}| = \kappa + 0(\alpha^2) \to \kappa \text{ at } \mathcal{H} \quad . \tag{5.22b}$$

Here κ is the "surface gravity" of the hole. Since $\alpha \to 0$, a must become infinite on $\mathcal{H}$: the ZAMO observers near the horizon must accelerate like hell to avoid falling into the hole. The unit spatial vector along $\underset{\sim}{a}$

$$\underset{\sim}{n} \equiv \underset{\sim}{a}/a \quad , \tag{5.23}$$

when viewed as a 4-vector $\mathbf{n}$, collapses into the 4-velocity $\mathbf{U}$ as one approaches the horizon:

$$\alpha\mathbf{n} \text{ and } \alpha\mathbf{U} \text{ both} \to \boldsymbol{\ell} \quad \text{at } \mathcal{H} \quad . \tag{5.24}$$

One can take, as a pair of well-behaved basis vectors in the $\mathbf{n}\wedge\mathbf{U}$ 2-flats, $\alpha\mathbf{U}$ and any vector of the form

$$\boldsymbol{\xi} \equiv (1/\alpha)(\mathbf{U} - \mathbf{n}) + (\text{const})\, \alpha \mathbf{U} \quad . \tag{5.25}$$

[That $\boldsymbol{\xi}$ is finite at $\mathcal{H}$ follows from (i) the finiteness of $\alpha\mathbf{U} = \boldsymbol{\ell}$ at $\mathcal{H}$, (ii) the fact that $(1/\alpha)(\mathbf{U}-\mathbf{n})$ is null, and (iii) the fact that $(1/\alpha)(\mathbf{U}-\mathbf{n})\cdot\boldsymbol{\ell} = -1$ at $\mathcal{H}$.] At the horizon $\alpha\mathbf{U} = \boldsymbol{\ell}$ is tangent to $\mathcal{H}$, while $\boldsymbol{\xi}$ points inward through $\mathcal{H}$ (cf. Figure 4). It is convenient to fix the constant in the definition (5.25) of $\boldsymbol{\xi}$ so that

$$\tilde{t}_{,\mu}\, \xi^{\mu} = 0 \tag{5.26}$$

(i.e., $\boldsymbol{\xi}$ lies in the 3-surfaces of constant $\tilde{t}$), where

$$\tilde{t} \equiv t + \kappa^{-1} \ln \alpha \quad . \tag{5.27}$$

One can verify that the scalar field $\tilde{t}$ has finite derivatives along $\boldsymbol{\ell}$ and along $\boldsymbol{\xi}$ at $\mathcal{H}$, and thus is well behaved there. This $\tilde{t}$ is the generic analog of the Eddington–Finkelstein $\tilde{t}$ (equation 5.20). For generic black holes, as for Schwarzschild black holes, all physical quantities must approach well-behaved limits as one approaches the horizon along $\boldsymbol{\xi}$. Formally, we define

"$\to$" means "becomes equal to, as one approaches the horizon along a curve to which $\boldsymbol{\xi}$ is tangent — i.e., along a curve of constant $\tilde{t}$ in the $\mathbf{n}\wedge\mathbf{U}$ plane". (5.28)

Near the horizon one can introduce spacetime coordinates t, φ, α, and λ with these properties: $\partial/\partial t \equiv \mathbf{k}$; $\partial/\partial\varphi \equiv \mathbf{m}$; α is the lapse function and is therefore equal to zero everywhere on the horizon; and λ measures proper distance along the horizon from the rotation axis down towards the equator. In an immediate neighborhood of the horizon the spacetime line element will read

$$ds^2 = -\alpha^2\, dt^2 + \varpi^2(d\varphi - \Omega^H dt)^2 + \kappa^{-2}\, d\alpha^2 + d\lambda^2 \tag{5.29a}$$

where

$$\alpha = 0 \text{ at horizon}, \quad \kappa = (\text{surface gravity}) = \text{constant}, \tag{5.29b}$$

$\varpi \equiv |\underset{\sim}{m}|$ is a function of λ, Ω^H = (angular velocity of hole) = constant.

