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Electromagnetic extraction of energy from Kerr black holes

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Summary. When a rotating black hole is threaded by magnetic field lines supported by external currents flowing in an equatorial disc, an electric potential difference will be induced. If the field strength is large enough, the vacuum is unstable to a cascade production of electron–positron pairs and a surrounding force-free magnetosphere will be established. Under these circumstances it is demonstrated that energy and angular momentum will be extracted electromagnetically. As a further consequence it is shown that charge can never contribute significantly to the geometry of a rotating hole. The fundamental equations describing a stationary axisymmetric magnetosphere are derived and the details of the energy and angular momentum balance are discussed. A perturbation technique is developed which can be used to provide approximate solutions for slowly rotating holes. Solutions appropriate when the field lines threading the hole lie on conical and paraboloidal surfaces at large distances are described to illustrate this mechanism.

These ideas are incorporated into a discussion of a model of active galactic nuclei containing a massive black hole surrounded by a magnetized accretion disc. In this model relativistic electrons can be accelerated at large distances from the hole and therefore will not incur serious losses, which is a defect of some existing models. In addition, if the field lines have paraboloidal shape, the energy will be beamed along antiparallel directions as observations of both compact and extended radio sources seem to require.

1 Introduction

One of the central problems of extragalactic astronomy concerns the nature of the 'prime mover' that is powering active galactic nuclei (including quasars). In discussions of this question, two analogies have been drawn with the Crab Nebula, which is clearly powered by a central pulsar – a spinning magnetized neutron star that is steadily liberating its rotational energy through relativistic particle and electromagnetic energy fluxes. Firstly it has been proposed that such a nucleus might contain a cluster of 10^{6-8} pulsars, the currently active ones forming several overlapping Crab Nebulae (e.g. Rees 1971). Alternatively, the central energy source might be a single magnetized star of mass $10^{6-8} M_\odot$ that acts like a giant pulsar (e.g. Morrison 1969). However, both of these models pose difficulties. On the one hand, there is the problem of the fate of stars of mass greater than the limiting neutron star mass ($\sim 2M_\odot$) and on the other it is doubtful whether a massive object can live sufficiently long to account for an object like 3C 236 (Fomalont & Miley 1975). Thus we must take seriously the possibility of one or more black holes, forming in the nucleus as a result of gravitational collapse – the seemingly inevitable evolutionary end-point for large masses.

It is therefore of interest to ask whether or not a spinning black hole can also liberate its rotational energy as a result of electromagnetic processes like those in a pulsar. Christodolou (1970) has shown that for a rotating hole described by the Kerr metric, a portion of the rest mass can be regarded as 'reducible' in the sense that it can be removed and extracted to infinity. The remaining 'irreducible' mass, which is proportional to the area of the event horizon, can be interpreted thermodynamically as the entropy of the hole (Bardeen, Carter & Hawking 1973; Hawking 1976). We are primarily interested in holes of mass $\gtrsim 1 M_\odot$ and in thermodynamic terms such holes are extremely cold bodies. Hawking's (1974) quantum mechanical spontaneous emission is negligible and the irreducible mass of such holes lives up to its name and never decreases. The possibility of extracting the reducible mass was first realized by Penrose (1969) who showed that the existence of negative energy orbits within the ergosphere surrounding a Kerr black hole (e.g. Misner, Thorne & Wheeler 1973) permits the mechanical extraction of energy via certain types of particle collision. Teukolsky & Press (1974) have investigated a similar process called super-radiant scattering which in its electromagnetic form has ingoing spherical vacuum waves being amplified and spinning down the hole. Unfortunately neither process seems likely to operate effectively in any astrophysical situation.

Material accreted by a black hole within a galactic nucleus will probably be magnetized and possess sufficient angular momentum to form a disc, as first proposed by Lynden-Bell (1969). The magnetic flux will be frozen into the accreting material and so the field close to the horizon can become quite large – much larger than the field at infinity. In this paper we consider the behaviour of a rotating black hole in the presence of a strong magnetic field supported by external currents flowing in an equatorial disc. It is usually assumed that the disc is Keplerian. However, this need not be the case. It is possible that a magnetic accretion disc could be supported by the field rather than by centrifugal forces (Bisnovatyi-Kogan & Ruzmaikin 1976). This could extend much closer to the event horizon than a centrifugally supported disc (Znajek 1976) and there is no fundamental relativistic objection to approaching 100 per cent efficiency in converting accreting mass into energy at infinity.

