Archives

Ref. TH.921

REGGE REPRESENTATION FOR FORWARD SCATTERING OF UNEQUAL MASS PARTICLES

W. Drechsler

CERN - Geneva

A B S T R A C T

The Sommerfeld-Watson transformation of a t channel helicity amplitude is investigated for $z_t = \pm 1$ corresponding to forward scattering of unequal mass particles in the s channel. It is shown that the Sommerfeld-Watson transform exists at these points under similar conditions as in the case of large $|z_t|$ and that it corresponds to a finite Regge pole contribution. This eliminates the infinities of the conventional Regge representation at $z_t = \pm 1$. A modified Regge representation for helicity amplitudes is given, which guarantees a continuous behaviour of the scattering amplitude for unequal mass particles near the forward direction at finite energies.

68/878/5-TH.921
17 July 1968

1. INTRODUCTION

The behaviour of forward scattering amplitudes in the framework of the Regge theory has recently attracted considerable interest in connection with the so-called conspiracy problem [1)-4)]. It seems that a rigorous treatment of this phenomenon requires elaborate group theoretical techniques [5)-7)]. It is, however, at present unclear how the SL(2C) or O(4) symmetry for vanishing momentum transfer t will influence the behaviour of the scattering amplitude at finite non-zero t. The forward direction in the scattering of unequal mass particles does not correspond to $t=0$. It approaches this value only in the limit of infinite energies s. It is therefore of interest to study the behaviour of a reggeized two-body amplitude in the forward direction for unequal mass kinematics and at finite s using more conventional means, i.e., the Sommerfeld-Watson transformation. We investigate in this paper the question under what conditions a Sommerfeld-Watson transformation of the amplitude is still possible at $\cos\theta_t = \pm 1$, and how a regular or at least continuous Regge representation can be obtained at these points. In a previous paper [8)], a reggeization procedure for general two-body amplitudes was given in terms of second kind rotation functions, $e^j_{\lambda,\mu}(z)$. One obtains from this representation an asymptotic expansion for the helicity amplitudes valid for large z. If, however, one does not use the asymptotic behaviour of the functions $e^j_{\lambda,\mu}(z)$ for large z and considers instead the representation at $z=\pm 1$ each term develops singularities at these points. Going back to the functions $d^j_{\lambda,\mu}(z)$ would not resolve the problem since also in this case singularities arise for general values of j at $z=\pm 1$, due to the signature combination of d^j- functions which enters.

It is well known that for the scattering of unequal mass particles the forward direction in the s channel corresponds to $|\cos\theta_t| = 1$ in the t channel, which would mean that the amplitude after reggeization in the t channel is singular for $\theta_s = 0$ and finite s. To remove this defect of the conventional Regge representation, we investigate in this paper more carefully the reggeization of two-body amplitudes for z_t near ± 1. This extends in a way the work done by Khuri [9)] in the case of potential scattering of spinless particles. Khuri used a special integral representation of Mehler's conic functions $P_{-\frac{1}{2}+ip}(-z)$ appearing in the background integral of the Regge representation in the spinless case and managed to eliminate the logarithmic singularity at $z=1$ of the Regge pole terms by introducing certain functions with a reduced cut. In the general spin case, the situation is more complicated since, in addition to the logarithmic singularity at $z=1$, there is in the case of the functions $\bar{d}^{j}_{\lambda,-\mu}(-z)$ for general j also a descending power series present at $z=1$ with the maximal power being $[2/(1-z)]^{\frac{1}{2}|\lambda-\mu|}$ *). Khuri's trick would not allow to remove the last mentioned singularities - leaving aside the difficulties which occur when Regge cuts are present.

One can, however, use a more direct way to see that these unwanted singularities in z of the d^{j} - functions do not enter into the Regge expressions. These singularities in z (a principal part and a logarithmic singularity) appear in the integrand of the contour integral in the j plane which replaces the partial wave sum. But they are not present in the integral. Hence, the formalism should assure their absence at each stage of the calculation and, in particular, also for the representation used in analyzing experimental data, where the full amplitude after performing the Sommerfeld-Watson transformation is approximated by a few Regge poles, neglecting the background integral and Regge cuts.

For $s \to \infty$ our method of analysis breaks down in the forward direction, since the domain of convergence of the original partial wave sum shrinks to nothing for t approaching zero. As is known from group theory [1], a different expansion has to be used in this case, involving the representation functions of $SL(2C)$ which is the little group corresponding to vanishing momentum transfer. We therefore regard in the following discussion the energy s as finite, in order not to confuse our problem with that of conspiracy which requires a different approach. From our investigation, it is not possible to establish the kind of singularity a single Regge pole would develop for $z_t = 1$ and $t = 0$.

*) Corresponding singularities are present in the functions $d^{j}_{\lambda,\mu}(z)$ at $z=-1$ for arbitrary j.

It is, however, known that the functions $d^{j\sigma}_{jj'm}(\zeta)$ appearing in the Lorentz pole description of the scattering amplitude are regular at $\zeta = 1$ for arbitrary σ. To include also this interesting limiting point ($t = 0$ or $u = 0$) for unequal mass kinematics, Freedman and Wang – in their now classical paper [5] – abandoned the variable z and the corresponding expansion in terms of Legendre functions and discussed the cancellation of singularities in the asymptotic contributions of Regge poles at $u = 0$, $z_u = 1$, by making use of a representation introduced by Khuri [10], involving power series expansions in the Mandelstam variables t and u.

As stated above, we do not touch in this paper the problem of conspiracy and daughters relevant for a complete discussion at vanishing momentum transfer. Our more modest aim is to establish a reggeized version of an exchange model leading at large but finite s and for unequal mass kinematics to Regge pole contributions free of singularities on the boundary of the physical region. We, however, think that our results have interesting implications also in connection with the conspiracy problem and a better understanding of differential cross-sections near the forward direction at non-asymptotic energies.

In Section 2, we briefly repeat the arguments involved in the conventional derivation of the Regge representation for helicity amplitudes in order to determine more clearly its limits of validity in the z plane. In Appendix A, we assemble the asymptotic expansions of the functions needed in this discussion. In Section 3, the proof of the Sommerfeld-Watson transformation for $z = 1$ is presented. The representations of the d^j- functions needed in this Section are given in Appendix B. In Section 4, we introduce new interpolating functions of the Wigner rotation functions which are continuous at $z = \pm 1$ for arbitrary complex j. We briefly discuss the difficulties occurring in the general spin case when functions with a reduced cut are introduced. Section 5 deals with the nonsense channel contributions which appear when the contour of integration in the j plane is shifted towards the left. Finally, Section 6 is devoted to a few concluding remarks.