The spatial geometry of the fiducial hypersurface $\mathcal{S}_t$ near $\mathcal{H}$ is

$$ds^2 = \varpi^2\, d\varphi^2 + \kappa^{-2}\, d\alpha^2 + d\lambda^2 \quad ; \tag{5.30}$$

and the 2-geometry of $\mathcal{H}$ is

$$\underset{\mathcal{H}}{ds^2} = \varpi^2\, d\varphi^2 + d\lambda^2 \quad . \tag{5.31}$$

The fiducial observers (ZAMOs) near $\mathcal{H}$ move with angular velocity $\omega = \Omega^H$:

$$\varphi = \Omega^H t \ , \quad \lambda = \text{const} \ , \quad \alpha = \text{const} \quad \text{is a fiducial world line near } \mathcal{H}. \tag{5.32}$$

5.4 BOUNDARY CONDITIONS ON THE ELECTRIC AND MAGNETIC FIELDS AT THE HORIZON

Znajek (1976, 1978b) and Damour (1978, 1979, 1980) have constructed a beautiful theory of electromagnetic boundary conditions at horizons of black holes; see Carter (1979) for an excellent review. In this section we translate that theory into our 3+1 language.

Znajek and Damour define electric and magnetic fields that live in the horizon by

$$E^H_\alpha = F_{\alpha\beta}\, \ell^\beta \ , \quad B^H_\alpha = -{}^*F_{\alpha\beta}\, \ell^\beta \quad . \tag{5.33}$$

Here $F_{\alpha\beta}$ is the electromagnetic field tensor and ${}^*F_{\alpha\beta}$ is its dual. For comparison, the electric and magnetic fields of our 3+1 formalism are $E_\alpha = F_{\alpha\beta}U^\beta$, $B_\alpha = -{}^*F_{\alpha\beta}U^\beta$. Equation (5.22a) reveals the relationship between the two types of fields:

$$\left.\begin{array}{l} \alpha \underset{\sim}{E} \rightarrow \underset{\sim}{E}^{H} \\ \alpha \underset{\sim}{B} \rightarrow \underset{\sim}{B}^{H} \end{array}\right\} \text{ on } \mathcal{H} . \tag{5.34}$$

Here $\underset{\sim}{E}$, $\underset{\sim}{B}$, $\underset{\sim}{E}^{H}$, and $\underset{\sim}{B}^{H}$ are viewed as spatial vectors lying in the hypersurfaces of simultaneity $\mathcal{S}_t$; as one approaches the horizon, these hypersurfaces become null and coincide with $\mathcal{H}$ itself, which is why $\alpha\underset{\sim}{E}$ and $\alpha\underset{\sim}{B}$ become vectors ($\underset{\sim}{E}^{H}$ and $\underset{\sim}{B}^{H}$) lying in $\mathcal{H}$.

Fiducial observers near the horizon can split their electric and magnetic fields into parts $\underset{\sim}{E}_{\parallel}$ and $\underset{\sim}{B}_{\parallel}$ parallel to the horizon, and parts $E_{\perp}$ and $B_{\perp}$ perpendicular to the horizon (i.e., along their acceleration direction $\underset{\sim}{n}$):

$$\underset{\sim}{E} = \underset{\sim}{E}_{\parallel} + E_{\perp}\underset{\sim}{n} , \qquad \underset{\sim}{B} = \underset{\sim}{B}_{\parallel} + B_{\perp}\underset{\sim}{n} . \tag{5.35}$$

Similarly, the horizon fields can be split into components $\underset{\sim}{E}^{H}_{\parallel}$ and $\underset{\sim}{B}^{H}_{\parallel}$ which lie in surfaces of constant $\tilde{t} = t + \kappa^{-1} \ln \alpha$, and components $E^{H}_{\perp}$ and $B^{H}_{\perp}$ which are orthogonal to these surfaces (i.e., which point along the null generator $\underset{\sim}{\ell}$):

$$\underset{\sim}{E}^{H} = \underset{\sim}{E}^{H}_{\parallel} + E^{H}_{\perp}\underset{\sim}{\ell} , \qquad \underset{\sim}{B}^{H} = \underset{\sim}{B}^{H}_{\parallel} + B^{H}_{\perp}\underset{\sim}{\ell} . \tag{5.36}$$

