# ELECTRODYNAMICS IN THE GENERAL RELATIVITY THEORY*

BY

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The restricted relativity theory resulted mathematically in the introduction of pseudo-euclidean four-dimensional space and the welding together of the electric and magnetic force vectors into the electromagnetic tensor.

Einstein's general relativity theory led to the assumption that the four-dimensional space mentioned above is a curved space and the curvature was made to account for the gravitational phenomena.

The Riemann tensor which measures the curvature and the electromagnetic tensor seem thus to play essentially different rôles in physics: the former reflects some properties of the space so that gravitation may be said to have been geometricized,—when the space is given all the gravitational features are determined; on the contrary, it seemed that the electromagnetic tensor is superposed on the space, that it is something external with respect to the space, that after space is given the electromagnetic tensor can be given in different ways. Several attempts were made to geometricize the electromagnetic forces, to find a geometric interpretation for the electromagnetic tensor, to incorporate this tensor into the space in the sense in which the gravitational forces had been incorporated.

It seemed that in order to do this it was necessary to change the geometry; to abandon the Riemann geometry and to adopt a more general space with a more complicated curvature tensor, one part of which would then account for the gravitational properties and the other would in the same way account for the electromagnetic phenomena.

H. Weyl arrived in a most natural way to such a generalization. His theory always will remain a brilliant mathematical feat, but it seems that it did not fulfil the expectations as a physical theory and the same seems to be true with respect to other attempts.

The electromagnetic tensor is, however, not entirely independent of the Riemann tensor in the ordinary general relativity theory; these two tensors are connected by the so called energy relation; it seemed to be desirable to try, without breaking the frame of the Riemann geometry, to study

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mathematically the connection between these two most important tensors of physics. This study forms the object of the present paper.

The result of this study is quite unexpected; it is that, under certain assumptions, the electromagnetic field is entirely determined by the curvature of space-time, so that there is no need of further generalizing the general relativity theory; it was only necessary to develop mathematically the consequences of well known relations in order to see that without any modifications it takes care of the electromagnetic field, as far as "classical electrodynamics" is concerned; whether the phenomena of emission and absorption of radiation and such features of the electron theory as equality of charges can be accounted for by the general relativity theory in its original form remains to be seen, but there are indications which show that they might.

As to the method of the study it seemed to me better to avoid, as far as possible, the introduction of things which have no intrinsic meaning, such as coördinates, the *g*'s, the three-indices symbols, the distinction between co- and contravariant quantities, etc. I believe that the present paper shows the advantages of this point of view which I expound at greater length elsewhere.* I also have not used the so called electromagnetic potential vector, which is, moreover, not fully determined; I believe that its use tends to conceal the fundamental properties of the really important things; if we use that vector, the fact that one of the sets of the Maxwell equations is satisfied seems to be granted beforehand and then the other set is a consequence of the general properties of the space;† but in reality the existence of the electromagnetic field imposes on the space additional conditions.

In writing the paper I endeavored not to recede very far from the notation now in general use; I start with components and also translate the results into the language of components, but I hope that the intrinsic meaning of the formulas remains sufficiently clear.

Part I is devoted to the study of the algebraic relations resulting from the energy relation; the electromagnetic tensor in each point is shown to be partly determined by the curvature tensor at that point, only one scalar remaining arbitrary. In Part II by the consideration of differential properties the indeterminateness is reduced to one constant of integration. In Part III it is shown to be possible to eliminate the remaining arbitrariness by consideration of certain integrals.

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* American Journal of Mathematics, April, 1924 and January, 1925.

† Cf. Einstein's paper *Bietet die Feldtheorie Möglichkeiten für die Lösung des Quantenproblems*, Berliner Sitzungsberichte, January 15, 1924, statement at the bottom of p. 362.



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The contents of Parts I and II were briefly presented in the Proceedings of the National Academy of Sciences in two notes under the title *Electrodynamics in the general relativity theory* in the April and July numbers, 1924. We shall cite them as "First Note" and "Second Note".

## PART I. ALGEBRAIC PROPERTIES

### 1. The invariable plane of a tensor of the second rank

We shall start with the usual form of the general relativity theory; we shall mostly have to consider two tensors of the second rank, the electromagnetic tensor, which is antisymmetric, and the energy tensor which is symmetric. In this first part we shall consider only the connection which exists between these two tensors at a given point, without taking into account the corresponding tensor fields; our considerations will belong, thus, to the algebra of tensors, not to the analysis of tensor fields. But before we consider the relation between our two tensors we shall have to study some geometric properties which belong to every tensor of the second rank.

We shall consider a tensor of the second rank as defining a transformation, and for that purpose it is convenient to use it in its mixed form, $f^i_j$; if $x^i$ are the contravariant components of a vector (we could also write $dx^i$) we form the expressions

$$f^i_\rho \, x^\rho \tag{1.1}$$

(we use throughout this paper Greek letters for umbral indices or dummy suffixes); these can be considered as contravariant components of a new vector; we see thus that a tensor of the second rank gives rise to a transformation of a vector into another vector, or to a linear vector function. In many cases it is much more convenient to refer to this linear vector function rather than to the components which depend upon the system of coördinates we are using; we shall simply write $x$ for the vector with the components $x^i$ and $f(x)$ for the transformed vector with the components (1.1); and we shall speak of the tensor $f$.

We shall write $\varrho x$ for the vector whose components are $\varrho x^i$ and we shall call the totality of vectors of the form $\varrho x$ with a fixed $x$ and a variable $\varrho$ a *direction*; two vectors belong, therefore, to the same direction if their components are proportional. The totality of vectors of the form $\varrho x + \sigma y$ with $x$ and $y$ fixed vectors and $\varrho$ and $\sigma$ variable numbers will be called a *plane*.

A direction or a plane is called an invariable direction or an invariable plane, respectively, of a tensor $f$ if vectors belonging to it are transformed

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by $f$ again into vectors belonging to it (compare, for a general theory of such regions, S. Pincherle and U. Amaldi, *Operazioni Distributive*). If a vector $a$ belongs to an invariable direction of $f$ we have

$$f(a) = \lambda a,$$

where $\lambda$ is a number which is called the *characteristic* number of this direction. If $a$ belongs to an invariable plane $\varrho x + \sigma y$, we have

$$a = \varrho x + \sigma y, \qquad f(a) = \varrho' x + \sigma' y;$$

applying $f$ to both sides of the second equality and writing $f^2(a)$ for $f[f(a)]$ we find

$$f^2(a) = \varrho'' x + \sigma'' y,$$

and it follows from the last three equalities that a relation of the form

$$f^2(a) - \alpha f(a) + \beta a = 0 \tag{1.2}$$

must hold for $a$; inversely, if $a$ does not belong to an invariable direction and a relation of the form (1.2) holds, $a$ belongs to an invariable plane defined by the vectors $a$ and $f(a)$.

It is known that a characteristic number $\lambda$ of an invariable direction satisfies the characteristic equation

$$|f_j^i - \lambda g_j^i| = 0. \tag{1.3}$$

The tensor $f$ itself satisfies a relation which for the four-dimensional space has the form

$$f^4(a) - \alpha f^3(a) + \beta f^2(a) - \gamma f(a) + \delta = 0, \tag{1.4}$$

$f^3(x)$ standing for $f[f^2(a)]$, etc., and the coefficients $\alpha, \beta, \gamma, \delta$ being equal to the coefficients of the corresponding characteristic equation*.

We shall have to use the following

THEOREM. *Every linear vector function of a four-dimensional space has at least one invariable plane.*

* A very simple proof of this proposition is given by L. E. Dickson, Journal de Mathématiques, ser. 9, vol. 2 (1923), p. 309, footnote.

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The proof depends upon the fact that the left hand side of equation (1.4) can be written in the form $h[k(a)]$ or $k[h(a)]$ with

$$h(a) = f^2(a) - \alpha_1 f(a) + \beta_1 a \quad \text{and} \quad k(a) = f^2(a) - \alpha_2 f(a) + \beta_2 a,$$

where $\alpha_1$, $\beta_1$, $\alpha_2$, $\beta_2$ are real numbers. Suppose now one of the functions $h$ and $k$, say $h$, never becomes zero for a non-zero argument; then every value of $h(a)$ makes $k$ zero; if among the values of $h(a)$ there are two which belong to different directions, they certainly give us at least one invariable plane; if they all have the same direction we have, e.g., $h(x) = \varrho a$, $h(y) = \sigma a$ and since $h$ never becomes zero $\varrho$ and $\sigma$ are different from zero; but then we have $h(\sigma x - \varrho y) = 0$, contrary to our assumption. If $h = k$ our equation (1.4) takes the form $h^2(a) = 0$ for every value of $a$, and from this follows $h(a) = 0$ for every value $a$.

