Electromagnetic energy density in a single-resonance chiral metamaterial

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We derive the electromagnetic energy density in a chiral metamaterial consisting of uncoupled single-resonance helical resonators. Both the lossless and absorptive cases are studied, and the energy density is shown to be positively definite. The key relation making the derivation successful is the proportionality between the magnetization and the rate of change of the electric polarization of the medium. The same time-domain formulation of energy density also applies to the bianisotropic medium proposed by Zhang *et al.* [Phys. Rev. Lett. **102**, 023901 (2009)]. This work may provide insights for studying time-dependent phenomena in metamaterials.

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"Metamaterial" usually refers to an artificial structure consisting of periodically arranged resonators that has unusual electromagnetic properties such as a negative index of refraction or strong anisotropy [1]. Fascinating phenomena associated with these unusual properties have been demonstrated, for example, superlensing and cloaking. To realize negative refraction, usually one unit cell of the periodic structure should contain both electric and magnetic resonators, giving rise to negative permittivity and negative permeability through electric and magnetic resonances, respectively. Recently, chiral metamaterials have attracted much interest. They consist of only one type of helical resonators and exhibit new properties going beyond conventional negative index metamaterials, such as negative reflection [2].

The resonance characteristics of metamaterials imply that they are inherently dispersive and absorptive. A fundamental problem of dispersive media is how to calculate the electromagnetic energy density, in particular for those with magnetic responses. For the lossless case without involving chirality, the answer is known [3]. However, when absorption is involved, such a problem is nontrivial [4,5], and quite some controversies exist in the literature [6,7]. For the wire-split ring resonator (wire-SRR) metamaterials, the frequency domain (time-averaged) formula [6] derived using the equivalent circuit approach and the time-domain formula [7] obtained using the electrodynamics (ED) approach did not agree. Recently, this apparent contradiction has been resolved [8]. According to [8], ED derivation is based on Maxwell's equations and the equations of motion for electric polarization $\mathbf{P}$ and magnetization $\mathbf{M}$. Besides, to find a unique expression of energy density, the correct form of power loss must be employed.

In this Letter, we derive the electromagnetic energy density in a single-resonance chiral metamaterial consisting of uncoupled helical resonators, following a similar procedure as that used in [8]. For our chosen model $\partial\mathbf{P}/\partial t$ is proportional to $\mathbf{M}$, and this is the key relation making the derivation possible. This feature makes the present derivation more subtle than the wire-SRR case, but the key steps are very close.

The constitutive relations for harmonic EM waves in an isotropic chiral medium are

$$\mathbf{D}_\omega = \epsilon_0\epsilon(\omega)\mathbf{E}_\omega + i\frac{\kappa(\omega)}{c}\mathbf{H}_\omega, \tag{1}$$

$$\mathbf{B}_\omega = -i\frac{\kappa(\omega)}{c}\mathbf{E}_\omega + \mu_0\mu(\omega)\mathbf{H}_\omega. \tag{2}$$

Here $c = 1/\sqrt{\epsilon_0\mu_0}$ is the speed of light in vacuum; ϵ, μ, and κ are the permittivity, permeability, and chirality parameters, respectively; and the complex representation (phasor) of the vector quantities are used. The suffix ω indicates that these relations are for harmonic fields. Our derivations are based on the following constitutive parameters of a single-resonance chiral metamaterial [9–12]:

$$\epsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2 - \omega_0^2 + i\Gamma\omega}, \tag{3}$$

$$\mu(\omega) = 1 - \frac{F\omega^2}{\omega^2 - \omega_0^2 + i\Gamma\omega}, \tag{4}$$

$$\kappa(\omega) = \frac{A\omega}{\omega^2 - \omega_0^2 + i\Gamma\omega}, \tag{5}$$

where $A = \pm\sqrt{F}\omega_p$, ω_p is a characteristic frequency of the medium, and F is the filling factor of the resonators in one unit cell satisfying $0 < F < 1$. The parameter ω_0 is the resonance frequency of the resonators, and Γ is the dissipation coefficient related to the power loss. Note that the chiral metamaterials defined above belong to a special kind of chiral media, thus the results derived in this Letter are not directly applicable if the constitutive parameters are of different form. The **D**, **B**, **E**, and **H** fields are defined by spatially averaging the corresponding "microscopic local fields" in one unit cell under the assumption that the relevant wavelength is much longer than the lattice constant.