2. REGGE REPRESENTATION OF HELICITY AMPLITUDES FOR $z \neq \pm 1$

We start by writing down a fixed t dispersion relation for the t channel helicity amplitude $\bar{f}_{\lambda_b \lambda_d; \lambda_a \lambda_c}(t, z_t)$ free of kinematical singularities in s [11),12)] which is assumed to be valid for negative t *)

$$\bar{f}_{\lambda_b\lambda_d;\lambda_a\lambda_c}(t,z) = \frac{1}{\pi}\int_{z_0(t)}^{\infty} \frac{g^d_{\lambda_b\lambda_d;\lambda_a\lambda_c}(t,z')}{z'-z}\,dz' + \frac{1}{\pi}\int_{z_0'(t)}^{\infty} \frac{g^e_{\lambda_b\lambda_d;\lambda_a\lambda_c}(t,z')}{z'+z}\,dz' \qquad (1)$$

The scattering angle in the t channel is given by

$$z = z_t = \cos\theta_t = \frac{1}{4tq_t q_t'}\left[(s-u)t + (m_a^2 - m_c^2)(m_b^2 - m_d^2)\right] \qquad (2)$$

with q_t and q_t' denoting the initial and final t channel momenta, respectively. The amplitudes $\bar{f}_{\lambda_b \lambda_d; \lambda_a \lambda_c}(t,z)$ for the t channel reaction $a+c \to b+d$ are defined in terms of the ordinary helicity amplitudes in the t channel by the relation **)

$$\bar{f}_{\lambda_b\lambda_d;\lambda_a\lambda_c}(t,z) = \left[\frac{1-z}{2}\right]^{-\frac{1}{2}|\lambda-\mu|}\left[\frac{1+z}{2}\right]^{-\frac{1}{2}|\lambda+\mu|} f_{\lambda_b\lambda_d;\lambda_a\lambda_c}(t,z) \qquad (3)$$

---

*) We do not consider subtractions in (1) since it is known that the partial wave amplitudes calculated from an m times subtracted dispersion relation are the same as those calculated from (1) for angular momenta equal to or bigger than the number of subtractions.

**) The notation is the same as in Ref. 8), except that we have interchanged the variables s and t, and, correspondingly, also the other quantities referring to the s and t channels.

The helicity differences in the initial and final states are $\lambda = \lambda_a - \lambda_c$ and $\mu = \lambda_b - \lambda_d$, respectively. We further define $a = |\lambda - \mu|$, $b = |\lambda + \mu|$, and $\lambda_{max} = \mathrm{Max}(|\lambda|,|\mu|) = \frac{1}{2}(a+b)$, $\lambda_{min} = \mathrm{Min}(|\lambda|,|\mu|) = \frac{1}{2}|a-b|$.

To obtain the partial wave decomposition for the amplitude $\bar{f}_{\lambda_b \lambda_d ; \lambda_a \lambda_c}(t,z)$ valid in an ellipse in the complex z plane, we use a generalization of Heine's formula [13)]

$$\frac{1}{z'-z} = \left[\frac{1-z'}{2}\right]^{a} \left[\frac{1+z'}{2}\right]^{b} \sum_{j=\lambda_{max}}^{\infty} (2j+1)\, \bar{d}^{\,j}_{\lambda,\mu}(z)\, \bar{e}^{\,j}_{\lambda,\mu}(z') \tag{4}$$

and, similarly,

$$\frac{1}{z'+z} = \left[\frac{1-z'}{2}\right]^{b} \left[\frac{1+z'}{2}\right]^{a} \sum_{j=\lambda_{max}}^{\infty} (2j+1)\, \bar{d}^{\,j}_{\lambda,-\mu}(-z)\, \bar{e}^{\,j}_{\lambda,-\mu}(z') \tag{5}$$

Here the functions $\bar{d}^{\,j}_{\lambda,\mu}(z)$ and $\bar{e}^{\,j}_{\lambda,\mu}(z)$ are the first and second kind rotation functions with the square root branch points in z removed *), i.e.,

$$\bar{d}^{\,j}_{\lambda,\mu}(z) = \left[\frac{1-z}{2}\right]^{-\frac{a}{2}} \left[\frac{1+z}{2}\right]^{-\frac{b}{2}} d^{\,j}_{\lambda,\mu}(z) \tag{6}$$

and analogously for $\bar{e}^{\,j}_{\lambda,\mu}(z)$. We prefer to use in the following discussion these reduced rotation functions because of their simpler analytic properties in the variable z.

The expressions (4) and (5) are uniformly convergent for z inside an ellipse going through z' and having its foci at ± 1. Inserting the expressions (4) and (5) into Eq. (1) and integrating term by term, we obtain the partial wave expansion

---

*) For the definition of the functions $d^{\,j}_{\lambda,\mu}(z)$ and $e^{\,j}_{\lambda,\mu}(z)$, see Ref. 8).

$$\bar{f}_{\lambda_b\lambda_d;\lambda_a\lambda_c}(t,z) = \sum_{j=\lambda_{max}}^{\infty}(2j+1)\Big\{\bar{d}^{\,j}_{\lambda,\mu}(z)\,K^{d}_{\lambda_b\lambda_d;\lambda_a\lambda_c}(t,j) + \bar{d}^{\,j}_{\lambda,-\mu}(-z)\,K^{e}_{\lambda_b\lambda_d;\lambda_a\lambda_c}(t,j)\Big\} \tag{7}$$

with

$$K^{d}_{\lambda_b\lambda_d;\lambda_a\lambda_c}(t,j) = \frac{1}{\pi}\int_{z_0(t)}^{\infty}\rho^{d}_{\lambda_b\lambda_d;\lambda_a\lambda_c}(t,z')\left[\frac{1-z'}{2}\right]^{a}\left[\frac{1+z'}{2}\right]^{b}\bar{e}^{\,j}_{\lambda,\mu}(z')\,dz' \tag{8.a}$$

and

$$K^{e}_{\lambda_b\lambda_d,\lambda_a\lambda_c}(t,j) = \frac{1}{\pi}\int_{z'_0(t)}^{\infty}\rho^{e}_{\lambda_b\lambda_d;\lambda_a\lambda_c}(t,z')\left[\frac{1-z'}{2}\right]^{b}\left[\frac{1+z'}{2}\right]^{a}\bar{e}^{\,j}_{\lambda,-\mu}(z')\,dz' \tag{8.b}$$