## 2. Some properties of the Minkowski space

The scalar product of two vectors $x$ and $y$ can be expressed through their contravariant components in the form

$$xy = x \cdot y = g_{\rho\sigma} x^\rho y^\sigma. \tag{2.1}$$

If we use geodesic coördinates with

$$g_{11} = -1, \; g_{22} = g_{33} = g_{44} = 1 \quad \text{and} \quad g_{ij} = 0 \qquad (i \neq j) \tag{2.11}$$

this gives

$$xy = -x^1 y^1 + x^2 y^2 + x^3 y^3 + x^4 y^4. \tag{2.2}$$

This shows that we have vectors of three kinds: those with negative square, those with positive square and those of zero length. We shall call the latter zero-vectors and the corresponding directions zero-directions. The elementary geometric properties of such a pseudo-euclidean bundle have been well known since the time of Minkowski. We shall only mention that we have three kinds of planes; those which have two zero-directions, those which have none and those which have one; a plane which contains a vector of negative square has two zero-directions.

Given a system of axes we introduce four vectors $i, j, k, l$ by their components

(2.3) $\quad 1, 0, 0, 0; \qquad 0, 1, 0, 0; \qquad 0, 0, 1, 0; \qquad 0, 0, 0, 1.$

We have

$$i^2 = -1, \qquad j^2 = k^2 = l^2 = 1; \tag{2.4}$$

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all the other products are zero. Vice versa, if we have four mutually perpendicular unit vectors (the square of the length of one will then necessarily be $-1$ and of each of the three others $+1$) we can introduce their directions as axes. We have the relations

$$(2.5)\quad x = ix^1 + jx^2 + kx^3 + lx^4 = -i(ix) + j(jx) + k(kx) + l(lx).$$

If we are given a plane with no zero-direction we can so choose the axes as to make it the $k, l$ plane; a plane with two zero-directions we can make the $i, j$ plane ($i+j$ and $i-j$ being two zero-vectors); a plane with just one zero-direction we can make the $i+j, k$ plane. The last statement may need a proof. Let us take any axes; on the zero-direction of our plane there will be a vector of the form $i + \alpha j + \beta k + \gamma l$ with $\alpha^2 + \beta^2 + \gamma^2 = 1$; we can change our "space axis" so as to make $\alpha j + \beta k + \gamma l$ our new $j$ vector; then our zero-vector already has the form $i+j$; now let $p$ be any unit vector of our plane; the vector of this plane $i + j - 2p(pi + pj)$ has a zero square and since there is but one zero-direction we must have $pi + pj = 0$; if we introduce now the vector $q = (pi)(i+j) + p$, we see that

$$q^2 = 2(pi)(pi + pj) + 1 = 1, \quad qi = -pi + pi = 0, \quad qj = pi + pj = 0;$$

we can, therefore, choose $q$ for our $k$.

Once the axes are chosen the formulas can be made more symmetrical, in many cases, by introducing imaginaries, but for the treatment of planes which have just one zero-direction the imaginaries present some difficulties; we shall therefore abstain from introducing them while we have yet to deal with such planes.

## 3. The antisymmetric tensor of the second rank

If a tensor of the second rank is given in its covariant form $f_{ij}$ or in its contravariant form $f^{ij}$ the property of antisymmetry is simply expressed respectively by the formulas

$$f_{ij} = -f_{ji}, \qquad f^{ij} = -f^{ji}.$$

In vector notations we have

$$f(x)\cdot y = f_{\rho\sigma} x^\rho y^\sigma, \qquad f(y)\cdot x = f_{\sigma\rho} x^\rho y^\sigma$$

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so that the property of antisymmetry is expressed by the formula

$$f(x)\cdot y = -f(y)\cdot x. \tag{3.1}$$

Incidentally, for a symmetric tensor we have

$$f(x)\cdot y = f(y)\cdot x. \tag{3.11}$$

But we have to use, at least temporarily, the mixed form and in this form the property under consideration has a more complicated expression; if we take geodesic coördinates (2.11) we find for the mixed components of an antisymmetric tensor

$$f_1^2 = f_2^1, \quad f_1^3 = f_3^1, \quad f_1^4 = f_4^1, \quad f_2^3 = -f_3^2, \quad f_2^4 = -f_4^2, \quad f_3^4 = -f_4^3.$$

The coefficients are *symmetric* in the indices if one of the indices is 1, and *antisymmetric* in other cases; they are zero when the two indices coincide.

We shall discuss now the question of invariable planes of an antisymmetric tensor. We know that there exists at least one invariable plane (§ 1). Suppose there exists an invariable plane which has no zero-directions; we can take this plane for the $k, l$ plane (§ 2); then we have $f(k) = \alpha k + \beta l$, $f(l) = \gamma k + \delta l$; that means that in the scheme of coefficients

$$\begin{Vmatrix} 0 & A & B & C \\ A & 0 & D & E \\ B & -D & 0 & F \\ C & -E & -F & 0 \end{Vmatrix} \tag{3.2}$$

$B = C = D = E = 0$. There only remains

$$\begin{Vmatrix} 0 & A & 0 & 0 \\ A & 0 & 0 & 0 \\ 0 & 0 & 0 & F \\ 0 & 0 & -F & 0 \end{Vmatrix}.$$

On the other hand, if there is an invariable plane with two zero-directions we can take it for the $i, j$ plane; we have then $f(i) = \alpha i + \beta j$, $f(j) = \gamma i + \delta j$, so that $B = C = D = E = 0$ with the same result as before. We have therefore in both cases considered

$$f(i) = A\cdot j, \quad f(j) = A\cdot i, \quad f(k) = -F\cdot l, \quad f(l) = F\cdot k.$$

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Using (2.5) we find

$$f(x) = -Aj(ix) + Ai(jx) - Fl(kx) + Fk(lx) = A\{i(jx) - j(ix)\} + F\{k(lx) - l(kx)\}. \tag{3.3}$$

This is the first canonical form of an antisymmetric linear vector function (the same expression holds also for the euclidean bundle where it is the *only* canonical form). The geometric meaning of a function of the form (3.3) is seen to be the following: a vector of each of the invariable planes (the $i, j$ plane and the $k, l$ plane) is transformed into another vector of the same plane, which is perpendicular to the original vector and whose length is, respectively, $A$ or $F$ times greater; the transformation of a vector which does not belong to one of the invariable planes is given by the transformation of its components in these planes.*

The case remains to be considered when the invariable plane or planes have only one zero-direction. According to § 2 we can take such a plane for the $i+j, k$ plane; then we have $f(i+j) = \alpha(i+j) + \beta k$, $f(k) = \gamma(i+j) + \delta k$; confronting this with the scheme (3.2) we find $C = E$, $B = D$, $F = 0$, so that

$$f(i) = Aj + Bk + Cl,\quad f(j) = Ai - Bk - Cl,\quad f(k) = Bi + Bj,\quad f(l) = Ci + Cj.$$

It is easy to see that unless $A = 0$ the vectors $f(i)$ and $f(j)$ determine an invariable plane which has two zero-directions, viz. that of the vector $i+j$ and that of the vector $(Aj + Bk + Cl)(B^2 + C^2 - A^2) + (Ai - Bk - Cl)(B^2 + C^2 + A^2)$; $A$ must, therefore, be zero. If now, without changing $i$ and $j$ we choose the unit vector of the direction $Bk + Cl$ for our new $k$ and denote the length of $Bk + Cl$ by $G$ we have

$$f(i) = Gk, \qquad f(j) = -Gk, \qquad f(k) = Gi + Gj, \qquad f(l) = 0,$$

and

$$f(x) = G\{i(kx) - k(ix) + j(kx) - k(jx)\} = G\{n(kx) - k(nx)\}, \tag{3.31}$$

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* This interpretation was given in a paper presented to the Society, February 24, 1923; compare also Comptes Rendus, vol. 176, p. 1294. A proof for the euclidean case is given by A. Mochoolsky in the Memoirs of the Research Institute, Odessa, February, 1924. It is interesting to note that Sommerfeld originally defined the six-vector as the set of two perpendicular planar quantities (Ebenenstücke), Annalen der Physik, vol. 32 (1910), p. 753; E. T. Whittaker also comes near to this interpretation in his paper on *The tubes of electromagnetic force,* Proceedings of the Royal Society of Edinburgh, vol. 42 (1922), pp. 1-23. See also S. R. Milner's paper in the Philosophical Magazine, ser. 6, vol. 44 (1922), p. 705.

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where $n = i+j$; this is the second canonical form of an antisymmetric linear vector function in a pseudo-euclidean bundle. Here we also have two perpendicular planes: $i+j, k$ and $i+j, l$ which are invariable, but this is a different kind of perpendicularity (in both cases we have a so called absolute perpendicularity, i. e., each vector of each of the two planes is perpendicular to each vector of the other plane, but in the first case the two planes have only a common point and in the second they have a common direction).