Equations (5.34) and (5.24) reveal the relationship between these decompositions:

$$\alpha \underset{\sim}{E}_{\parallel} \rightarrow \underset{\sim}{E}^{H}_{\parallel} , \qquad \alpha \underset{\sim}{B}_{\parallel} \rightarrow \underset{\sim}{B}^{H}_{\parallel} , \tag{5.37a}$$

$$E_{\perp} \rightarrow E^{H}_{\perp} , \qquad B_{\perp} \rightarrow B^{H}_{\perp} . \tag{5.37b}$$

Notice that the tangential fields $\underset{\sim}{E}_{\parallel}$ and $\underset{\sim}{B}_{\parallel}$ diverge as one approaches the horizon ($\alpha \rightarrow 0$), but the radial fields remain finite. Physically this comes about because the ZAMOs near the horizon are moving outward at nearly the speed of light relative to physically more reasonable infalling observers, who see finite fields at the horizon. This motion converts the tangential fields, whatever they may be in physically reasonable frames, into inward propagating plane waves as seen by the ZAMOs. By pursuing this line of reasoning one can derive the following plane-wave relationship between $\underset{\sim}{E}_{\parallel}$ and $\underset{\sim}{B}_{\parallel}$:

$$|\underset{\sim}{E}_{\parallel} - \underset{\sim}{n} \times \underset{\sim}{B}_{\parallel}| \text{ goes to zero proportionally to } \alpha \text{ at } \mathcal{H} \quad . \tag{5.38}$$

Hajicek (1973, 1974) and Hanni and Ruffini (1973) have introduced the concept of surface density of electric charge on the horizon

$$\sigma^H \equiv (1/4\pi)E_{\perp}^H \quad . \tag{5.39}$$

This charge does not really exist physically on the horizon; rather, it is the charge per unit area which would precisely terminate the perpendicular electric field lines $E_{\perp}\underset{\sim}{n}$ at the horizon. If one pretends that this charge really exists, then one can ignore the actual fate of the electric field lines inside the horizon.

Damour (1978) has pursued this viewpoint further: He introduces a (fictitious) surface current density $\underset{\sim}{\mathscr{J}}^H$ (charge per unit time $\tilde{t}$ crossing a unit length perpendicular to $\underset{\sim}{\mathscr{J}}^H$), which is perfectly contrived to "complete the circuit" of all currents $\underset{\sim}{j}$ entering and leaving the horizon. Damour goes on to show that the hole behaves as though it had a surface resistivity

$$R^H \equiv 4\pi = 377 \text{ ohms} \tag{5.40}$$

(first inferred by Znajek (1976,1978b) in a different manner), in the sense that

$$\underset{\sim}{\mathscr{J}}^H = \underset{\sim}{E}_{\parallel}^H / R^H \quad . \tag{5.41}$$

Damour's properties of the surface charge and current, reexpressed in our 3+1 language, are the following:

$$E_\perp \to 4\pi\sigma^H \quad \text{at } \mathscr{H} \ , \tag{5.42}$$

[Gauss's law: normal component of electric field is 4π times horizon's surface charge density; derivable from (5.37b) and (5.39).]

$$\alpha \underset{\sim}{B}_{\parallel} \to \underset{\sim}{B}^H = 4\pi\, \underset{\sim}{\mathscr{J}}^H \times \underset{\sim}{n} \quad \text{at } \mathscr{H} \ , \tag{5.43}$$

[Ampere's law: tangential magnetic field $\alpha \underset{\sim}{B}_{\parallel}$ is produced by surface current; derivable from (5.38), (5.37a), (5.40), and (5.41).]

$$\alpha\, \underset{\sim}{j} \cdot \underset{\sim}{n} \to -{}^{(2)}\underset{\sim}{\nabla} \cdot \underset{\sim}{\mathscr{J}}^H - \frac{d}{d\tilde{t}}\sigma^H \quad \text{at } \mathscr{H} \ . \tag{5.44}$$