General stationary axisymmetric vacuum electromagnetic fields around a Kerr hole have been discussed by Petterson (1975) and King, Lasota & Kundt (1975). Wald (1974) argued that when a rotating hole was situated in an externally supported field *in vacuo*, an electric field is 'induced' and the lowest energy state has a finite charge on the hole. There is of course no radiation of energy from this configuration. However, the electric field generally has a non-zero component parallel to the magnetic field. In Section 2 we show that if the magnetic field and angular momentum are large enough, the vacuum surrounding the hole is unstable because any stray charged particles will be electrostatically accelerated and will radiate, and this radiation will in turn produce further charged particles in the form of electron–positron pairs. When charges are produced so freely, the electromagnetic field in the vicinity of the horizon will become approximately force-free. (We emphasize that this is a purely electromagnetic effect and is quite distinct from Hawking's spontaneous emission.)

The corresponding situation with a spherical conductor rotating in flat space and endowed with an axisymmetric magnetic field was investigated by Goldreich & Julian (1969) who argued that just such a stationary force-free magnetosphere would be established with space charges supporting a poloidal electric field and currents creating additional poloidal and toroidal magnetic fields. Energy and angular momentum are transported outwards electromagnetically. In their treatment, and in many subsequent papers dealing with pulsar magnetospheres, the sole function of the plasma is to support the electromagnetic fields. The plasma has no mechanical role. The limitations of this assumption for models of pulsar magnetospheres are discussed in Mestel, Wright & Westfold (1976). As these force-free solutions contain sources all the way out to infinity they do not correspond to the familiar multipole solutions of the vacuum Maxwell equations and in particular do not require an $e^{im\phi}$ variation in azimuth in order to transport angular momentum. In Blandford (1976), similar magnetospheric effects occur above and below a Keplerian disc surrounding a compact object are investigated. It is shown that if energy and angular momentum are extracted solely electromagnetically, then stationarity requires that the field strength vary inversely with radius. In this way it is possible to remove the gravitational energy of infalling material without thermal dissipation in the disc and also to collimate this energy parallel to the rotation axis (*cf.* Lovelace 1976).

In Section 3 of this paper we derive the equations governing stationary axisymmetric force-free electromagnetic fields in Kerr space-time. In Section 4 it is shown that energy and angular momentum from a rotating hole can indeed be extracted by a mechanism directly analogous to that of Goldreich & Julian (1969). This process is rather similar to that of Penrose (1969) if we think of it in terms of particles inside the event horizon interacting with particles a long way away from the hole through the agency of the magnetic field. A related mechanism for extracting rotational energy has been discussed by Ruffini & Wilson (1975). They describe a magnetohydrodynamic flow in which the field is too weak to prevent the fluid from flowing along geodesics, but in which there is an outwardly directed Poynting flux. However, this solution is unlikely to be of direct practical application as the overall efficiency of energy extraction is extremely low and in any case one would expect that the accretion would be able to amplify the field until it became dynamically significant. In Section 5 we describe a perturbation technique that can be used to calculate approximate solutions under certain circumstances. Two such solutions are derived in Sections 6 and 7. Finally in Section 8 we outline the application of these ideas to a model of active galactic nuclei.

2 Pair production discharge mechanism

Before we discuss the electromagnetic properties of force-free magnetospheres in the vicinity of black holes, we must first tackle an important problem: there must be some source of particles within the near magnetosphere. The currents that pervade the magnetosphere as sources of the magnetic field are presumably carried by charged particles that are flowing outwards at large distances. (It is in principle possible to supply inflowing particles if the field lines crossing the horizon also intersect the disc but if, as discussed further in Section 8, particle inertia eventually becomes important and an electromagnetically driven wind is produced, positive outflow at large radii seems unavoidable.) We also know that the particle flux must be directed inwards through the event horizon, and so it cannot be conserved.