The expansion (7) converges for z within an ellipse with semi-major axes $\bar{z}_0(t) = \mathrm{Min}(z_0(t), z'_0(t))$ and foci ± 1. The amplitudes of definite signature σ are given by *)

$$F^{\sigma}_{\lambda_b\lambda_d;\lambda_a\lambda_c}(t,j) = K^{d}_{\lambda_b\lambda_d;\lambda_a\lambda_c}(t,j) + \sigma(-1)^{\lambda-\vartheta}K^{e}_{\lambda_b\lambda_d;\lambda_a\lambda_c}(t,j) \tag{9}$$

The positive signature amplitudes, $F^{\sigma=1}$, interpolate uniquely the physical partial wave amplitudes for even values of $(j-\vartheta)$, whereas

---

*) The amplitudes of definite parity which we do not introduce here are linear combinations of

$$F^{\sigma}_{\lambda_b\lambda_d;\lambda_a\lambda_c}(t,j) \quad \text{and} \quad F^{\sigma}_{-\lambda_b-\lambda_d;\lambda_a\lambda_c}(t,j).$$

the negative signature amplitudes, $F^{\sigma=-1}$, give the unique interpolation for odd values of $(j-\vartheta)$, where $\vartheta = 0$ for λ, μ integer and $\vartheta = \frac{1}{2}$ for λ, μ half-integer.

To obtain the Regge representation of $\bar{f}(t,z)$ by means of the Sommerfeld-Watson transformation, one first rewrites Eq. (7) in the familiar way as a contour integral in the complex j plane

$$\bar{f}_{\lambda_b\lambda_d;\lambda_a\lambda_c}(t,z) = -\frac{1}{2i}\sum_{\sigma=\pm 1}\int_C dj\,(2j+1)\, F^{\sigma}_{\lambda_b\lambda_d;\lambda_a\lambda_c}(t,j) \cdot \frac{\frac{1}{2}\left[\bar{d}^{j}_{\lambda,-\mu}(-z)+\sigma(-1)^{\lambda-\vartheta}\,\bar{d}^{j}_{\lambda,\mu}(z)\right]}{\sin\pi(j-\lambda)} \tag{10}$$

The contour C encircles the sense-sense points $j = \lambda_{max}$, $j = \lambda_{max}+1, \dots$ in the clockwise direction.

The proof [14] that (10) allows a Sommerfeld-Watson transformation for large z and $\bar{z}_o(t) > 1$ now uses the asymptotic formulae collected in Appendix A. This leads, as is well known, to the expression (11) for the helicity amplitudes given below. In the familiar manner one has opened up the contour C until it coincides with the curve C' running parallel to the imaginary j axes at $\mathrm{Re}\, j = \lambda_{max} - \epsilon$, avoiding possible branch cuts, and picking up the Regge poles of signature σ at $j = \alpha_i$ with residue $\beta^{\sigma}_{\lambda_b\lambda_d;\lambda_a\lambda_c}(t,\alpha_i)$

$$\bar{f}_{\lambda_b\lambda_d;\lambda_a\lambda_c}(t,z) = -\frac{1}{2i}\sum_{\sigma=\pm 1}\int_{C'} dj\,(2j+1)\, F^{\sigma}_{\lambda_b\lambda_d;\lambda_a\lambda_c}(t,j) \cdot \frac{\frac{1}{2}\left[\bar{d}^{j}_{\lambda,-\mu}(-z)+\sigma(-1)^{\lambda-\vartheta}\,\bar{d}^{j}_{\lambda,\mu}(z)\right]}{\sin\pi(j-\lambda)} \tag{11}$$

$$-\pi\sum_{\substack{i\\ \mathrm{Re}\,\alpha_i>\lambda_{max}}}(2\alpha_i+1)\,\beta^{\sigma}_{\lambda_b\lambda_d;\lambda_a\lambda_c}(t,\alpha_i)\,\frac{\frac{1}{2}\left[\bar{d}^{\alpha_i}_{\lambda,-\mu}(-z)+\sigma(-1)^{\lambda-\vartheta}\,\bar{d}^{\alpha_i}_{\lambda,\mu}(z)\right]}{\sin\pi(\alpha_i-\lambda)}$$

We do not discuss for the moment the shift of the curve C' towards the left and the appearance of the nonsense channel contributions. This will be done later in Section 5. Our main point is to observe that the Regge representation (11) is valid in the domain $|\arg(z-1)| > 0$, $|\arg(z+1)| < \pi$. As shown in Appendix A, a special boundedness condition in j for the function $I^{\sigma}_{\lambda_b \lambda_d; \lambda_a \lambda_c}(t,j)$ as defined in Eq. (A.5) is needed to obtain a valid representation (11) also in the case of z approaching a real value $(|z| > 1)$ on the cuts extending from ± 1 outwards. The points $z = \pm 1$ play a special role as in already clear from the asymptotic formula (A.1). These points require a separate investigation. Both terms on the right-hand side of (11) are in fact singular at $z = \pm 1$ whereas $\bar{f}_{\lambda_b \lambda_d; \lambda_a \lambda_c}(t,z)$ is known to be regular there. The type of singularity of the d^j - functions at these points can be seen from Eq. (B.3) of Appendix B, where an expansion of the function $\bar{d}^j_{\lambda,-\mu}(-z)$ for z near $+1$ is given. A corresponding expansion holds for $\bar{d}^j_{\lambda,\mu}(z)$ near $z = -1$. In the following Section, we study in some more detail the Sommerfeld-Watson transformation at $z = +1$ to see how the infinity of the Regge representation can be eliminated at that point. We assume, for instance, that the particles are numbered in such a way that $\theta_s = 0$ corresponds to $z = z_t = +1$. The point $z = -1$ can be treated in a similar manner. In the final result it is of course essential that the modifications are made for both points $z = +1$ and $z = -1$ in a symmetric way.

3. SOMMERFELD-WATSON TRANSFORMATION FOR $z = 1$

We have seen in the last Section that the usual Regge representation is not valid for $z = \pm 1$ and that singularities show up when one considers the formulae at these points. A logarithmic singularity and a series of poles in z at $z = \pm 1$ are introduced in the integrand of Eq. (10) in rewriting the partial wave sum (7) as a contour integral in the complex j plane. These singularities disappear again in performing the integration along the path C.