[Charge conservation: $\alpha\, \underset{\sim}{j} \cdot \underset{\sim}{n}$ is the charge emerging from the horizon per unit area of horizon and per unit of global time t (or $\tilde{t}$); $(d/d\tilde{t})\sigma^H \equiv \sigma^H{}_{,\mu}\ell^\mu$ is the rate of change of the surface charge density with respect to global time; and ${}^{(2)}\underset{\sim}{\nabla} \cdot \underset{\sim}{\mathscr{J}}^H$ is the divergence of the surface current — the divergence ${}^{(2)}\underset{\sim}{\nabla}$ being taken with respect to the intrinsic 2-geometry of a $\tilde{t}$ = constant slice through $\mathscr{H}$. This law of charge conservation can be derived by projecting the Maxwell equation (3.4c) along $\underset{\sim}{n}$, and by invoking $D_\tau \underset{\sim}{n} = -\underset{\sim}{\sigma} \cdot \underset{\sim}{n}$ together with the horizon's Gauss and Ampere laws (5.42) and (5.43).]

Equations (5.42) - (5.44) allow one to regard the horizon as a thin surface with finite electrical conductivity, surrounding a rather peculiar interior: The interior cannot support any charges ρ_e or currents $\underset{\sim}{j}$ or perpendicular electric fields $E_\perp$ or tangential magnetic fields $\underset{\sim}{B}_{\parallel}$, but it can support tangential electric fields $\underset{\sim}{E}_{\parallel}$ and perpendicular magnetic fields $B_\perp$. As a consequence, the horizon is forced to acquire just the right surface charge density σ^H and current density $\underset{\sim}{\mathscr{J}}^H$ to (i) satisfy Ohm's law (equation 5.41), (ii) complete the circuit of external currents (equation 5.44), (iii) annul $E_\perp$ (equation 5.42), and (iv) annul $\underset{\sim}{B}_{\parallel}$ (equation 5.43); but the horizon permits $\underset{\sim}{E}_{\parallel}$ and $B_\perp$ to extend into the hole's interior. This description of a black hole is due to Damour (1978) and Carter (1979).

Znajek (1978b) describes the horizon in a somewhat different manner from this: He endows it with magnetic charge as well as electric charge, and with very high <u>volume</u> conductivities for both magnetic current and electric current. The resulting charges and currents annul <u>all</u> external quantities ($\underset{\sim}{E}_{\parallel}$, $E_{\perp}$, $\underset{\sim}{B}_{\parallel}$, $B_{\perp}$, ρ_e, $\underset{\sim}{j}$) in a thin skin just below the horizon. Znajek's description has the beauty and advantage of treating $\underset{\sim}{E}$ and $\underset{\sim}{B}$ on equal footings and of not attributing peculiar properties to the hole's interior. Nevertheless, we have adopted the Damour-Carter description instead of Znajek's because we want a formalism which so far as possible meshes with one's flat-spacetime, laboratory experience, where magnetic monopoles are nonexistent.

Since the hypersurfaces $\check{\mathcal{S}}_t$ do not extend inside the horizon, Gauss's law $\int_{\partial\mathcal{V}} \underset{\sim}{B} \cdot d\underset{\sim}{\Sigma} = 0$ [which relies on $\mathcal{V}$ lying entirely in $\mathcal{S}_t$] cannot be applied to 2-surfaces $\partial\mathcal{V}$ that enclose the horizon. On the other hand, Faraday's law (4.3) can be applied to such 2-surfaces [with $\mathcal{A}(t) = \partial\mathcal{V}$, $\partial\mathcal{A}(t) = 0$]. It says that

$$\frac{d}{dt} \int_{\partial\mathcal{V}(t)} \underset{\sim}{B} \cdot d\underset{\sim}{\Sigma} = 0 \qquad (5.45)$$

for any 2-surface $\partial\mathcal{V}(t)$ enclosing the horizon — i.e., the total magnetic flux down the hole can never change. If the hole was created in the big bang, it conceivably could have been born with nonzero total magnetic flux; but if it was created by the collapse of a star, its total flux would have to be zero.