Fortunately, this is a situation that is familiar from studies of pulsars. The positive ion work function on the surface of a neutron star may be so large that only electrons can escape. For those open field lines on which positive charges have to stream outwards, a mechanism has been postulated which allows the vacuum to break down as a result of the creation of sufficient electron–positron pairs. This has been analysed in some detail by Ruderman & Sutherland (1975). The salient features are that a spark gap will develop containing a parallel electric field. A single electron (or positron) accelerated by this potential difference will reach a high enough energy to radiate gamma-ray photons by the curvature process. These photons, emitted initially tangentially to the field can, after traversing most of the gap, encounter a significant perpendicular component of the curving magnetic field and create an electron–positron pair which will in turn be accelerated leading to a cascade. The gap width presumably adjusts so that the 'gain' is slightly greater than unity and sufficient charged particles of both signs are created to supply the currents.

In the case of a pulsar, the natural location of such a gap is on the neutron star surface. However, for a black hole we only require that it be located outside the event horizon. In fact there is no reason to believe that its position is stationary. Provided that the potential difference necessary to produce breakdown is much less than the total across the open field lines, an electromagnetic force-free solution should provide a reasonable approximation to the time-averaged structure of such a magnetosphere. Pair creation is of course consistent with charge conservation. The macroscopic Maxwell equations do not involve details of the charge transport which are presumably determined by the equations of motion involving inertial terms and the small residual electric field components parallel to the magnetic field. In fact any electromagnetically required values of current, $\mathbf{j}$, and charge, ρ, can be created without restriction on the direction of motion of either of the charge carriers. Hence $\mathbf{j}/\rho$ can be outwardly directed at the event horizon with both electrons and positrons falling inwards.

We now estimate the field strengths necessary for this mechanism to operate effectively near the event horizon of a rotating Kerr hole. As usual, the mass, M, and specific angular momentum, a, of the hole are measured in units of c^2/G and c respectively so that they both have dimensions of length. Then M and ac/M^2 respectively provide a length scale and a characteristic angular frequency for the hole. For a field of strength $\mathbf{B}$, the charge density necessary to ensure that $\mathbf{E}.\mathbf{B} = 0$ is

$$\rho \sim \epsilon_0 \frac{a}{M^2} cB. \tag{2.1}$$

In the absence of this charge, the potential difference across a gap of height h is

$$\Delta V \sim \frac{a}{M^2} cBh^2 \tag{2.2}$$

giving electron or positron energies $\gamma m_e c^2$, where

$$\gamma \sim \left(\frac{h}{M}\right)^2 \left(\frac{\omega_G a}{c}\right) \tag{2.3}$$

with ω_G the electron gyro frequency. We expect that the field lines will have curvature $\sim M^{-1}$ and so the peak energy of the radiated photons will be

$$\epsilon \sim \gamma^3 \left(\frac{\hbar c}{M}\right) \tag{2.4}$$

The photon mean free path to pair creation in a direction making an angle ξ to the magnetic field is

$$l \sim 600 \frac{c}{\omega_G \sin \xi} \exp\left[\frac{8}{3}\left(\frac{m_e c^2}{\epsilon}\right)\left(\frac{m_e c^2}{\hbar \omega_G \sin \xi}\right)\right]; \quad \epsilon \gg 2m_e c^2 \tag{2.5}$$

(Erber 1966). The number of photons with energy $\sim \epsilon$ radiated per electron per unit length is

$$\frac{dn_\gamma}{dx} \sim \frac{1}{4\pi\epsilon_0}\frac{e^2}{c^3}\gamma^4\left(\frac{c^2}{M}\right)^2\left(\frac{1}{c\epsilon}\right) \sim \frac{\gamma}{137M}. \tag{2.6}$$