For a lossless chiral medium, i.e., $\Gamma = 0$, the time-averaged energy density can be derived using the method formulated in [3] (Chap. 9, pp. 274–276), as

$$\langle W \rangle = \frac{1}{4} V^{\dagger} \mathcal{M}_0 V, \tag{6}$$

$$\mathcal{M}_0 = \begin{pmatrix} \frac{\partial(\omega\epsilon)}{\partial\omega} & i\frac{\partial(\omega\kappa)}{\partial\omega} \\ -i\frac{\partial(\omega\kappa)}{\partial\omega} & \frac{\partial(\omega\mu)}{\partial\omega} \end{pmatrix}, \qquad V = \begin{pmatrix} \sqrt{\epsilon_0}\mathbf{E}_\omega \\ \sqrt{\mu_0}\mathbf{H}_\omega \end{pmatrix}, \tag{7}$$

where $V^{\dagger}$ is the Hermitian conjugate of V. The same expression can also be found in [13]. For a physically meaningful medium, the energy density must be positively definite, which implies both the trace and determinant of $\mathcal{M}_0$ are positive:

$$\frac{\partial(\omega\epsilon)}{\partial\omega} + \frac{\partial(\omega\mu)}{\partial\omega} > 0, \qquad \frac{\partial(\omega\epsilon)}{\partial\omega}\frac{\partial(\omega\mu)}{\partial\omega} > \left(\frac{\partial(\omega\kappa)}{\partial\omega}\right)^2. \tag{8}$$

The two inequalities can be confirmed by substituting Eq. (3)–(5) into Eq. (8) and setting $\Gamma = 0$. Similar inequalities for the absorptive case will be proved later.

We now turn to the time-domain description and discuss the absorptive case. The fundamental constitute relations

$$\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P}, \qquad \mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}), \tag{9}$$

and Eqs. (3)–(5) lead to the following equations:

$$\frac{\partial^2\mathbf{P}}{\partial t^2} + \Gamma\frac{\partial\mathbf{P}}{\partial t} + \omega_0^2\mathbf{P} = \epsilon_0\omega_p^2\mathbf{E} + \frac{A}{c}\frac{\partial\mathbf{H}}{\partial t}, \tag{10}$$

$$\frac{\partial\mathbf{M}}{\partial t} + \Gamma\mathbf{M} + \omega_0^2\int\mathbf{M}dt = -F\frac{\partial\mathbf{H}}{\partial t} - \frac{A\mathbf{E}}{\mu_0 c}. \tag{11}$$

Both equations are derived from the same circuit equation

$$L\frac{dI}{dt} + RI + \frac{q}{C} = V_e - \frac{d\Phi}{dt} \tag{12}$$

associated with the helical RLC circuit in a unit cell [12]. Here L, R, and C represent the inductance, resistance, and capacitance of the circuit, respectively [9,10,12]. The term $V_e - d\Phi/dt$ represents the total voltage difference across the helical circuit contributed from the electric field and the time-varying magnetic flux. The fact that both Eqs. (10) and (11) are derived from Eq. (12) implies

$$A = \pm\sqrt{F}\omega_p, \qquad \frac{\partial\mathbf{P}}{\partial t} = -\frac{\omega_p^2}{Ac}\mathbf{M}. \tag{13}$$

Note that the proportionality between $\partial\mathbf{P}/\partial t$ and $\mathbf{M}$ does not necessarily imply that $\mathbf{P}$ and $\mathbf{M}$ are parallel. An example for clarifying this point will be discussed later.

Now we derive the energy density formula using the time-domain (ED) approach [8]. Using Maxwell's equations and Eq. (9), we can derive

$$-\nabla\cdot\mathbf{S} = \frac{\partial W_0}{\partial t} + \mathbf{E}\cdot\frac{\partial\mathbf{P}}{\partial t} - \mu_0\mathbf{M}\cdot\frac{\partial\mathbf{H}}{\partial t}, \tag{14}$$

where

$$W_0 = \frac{\epsilon_0}{2}\mathbf{E}^2 + \frac{\mu_0}{2}\mathbf{H}^2 + \mu_0\mathbf{H}\cdot\mathbf{M} \tag{15}$$

is a temporary quantity having the dimension of energy density. We will show that $\mathbf{E}\cdot\partial\mathbf{P}/\partial t - \mu_0\mathbf{M}\cdot\partial\mathbf{H}/\partial t$ can be written as $\partial U/\partial t + P_{\text{loss}}$, and thus the right-hand side of Eq. (14) can be identified as $\partial W/\partial t + P_{\text{loss}}$, where $W = W_0 + U$ is the energy density and P_{loss} is the power loss. Just like in the wire-SRR case [8], the power loss corresponds to the joule heat I^2R. In the present chiral structure, the current flowing in the helical circuit corresponds to both $\partial\mathbf{P}/\partial t$ and $\mathbf{M}$, thus the power loss should be expressed as $P_{\text{loss}} = \alpha(\partial\mathbf{P}/\partial t)^2 = \beta\mathbf{M}^2$. Here α and β are two appropriate constants to be identified.