For the components we have in the first case

$$f_2^1 = f_1^2 = A, \qquad f_4^3 = -f_3^4 = F, \qquad \text{all the others zero;}$$

in the second case

$$f_3^1 = f_1^3 = -f_3^2 = f_2^3 = G, \qquad \text{all the others zero.}$$

Every antisymmetric tensor is known to have two invariants

$$(3.4) \qquad \begin{aligned} I_1 &= f_1^2 f_2^1 + f_1^3 f_3^1 + f_1^4 f_4^1 + f_2^3 f_3^2 + f_2^4 f_4^2 + f_3^4 f_4^3 \qquad \text{and} \\ I_2 &= f_1^2 f_4^3 + f_1^3 f_2^4 + f_1^4 f_2^3 . \end{aligned}$$

In the first case their values are $A^2 - F^2$ and $AF$; in the second case both invariants vanish.

We conceive of an electromagnetic field as of something of the nature of an analytic function (compare Part III); it is natural to assume, therefore, that the invariants of an electromagnetic field cannot be *strictly* zero in one region without being zero all over; and since there are regions where they are different from zero we shall assume that they are different from zero everywhere with the exception only of points. From this point of view a field for which both invariants are strictly zero (a self-conjugate field, using the terminology of H. Bateman*) does not exist in nature and must be considered only as an approximation, this approximation not being, incidentally, an intrinsic quality because it depends on the separation of space and time.

Instead of considering the vanishing of the two invariants $I_1$ and $I_2$ as characteristic for the self-conjugate field, we may consider as such the vanishing of one number

$$(3.5) \qquad 4\omega^4 = I_1^2 + 4 I_2^2;$$

* *Electrical and Optical Wave-Motion,* p. 5.

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in the case when $\omega$ does not vanish, i. e., in the case when the tensor may be presented in the first canonical form, we have

$$\omega^2 = \frac{A^2 + F'^2}{2}. \tag{3.51}$$

The number $\omega\sqrt{2}$ is considered by Milner (paper cited above) who designates it $R$.

It is often inconvenient, as already mentioned, to have the square of one of our unit vectors negative while the others have positive squares. In order to avoid this we shall consider henceforth instead of the vector $i$ this vector multiplied by $\sqrt{-1}$, but we shall designate this new vector by the same letter $i$; this change necessitates the substitution of $-\sqrt{-1}\cdot A$ for $A$ in the formula (3.3); we shall call this imaginary number $\lambda$ and instead of $F$ we shall write $\mu$. The electromagnetic tensors with which we shall have to deal will, therefore, have the form

$$f(x) = \lambda\{i(jx) - j(ix)\} + \mu\{k(lx) - l(kx)\}, \tag{3.6}$$

with

$$i^2 = j^2 = k^2 = l^2 = 1, \qquad i\cdot j = i\cdot k = \cdots = 0; \tag{3.7}$$

$\lambda$ is an imaginary and $\mu$ a real number.

We shall say that the planes $i, j$ and $k, l$ form the *skeleton* of the tensor; in order to know the tensor it is necessary to know the skeleton and the two numbers $\lambda$ and $\mu$.

It must be noticed that, whereas $\lambda$ and $\mu$ are entirely determined by the tensor, the vectors $i, j, k, l$ are not; the vectors $k, l$ may be turned in their plane through an arbitrary angle $\psi$, i. e. we may introduce in their stead two vectors $K$ and $L$ connected with them by the relations

$$k = K\cos\psi - L\sin\psi, \qquad l = K\sin\psi + L\cos\psi; \tag{3.8}$$

the substitution of these expressions in (3.6) will show that the vectors $K, L$ play exactly the same part as $k, l$. The same can be said with reference to the couple $i, j$ with a little modification necessitated by the fact that $i$ is imaginary; we shall have here the transformation

$$i = I\cos\chi + J\sin\chi\cdot\sqrt{-1}, \qquad j = I\sin\chi\cdot\sqrt{-1} + J\cos\chi. \tag{3.9}$$

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I would not say that the consideration of these vectors $i$, $j$ and $k$, $l$ instead of the planes which they determine is entirely satisfactory from the point of view of mathematical elegance; it introduces elements which have no intrinsic significance and it is to be hoped that it will eventually be possible to do without them, to operate directly on the planes. But as things stand now, we have to use the vectors. If the two vectors $i$, $j$ are given they determine also the plane $k$, $l$ (and vice versa) because there is only one plane perpendicular to a given plane in a four-dimensional bundle. We could, therefore, use only one couple of vectors but this would necessitate the introduction of a new operation and would make our formulas less symmetric.

## 4. The energy relation

The electromagnetic energy (and momentum) tensor is usually given in the form $f^i_\rho f^\rho_j - \frac{1}{4} g^i_j f^\sigma_\rho f^\rho_\sigma$, but a simpler form* can be obtained if we use the dual or reciprocal tensor $d$ together with $f$, viz.,

$$\frac{1}{2}\{f^i_\rho f^\rho_j - d^i_\rho d^\rho_j\}. \tag{4.1}$$

Now $f^i_\rho f^\rho_\sigma x^\sigma$ is in vector notation simply $f^2(x)$ because it is the result of the transformation $f$ applied twice; we can write therefore for the energy tensor

$$\frac{1}{2}\{f^2(x) - d^2(x)\}. \tag{4.11}$$

The dual tensor of an antisymmetric tensor $f^i_j$ is defined by

$$d^1_2 = f^3_4, \quad d^1_3 = f^4_2, \quad d^1_4 = f^2_3, \quad d^3_4 = f^1_2, \quad d^4_2 = f^1_3, \quad d^2_3 = f^1_4.$$

If we take $f$ in the canonical form (3.6) its components are

$$f^1_2 = \lambda, \qquad f^3_4 = \mu \qquad \text{and} \qquad f^1_3 = f^1_4 = f^4_2 = f^2_3 = 0. \tag{4.2}$$

The components of $d$ will, therefore, be

$$d^1_2 = \mu, \quad d^3_4 = \lambda, \qquad d^1_3 = d^1_4 = d^4_2 = d^2_3 = 0, \tag{4.21}$$

* See, e. g., J. Rice, *Relativity*, London, 1923, p. 224. This form is due to Laue; see Sommerfeld, loc. cit., p. 768.

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so that

$$d(x) = \mu\{i(jx)-j(ix)\}+\lambda\{k(lx)-l(kx)\}. \tag{4.3}$$

It should be noted that with our convention this is an imaginary tensor, i. e., it gives us vectors multiplied by $\sqrt{-1}$; in the formula (4.11) nothing imaginary remains because $d$ is there applied twice in succession. We could of course easily introduce instead of $d$ a real tensor, but we see no harm in its remaining imaginary and in some cases it is even of some advantage (see §§ 6 and 9).

We have

$$f^2(x) = -\lambda^2\{i(ix)+j(jx)\}-\mu^2\{k(lx)+l(kx)\},$$
$$d^2(x) = -\mu^2\{i(ix)+j(jx)\}-\lambda^2\{k(kx)+l(lx)\},$$

so that the expression for the electromagnetic energy tensor (4.11) becomes

$$\omega^2\{i(ix)+j(jx)-k(kx)-l(lx)\}, \tag{4.4}$$

if we put, in accord with (3.51),

$$\omega^2 = \frac{\mu^2-\lambda^2}{2}. \tag{4.5}$$

Now in the general relativity theory the energy tensor at a given point can be calculated from the Riemann tensor. If $R_j^i$ is the contracted Riemann tensor, then the energy tensor is usually assumed to be $R_j^i - \frac{1}{2} g_j^i R_\rho^\rho$. In a region which is free from matter the whole energy is electromagnetic, so that this expression must be equal to the electromagnetic energy tensor and we have the equation

$$R_j^i - \frac{1}{2} g_j^i R_\rho^\rho = f_\sigma^i f_j^\sigma - \frac{1}{4} g_j^i f_\sigma^\tau f_\tau^\sigma. \tag{4.6}$$

Contracting, we see that $R_\rho^\rho$ must be in this case equal to zero, so that the electromagnetic energy tensor is equal to the contracted Riemann tensor. It is also possible to suppose that $R_\rho^\rho$ is a constant different from zero—this would correspond to the cosmological equations. In this case we have to take for the energy tensor the expression $R_j^i - \frac{1}{4} g_j^i R_\rho^\rho$; in both cases, we see, the electromagnetic energy tensor is equal to an expression which can be obtained from the Riemann tensor, i. e., which can be found

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if the space-time is given. If we denote this tensor, which is obtained from the curvature of the space-time, by $F^i_j$ we have, therefore, the relation

$$(4.12)\quad F^i_j = \frac{1}{2}\{f^i_\rho f^\rho_j - d^i_\rho d^\rho_j\} \quad \text{or} \quad F(x) = \frac{1}{2}\{f^2(x) - d^2(x)\}.$$

This relation which we call the energy relation connects the curvature field and the electromagnetic field. We are going to find out what information concerning each of these fields can be obtained from this relation. We shall start by investigating what restrictions are imposed on $F(x)$ by the existence of the relation (4.12). From the general theory of curved space we only know that $F(x)$ is a symmetric linear vector function and that any symmetric linear vector function can be taken for $F$, as far as general properties of space are concerned; but if we write our relation in the form

$$(4.41)\qquad F(x) = \omega^2\{i(ix) + j(jx) - k(kx) - l(lx)\},$$

we see that $F(x)$ must be a linear vector function of a special form. We are going to ask ourselves how, given a tensor of the second rank, we can know whether it has the form (4.41) or not. First or all, substituting in (4.41) in turn $x = i, j, k, l$, we find

$$(4.42)\quad F(i) = \omega^2 i, \quad F(j) = \omega^2 j, \quad F(k) = -\omega^2 k, \quad F(l) = -\omega^2 l.$$