An approximate criterion for breakdown is that each electron produces one pair within the gap. Combining equations (2.3)–(2.6) we obtain

$$\left(\frac{h}{M}\right) \sim \left(\frac{\hbar\omega_G}{m_e c^2}\right)^{-4/7}\left(\frac{a}{M}\right)^{-3/7}\left(\frac{\hbar}{Mm_e c}\right)^{2/7}\left[\ln\left(10^{-5}\left(\frac{h}{M}\right)^5\left(\frac{\omega_G a}{c}\right)\left(\frac{\omega_G M}{c}\right)\right)\right]^{-1/7} \tag{2.7}$$

using $\sin\xi \sim h/M$. The logarithm has a value ~ 10–100 for parameters of interest and for present purposes it is sufficient to set it as 30. Breakdown will occur as long as $h \lesssim M$, i.e. for

$$B \gtrsim 20\left(\frac{a}{M}\right)^{-3/4}\left(\frac{M}{M_\odot}\right)^{-1/2} T; \qquad \left(\frac{M}{M_\odot}\right) \gtrsim \text{Max}\left[10^{-14}\left(\frac{a}{M}\right)^{-3/2}, 10^{-11}\left(\frac{a}{M}\right)^{-1/2}\right] \tag{2.8}$$

where the second inequality follows from the requirements $\epsilon \geqslant 2m_e c^2$, $n_\gamma > 1$. If this inequality is satisfied by a large factor, the conductivity along the field lines is effectively infinite.

In deriving inequality (2.8) it has been assumed that gamma rays are only produced by the curvature process. In fact the radiation of transverse gyrational energy, Doppler-shifted by the longitudinal motion, inverse Compton scattering, and for very small holes free–free emission could be alternative sources of hard photons. The efficiency of these processes cannot be estimated so easily, but if they are important, field strengths significantly lower than that given by equation (2.8) may be adequate to ensure breakdown of the vacuum.

In fact in the principal application of this mechanism to a massive black hole in a galactic nucleus (Section 8) inverse Compton scattering of the ambient radiation field may prevent the electrons from achieving sufficiently high energies, $\gamma \gtrsim m_e c^2/\hbar\omega_G$, for subsequent pair creation in the magnetic field. However, within this environment there is a much more efficient breakdown mechanism. A relativistic electron can inverse Compton scatter a photon from the ambient radiation field as a γ-ray which can then pair create with the assistance of a second background photon. If the ambient radiation energy density is ϵ with characteristic frequency, ω, then the vacuum will break down provided that

$$(\epsilon/\hbar\omega)\,\sigma_T M \gtrsim 1, \quad \omega_G \lesssim m_e c^2/\hbar$$

and

$$\gamma \sim (m_e c^2/\hbar\omega).$$

This requires that the component of electric field parallel to the magnetic field be sufficiently strong to maintain an electron at this energy against radiative losses. Hence necessary conditions for breakdown by this method are

$$B \gtrsim \left(\frac{a}{M}\right)^{-1}\left(\frac{M}{M_\odot}\right)^{-1}\left(\frac{\omega}{10^{15}\,\text{rad/s}}\right)^{-1} \text{T},$$

$$\epsilon \gtrsim 10^8\left(\frac{M}{M_\odot}\right)^{-1}\left(\frac{\omega}{10^{15}\,\text{rad/s}}\right) \text{Jm}^{-3} \tag{2.9}$$

where we have ignored contributions to the radiation energy density from frequencies below the maximum for which these inequalities are satisfied.

In a general Kerr–Newman black hole (e.g. Misner *et al.* 1973) both spin and charge contribute significantly to the geometry. As a corollary to the above discussion, we now show that such objects are unlikely to exist. For a charge, Q, in units of length, $(4\pi\epsilon_0/G)^{1/2}c^2$, the magnetic and electric fields near the horizon are given by

$$B \sim \left(\frac{a}{M}\right)\frac{E}{c} \sim 5\times 10^{14}\left(\frac{Q}{M}\right)\left(\frac{a}{M}\right)\left(\frac{M}{M_\odot}\right)^{-1} \text{T}.$$

From an analogous calculation we find that breakdown will occur in a Kerr–Newman black hole if

$$\left(\frac{Q}{M}\right) \gtrsim 10^{-13}\left(\frac{a}{M}\right)^{-1/2}\left(\frac{M}{M_\odot}\right)^{1/2}; \quad \text{and} \quad \left(\frac{M}{M_\odot}\right) \gtrsim 10^{-12}\left(\frac{a}{M}\right)^{-1}. \tag{2.10}$$

If this inequality is satisfied, the hole will rapidly discharge. Condition (2.10) effectively rules out the possibility of a hole with gravitationally significant spin and charge.