Using Eq. (10) and (13), we find

$$\begin{aligned}\mathbf{E}\cdot\frac{\partial\mathbf{P}}{\partial t} &= \frac{1}{\epsilon_0\omega_p^2}\left(\frac{\partial\dot{\mathbf{P}}}{\partial t} + \Gamma\dot{\mathbf{P}} + \omega_0^2\mathbf{P} - \frac{A}{c}\dot{\mathbf{H}}\right)\cdot\dot{\mathbf{P}} \\ &= \frac{\partial}{\partial t}\left(\frac{\dot{\mathbf{P}}^2}{2\epsilon_0\omega_p^2} + \frac{\omega_0^2\mathbf{P}}{2\epsilon_0\omega_p^2}\right) + \frac{\Gamma\dot{\mathbf{P}}^2}{\epsilon_0\omega_p^2} + \mu_0\mathbf{M}\cdot\frac{\partial\mathbf{H}}{\partial t},\end{aligned} \tag{16}$$

thus the energy density is given by

$$W = \frac{\epsilon_0\mathbf{E}^2}{2} + \frac{\mu_0\mathbf{H}^2}{2} + \mu_0\mathbf{H}\cdot\mathbf{M} + \frac{\dot{\mathbf{P}}^2 + \omega_0^2\mathbf{P}^2}{2\epsilon_0\omega_p^2}, \tag{17}$$

$$= \frac{\epsilon_0\mathbf{E}^2}{2} + \frac{\omega_0^2\mathbf{P}^2}{2\epsilon_0\omega_p^2} + \frac{\mu_0(1-F)\mathbf{H}^2}{2} + \frac{\mu_0(\mathbf{M}+F\mathbf{H})^2}{2F}. \tag{18}$$

Note that the power loss

$$P_{\text{loss}} = \frac{\Gamma}{\epsilon_0\omega_p^2}\left(\frac{\partial\mathbf{P}}{\partial t}\right)^2 = \frac{\mu_0\Gamma}{F}\mathbf{M}^2 \tag{19}$$

is indeed the expected form. According to Eq. (18), the energy density of electromagnetic field in a chiral medium is always positive. Equations (17)–(19) are the main results of this Letter.

We next calculate the time-averaged energy density $\langle W \rangle$ for a harmonic wave. Applying the formula $\langle a(t)b(t)\rangle = (1/2)\mathrm{Re}(a_\omega b_\omega^*)$ to Eq. (18) and (19), we have

$$\langle W \rangle = \frac{1}{4}V^{\dagger}\mathcal{M}V, \qquad \langle P_{\text{loss}} \rangle = \frac{1}{2}V^{\dagger}\mathcal{P}V, \tag{20}$$

where

$$\mathcal{M} = \begin{pmatrix} 1 + \frac{\omega_p^2(\omega_0^2+\omega^2)}{(\omega_0^2-\omega^2)^2+\Gamma^2\omega^2} & -\frac{A\omega(\Gamma\omega+2i\omega_0^2)}{(\omega_0^2-\omega^2)^2+\Gamma^2\omega^2} \\ -\frac{A\omega(\Gamma\omega-2i\omega_0^2)}{(\omega_0^2-\omega^2)^2+\Gamma^2\omega^2} & 1 + \frac{F\omega^2(3\omega_0^2-\omega^2)}{(\omega_0^2-\omega^2)^2+\Gamma^2\omega^2} \end{pmatrix}, \tag{21}$$

$$\mathcal{P} = \frac{\Gamma\omega^2}{(\omega_0^2-\omega^2)^2+\Gamma^2\omega^2}\begin{pmatrix} \omega_p^2 & -iA\omega \\ iA\omega & F\omega^2 \end{pmatrix}. \tag{22}$$

In the $\Gamma \to 0$ limit, the matrix $\mathcal{M}$ becomes $\mathcal{M}_0$, and Eq. (7) is recovered. To prove the positiveness of $\langle W \rangle$ in the absorptive case, we need to check the conditions $\mathrm{tr}\mathcal{M} > 0$ and $\det \mathcal{M} > 0$. The $\mathrm{tr}\mathcal{M} > 0$ condition is obvious. The inequality $\mathcal{M}_{22} - (1 - F) > 0$ is also easy to check. Thus we have $\det \mathcal{M} = \mathcal{M}_{11}\mathcal{M}_{22} - |\mathcal{M}_{12}|^2 > (\mathcal{M}_{11} - 1)(\mathcal{M}_{22} - 1 + F) - |\mathcal{M}_{12}|^2 = F\omega_0^2\omega_p^2/[(\omega_0^2 + \omega^2)^2 + \Gamma^2\omega^2] > 0$ (Condition $0 < F < 1$ is used.)