We see that the vectors $i, j, k, l$, belong to invariable directions, their characteristic numbers being $\omega^2$, $\omega^2$, $-\omega^2$, $-\omega^2$. It is easy to see that every direction of each of the planes $i, j$ and $k, l$ is an invariable direction with the characteristic number $\omega^2$, $-\omega^2$, respectively. Here we have a full geometric characterization of $F$:

(4.43) *It has two planes of invariable directions with characteristic numbers of opposite signs; these planes are (absolutely) perpendicular with one common point.*

If we want to find a characterization of $F$ in terms of its components, the best way is to start with the remark that

$$(4.7)\quad \begin{aligned} F^2(x) = F[F(x)] &= \omega^2\{F(i)(ix) + F(j)(jx) - F(k)(kx) - F(l)(lx)\} \\ &= \omega^4\{i(ix) + j(jx) + k(kx) + l(lx)\} = \omega^4 x. \end{aligned}$$

In components we may write for the left hand part (as we did before for $f$) $F^i_\rho F^\rho_\sigma x^\sigma$, and the right hand part may be written as $\omega^4 g^i_\rho x^\rho$; we therefore have

$$(4.71)\qquad F^i_\rho F^\rho_j = g^i_j \omega^4.$$

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This is a necessary condition for $F$ but it is not sufficient, as, e. g., the function

$$F(x) = \omega^2\{i(ix) - j(jx) - k(kx) - l(lx)\}$$

also satisfies it. But together with the condition

$$F^\rho_\rho = 0, \tag{4.8}$$

which was obtained by contracting (4.6), the equation (4.71) gives a full characterization of $F$. The proof of this statement will be a little cumbersome because we have not studied the geometric properties of a symmetric function in a pseudo-euclidean bundle. We know, however, that $F(x)$, like every linear vector function, has at least one invariable plane; if there is such a plane with two zero-directions we take it for the $i, j$ plane; we have $F(i) = \alpha i + \beta j$, $F(j) = \gamma i + \delta j$. Comparing this with the scheme for a symmetric linear vector function, viz.,

$$F(i) = Ai + Bj + Ck + Dl, \qquad F(j) = Bi + Ej + Gk + Hl,$$
$$F(k) = Ci + Gj + Kk + Ll, \qquad F(l) = Di + Hj + Lk + Ml,$$

we find $C = D = G = H = 0$; we thus have two perpendicular invariable planes and we shall show that each of them has two perpendicular invariable directions with the characteristic numbers $\pm\omega^2$. Take, e. g., the plane $i, j$; writing that $F^2(i) = \omega^4 i$, $F^2(j) = \omega^4 j$, we find $A^2 + B^2 = B^2 + E^2 = \omega^4$, $B(A+E) = 0$; if $B = 0$, $A^2 = E^2 = \omega^4$ and the vectors $i$ and $j$ give us the directions we want. If $B \neq 0$, $E = -A$ and a simple calculation shows that the vectors $Bi - (A - \omega^2)j$ and $(A - \omega^2)i + Bj$ belong to two invariable directions with the characteristic numbers $\pm\omega^2$. We have thus established that there are four mutually perpendicular directions with characteristic numbers $\pm\omega^2$. The equation (4.8) shows that the sum of the characteristic numbers is zero; two of them must therefore be positive and two negative. It remains to show that there always is a plane with two zero-directions; if all time-directions are invariable a plane defined by two of them is certainly invariable and it contains two zero-directions (§ 2); if there is a time-direction which is not invariable let the vector $i$ belong to it; the plane determined by $i$ and $F(i)$ is invariable because $F[F(i)] = F^2 i = \omega^4 i$, and it has two zero-directions.

The above discussion leaves open the possibility $F(i) = -\omega^2 i$; if we want to exclude this we have to put down the additional condition

$$F^1_1 > 0. \tag{4.9}$$

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We know now the necessary and sufficient conditions which have to be satisfied if our tensor $F$ is to have the form (4.41); if these conditions are satisfied we can find the vectors $i, j, k, l$ and the number $\omega$. Once we have found them we put, to satisfy (4.5),

$$\lambda = \omega\sqrt{-2}\sin\varphi, \qquad \mu = \omega\sqrt{2}\cos\varphi, \tag{4.51}$$

where $\varphi$ is an arbitrary (real) angle and have in

$$f(x) = \lambda\{i(jx) - j(ix)\} + \mu\{k(lx) - l(kx)\}$$

an electromagnetic tensor which satisfies the energy relation with the given tensor $F$. We see thus that the electromagnetic tensor is not entirely determined by the curvature tensor of space-time at the same point; after the curvature tensor is given there are an infinity of electromagnetic tensors which are possible from the point of view of the energy relation. To complete the determination of the electromagnetic tensor we must know, besides the curvature tensor, the number $\varphi$. From the geometric point of view we may say that the curvature tensor gives the skeleton of the electromagnetic tensor, but instead of giving the two numbers $\lambda$ and $\mu$ it gives only their combination $\mu^2 - \lambda^2$.

It would, however, be wrong to conclude from this that the curvature of space-time does not determine the electromagnetic *field*. So far we have considered only the relation between the two tensors *in a point*. We shall now take into account their differential properties.

## PART II. DIFFERENTIAL PROPERTIES

### 5. Preliminary remarks

We shall proceed to study a region of space-time, in each point of which we consider the electromagnetic tensor; in each point the energy relation holds, so that the results of Part I are applicable, but we shall now take into account also the Maxwell equations which are satisfied by the electromagnetic tensor. We shall ask ourselves, first, what additional information with respect to the field $F$ can be obtained from the fact that $f$, which is connected with $F$ by the energy relation, is, at the same time, subjected to the Maxwell* equations. After we have found the restrictions which have to be imposed on the field of $F$ we shall, secondly, take up again the question of how far the field $f$ is determined by the field $F$; and finally

* When we say Maxwell equation in the following we always imply *in empty space.*

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we shall translate the conditions for the field $F$ into the language of components.

The usual form of the Maxwell equations in regions where matter is absent is

$$(5.1) \qquad f_{i,\rho}^{\rho} = 0, \qquad d_{i,\rho}^{\rho} = 0,$$

where $d$, as before, means the dual tensor of $f$, and the index after the comma corresponds to covariant differentiation. It will not, however, be convenient for us to deal with the components of tensors; the results of § 3 permit us, it is true, to choose the coördinates for a given point in such a way as to bring the components of $f$ into the simple form (see 4.2)

$$(5.2) \qquad \begin{Vmatrix} 0 & \lambda & 0 & 0 \\ \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu \\ 0 & 0 & -\mu & 0 \end{Vmatrix},$$

but we shall not be able to use this form where differentiation is involved, because this holds only for the point in which the system of coördinates is geodesic and if it is geodesic for one point it cannot be geodesic in its neighborhood. We therefore take the form (see (3.6) and (4.3))

$$(5.3) \qquad \begin{aligned} f(x) &= \lambda\{i(jx)-j(ix)\}+\mu\{k(lx)-l(kx)\}, \\ d(x) &= \mu\{i(jx)-j(ix)\}+\lambda\{k(lx)-l(kx)\}. \end{aligned}$$

We can consider $f$ and $d$ as given in this form for all points (of a certain region). Of course, the vectors $i, j, k, l$ will not be the same in different points; in a curved space there is no such thing as equality—and still less identity—between vectors of different bundles. The values of the numbers $\lambda$ and $\mu$ may also change from point to point. The vectors $i, j, k, l$ and the numbers $\lambda$ and $\mu$ will therefore be point functions. If a definite system of coördinates is introduced, the numbers $\lambda$ and $\mu$ and the components of the vectors $i, j, k, l$ will be functions of coördinates; they will constitute tensor fields of rank zero and one, respectively. Of course the tensor analysis can be developed from the beginning independently of coördinates (compare the author's papers cited in the introduction), but here we shall translate into vector language only the things which we are going to use.



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If we have a tensor of rank zero, i. e., a point function $\lambda$, it has in each point four derivatives $\lambda_i$; we may consider them as the covariant components of a vector, which is called the gradient of $\lambda$ and denoted by grad $\lambda$. We may, on the other hand, consider instead of the derivatives the differential; if we denote the differentials of coördinates by $h^i$ and the vector which has $h^i$ for its components by $h$, the differential can be written as $\lambda_\rho h^\rho$. This is the scalar product of grad $\lambda$ by $h$; it is also a scalar linear function of $h$ which we shall designate $\lambda'(h)$; in short, the differential of a scalar field $\lambda$ is

$$(5.4) \qquad \lambda'(h) = \operatorname{grad} \lambda \cdot h = \lambda_\rho h^\rho;$$

and we have, using (2.5) (with — changed into + according to (3.7)),

$$(5.5) \qquad \operatorname{grad} \lambda = i \cdot \lambda'(i) + j \cdot \lambda'(j) + k \cdot \lambda'(k) + l \cdot \lambda'(l),$$

if $i, j, k, l$ are any four perpendicular unit vectors.