Because Damour's fictitious surface current and charge densities satisfy Maxwell's equations in the way described above, we are guaranteed that they will also lead, in the usual manner, to an electromagnetic torque on the horizon, $(\sigma^H \underset{\sim}{E}^H_{\parallel} + \underset{\sim}{\mathcal{J}}^H \times B^H_{\perp} \underset{\sim}{n}) \cdot \underset{\sim}{m}$, which precisely equals the flux of electromagnetic angular momentum down the hole, and a Joule heating $\underset{\sim}{E}^H_{\parallel} \cdot \underset{\sim}{\mathcal{J}}^H$ of the horizon which precisely equals the hole's temperature times its rate of increase of entropy [Znajek (1978b), Damour (1978), Carter (1979)]. Specifically, the inward flux of angular momentum $-\underset{\sim}{S}_L \cdot \underset{\sim}{n}$ (equation 5.16b), when multiplied by $\alpha = d\tau/dt$ to convert to a "per unit global time" basis, and when combined with Gauss's law (5.42) and Ampere's law (5.43), becomes

$$-\alpha\, \underset{\sim}{S}_L \cdot \underset{\sim}{n} \rightarrow \frac{d(\text{angular momentum of hole})}{d(\text{area of horizon})\, dt} \equiv \frac{dL^H}{d\Sigma^H dt}$$

$$= (\sigma^H \underset{\sim}{E}^H_{\parallel} + \underset{\sim}{\mathcal{J}}^H \times B^H_{\perp} \underset{\sim}{n}) \cdot \underset{\sim}{m} \quad . \tag{5.46}$$

Similarly, the inward flux of redshifted energy, $-\underset{\sim}{S}_E \cdot \underset{\sim}{n}$ (equation 5.15b), when multiplied by α to convert to "per unit global time", and when combined with Gauss's law (5.42) and Ampere's law (5.43), and with $\omega \rightarrow \Omega^H$, becomes

$$-\alpha\, \underset{\sim}{S}_E \cdot \underset{\sim}{n} \rightarrow \frac{d(\text{mass of hole})}{d(\text{area of horizon})\, dt} \equiv \frac{dM^H}{d\Sigma^H dt}$$

$$= \underset{\sim}{E}^H_{\parallel} \cdot \underset{\sim}{\mathcal{J}}^H + \Omega^H(\sigma^H \underset{\sim}{E}^H_{\parallel} + \underset{\sim}{\mathcal{J}}^H \times B^H_{\perp} \underset{\sim}{n}) \cdot \underset{\sim}{m} \quad . \tag{5.47}$$

Finally, combining expressions (5.46) and (5.47) with the first law of thermodynamics $dM^H = \Omega^H dL^H + \Theta^H dS^H$ where $\Theta^H = (\hbar/2\pi k)\kappa$ is the black-hole temperature and S^H is its entropy (Hawking 1976) we obtain the Joule-heating relation

$$\Theta^H \frac{dS^H}{d\Sigma^H dt} = \frac{dM^H}{d\Sigma^H dt} - \Omega^H \frac{dL^H}{d\Sigma^H dt} = \underset{\sim}{\mathcal{J}}^H \cdot \underset{\sim}{E}^H_{\parallel} \quad . \tag{5.48}$$

See Znajek (1978b), Damour (1978, 1979, 1980), and Carter (1979) for the original derivations and discussions of these relations in the 4-dimensional language.

6 Explicit Solutions of the Maxwell Equations

Since 1972 relativity theorists have put much effort into analytic solutions of Maxwell's equations for stationary, axially symmetric electromagnetic fields in black-hole spacetimes; and Wilson (1977) has initiated numerical studies of nonstationary fields in the magnetohydrodynamic approximation. This work has been motivated in large measure by Ruffini's (1973) early recognition that electrodynamic phenomena around black holes will have important astrophysical consequences. Ruffini (1979) reviews many of the studies that have been made.