This breakdown mechanism may also operate for a non-rotating, charged Reissner–Nordstrom hole. However, a necessary condition for a cascade to develop is that each charged particle produces at least one sufficiently hard photon. As the acceleration in a radial electric field is purely linear, a more careful calculation must be performed to determine when this requirement is satisfied. An approximate calculation leaves the question unresolved.

Gibbons (1974) has analysed pair creation occurring near the horizon of a Reissner–Nordstrom hole and its spontaneous loss of charge. He assumes that the 'Schwinger process' operates, which is efficient whenever the electric potential difference across a Compton wavelength exceeds the rest mass of an electron (in eV). The present process in which the Compton wavelength is effectively replaced by M/γ will generally set in at much lower electromagnetic field strengths, at least when the hole is rotating. It is for this reason that the upper limit (2.10) is much lower than that implied by Gibbons' analysis. However, both these conditions allow the charge on the hole to exceed that value for which the electric force on individual charged particles is much greater than the gravitational force and so in the absence of any other electromagnetic effects selective accretion of charges of opposite sign to the hole will occur, further reducing Q.

3 Force-free magnetosphere in Kerr spacetime

We now derive the fundamental equations governing a force-free magnetosphere, generalizing the flat space treatment of, for example, Okamoto (1974). The electromagnetic field tensor $F^{\mu\nu}$ in a force-free magnetosphere satisfies

$$F_{\mu\nu}J^\nu = 0 \tag{3.1}$$

where J^ν is the current 4-vector.* This condition is expected to hold everywhere except in the $\theta = \pi/2$ plane occupied by the disc. The homogeneous Maxwell equations are automatically satisfied if we describe the field by a vector potential A_μ so that

$$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}.$$

* This expression is more appropriate in the present context than the usual MHD condition ($F_{\mu\nu}U^\nu = 0$), because the plasma will not necessarily possess a well-defined fluid velocity, U^ν. It follows directly from the requirement that inertial and collisional terms be insignificant for each individual particle species. Formally similar results can be derived using MHD (e.g. Damour 1975). It can be shown that a necessary and sufficient condition for the *individual* particle speeds to be less than c is that $F_{\mu\nu}F^{\mu\nu} > 0$. This is true for the examples presented below in Sections 6 and 7, as a direct calculation confirms.

In Boyer–Lindquist coordinates the metric around a Kerr black hole is (with $c=G=1$)

$$ds^2 = \left(1 - \frac{2Mr}{\Sigma}\right) dt^2 + \frac{4Mar\sin^2\theta}{\Sigma} dt\, d\phi - \frac{\Sigma}{\Delta} dr^2 - \Sigma\, d\theta^2 - \frac{A\sin^2\theta}{\Sigma} d\phi^2 \tag{3.2}$$

where $\Sigma = r^2 + a^2\cos^2\theta$, $\Delta = r^2 - 2Mr + a^2 \equiv (r-r_+)(r-r_-)$ and $A = (r^2+a^2)^2 - \Delta a^2 \sin^2\theta$. $r_+ (= M + \sqrt{M^2 - a^2})$ is the radius of the event horizon. We assume the field and vector potential are stationary ($\partial/\partial t = 0$) and axisymmetric with the same symmetry axis as the hole ($\partial/\partial\phi = 0$) and also symmetric about $\theta = \pi/2$. These conditions reduce the gauge freedom of A_0 and A_ϕ to an additive constant.

In Boyer–Lindquist coordinates (3.1) becomes

$$A_{0,r} J^r + A_{0,\theta} J^\theta = 0 \tag{3.3a}$$

$$A_{0,r} J^0 + A_{\phi,r} J^\phi + B_\phi J^\theta = 0 \tag{3.3b}$$

$$A_{0,\theta} J^0 + A_{\phi,\theta} J^\phi - B_\phi J^r = 0 \tag{3.3c}$$

$$A_{\phi,r} J^r + A_{\phi,\theta} J^\theta = 0 \tag{3.3d}$$

where $B_\phi = F_{r\theta} = A_{0,r} - A_{r,0}$ is gauge invariant. From equations (3.3a) and (3.3d) we can define a function $\omega(r,\theta)$ such that

$$\frac{A_{0,r}}{A_{\phi,r}} = -\omega = \frac{A_{0,\theta}}{A_{\phi,\theta}}. \tag{3.4}$$