If we have a vector field $v$, i. e., a tensor field of rank one with contravariant components $v^i$, the absolute derivatives $v^i_{,j}$ of these components can be considered as mixed components of a tensor of the second rank; if instead of the derivatives we consider the differential $v^i_{,\rho} h^\rho$ we can interpret this as a transformation applied to the vector $h$, i. e., a linear vector function, which we shall designate by $v'(h)$.

If we have a tensor field of the second rank given by its mixed components $f^i_j$ the absolute derivatives will be $f^i_{j,k}$ and if we consider instead of the components $f^i_j$ the transformation $f^i_\rho x^\rho$, the differential will be $f^i_{\rho,\sigma} x^\rho h^\sigma$, i. e., a bilinear vector function with the arguments $x$ and $h$; we shall in vector notations write for the differential of the linear vector function $f(x)$ simply $f'(x, h)$.

The result of contracting $f^i_{j,k}$ with respect to the indices $i$, $k$ is $f^\rho_{j,\rho} = f^1_{j,1} + f^2_{j,2} + f^3_{j,3} + f^4_{j,4}$; these are components of a vector, say $v_j$. It is easy to see that in vector notations this becomes

$$(5.6) \qquad v = f'(i, i) + f'(j, j) + f'(k, k) + f'(l, l).$$

We shall not go farther in this direction; that is all we need for the translation of the Maxwell equations. But before we start the work on them we notice that, differentiating the identities (3.7), we find

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$$i'(h)\cdot i = j'(h)\cdot j = k'(h)\cdot k = l'(h)\cdot l = 0; \tag{5.7}$$

$$\begin{aligned} i'(h)\cdot k + k'(h)\cdot i = j'(h)\cdot k + k'(h)\cdot j &= i'(h)\cdot l + l'(h)\cdot i \\ &= j'(h)\cdot l + l'(h)\cdot j = 0; \end{aligned} \tag{5.8}$$

$$i'(h)\cdot j + j'(h)\cdot i = k'(h)\cdot l + l'(h)\cdot k = 0. \tag{5.9}$$

## 6. Geometric properties of a Maxwell field

In order to write down the first set of Maxwell's equations in vector form we have to write that the vector $v$ (5.6) is zero when $f$ is the tensor given by (5.3). The differential of $f(x)$ is

$$\begin{aligned} f'(x, h) = {} & \lambda'(h)\cdot i\cdot(jx) - \lambda'(h)\cdot j\cdot(ix) + \mu'(h)\cdot k\cdot(lx) - \mu'(h)\cdot l\cdot(kx) \\ & + \lambda\cdot i'(h)\cdot(jx) - \lambda\cdot j'(h)\cdot(ix) + \mu\cdot k'(h)\cdot(lx) - \mu\cdot l'(h)\cdot(kx) \\ & + \lambda\cdot i\cdot[j'(h)\cdot x] - \lambda\cdot j\cdot[i'(h)\cdot x] + \mu k[l'(h)\cdot x] - \mu l[k'(h)\cdot x]. \end{aligned} \tag{6.1}$$

Instead of writing that the vector $v$ is zero we shall write that its components, i. e., the products $v\cdot i$, etc., are zero. In order to form, e. g., $v\cdot i$, we consider $f'(x, h)\cdot i$; the multiplication of (6.1) by $i$ destroys on its left hand side all the terms which are perpendicular to $i$, i. e., those which have the directions of $j, k, l, i'$. There remains

$$\begin{aligned} f'(x, h)\cdot i = {} & \lambda'(h)\cdot(jx) - \lambda\cdot[j'(h)\cdot i]\cdot(ix) + \mu\cdot[k'(h)\cdot i]\cdot(lx) \\ & - \mu\cdot[l'(h)\cdot i]\cdot(kx) + \lambda[j'(h)\cdot x]. \end{aligned} \tag{6.2}$$

To obtain $v\cdot i$ we have to put here $x = h$, to substitute for this vector in turn $i, j, k, l$ and to add the results. The first term of (6.2) gives a vector different from zero only for $h = j$; the second for $h = i$, the third for $h = l$, the fourth for $h = k$ and the last for $h = i$, or $k$ or $l$. We have thus, since the second and the last terms of the result destroy each other, using (5.8)

$$v\cdot i = \lambda'(j) - \lambda\{k'(k)\cdot j + l'(l)\cdot j\} - \mu\{l'(k)\cdot i - k'(l)\cdot i\} = 0. \tag{6.3}$$

This is a scalar equation and we shall have three more similar equations for the other components of $v$ from the first set of the Maxwell equations (5.1) and four more from the second set. We can obtain them from (6.3) by interchanging $i$ and $j$, $k$ and $l$, and $\lambda$ and $\mu$. But we need now only the one which we obtain by interchanging $\lambda$ and $\mu$, viz.

$$\mu'(j) - \mu\{k'(k)\cdot j + l'(l)\cdot j\} - \lambda\{l'(k)\cdot i - k'(l)\cdot i\} = 0. \tag{6.31}$$



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Eliminating from (6.3) and (6.31) first the third terms and then the second terms, we obtain

$$(6.41) \qquad \mu\mu'(j) - \lambda\lambda'(j) = (\mu^2 - \lambda^2)\{k'(k)\cdot j + l'(l)\cdot j\},$$

$$(6.42) \qquad \lambda\mu'(j) - \mu\lambda'(j) = (\lambda^2 - \mu^2)\{k'(l)\cdot i - l'(k)\cdot i\}.$$

Now we have from (4.5) and (4.51), remembering that $\omega \neq 0$,

$$\frac{\mu\mu' - \lambda\lambda'}{\mu^2 - \lambda^2} = \frac{1}{2}\,\frac{(\mu^2 - \lambda^2)'}{\mu^2 - \lambda^2} = \frac{1}{2}\,\frac{(\omega^2)'}{\omega^2} = \frac{\omega'}{\omega},$$

$$\frac{\lambda\mu' - \mu\lambda'}{\mu^2 - \lambda^2} = \frac{1}{2\omega^2}\begin{vmatrix} -\omega\sqrt{2}\sin\varphi\cdot\varphi' + \omega'\sqrt{2}\cos\varphi & \omega\sqrt{-2}\cos\varphi\cdot\varphi' + \omega'\sqrt{-2}\sin\varphi \\ \omega\sqrt{2}\cos\varphi & \omega\sqrt{-2}\sin\varphi \end{vmatrix}$$

$$= -\varphi'\sqrt{-1}.$$

This permits us to write (6.41) and (6.42) in the form

$$\frac{\omega'(j)}{\omega} = k'(k)\cdot j + l'(l)\cdot j = [k'(k) + l'(l)]\cdot j,$$

$$\varphi'(j)\cdot\sqrt{-1} = k'(l)\cdot i - l'(k)\cdot i = [k'(l) - l'(k)]\cdot i,$$

and we have three more of each type. If we put

$$(6.51) \qquad \begin{aligned} p = {} & i\cdot[k'(k)\cdot i + l'(l)\cdot i] + j\cdot[k'(k)\cdot j + l'(l)\cdot j] \\ & + k\cdot[i'(i)\cdot k + j'(j)\cdot k] + l\cdot[i'(i)\cdot l + j'(j)\cdot l], \end{aligned}$$

$$(6.52) \qquad \begin{aligned} q = {} & i\cdot[k'(l)\cdot j - l'(k)\cdot j] + j\cdot[l'(k)\cdot i - k'(l)\cdot i] \\ & + k\cdot[i'(j)\cdot l - j'(i)\cdot l] + l\cdot[j'(i)\cdot k - i'(j)\cdot k], \end{aligned}$$

and use (5.5) we find *as the equivalents of Maxwell's equations*

$$(6.61) \qquad \operatorname{grad}\omega = \omega p \qquad \text{or} \qquad \operatorname{grad}\log\omega = p,$$

$$(6.62) \qquad \sqrt{-1}\operatorname{grad}\varphi = q.$$

In § 3, we called the two invariable planes of an antisymmetric tensor the skeleton of this tensor. We shall now call skeleton of an antisymmetric

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*field the totality of the skeletons of its tensors.* It is easy to show that the vectors $p$ and $q$ defined by (6.51) and (6.52) are entirely determined (for each point) by the skeleton of the field (in the neighborhood of that point); in order to do so it is enough to notice that the form of the expressions (6.51), (6.52) is not changed by the transformations (3.8) and (3.9). The equations (6.61) and (6.62) give therefore a property of the skeleton of an antisymmetric field which satisfies Maxwell's equations, which may be stated as follows:

THEOREM. *If an antisymmetric field satisfies Maxwell's equations, the vectors p and q defined by its skeleton are gradients of scalar functions.*

The converse is also true. Suppose we are given two perpendicular planes in each point of a region and we want to know whether there exists a Maxwellian field which has these planes for its skeleton. We choose in each plane two perpendicular unit vectors $i$, $j$ and $k$, $l$ respectively, and form according to the formulas (6.51) and (6.52) the vectors $p$ and $q$; if these vectors are gradients of scalar functions there exists an $\infty^2$ of different Maxwellian fields with these planes as skeletons. In fact, we can determine two functions $\omega$ and $\varphi$ (each containing an arbitrary additive constant), satisfying (6.61) and (6.62); if we now form $\lambda$ and $\mu$ according to the expressions (4.51) and use them in (5.3) we have the fields in question.