To demonstrate that, under reasonable conditions, the expression (10) still allows a Sommerfeld-Watson transformation at $z = 1$, we insert the expansion (B.3) of $\bar{d}^{\,j}_{\lambda,-\mu}(-z)$ for z near 1 into the right-hand side of Eq. (10)

$$
\begin{aligned}
\bar{f}_{\lambda_b\lambda_d;\lambda_a\lambda_c}(t,z) = & -\frac{1}{2i} \sum_{\sigma=\pm 1} \int_C dj\,(2j+1)\, F^{\sigma}_{\lambda_b\lambda_d;\lambda_a\lambda_c}(t,j) \,\cdot \\
& \cdot \frac{1}{2}\Bigg\{ \frac{1}{\pi}\,\mathrm{sign}(\lambda,\mu) \Bigg[-\frac{\Gamma(a)}{\phi_{\lambda,\mu}(j)} \sum_{n=0}^{a-1} \frac{(j+1+\lambda_{max}-a)_n(-j+\lambda_{max}-a)_n}{n!\,(1-a)_n} \left(\frac{2}{1-z}\right)^{a-n} \\
& + \phi_{\lambda,\mu}(j) \sum_{n=0}^{\infty} \frac{(j+1+\lambda_{max})_n(-j+\lambda_{max})_n}{n!\,(a+n)!} \left(\frac{1-z}{2}\right)^{n} \cdot \Bigg(\log\left(\frac{1-z}{2}\right) - \\
& - \psi(1+n) - \psi(1+a+n) + \psi(-j+\lambda_{max}+n) + \psi(j+1+\lambda_{max}+n)\Bigg)\Bigg] \\
& + \sigma(-1)^{\lambda-\nu} \frac{\mathrm{sign}(\lambda,\mu)\,\phi_{\lambda,\mu}(j)}{\sin\pi(j-\lambda)} \frac{1}{\Gamma(a+1)} F\left(-j+\lambda_{max}, j+1+\lambda_{max}; a+1; \frac{1-z}{2}\right)\Bigg\}
\end{aligned}
\tag{12}
$$

Only the terms involving $\psi(-j+\lambda_{max}+n)$ in the part originating from $\bar{d}^{j}_{\lambda,-\mu}(-z)$ have singularities (simple poles) inside the curve C and hence contribute, whereas all the rest in the square brackets is regular in j and disappears when the integration along C is performed. We therefore write Eq. (12) as

$$\bar{f}_{\lambda_b\lambda_d,\lambda_a\lambda_c}(t,z) = -\frac{1}{2i}\sum_{\sigma=\pm 1}\int_C dj\,(2j+1)\,F^{\sigma}_{\lambda_b\lambda_d;\lambda_a\lambda_c}(t,j)\,\cdot$$

$$\cdot\,\frac{1}{2}\left[\left(\bar{d}^{\,j}_{\lambda,-\mu}(-z)\right)_{reg}+\sigma(-1)^{\lambda-\nu-j}\,\bar{d}^{\,j}_{\lambda,\mu}(z)\right]\frac{1}{\sin\pi(j-\lambda)} \qquad (12')$$

Here we have defined the regularized function $\bar{d}^{j}_{\lambda,-\mu}(-z)$ for $|z-1| < 2$ by *)

$$\left(\bar{d}^{\,j}_{\lambda,-\mu}(-z)\right)_{reg} = \frac{1}{\pi}\sin\pi(j-\lambda)\,\mathrm{sign}(\lambda,\mu)\,\phi_{\lambda,\mu}(j)\,\cdot$$

$$\cdot\sum_{n=0}^{\infty}\frac{(j+1+\lambda_{max})_n\,(-j+\lambda_{max})_n}{n!\,(\ell+n)!}\left(\frac{1-z}{2}\right)^{n}\psi(-j+\lambda_{max}+n) \qquad (13)$$

For $j=\lambda_{max}+k$, $k=0,1,2,\ldots$ the right-hand side of (13) reduces to a polynomial of degree k, and one obtains in this case the familiar formula (B.4) of Appendix B for the Wigner functions.

---

*) To retain the formula $\bar{d}^{j}_{\lambda,\mu}(z) = \bar{d}^{-j-1}_{-\lambda,-\mu}(z)$ typical for functions of the first kind, one would have to replace $\psi(-j+\lambda_{max}+n)$ by $\psi(-j+\lambda_{max}+n)+\psi(j+1+\lambda_{max}+n)$ in Eq. (13), corresponding to the last two terms in Eq. (B.3). The following argument is not changed by such a replacement.

We now show that the infinite semicircles do not contribute in (12') when the contour C is opened up to run parallel to the imaginary axes (avoiding possible branch cuts). One has to be careful in calculating the contribution of these semicircles, since one has to go to the limit $j \to \infty$ and $z \to 1$ at the same time.

The behaviour of the Jacobi polynomials $P_{\varkappa}^{(a,b)}(z)$ for $\varkappa \to \infty$ and z near 1 have been studied in detail by Tricomi [15),16)] by an expansion in terms of confluent hypergoemetric functions. From Eq. (23) of Ref. 15), we calculate for the limit $j \to \infty$ and $z \to 1$, such that $(2j+1)(\sqrt{(1-z)/2}) = X$ stays constant, the following behaviour

$$P^{(a,b)}_{j-\lambda_{max}}(z) \underset{(2j+1)\sqrt{\frac{1-z}{2}} = \text{Const}}{\overset{\substack{j \to \infty \\ |\arg j| < \pi \\ z \to 1}}{\sim}} \frac{j^a J_a\left((2j+1)\sqrt{\frac{1-z}{2}}\right)}{\left[\frac{1}{2}(2j+1)\sqrt{\frac{1-z}{2}}\right]^a} \tag{14}$$

Here $J_a(X)$ denotes the Bessel function of order a. The relation (14) is a generalization of the familiar formula

$$P_\ell(z) \underset{(2\ell+1)\sqrt{\frac{1-z}{2}} = \text{const}}{\overset{\substack{\ell \to \infty \\ z \to 1}}{\sim}} J_0\left((2\ell+1)\sqrt{\frac{1-z}{2}}\right)$$