Differentiating equations (3.4),

$$\frac{\omega_{,r}}{\omega_{,\theta}} = \frac{A_{\phi,r}}{A_{\phi,\theta}} = \frac{A_{0,r}}{A_{0,\theta}}. \tag{3.5}$$

The poloidal field surfaces can be defined by A_ϕ = constant: A_ϕ is a suitable stream function for the magnetic field (*cf.* Scharlemann & Wagoner 1973). Equation (3.5) is the relativistic generalization of Ferraro's (1937) Law of Isorotation: ω and the 'electrostatic' potential A_0 are constant along field lines.

Equations (3.3a) and (3.3d) imply the equivalence of (3.3b) and (3.3c). From equation (3.3d) we can deduce the existence of a function $\mu(r,\theta)$ satisfying

$$\frac{J^r}{\epsilon_0} = \frac{\mu}{g^{1/2}} A_{\phi,\theta}, \quad \frac{J^\theta}{\epsilon_0} = -\frac{\mu}{g^{1/2}} A_{\phi,r}, \tag{3.6}$$

where

$$g = -\det(g_{\alpha\beta})$$
$$= \Sigma^2 \sin^2\theta$$

in Boyer–Lindquist coordinates. (In fact most of this section can be generalized to any alternative stationary, axisymmetric metric.) Equation (3.3b) becomes

$$J^\phi = \omega J^0 + \frac{\mu\epsilon_0}{g^{1/2}} B_\phi. \tag{3.7}$$

ω can be interpreted as an electromagnetic angular velocity. It is not necessarily equal to any material angular velocity. The current conservation equation $g^{-1/2}(g^{1/2}J^\mu)_{,\mu} = 0$ reduces to

$$(\mu A_{\phi,\theta})_{,r} = (\mu A_{\phi,r})_{,\theta} \tag{3.8}$$

which shows that μ is also constant on field surfaces A_ϕ = const.

The inhomogeneous Maxwell equations

$$\epsilon_0^{-1} J^\mu = F^{\mu\nu}_{;\nu} = g^{-1/2}(g^{1/2} g^{\mu\alpha} g^{\nu\beta}(A_{\beta,\alpha} - A_{\alpha,\beta}))_{,\nu}$$

can be written explicitly as

$$-\epsilon_0^{-1} g^{1/2} J^0 = (g^{1/2} g^{00} g^{rr} A_{0,r} + g^{1/2} g^{0\phi} g^{rr} A_{\phi,r})_{,r} + (g^{1/2} g^{00} g^{\theta\theta} A_{0,\theta} + g^{1/2} g^{0\phi} g^{\theta\theta} A_{\phi,\theta})_{,\theta} \quad (3.9a)$$

$$\epsilon_0^{-1} g^{1/2} J^r = B_{T,\theta} \quad (3.9b)$$

$$\epsilon_0^{-1} g^{1/2} J^\theta = -B_{T,r} \quad (3.9c)$$

$$-\epsilon_0^{-1} g^{1/2} J^\phi = (g^{1/2} g^{\phi\phi} g^{rr} A_{\phi,r} + g^{1/2} g^{\phi 0} g^{rr} A_{0,r})_{,r} + (g^{1/2} g^{\phi\phi} g^{\theta\theta} A_{\phi,\theta} + g^{1/2} g^{\phi 0} g^{\theta\theta} A_{0,\theta})_{,\theta} \quad (3.9d)$$

where $B_T = (\Delta/\Sigma)\sin\theta B_\phi$ and the $g^{\alpha\beta}$ can be obtained from equation (3.2). Substituting equations (3.6) into (3.9b) and (3.9c),

$$\mu A_{\phi,\theta} = B_{T,\theta} \quad \text{and} \quad \mu A_{\phi,r} = B_{T,r}.$$