The published analytic solutions include: the electric field of a point charge at rest in the Schwarzschild geometry [solution in closed form by Copson (1928) and Linet (1976); solution as a multipole expansion by Cohen and Wald (1971); field lines plotted by Hanni and Ruffini (1973); force of hole on particle studied by Smith and Will (1980)]; the electric and magnetic fields of a point charge at rest on the symmetry axis of a Kerr black hole [Misra (1977), Léauté (1977), Linet (1979)]; the electric and magnetic fields of charged current loops around Kerr black holes [Petterson (1975), Chitre and Vishveshwara (1975), and Linet (1979) for loop in equatorial plane; Znajek (1978a) for loop out of equatorial plane]; the distortion of a uniform magnetic field by the gravity of a black hole [Ginzburg (1964) for formulas, and Hanni and Ruffini (1976) for pictures in the Schwarzschild case; Wald (1974) for Kerr hole; Znajek (1977) for Kerr hole with external magnetic field in a state of slow rotation; King and Lasota (1977) for Kerr hole with the field oblique to the axis of rotation]; and a magnetohydrodynamic solution, in the limit of very weak magnetic field, for the magnetic field dragged onto a Kerr hole by a geodesically moving, charged fluid (Ruffini and Wilson 1975).

Though none of these analytic solutions were written in 3+1 language, they can all be translated easily into that language. We give two examples. In these examples we use units in which the speed of light c and Newton's gravitational constant G are both equal to unity.

6.1 POINT CHARGE AT REST OUTSIDE A SCHWARZSCHILD HOLE

For a Schwarzschild hole the spatial geometry, lapse, and fiducial angular velocity are

$$ds^2 = \frac{dr^2}{1-2M/r} + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2) \quad , \tag{6.1a}$$

$$\alpha = (1-2M/r)^{1/2} \quad , \qquad \omega = 0 \quad . \tag{6.1b}$$

For a point charge q at rest at $r = b$, $\theta = 0$ Copson (1928) as corrected by Linet (1976) gives the potentials

$$A_0 = -\frac{q}{br}\,\frac{(r-M)(b-M) - M^2\cos\theta}{[(r-M)^2 + (b-M)^2 - M^2 - 2(r-M)(b-M)\cos\theta + M^2\cos^2\theta]^{1/2}} - \frac{qM}{br} \quad ,$$

$$\underset{\sim}{A} = A_\varphi = 0 \quad . \tag{6.2}$$

The electric field $\underset{\sim}{E} = \alpha^{-1}\underset{\sim}{\nabla} A_0$ (equation 5.9a), in terms of physical basis vectors $\underset{\sim}{e}_{\hat{r}} = (1-2M/r)^{1/2}\,\partial/\partial r$ and $\underset{\sim}{e}_{\hat{\theta}} = r^{-1}\,\partial/\partial\theta$, is

$$\underset{\sim}{E} = \frac{q}{br^2}\left\{M\left[1 - \frac{b-M+M\cos\theta}{D}\right] + \frac{r[(r-M)(b-M) - M^2\cos\theta]\,[r - M - (b-M)\cos\theta]}{D^3}\right\}\underset{\sim}{e}_{\hat{r}}$$

$$+ \frac{q(b-2M)(1-2M/r)^{1/2}\sin\theta}{D^3}\,\underset{\sim}{e}_{\hat{\theta}} \quad , \tag{6.3a}$$

where

$$D \equiv [(r-M)^2 + (b-M)^2 - M^2 - 2(r-M)(b-M)\cos\theta + M^2\cos^2\theta]^{1/2} \quad . \tag{6.3b}$$

The electric field lines intersect the horizon $r = 2M$ orthogonally, producing a surface charge density

$$\sigma^H = \frac{q[M(1+\cos^2\theta) - 2(b-M)\cos\theta]}{8\pi b[b-M(1+\cos\theta)]^2} \tag{6.4}$$

but no surface current. The total induced surface charge is zero. Hanni and

Ruffini (1973) plot the field lines.