We now prove that Eq. (18) also applies to the bianisotropic metamaterial proposed in [14]. Unlike the previous case defined by Eq. (3)–(5), $\mathbf{P}$ and $\mathbf{M}$ are no longer parallel to each other. According to the constitutive relations in [14], $\mathbf{P}$ and $\mathbf{M}$ fields satisfy

$$\frac{\partial^2 \mathbf{P}}{\partial t^2} + \Gamma \frac{\partial \mathbf{P}}{\partial t} + \omega_0^2 \mathbf{P} = \epsilon_0 \omega_p^2 \mathbf{E} + \frac{AR}{c}\frac{\partial \mathbf{H}}{\partial t}, \qquad (23)$$

$$\frac{\partial \mathbf{M}}{\partial t} + \Gamma \mathbf{M} + \omega_0^2 \int \mathbf{M} dt = -F\frac{\partial \mathbf{H}}{\partial t} - \frac{AR^{-1}\mathbf{E}}{\mu_0 c}. \qquad (24)$$

Here $R = R(\alpha)$ is a rotation matrix belonging to orthogonal group, and α is a parameter characterizing the angle between $-\partial \mathbf{P}/\partial t$ and $\mathbf{M}$. The inverse of R is $R^{-1} = R(-\alpha) = R^t$, the transpose of R. The proportionality between Eq. (23) and (24) yields

$$\frac{\partial \mathbf{P}}{\partial t} = -\frac{\omega_p^2}{Ac} R(\alpha)\mathbf{M}. \qquad (25)$$

The term $-(A/\epsilon_0\omega_p^2 c)\dot{\mathbf{H}} \cdot \dot{\mathbf{P}}$ in Eq. (16) is now replaced by

$$-\frac{A(R\dot{\mathbf{H}}) \cdot \dot{\mathbf{P}}}{\epsilon_0 \omega_p^2 c} = \mu_0 (R\dot{\mathbf{H}}) \cdot (R\mathbf{M}) = \mu_0 \dot{\mathbf{H}} \cdot (R^t R)\mathbf{M}$$
$$= \mu_0 \frac{\partial \mathbf{H}}{\partial t} \cdot \mathbf{M}. \qquad (26)$$

The energy density and power loss in the bianisotropic medium [14] are thus still given by Eqs. (17)–(19).

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References

1. L. Solymar and E. Shamonina, *Waves in Metamaterials* (Oxford University, 2009).
2. B. Wang, J. Zhou, T. Koschny, M. Kafesaki, and C. M. Soukoulis, J. Opt. A **11**, 114003 (2009).
3. L. D. Landau and E. M. Lifshitz, *Electrodynamics of Continuous Media*, 2nd. ed. (Pergamon 1984).
4. R. Loudon, J. Phys. A **3**, 233 (1970).
5. R. Ruppin, Phys. Lett. A **299**, 309 (2002).
6. S. A. Tretyakov, Phys. Lett. A **343**, 231 (2005).
7. A. D. Boardman and K. Marinov, Phys. Rev. B **73**, 165110 (2006).
8. P. G. Luan, Phys. Rev. E **80**, 046601 (2009).
9. L. Jelinek, R. Marqués, F. Mesa, and J. D. Baena, Phys. Rev. B **77**, 205110 (2008).
10. I. V. Semchenko, S. A. Khakhomov, and S. A. Tretyakov, Eur. Phys. J. Appl. Phys. **46**, 32607 (2009).
11. S. A. Tretyakov, F. Mariotte, C. R. Simovski, T. G. Kharina, and J.-P. Heliot, IEEE Trans. Antennas Propagat. **44**, 1006 (1996).
12. R.-K. Zhao, T. Koschny, and C. M. Soukoulis, Opt. Express **18**, 14553 (2010).
13. S. Tretyakov, I. Nefedov, A. Sihvola, S. Maslovski, and C. Simovski, "Waves and energy in chiral nihility," arXiv: cond-mat/0211012 (2002).
14. S. Zhang, Y. S. Park, J. Li, X. Lu, W. Zhang, and X. Zhang, Phys. Rev. Lett. **102**, 023901 (2009).