It is interesting to notice that $q$ is an imaginary vector, i. e. a vector of our space multiplied by $\sqrt{-1}$, because $i$ enters in every term once as a factor ($p$ is real because $i$ enters in some of its terms twice and does not enter in other terms at all). If we consider, in a purely formal way, the sum $p+q$ as a complex vector we can say that it is the gradient of

$$\log \omega + \varphi \sqrt{-1} = \log \omega e^{\varphi\sqrt{-1}}.$$

The formulas (4.51) show that

$$\omega e^{\varphi\sqrt{-1}} = \mu + \lambda.$$

Following this line and introducing complex tensors we could considerably simplify our calculations but as the purpose of this paper is only to show how the electromagnetic field is determined by the curvature it does not appear desirable to make the calculations depend on these concepts because this would tend to obscure the principal point at issue. (See, however, § 9.)

A different expression is given for the vector $p$ (with the sign changed) in the "First Note" (formula 4). This expression holds only if the vectors $i, j, k, l$ are chosen in a special way indicated there and does not seem to have any essential advantage over (6.51).

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## 7. Differential properties of the energy tensor

We saw (end of § 4) that the curvature tensor gives the skeleton of the electomagnetic tensor and the number $\omega^2$ in each point. We can restate this now saying that the curvature *field* determines the skeleton of the electromagnetic *field* and the scalar *function* $\omega^2$. In order to complete the determination of the electromagnetic field it remains for us to determine the function $\varphi$, but we shall take this question up a little later. For the present we emphasize the fact that the equations (6.61) and (6.62) must furnish us some properties of space-time in which there is an electromagnetic field, because the vectors $p$, $q$ and the function $\omega^2$ are determined by the curvature of space-time.

As for the equation (6.61) both $p$ and $\omega$ are given by the curvature so that it directly gives us a property of space-time. *This property is, however, not new*; it is a consequence of the known relation

$$\left(R_i^\rho - \frac{1}{2} g_i^\rho R_\sigma^\sigma\right)_{i,\rho} = 0, \tag{7.1}$$

which holds in every curved space.* In our case this relation takes the simpler form

$$F_{i,\rho}^\rho = 0. \tag{7.2}$$

Using the expression (4.41) for $F$ and proceeding in the same way as we did in the beginning of § 6 when we were about to translate Maxwell's equations, which have the same form as (7.2), we find

$$\begin{aligned} F'(x,h)\cdot i &= 2\omega\cdot\omega'(h)\cdot(ix) \\ &\quad + \omega^2\{[j'(h)\cdot i](jx) - [k'(h)\cdot i](kx) - [l'(h)\cdot i](lx) + i'(h)\cdot x\} \end{aligned} \tag{7.3}$$

and

$$\begin{aligned} v\cdot i &= 2\omega\cdot\omega'(i) \\ &\quad + \omega^2\{j'(j)\cdot i - k'(k)\cdot i - l'(l)\cdot i + i'(j)\cdot j + i'(k)\cdot k + i'(l)\cdot l\} = 0. \end{aligned}$$

The relations (5.9) show that the first and the fourth terms in the brackets give a sum zero, and the relations (5.8) that the fifth is equal to the second and the sixth to the fourth; we can write therefore, remembering that $\omega \neq 0$,

$$\frac{\omega'(i)}{\omega} = k'(k)\cdot i + l'(l)\cdot i,$$

* Cf. J. A. Schouten and D. J. Struik, Philosophical Magazine, vol. 47 (1924), p. 584.

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and, with the three other similar equations, this is equivalent to (6.61); this proves our assertion that this last equation does not impose any new restrictions on space-time. It may, however, be argued that the choice of the expressions for the energy tensor in terms of the curvature tensor, viz. $R^i_j$ or $R^i_j - \frac{1}{2} g^i_j R^\sigma_\sigma$ was influenced by the consideration that for the energy tensor the equation (7.2) must be satisfied.

We shall try now to find whether equation (6.62) gives us some property of space-time containing an electromagnetic field. We know that the point function $\varphi$ which enters in (6.62) is *not* determined by the tensor $F$ in the corresponding point; we have, therefore, to eliminate $\varphi$ from this equation and this we can do simply saying that $q$ must be a gradient of a scalar field; or we may write

(7.4) $$\operatorname{rot} q = 0, \qquad \text{or} \qquad q_{i,j} = q_{j,i}.$$

This property of space-time containing an electromagnetic field does not seem to be a consequence of general properties of curved space; it seems to be an additional restriction imposed on our space-time. However this may be, we suppose henceforth that this condition is satisfied.

We return now to the question of how far the electromagnetic field is determined by space-time. We stated at the beginning of this section that we had still to determine the function $\varphi$; *but that is just what equation* (6.62) *does*; it determines the function $\varphi$, the only remaining arbitrariness being in a constant of integration. If $\varphi$ is a solution of (6.62) the general solution is $\varphi + \gamma$, $\gamma$ being a constant. From (4.51) we obtain

(7.5) $$\lambda = \omega\sqrt{-2}\sin(\varphi+\gamma), \qquad \mu = \omega\sqrt{2}\cos(\varphi+\gamma),$$

and, if by $\lambda_0$ and $\mu_0$ we designate the values of $\lambda, \mu$ which correspond to $\gamma = 0$, we have

$$\lambda = \lambda_0\cos\gamma + \mu_0\sin\gamma\sqrt{-1}, \qquad \mu = \mu_0\cos\gamma + \lambda_0\sin\gamma\sqrt{-1};$$

if, further, by $f_0$ and $d_0$ we designate the tensor fields which are obtained from (5.3) for $\lambda = \lambda_0$, $\mu = \mu_0$, we can write *the general electromagnetic field which is compatible with the given space-time in the form*

(7.6) $$f = f_0\cos\gamma + d_0\sin\gamma\cdot\sqrt{-1}, \qquad d = d_0\cos\gamma + f_0\sin\gamma\cdot\sqrt{-1},$$

the vectors $i, j, k, l$ and the number $\omega$ being determined by the tensor $F$ in the point considered and the function $\varphi$ by the field $F$ in the neighborhood of that point.

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It is not the place here to treat the connection of the results obtained with the question of radiation, which was briefly indicated in our "Second Note."

## 8. Second order property in components

The field $F$ determines the skeleton, the skeleton determines the vector field $q$; equation (7.4) expresses, therefore, a property of the tensor field $F$. We shall show now how this property can be expressed in terms of the components of $F$.

Multiplying both sides of the relation $F^2(x) = \omega^4 x$ (see (4.7)) by $y$ and using the symmetry of $F$ (3.11), we obtain

$$F(x)\cdot F(y) = \omega^4\cdot(xy).$$

In what follows we will consider only the vectors $i, j, k, l$, which are mutually perpendicular, as the values of $x, y$; therefore if $x, y$ are different we will have

$$F(x)\cdot F(y) = 0.$$

Differentiating this we obtain

$$F'(x,h)\cdot F(y) + F'(y,h)\cdot F(x) = 0. \tag{8.1}$$

We now form

$$P(x,y,z) = F'(x,y)\cdot F(z) + F'(y,z)\cdot F(x) + F'(z,x)\cdot F(y). \tag{8.2}$$

Using (8.1) we easily see that $P(x, y, z)$ changes its sign when two of its arguments are interchanged (always supposing $x, y, z$ to be three different vectors from among $i, j, k, l$); it has, therefore, only four essentially different values, but they can be obtained from one by interchanging $i, j, k, l$. Let us calculate, e. g., $P(i,j,k)$, or, according to (4.42),

$$\omega^2\{-F'(i,j)\cdot k + F'(j,k)\cdot i + F'(k,i)\cdot j\};$$

the middle term can be obtained from (7.3), making $x = j$, $h = k$. Taking in consideration (5.9) we see that it vanishes. To obtain the last term, we interchange in (7.3) $i$ and $j$, and make then $x = k$, $h = i$; there remains

$$\omega^2\{-k'(i)\cdot j + j'(i)\cdot k\} = 2\omega^2[j'(i)\cdot k]$$

according to (5.8). With the aid of (8.1) we see that $F'(i,j)\cdot k$ can be obtained from this interchanging $i$ and $j$. We have thus

$$P(i,j,k) = 2\omega^4[j'(i)\cdot k - i'(j)\cdot k].$$

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Confronting this with (6.52) we see that this is the product $q \cdot l$ multiplied by the factor $2\omega^4$, or that $P(i, j, k)$ is but for this factor the $l$-component of the vector $q$.