6.2 KERR HOLE IMMERSED IN A UNIFORM MAGNETIC FIELD

For a Kerr hole the spatial geometry, lapse, fiducial angular velocity, and angular Killing vector are

$$ds^2 = (\rho^2/\Delta)dr^2 + \rho^2 d\theta^2 + (A\sin^2\theta/\rho^2)d\varphi^2 \ , \tag{6.5a}$$

$$\rho^2 \equiv r^2 + a^2\cos^2\theta \ , \quad \Delta = r^2 - 2Mr + a^2 \ , \quad A = (r^2 + a^2)^2 - \Delta a^2 \sin^2\theta \ ;$$

$$\alpha = (\rho^2\Delta/A)^{1/2} \ ; \tag{6.5b}$$

$$\omega = 2aMr/A \ ; \tag{6.5c}$$

$$\underset{\sim}{m} = \partial/\partial\varphi \ . \tag{6.5d}$$

Here a is the angular momentum per unit mass of the black hole, and should not be confused with the acceleration of the fiducial congruence. Wald (1974) derives the 4-vector potential $\mathfrak{A}^\alpha = \frac{1}{2} B_o(m^\alpha + 2ak^\alpha)$ for a source-free magnetic field which is asymptotically uniform with strength B_o far from the hole. From equations (3.3) and (5.10a,b) we compute the corresponding 3+1 potentials

$$A_0 = -B_o[a\alpha^2 + \omega\, \underset{\sim}{m}^2(1/2 - a\omega)] \ , \tag{6.6a}$$

$$A_\varphi = B_o(1/2 - a\omega)\underset{\sim}{m}^2 \ , \tag{6.6b}$$

$$\underset{\sim}{A} = B_o(1/2 - a\omega)\underset{\sim}{m} \ . \tag{6.6c}$$

From equations (5.9a,b) we derive the magnetic and electric fields, which reside in the Kerr spatial geometry (6.5a)

$$\underset{\sim}{B} = \frac{B_o}{2\rho \sin\theta}\left(\frac{\Delta}{A}\right)^{1/2}\left[\frac{\partial X}{\partial\theta}\frac{\partial}{\partial r} - \frac{\partial X}{\partial r}\frac{\partial}{\partial\theta}\right] \tag{6.7a}$$

where $X \equiv (\sin^2\theta/\rho^2)(A - 4a^2Mr)$,

$$\underset{\sim}{E} = \frac{-B_o a A^{1/2}}{\rho^3}\left\{\Delta^{1/2}\left[\frac{\partial(\alpha^2)}{\partial r} + \frac{M\sin^2\theta}{\rho^2}(A - 4a^2Mr)\frac{\partial}{\partial r}\left(\frac{r}{A}\right)\right]\frac{\partial}{\partial r} + \Delta^{-1/2}\left[\frac{\partial(\alpha^2)}{\partial\theta} + \frac{Mr\sin^2\theta}{\rho^2}(A - 4a^2Mr)\frac{\partial}{\partial\theta}\left(\frac{1}{A}\right)\right]\frac{\partial}{\partial\theta}\right\} . \tag{6.7b}$$

The electric field is induced by the hole's dragging of inertial frames: note that $\underset{\sim}{E} = 0$ if $a = 0$. The 2-geometry of the horizon ($\Delta = 0$) is

$$ds^2 = (r_+^2 + a^2\cos^2\theta)d\theta^2 + \frac{(r_+^2 + a^2)^2\sin^2\theta}{r_+^2 + a^2\cos^2\theta}d\varphi^2 , \tag{6.8}$$

where $r_+ \equiv M + (M^2 - a^2)^{1/2}$ is the radius of the horizon. The magnetic and electric fields and the charge and current densities on this horizon are

$$\underset{\sim}{B}^H_{\parallel} = \underset{\sim}{E}^H_{\parallel} = \underset{\sim}{\mathcal{J}}^H = 0 , \tag{6.9a}$$

$$B^H_{\perp} = \frac{4B_o Mr_+^2(r_+ - M)\cos\theta}{(r_+^2 + a^2\cos^2\theta)^2} , \tag{6.9b}$$

$$E^H_{\perp} = 4\pi\sigma^H = -\frac{B_o a(r_+ - M)}{r_+^2 + a^2\cos^2\theta}(1 + \cos^2\theta) .$$

In this example the absence of tangential fields and currents at the horizon implies that no torque acts to slow the horizon's rotation. If the external magnetic field were inclined obliquely to the rotation axis instead of aligned with it, a slowing torque would act; cf. King and Lasota (1977).

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