Using now Eq. (B.2) and the formula (A.4) for the asymptotic behaviour of the partial wave amplitudes for large j and $\bar{z}_0(t) > 1$, we have for the second summand in Eq. (12') the following result

$$(2j+1)\frac{F^{\sigma}_{\lambda_b\lambda_d;\lambda_a\lambda_c}(t,j)\, \bar{d}^{\,j}_{\lambda,\mu}(z)}{\sin\pi(j-\lambda)} \underset{X = \text{const}}{\overset{\substack{j \to \infty \\ |\arg j| < \pi \\ z \to 1}}{\sim}} C\, j^{a+\frac{1}{2}}\, e^{-\pi|\mathrm{Im}\, j|} \cdot$$

$$\cdot\, e^{-(\mathrm{Re}\, j+\frac{1}{2})\log\left(\bar{z}_0+\sqrt{\bar{z}_0^2-1}\right)}\, I^{\sigma}_{\lambda_b\lambda_d;\lambda_a\lambda_c}(t,j) \tag{15}$$

Here C is a constant depending on λ. From this expression, it follows that the second contribution of the right-hand side in Eq. (12') allows a Sommerfeld-Watson transformation at $z=1$.

Treating now the first contribution in Eq. (12'), we observe that

$$\frac{\left(\bar{d}^{\,j}_{\lambda-\mu}(-z)\right)_{reg}}{\sin\pi(j-\lambda)} = \frac{1}{\pi}\,\psi(-j+\lambda_{max})\,\bar{d}^{\,j}_{\lambda\mu}(z)\left[1+O\left(\frac{1}{j}\right)\right] \tag{16}$$

From (16) it follows, along similar lines as in the above discussion, that the first term on the right-hand side of Eq. (12') behaves for $j\to\infty$ and $z\to 1$ as

$$(2j+1)\;\frac{F^{\sigma}_{\lambda_b\lambda_d;\lambda_a\lambda_c}(t,j)\left(\bar{d}^{\,j}_{\lambda-\mu}(-z)\right)_{reg}}{\sin\pi(j-\lambda)} \underset{\lambda=const}{\overset{\substack{j\to\infty\\ 0<|\arg j|<\pi\\ z\to 1}}{\sim}} \frac{C}{\pi}\,j^{a+\frac{1}{2}}\,.$$

$$\cdot\left(\log(-j+\lambda_{max})+O\left(\frac{1}{j}\right)\right)e^{-(\mathrm{Re}\,j+\frac{1}{2})\ln\left(z_0+\sqrt{z_0^2-1}\right)}\,I^{\sigma}_{\lambda_b\lambda_d;\lambda_a\lambda_c}(t,j) \tag{17}$$

It was found in Section 2, in discussing the behaviour of the integrand in Eq. (10) for $|j|\to\infty$ and $\varphi=\arg j=\pi/2$, that the quantity (A.6) ceased to vanish exponentially for real z, $|z|>1$. To allow a Sommerfeld-Watson transformation in this case $I^{\sigma}(t,j)$ had to vanish like a power in j bigger than one for j going to infinity (see Appendix A).

A similar situation appears in Eq. (17). If we want to open up the contour C not only to an angle $\varphi=|\pi/2-\epsilon|$, $\epsilon>0$, but to a parallel to the imaginary axes, we have to impose here the slightly stronger condition that $I^{\sigma}(t,j)$ should vanish at least like j^{-a-2} for large j. Under this condition, Eq. (10) allows a Sommerfeld-Watson transformation at $z=1$. From Eq. (12'), one obtains the following regularized Regge representation at this point

$$\bar{f}_{\lambda_b\lambda_d;\lambda_a\lambda_c}(t',z=1) = -\frac{1}{2i}\sum_{\sigma=\pm1}\int_{C'} dj\,(2j+1)\,F^{\sigma}_{\lambda_b\lambda_d;\lambda_a\lambda_c}(t',j)\,\cdot$$

$$\cdot\;\frac{\frac{1}{2}\left[\left(\bar{d}^{j}_{\lambda,-\mu}(-1)\right)_{reg}+\sigma(-1)^{\lambda-\nu}\,\bar{d}^{j}_{\lambda,\mu}(1)\right]}{\sin\pi(j-\lambda)} \qquad (18)$$

$$-\pi\sum_{i,\;Re\,\alpha_i>\lambda_{max}}(2\alpha_i+1)\,\beta^{\sigma}_{\lambda_b\lambda_d;\lambda_a\lambda_c}(t,\alpha_i)\,\frac{\frac{1}{2}\left[\left(\bar{d}^{\alpha_i}_{\lambda,-\mu}(-1)\right)_{reg}+\sigma(-1)^{\lambda-\nu}\,\bar{d}^{\alpha_i}_{\lambda,\mu}(1)\right]}{\sin\pi(\alpha_i-\lambda)}$$

Each term in Eq. (18) is finite and well defined. The momentum transfer $t' = t'(s)$ corresponds to $z = z_t(s,t') = 1$ and forward scattering in the s channel at this particular energy s.

Up to now we have treated only the point $z = 1$. It will be discussed below that the contribution in Eq. (18) originating from the function $\bar{d}^{j}_{\lambda,\mu}(z)$ which is regular at $z = 1$ will have to be modified when the singularities at $z = -1$ are treated at the same time. The result is given in formula (28) below, which leads to a finite, non-zero contribution of a single Regge pole to the amplitudes $\bar{f}(t,z)$ for forward scattering or unequal mass particles in the s channel.

4. REMOVAL OF INFINITIES AT $z=\pm 1$ IN THE CONVENTIONAL REGGE REPRESENTATION

From the above discussion, it follows that a single Regge pole in the t channel gives a finite contribution in the forward direction to the s channel amplitude for the scattering of unequal mass particles at finite energies. As was already mentioned in the Introduction, the proof of Eqs. (11) and (18) requires $\bar{z}_0(t) > 1$. For s going to infinity, $\bar{z}_0(t)$ approaches 1 for forward scattering in the s channel and our discussion breaks down in this limit.

We come now to the problem of regularizing the standard Regge representation at $z=\pm 1$. As before, we treat explicitly only the amplitude at the point $z=+1$ and generalize the result in an obvious way to $z=-1$.