If we make $x = i$, $y = j$, $z = k$ in (8.2) we obtain for $P(i, j, k)$ an expression which, translated in the usual language of coördinates, is

$$F^{\rho}_{1,2}\, F_{\rho 3} + F^{\rho}_{2,3}\, F_{\rho 1} + F^{\rho}_{3,1}\, F_{\rho 2};$$

this is equal to $\omega^4 q_4$ but it is a component of a tensor of the third rank, which, according to our remark following (8.2), is completely alternating. It is, therefore, more convenient to introduce instead of $q$ a completely alternating tensor of the third rank $q_{ijk}$ defined for geodesic coördinates by the equalities

$$(8.3) \qquad q_{123} = q_4, \qquad q_{234} = -q_1, \qquad q_{341} = q_2, \qquad q_{412} = -q_3,$$

and which is sometimes referred to as complement of $q_i$. For this tensor we have then*

$$(8.4) \qquad 2\omega^4 \cdot q_{ijk} = F^{\rho}_{i,j}\, F_{\rho k} + F^{\rho}_{j,k}\, F_{\rho i} + F^{\rho}_{k,i}\, F_{\rho j}.$$

It remains to write in terms of the tensor $q_{ijk}$ the equations (7.4), which express the condition that $q$ must be a gradient. Take, e. g., the equation $q_{1,2} = q_{2,1}$; using (8.3) we obtain $-q_{234,2} = q_{341,1}$ or, on account of the alternating property, $q_{341,1} + q_{342,2} = 0$; and finally since $q_{ijk}$ vanishes when two indices are equal,

$$q^{341}{}_{,1} + q^{342}{}_{,2} + q^{343}{}_{,3} + q^{344}{}_{,4} = 0,$$

where we use contravariant components, which makes no difference while we are using geodesic coördinates but permits us to write the result in a form which is independent of the system of coördinates, viz.,

$$(8.5) \qquad q^{ij\tau}{}_{,\tau} = 0.$$

This together with the formula (8.4) defining $q_{ijk}$ gives us the differential conditions to which the curvature tensor is subjected as a consequence of the presence of the electromagnetic field.

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* In the "Second Note", formula (11), $\omega^4$ must stand instead of $\omega^2$; this is obvious because $q$ must not change when $F$ is multiplied by a constant.

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# PART III. INTEGRAL PROPERTIES AND SINGULARITIES

## 9. Analogy with analytic functions

In order to find the significance of the fact that the curvature of space-time seems to leave undetermined a constant in the expression for the electromagnetic field we shall have to touch upon the question of *matter*, which we consider as constituted by the singularities of the field. In discussing these singularities much help can be derived from the consideration of both points of striking *analogy* and points of *difference* between the theory of the electromagnetic field and the theory of analytic functions of a complex variable.

We begin with the *analogy*, which can also be stated by saying that both the theory of analytic functions and the theory of the electromagnetic field are special cases, corresponding to $r = 2$, and $r = 4$, respectively, of a general theory of conjugate functions, imagined by Volterra as early as 1889.* From this point of view the Maxwell equations are analogous to the Cauchy-Riemann equations of the theory of functions. They can also be replaced by an equivalent integral relation which is analogous to the Cauchy-Morera theorem of the theory of functions. Before we write down this integral form of the Maxwell equations, we go one step farther than is usually done (so far as we know†) and introduce, instead of the tensors $f$ and $d$, their sum

$$(9.1)\qquad w(x) = f(x) + d(x) = \nu\{i(jx) - j(ix) + k(lx) - l(kx)\}.$$

Since the tensor $d$ is an imaginary tensor (cf. the statement after (4.3)) the tensor $w$ is to be considered as a *complex* tensor, i. e., if $x$ is a vector of our space, $w(x)$ is the sum of a vector of our space and of a vector of our spac multiplied by $\sqrt{-1}$; the number $\nu$, being the sum of a real number $\mu$ and an imaginary number $\lambda$, is also a complex number. Incidentally, from this point of view the tensor $F$ is the product of $w$ by the conjugate tensor $\bar{w}$, or the square of the modulus of the tensor $w$, and the number $\omega^2$ is half the square of the modulus of $\nu$; using these notations we could have simplified our calculations in the §§ 6 and 7, grad log $\nu$ would furnish us the complex vector $p + q$, etc.

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* Lincei Rendiconti, 1889, 1st semester, pp. 599–611 and 630–640. The analogy in question has been already noticed. See, e. g., F. Kottler, *Maxwell'sche Gleichungen und Metrik*, Wiener Sitzungsberichte, IIa, vol. 131, No. 2, pp. 119–146. This paper contains full bibliographical references.

† Compare, however, L. Silberstein, Annalen der Physik, ser. 4, vol. 22 (1907), p. 579 and H. Weber, *Partielle Differentialgleichungen der Mathematischen Physik*, vol. 2, 1901, p. 348.

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We can formulate now the analogue of the Cauchy-Morera theorem as follows; the statement that Maxwell's equations for empty space hold in a certain region is equivalent to the statement that the integral

$$\int w_n \, d\sigma, \tag{9.2}$$

taken over any two-dimensional surface which belongs to the region and can be continuously transformed into a point without leaving that region, vanishes.* An immediate consequence of this is that the value of integral (9.2) when it does not vanish—this value is a complex number—does not change if, instead of one closed surface, we take another into which the former can be continuously transformed without leaving the region where Maxwell's equations are satisfied.

The question naturally arises: what is it in this theory, that takes the place of singular points of the theory of analytic functions? Many considerations, both physical and mathematical, lead us to believe that the most interesting objects of this kind are singular lines (having time-direction). If we consider a two-dimensional surface $\Sigma$ which surrounds such a singular line $\Gamma$ (much as a circle surrounds a straight line in our three-dimensional space), the value of the integral (9.2) taken over $\Sigma$ is not necessarily zero, but it follows from what was said after (9.2) that we may change $\Sigma$ as we want; so long as it surrounds $\Gamma$ and remains in a simply connected region in which $\Gamma$ is the only singularity, the value of the integral will not change. In other words this value is entirely determined by the singular line. This value, which is a complex number, is obviously an analogue of the residue, and we shall use for it this word.

Now it so happens that, if we look for the physical interpretation of (9.2), we find that its real part gives the electric charge which is present in some three-dimensional volume enclosed by our surface, and the imaginary part would correspond to a magnetic charge, *but this magnetic charge is always zero.* This last fact seems to be inexplicable from the point of view of the electromagnetic field considered independently of the curvature of space-time, or, let us say, in the space-time of special relativity theory. But it is different from the point of view of general relativity theory on which we stood in the first two parts of the present paper, and to which we shall revert presently.

* A formulation of Maxwell's equations involving integrals over two-dimensional surfaces in time-space was given by R. Hargreaves as early as 1908 (contemporaneously with the famous publications of Minkowski) in the Cambridge Philosophical Society Transactions, vol. 21, p. 116. For a comprehensive presentation see F. D. Murnaghan's book *Vector Analysis and the Theory of Relativity*, Baltimore, 1922, especially p. 72 sqq.

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## 10. Consequences of the curvature of space-time

The considerations regarding integral properties of the electromagnetic field and the residue are independent of the metrical structure of the space. They are, therefore, applicable in the case when the space is the Riemann space of the general relativity theory (in fact, Volterra's general theory holds in much more general spaces). If we consider space-time as originally given, the electromagnetic field is, as we saw in § 7, not completely determined by it; we may say that there is an infinity of possible electromagnetic fields which are given by the expressions (7.6) involving the arbitrary constant $\gamma$. All these "associated" fields will have, obviously, the same singular lines but the residue of such a line will be different for different fields; it will depend on the constant $\gamma$; if $\varrho = \varepsilon + \varkappa\sqrt{-1}$ is its value for $\gamma = 0$ its value for an arbitrary $\gamma$ will be, in consequence of (7.6).

$$\begin{aligned}(10.1)\qquad &\varepsilon\cos\gamma + \varkappa\sqrt{-1}\sin\gamma\sqrt{-1} + \varkappa\sqrt{-1}\cos\gamma + \varepsilon\sin\gamma\sqrt{-1} \\ &= (\varepsilon + \varkappa\sqrt{-1})(\cos\gamma + \sin\gamma\sqrt{-1}) = \varrho e^{i\gamma}.\end{aligned}$$

All these numbers have the same modulus $|\varrho| = \sqrt{\varepsilon^2 + \varkappa^2}$, so that we may say that only the modulus of the residue is determined by the curvature field.

If we have but one singular line (one electron) we can so choose the constant $\gamma$ as to make the residue real; or, we may say, among the *possible* fields there is just one (or, more precisely, two of opposite signs) for which the magnetic charge vanishes. We can agree always to choose this field as the *existing* electromagnetic field; by this two difficulties would be solved at one stroke; the electromagnetic field would be entirely determined by the curvature field, and the fact that the nagnetic charge is zero would be explained, as the result of our agreement.