The question is how one can eliminate the singularities at the points $z=\pm 1$ for complex angular momenta inherent to a reggeized amplitude expressed in terms of first kind rotation functions. One could think of repeating in the absence of Regge cuts for $\mathrm{Re}\, j > -\frac{1}{2}$ the arguments of Ref. 9), and introducing generalized Khuri-type functions with a reduced cut, ${}^{z_0}\bar{K}^{j}_{\lambda,-\mu}(-z)$, by the following definition

$$ {}^{z_0}\bar{K}^{j}_{\lambda,-\mu}(-z) = \mathrm{sign}(\lambda,-\mu)\,\phi_{\lambda,-\mu}(j)\; \underset{-z}{\mathrm{I}}^{(b,a)} \left\{ {}^{z_0}K_j(z) \right\} \tag{19} $$

Here ${}^{z_0}K_j(z)$ is the Khuri function in the spinless case which, for $-1 < \mathrm{Re}\, j < \infty$, is given by 9)

$$ {}^{z_0}K_j(z) = P_j(z) + \frac{\sin\pi j}{\pi\sqrt{2}} \int_{-\infty}^{\operatorname{arccosh} z_0} \frac{e^{(j+\frac{1}{2})x}}{[\cosh x + z]^{1/2}}\, dx \tag{20} $$

${}^{z_o}K_j(z)$, as defined by Eq. (20), is regular at $z=1$ and has a cut in z extending from z_o to $+\infty$. The operator $I_{-z}^{(b,a)}$ in Eq. (19) is a generalized diffentiation operator [13] involving the Riemann-Liouville fractional integrals. It is defined for $\lambda_{max} \neq 0$ by

$$I_{-z}^{(b,a)}\left\{f(z)\right\} = \begin{cases} \dfrac{1}{(z+1)^b}\left(\dfrac{d}{dz}\right)^{\lambda_{max}-b}\left[\left(\dfrac{1-z}{2}\right)^{-\lambda_{max}}\dfrac{1}{\Gamma(\lambda_{max})}\displaystyle\int_{-1}^{z} f(-z')(z-z')^{\lambda_{max}-1}dz'\right], & \text{for } \lambda_{max}-b = 0,1,2,\ldots \\ \dfrac{1}{(z+1)^b}\dfrac{1}{\Gamma(b-\lambda_{max})}\displaystyle\int_{-1}^{z}\left[\left(\dfrac{1-z''}{2}\right)^{-\lambda_{max}}\dfrac{1}{\Gamma(\lambda_{max})}\int_{-1}^{z''} f(-z')\cdot \right. & \\ \qquad\left.\cdot(z''-z')^{\lambda_{max}-1}dz'\right](z-z'')^{b-\lambda_{max}-1}dz'', & \text{for } \lambda_{max}-b \neq 0,1,2,\ldots \end{cases} \tag{21}$$

with $(z'+1) = t'(z''+1)$, $(z''+1) = t''(z+1)$ and $0 \leq t', t'' \leq 1$.

In the definition (19) of the generalized Khuri function, one has to regard z at first in the circle $|z+1| < 2$ where the z and x integrations are interchangeable and later, after the performance of the z integrations, continue the result analytically in z. One finds in this way that the functions ${}^{z_o}\overline{K}^j_{\lambda,-\mu}(-z)$ although having no unphysical cut for $1 < z < z_o$ still have unwanted poles at $z=1$ of order $|\lambda-\mu|$ and lower. We have not been able to find generalized Khuri-type functions with the properties that they :

i) interpolate the Wigner rotation functions for $(j-\lambda_{max}) = 0,1,2,\ldots$ for integer as well as half-integer helicities;

ii) have a reduced cut starting at z_o, and;

iii) have no unphysical poles at $z=1$.

passing that for $\rho^{su} = 0$ the nonsense channel contribution $N(t,z)$ is given by an expression involving only the first sum of (30) and (31), respectively, which in this case has to be extended over w.s. points as well.

One observes that each term in the representation (28) is continuous and well-defined at $z = \pm 1$. We note in concluding this Section that the functions which enter the helicity amplitudes $f^{p}_{\lambda_b \lambda_d ; \lambda_a \lambda_c}(t,z)$ corresponding to a definite parity $p = \pm 1$ exchanged in the t channel are given by

$$\mathcal{H}^{\sigma,p}_{\lambda,-\mu}(-z,j) = \frac{1}{2}\left[\mathcal{H}^{\sigma}_{\lambda,-\mu}(-z,j) + p(-1)^{\lambda+\lambda_{max}} \mathcal{H}^{\sigma}_{\lambda,\mu}(-z,j)\right] \qquad (32)$$

6. CONCLUSIONS

We have shown in this paper that under certain boundedness conditions for the functions $I^{\sigma}(t,j)$ as j goes to infinity, which are only slightly stronger than what is usually needed for a proof of the Regge representation for $|z| > 1$, a Sommerfeld-Watson transformation is still possible for a general two-body amplitude at $z = \pm 1$. This led to the conclusion that a single Regge pole in the t channel produces a finite, non-zero contribution to the s channel amplitude for forward scattering at finite s corresponding to $\cos\theta_t = \pm 1$. A Regge representation for a general two-body scattering amplitude was given which smoothly joins the value for the forward scattering. The conventional Regge representation was modified by the introduction of counter-terms which eliminated the infinities at $z = \pm 1$. It was shown that the new interpolating functions for the Wigner rotation functions which are continuous at $z = \pm 1$ for arbitrary j still allow a Sommerfeld-Watson transformation of the scattering amplitude. We discussed briefly the difficulties connected with the introduction of functions with a reduced cut which would solve the problem of fully regularizing the Regge representation at $z = \pm 1$. In a separate paper, the implications of the proposed Regge representation for the photoproduction of mesons near the forward direction will be studied in more detail.

ACKNOWLEDGEMENTS

I should like to thank Professor D. Amati, Dr. J.S. Bell, Dr. J. Finkelstein and Dr. M. Toller for discussions.

APPENDIX A

The asymptotic behaviour of the functions $e^{j}_{\lambda,\mu}(z)$ for large j is given by the formula

$$e^{j}_{\lambda,\mu}(z) \underset{\substack{|j|\to\infty \\ |\arg j|<\pi}}{\sim} \operatorname{sign}(\lambda,\mu)\, e^{\eta i\pi\frac{a}{2}} \sqrt{\frac{\pi}{2}}\; \frac{1}{j^{1/2}}\, \frac{(z+\sqrt{z^2-1})^{-j-\frac{1}{2}}}{(z^2-1)^{1/4}} \cdot \tag{A.1}$$

$$\cdot \left[1+O\left(\frac{1}{j}\right)\right]$$

$\eta = \operatorname{sign} \operatorname{Im} z$

The factor $\operatorname{sign}(\lambda,\mu)$ is defined in Appendix B. From (A.1) and the relation

$$d^{j}_{\lambda,\mu}(z) = \frac{1}{\pi} \operatorname{tg} \pi(j-\lambda)\left[e^{j}_{\lambda,\mu}(z) - e^{-j-1}_{-\lambda,-\mu}(z)\right] \quad ; \; |z-1|>2$$

one obtains the asymptotic behaviour of the functions $d^{j}_{\lambda,\mu}(z)$ which we would like to quote in the form most suited for the following discussion

$$\frac{d^{j}_{\lambda,\mu}(z)}{\sin \pi(j-\lambda)} \underset{\substack{|j|\to\infty \\ 0<|\arg j|<\pi}}{\sim} \sqrt{\frac{2}{\pi}}\; \frac{1}{(z^2-1)^{1/4}}\; \frac{e^{\pi|\operatorname{Im} j|}\; e^{|(j+\frac{1}{2})\xi|}}{j^{1/2}} \tag{A.2}$$

In Eq. (A.2) sign factors on the right-hand side have been neglected and $\log(z+\sqrt{z^2-1})$ has been called ξ. Similarly, one obtains, using the relation

$$e^{j}_{\lambda,-\mu}(-z) = -e^{\eta i\pi(j-\lambda)}\, e^{j}_{\lambda,\mu}(z)$$

for the asymptotic behaviour of the function $d^{j}_{\lambda,-\mu}(-z)$ the formula

$$\frac{d^{j}_{\lambda,-\mu}(-z)}{\sin\pi(j-\lambda)} \underset{\substack{|j|\to\infty \\ 0<|\arg j|<\pi}}{\sim} \sqrt{\frac{2}{\pi}}\ \frac{1}{(z^2-1)^{1/4}}\ \frac{e^{-\pi|\mathrm{Im}\, j|}\ e^{|i\pi j-(j+\frac{1}{2})\xi|}}{j^{1/2}} \tag{A.3}$$

Using relation (A.1), one derives from Eqs. (8) and (9) that the partial wave amplitudes are bounded by

$$F^{\sigma}_{\lambda_b\lambda_d;\lambda_a\lambda_c}(t,j) \underset{\substack{|j|\to\infty \\ |\arg j|<\pi}}{\sim} \frac{\mathrm{Sign}(\lambda,\mu)}{\sqrt{2\pi}}\ \frac{1}{j^{1/2}}\ \frac{e^{-(\mathrm{Re}\, j+\frac{1}{2})\bar{\xi}_0}}{(\bar{z}_0^{\ 2}-1)^{1/4}}\ I^{\sigma}_{\lambda_b\lambda_d;\lambda_a\lambda_c}(t,j) \tag{A.4}$$

with $I^{\sigma}(t,j)$ given by

$$I^{\sigma}_{\lambda_b\lambda_d;\lambda_a\lambda_c}(t,j) = \int_{\bar{z}_0(t)}^{\infty} \Big\{ \rho^{d}_{\lambda_b\lambda_d;\lambda_a\lambda_c}(t,z) \left[\frac{z-1}{2}\right]^{\frac{a}{2}} \left[\frac{z+1}{2}\right]^{\frac{b}{2}}$$

$$+\ \sigma(-1)^{\lambda_{max}-\nu} \rho^{e}_{\lambda_b\lambda_d;\lambda_a\lambda_c}(t,z) \left[\frac{z-1}{2}\right]^{\frac{b}{2}} \left[\frac{z+1}{2}\right]^{\frac{a}{2}} \Big\} e^{-i\,\mathrm{Im}\, j\,\xi}\, dz \tag{A.5}$$

where $\xi = \log(z+\sqrt{z^2-1})$ and $\bar{\xi}_0 = \log(\bar{z}_0+\sqrt{\bar{z}_0^2-1}) > 0.$

We assume that the properties of the functions ρ^{d} and ρ^{e} are such that $I^{\sigma}(t,j)$ behaves for large j like $j^{\varkappa}$ with $\varkappa < 0$. As will be shown below, such a behaviour is needed in order to guarantee the Sommerfeld-Watson transformation also for real z larger than 1 or smaller than -1.

From (A.3) and (A.4), one obtains the asymptotic behaviour in j of the first term in Eq. (10). Neglecting all inessential factors, one has for $z \neq 1$

$$(2j+1)\,\frac{F^{\sigma}_{\lambda_b\lambda_d;\lambda_a\lambda_c}(t,j)\;\bar{d}^{\,j}_{\lambda,-\mu}(-z)}{\sin\pi(j-\lambda)} \underset{\substack{|j|\to\infty \\ 0<|\arg j|<\pi}}{\sim} I^{\sigma}_{\lambda_b\lambda_d;\lambda_a\lambda_c}(t,j)\,\cdot$$

$$\cdot\; e^{-|j|\cos\varphi\,\bar{\xi}_0}\;\frac{e^{|(\frac{1}{2}+|j|\cos\varphi)\log|\zeta|+|j|\sin\varphi(\eta\pi-\arg\zeta)|}}{e^{\pi|j||\sin\varphi|}} \qquad \text{(A.6)}$$

where $\zeta = z+\sqrt{z^2-1}$ and $\varphi = \arg j$. A similar expression is obtained for the second term in Eq. (10).

The right-hand of Eq. (A.6) defines, in a complicated way for arbitrary φ in the interval $0 < \varphi \leq \pi/2$, the domain in z for which one finds a decreasing exponential, as $|j|$ goes to infinity. This domain is largest for $|\varphi| = \pi/2$. The right-hand side of (A.6) vanishes exponentially in this case for large $|j|$, provided that $\arg\zeta$ is not zero, i.e., $|\arg(z-1)| > 0$. Similarly, one obtains for the second term in Eq. (10) for $\varphi = \pi/2$ the restriction $|\arg(z+1)| < \pi$. If one wants to obtain a convergent background integral in Eq. (11) also in the limiting case of z approaching the real axes from one side, one has to require that $I^{\sigma}(t,j)$ behaves at least like some negative power of j bigger than 1 as j goes to infinity.