But there exists more than one electron; if we have several singular lines the situation is not as simple as in the case of one singular line. If we choose our constant $\gamma$ so as to make the imaginary part of the residue of one line zero we do not see immediately why the imaginary parts of the residues of other lines also should vanish; in other words, why the arguments of all residues should have values differing only by multiples of $\pi$. But we know from experimental physics that there are no magnetic charges; that means, that the existing electromagnetic field (i. e., one of the possible electromagnetic fields) has only real residues and from (10.1) it follows then, since $\varrho$ is real, that for the possible field which corresponds to the value $\gamma$ of the constant the argument is either $\gamma$ or $\pi + \gamma$. It is important to notice that this experimental fact that *the differences between the arguments of the*

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*residues of different singular lines in every possible field is a multiple of* $\pi$ is a property of the space-time, because the totality of possible fields is given by the curvature field of space-time. There may be a question whether this fact can be accounted for on the general theory of relativity as it is now (i. e., whether it is a consequence of the conditions which must be satisfied by the curvature field of space-time, and which we found by eliminating the electromagnetic tensor from the energy relation and the Maxwell equations, viz. (4.71), (4.8), (8.4) and (8.5)) or whether it must be taken as an additional assumption; however this may be (see the next section) we have to consider the underlined statement as expressing an established property of space-time; but then we can determine the electromagnetic field which corresponds to a given space-time by the condition that its residues must be real. The result is the same as in the case of only one singular line. We have thus proved our contention that, under the assumptions which we have made, the electromagnetic field is entirely determined by the curvature field of space-time. These assumptions are the following:

1. In no region do the invariants of the electromagnetic field *strictly* vanish.
2. The underlined statement above.

### 11. Non-linearity of the field and possible consequences

Before we treat in the next section a simple example illustrating the foregoing general discussion, we cannot help indicating some speculative reasonings which bear on the assumptions just mentioned.

Considering the *analogy* with the theory of functions it may be hoped that the first of these assumptions will be deduced from the equations of the field (besides, this assumption may not be necessary because the treatment of the second canonical form of the electromagnetic tensor (3.31) may lead to the same results).

As to the second assumption there may be hope of throwing some light on it by the consideration of an essential *difference* which exists between the theory of the electromagnetic field and the theory of analytic functions. This difference is given by the fact that the conditions which define the electromagnetic field of the general relativity theory are *not linear* (see "First Note", p. 125); therefore we connot, if two different fields are given, obtain, in general, a new field by adding, say, the components in the corresponding points. In the case of analytic functions, and also in the case of electromagnetic fields of special relativity theory, we may obtain a field with two singularities by adding two fields, each of which has one singularity; there can be, in this case, no necessary connection, no interdependence between two singularities of a field, because we can choose the constants characterizing the singularities in the two fields, which are being

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added, quite arbitrarily, independently each of the other. Not so in the case of the electromagnetic field of the general relativity theory; we cannot add here two fields with given singularities and be sure that the result is again a field which satisfies our conditions; the fields which are being added must satisfy some additional condition if their sum is to be such a field, and there seems to be nothing impossible in the assumption that this additional condition may bear on the constants which characterize the singularities, for instance, that it may lead to the result that the arguments of the residues can differ only by a multiple of $\pi$—and, moreover, that the moduli of the residues are equal; this would account for the equality of charges of different electrons. This additional condition may even affect the paths, i. e., the shape of singular lines.

To make this speculation more concrete we may consider two spaces given by their $g$'s; the equations (4.7), (4.81), (8.4), (8.5) which must be satisfied by the curvature tensor field will give us equations of the second and fourth order in the $g$'s and these equations are *not linear* in the $g$'s. Suppose now each system of the $g$'s defines a space with one singularity but involves arbitrary constants; if we add the corresponding $g$'s and determine a new space by the sums, we will have some additional condition which must be satisfied by the two systems of the $g$'s and this condition may result in relations between the constants of the two systems of the $g$'s which are being added. Of course all this must be worked out in full detail and cannot be considered at the present time as being more than a vague suggestion.

Meanwhile we are able to treat by the preceding method only the simplest case of one singular line; we will see in the next section that we come thus to a solution which has already been obtained several times by different methods.

## 12. The centro-symmetric solution

We shall try to find a centrosymmetric field which satisfies our equations. In this case the expression for the line element can be taken in the form

$$-ds^2 = \xi(r)\,dr^2 + r^2 d\vartheta^2 + r^2 \sin\vartheta \cdot d\psi^2 - \eta(r)\cdot dt^2,$$

and the mixed components of the contracted Riemann tensor are, according to the calculations of F. Kottler (Annalen der Physik, vol. 56 (1918), p. 433),

(12.1)
$$F_1^1 = \frac{1}{r^2}\left(1-\frac{1}{\xi}\right) - \frac{\eta'}{\xi\eta}\cdot\frac{1}{r}, \qquad F_4^4 = \frac{1}{r^2}\left(1-\frac{1}{\xi}\right) + \frac{\xi'}{\xi^2}\cdot\frac{1}{r},$$

$$F_2^2 = F_3^3 = \frac{\xi'}{2\xi^2}\cdot\frac{1}{r} - \frac{\eta'}{2\xi\eta}\cdot\frac{1}{r} - \frac{\eta''}{2\xi\eta} + \frac{\eta'^2}{4\xi\eta^2} + \frac{\xi'\eta'}{4\xi^2\eta},$$

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all the other components being zero. We see that the 2,3 plane is a plane of invariable directions with the characteristic number $F_2^2 = F_3^3$; this plane being a space-plane we must have $F_2^2 = F_3^3 = -\omega^2$ and, if our geometric conditions (4.43) are to be satisfied, the perpendicular plane must also be a plane of invariable directions and have $+\omega^2$ for its characteristic number; both $F_1^1$ and $F_4^4$ must therefore be equal to $\omega^2$ so that we have

$$(12.2) \qquad F_1^1 = F_4^4, \qquad F_1^1 + F_2^2 = 0.$$

The same result could have been obtained algebraically: equation (4.71) shows that the square of each of the numbers $F_i^i$ is equal to $\omega^4$ and (4.8) that the sum of these numbers is zero; since we know that $F_2^2 = F_3^3$, we conclude that $F_1^1$ and $F_4^4$ must be equal to each other and have the sign opposite to that of $F_2^2 = F_3^3$. The equation $F_1^1 = F_4^4$ gives

$$(12.3) \qquad \frac{\xi'}{\xi} + \frac{\eta'}{\eta} = 0,$$

whence

$$(12.4) \qquad \xi\eta = 1,$$

where we gave the value 1 to the constant of integration by choosing appropriately the unit of time. If we use (12.3) and (12.4), the last two terms of the expression for $F_2^2$ (see (12.1)) destroy each other and the first two can be written as $-\eta'/r$; the equation $F_1^1 + F_2^2 = 0$ takes the form

$$\frac{1}{r^2}(1-\eta) - \eta' \cdot \frac{1}{r} - \eta' \cdot \frac{1}{r} - \frac{1}{2}\eta'' = 0 \qquad \text{or} \qquad \left(\frac{\eta r^2}{2}\right)'' = 1,$$

whence

$$(12.5) \qquad \eta = 1 + \frac{b}{r} + \frac{a}{r^2}, \qquad \xi = \frac{1}{1 + \frac{b}{r} + \frac{a}{r^2}}.$$

We obtain thus the known solution representing the line-element corresponding to a point charge, found for the first time by Weyl (Annalen der Physik, vol. 54 (1917), p. 117) and then by Nordstrom, Jeffery and others.

Substituting the expressions for $\xi$ and $\eta$ in the first of (12.1) we find

$$\omega^2 = F_1^1 = \frac{a}{r^4}.$$

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The next task is to find the vector $q$. Somewhat lengthy but elementary calculations lead to the result, which is practically evident geometrically, that in our case $q = 0$; $\varphi$ is therefore an arbitrary constant and the electromagnetic tensor is

$$f(x) = \frac{\sqrt{-2a}}{r^2}\sin\varphi\{i(jx) - j(ix)\} + \frac{\sqrt{2a}}{r^2}\cos\varphi\{k(lx) - l(kx)\},$$

where $k, l$ are two mutually perpendicular unit vectors which are perpendicular to the line joining the point considered with the electron, $j$ is a unit vector in the direction of that line and $i$ the unit time-vector; it is clear that the residue will be real if we choose $\varphi = 0$; in this case the only components which are different from zero are

$$f_2^3 = -f_3^2 = \frac{\sqrt{2a}}{r^2},$$

the indices 2 and 3 corresponding, as above, to the coördinates $\psi$ and $\vartheta$, and an integration over a sphere shows that $a$ is proportional to the square of the charge, but this, of course, is very well known. Incidentally, $a$ is thus proportional to the square of the modulus of the residue.

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