A P P E N D I X   B

The functions $\bar{d}^{j}_{\lambda,\mu}(z)$ for complex j and z, and with $|z-1| < 2$, are defined by

$$\bar{d}^{j}_{\lambda,\mu}(z) = \mathrm{Sign}(\lambda,\mu)\, \phi_{\lambda,\mu}(j)\, \frac{1}{\Gamma(a+1)}\, F\left(-j+\lambda_{max}, j+1+\lambda_{max}; a+1; \frac{1-z}{2}\right) \qquad (B.1)$$

where

$$\phi_{\lambda,\mu}(j) = \left[\frac{\Gamma(j+1+\lambda_{max})\,\Gamma(j+1-\lambda_{max}+a)}{\Gamma(j+1-\lambda_{max})\,\Gamma(j+1+\lambda_{max}-a)}\right]^{1/2}$$

$$a = |\lambda-\mu| \;; \qquad \lambda_{max} = \tfrac{1}{2}(a+b)$$
$$b = |\lambda+\mu| \;; \qquad \lambda_{min} = \tfrac{1}{2}|a-b|$$

and

$$\mathrm{Sign}(\lambda,\mu) = \begin{cases} 1 & \text{for } \lambda-\mu \geq 0 \\ (-1)^{\lambda-\mu} & \text{for } \lambda-\mu \leq 0 \end{cases}$$

Introducing the functions $P^{(a,b)}_{j-\lambda_{max}}(z)$, which reduce to the Jacobi polynomials for $j-\lambda_{max}$ being a positive integer or zero, Eq. (B.1) can also be written

$$\bar{d}^{j}_{\lambda,\mu}(z) = \mathrm{Sign}(\lambda,\mu) \left[\frac{\Gamma(j+1+\lambda_{max})\,\Gamma(j+1-\lambda_{max})}{\Gamma(j+1+\lambda_{min})\,\Gamma(j+1-\lambda_{min})}\right]^{1/2} P^{(a,b)}_{j-\lambda_{max}}(z) \qquad (B.2)$$

The functions $\bar{d}^{j}_{\lambda,\mu}(z)$ are singular at $z=-1$ for arbitrary j with $j-\lambda_{max}$ not equal to zero or a positive integer (or the corresponding mirror points with respect to $j=-\frac{1}{2}$). This can be seen from the following expansion *) of the function $\bar{d}^{j}_{\lambda,-\mu}(-z)$ valid for $|z-1|<2$, $|\arg(z-1)|>0$ and $a\geq 1$ **)

$$\bar{d}^{j}_{\lambda,-\mu}(-z) = \operatorname{sign}(\lambda,\mu)\frac{1}{\pi}\sin\pi(j-\lambda)\Bigg\{-\frac{\Gamma(a)}{\phi_{\lambda,\mu}(j)}\cdot\sum_{n=0}^{a-1}\frac{(j+1+\lambda_{max}-a)_n}{n!}\cdot\frac{(-j+\lambda_{max}-a)_n}{(1-a)_n}\left(\frac{2}{1-z}\right)^{a-n}$$
$$+\phi_{\lambda,\mu}(j)\sum_{n=0}^{\infty}\frac{(j+1+\lambda_{max})_n\,(-j+\lambda_{max})_n}{n!\;(a+n)!}\left(\frac{1-z}{2}\right)^{n}\Bigg[\log\left(\frac{1-z}{2}\right)$$
$$-\psi(1+n)-\psi(1+a+n)+\psi(-j+\lambda_{max}+n)+\psi(j+1+\lambda_{max}+n)\Bigg]\Bigg\} \qquad \text{(B.3)}$$

Here $\psi(x)$ is the logarithmic derivative of the gamma function, and $(x)_n = \Gamma(x+n)/\Gamma(x)$. For sense-sense values of j and the corresponding mirror points with respect to $j=-\frac{1}{2}$ the right-hand side of Eq. (B.3) reduces to a polynomial in $(1-z)/2$ of degree $j-\lambda_{max}$ obeying :

$$\bar{d}^{j}_{\lambda,-\mu}(-z) = (-1)^{j-\lambda}\,\bar{d}^{j}_{\lambda,\mu}(z)\,,\quad \begin{matrix} j-\lambda_{max}=0,1,2,\dots \\ j+\lambda_{max}=-1,-2,-3,\dots\end{matrix} \qquad \text{(B.4)}$$

---

*) Formula (B.3) follows from Eq. (15.3.12) on p.560 of Ref. [19].

**) For $a=0$, the principal part is absent in (B.3) and the rest of the formula gives the correct expression in this case.

R E F E R E N C E S

1) M. Toller, Nuovo Cimento 53A, 671 (1968).

2) G. Cosenza, A. Sciarrino and M. Toller, Istituto di Fisica "G. Marconi" Università di Roma, preprint 159, February 1968.

3) G. Domokos and G.L. Tindle, Phys.Rev. 165, 1906 (1968).

4) E. Leader, Phys.Rev. 166, 1599 (1968).

5) D.Z. Freedman and J.M. Wang, Phys.Rev. 160, 1560 (1967).

6) M. Toller, Nuovo Cimento 54A, 295 (1968).

7) A. Sciarrino and M. Toller, J.Math.Phys. 8, 1252 (1967).

8) W. Drechsler, Nuovo Cimento 53A, 115 (1968).

9) N.N. Khuri, Phys.Rev. 130, 429 (1963).

10) N.N. Khuri, Phys.Rev. 132, 914 (1963).

11) L.L. Wang, Phys.Rev. 142, 1187 (1966).

12) G. Cohen-Tannoudji, A. Morel and H. Navelet, Ann.Phys.(N.Y.) 46, 111 (1968).

13) M. Andrews and J. Gunson, J.Math.Phys. 5, 1391 (1964).

14) F. Calogero, J.M. Charap and E.J. Squires, Ann.Phys.(N.Y.) 25, 325 (1963).

15) F.G. Tricomi, Comm.Math.Helvetici 25, 196 (1959).

16) F.G. Tricomi, "Vorlesungen über Orthogonalreihen", Springer-Verlag, p.174, Berlin-Göttingen-Heidelberg (1955).

17) S. Mandelstam and L.L. Wang, Phys.Rev. 160, 1490 (1967).

18) W. Drechsler, CERN preprint TH.841 (1967) - unpublished. Talk presented at the Heidelberg International Conference on Elementary Particles, Heidelberg (1967).

19) Handbook of Mathematical Functions, edited by M. Abramowitz and I.A. Stegum, National Bureau of Standards, AMS 55 (1964).