This shows that B_T is a fourth quantity that is a function of A_ϕ only, and that

$$\mu = dB_T/dA_\phi. \quad (3.10)$$

Equation (3.6) shows that current does not cross the poloidal field surfaces. The outward current between surfaces A_ϕ and $A_\phi + dA_\phi$ (counting both hemispheres), as measured by an observer at infinity, is

$$dI = 4\pi\epsilon_0 \mu(A_\phi)\, dA_\phi. \quad (3.11)$$

We do not expect a line-current to be present on the $\theta = 0$ axis, and so $B_T(A_p) = 0$ where $A_\phi = A_p$ on $\theta = 0$. Integrating equation (3.11) using (3.10) we find that the net current from the hole to the magnetosphere is

$$I = 4\pi\epsilon_0 B_T(A_\mathrm{e}) \quad (3.12)$$

where the subscript e labels the field line touching the event horizon at $\theta = \pi/2$. This must be balanced by an equal current flowing radially inwards in the disc which supports the discontinuity in the toroidal magnetic field across the disc.

Suppose we know $B_T(A_\phi)$ and $\omega(A_\phi)$ and require a function $A_\phi(r, \theta)$ that will give us a self-consistent solution of the force-free and Maxwell equations. Equations (3.6) and (3.10) ensure that (3.9b) and (3.9c) are satisfied. Equation (3.4) enables us to determine A_0 using

$$dA_0 = -\omega(A_\phi)\, dA_\phi. \quad (3.13)$$

Equations (3.6), (3.7) and (3.13) imply that the force-free equations are satisfied. J^0 and J^ϕ are given in terms of A_ϕ by equations (3.9a), (3.9d) and (3.13); the only constraint they must satisfy is equation (3.7). Thus by substituting (3.9a), (3.9d) and (3.13) into (3.7) we obtain the fundamental differential equation for the potential A_ϕ

$$\begin{aligned}\frac{B_T(A_\phi)\, dB_T/dA_\phi}{g^{1/2} g^{rr} g^{\theta\theta}} = &-\omega[g^{1/2} g^{rr} g^{00} A_{\phi,r}\omega]_{,r} - [g^{1/2} g^{rr} g^{\phi\phi} A_{\phi,r}]_{,r} \\ &+ \omega[g^{1/2} g^{rr} g^{0\phi} A_{\phi,r}]_{,r} + [g^{1/2} g^{rr} g^{0\phi} A_{\phi,r}\omega]_{,r} \\ &- \omega[g^{1/2} g^{\theta\theta} g^{00} A_{\phi,\theta}\omega]_{,\theta} - [g^{1/2} g^{\theta\theta} g^{\phi\phi} A_{\phi,\theta}]_{,\theta} \\ &+ \omega[g^{1/2} g^{\theta\theta} g^{0\phi} A_{\phi,\theta}]_{,\theta} + [g^{1/2} g^{\theta\theta} g^{0\phi} A_{\phi,\theta}\omega]_{,\theta} \end{aligned} \quad (3.14)$$

which for a Kerr metric can be put into the form

$$\frac{\Sigma B_T B_T'}{\Delta \sin\theta} = \omega^2\alpha + 2\omega\beta + \gamma + \frac{\sin\theta}{\Sigma\Delta}(A\omega - 2Mar)(\Delta(A_{\phi,r})^2 + (A_{\phi,\theta})^2)\,\omega'$$

where the prime denotes differentiation with respect to A_ϕ and

$$\alpha = \sin\theta\left(\frac{A}{\Sigma}A_{\phi,r}\right)_{,r} + \Delta^{-1}\left(\frac{A\sin\theta}{\Sigma}A_{\phi,\theta}\right)_{,\theta}$$

$$\beta = -2Ma\left[\sin\theta\left(\frac{rA_{\phi,r}}{\Sigma}\right)_{,r} + \frac{r}{\Delta}\left(\frac{\sin\theta A_{\phi,\theta}}{\Sigma}\right)_{,\theta}\right]$$

$$\gamma = -\left(\frac{\Sigma - 2Mr}{\Sigma\sin\theta}A_{\phi,r}\right)_{,r} - \left(\frac{\Sigma - 2Mr}{\Sigma\Delta\sin\theta}A_{\phi,\theta}\right)_{,\theta}.$$

Equation (3.14) is the relativistic generalization of the flat space equation derived by Scharlemann & Wagoner (1973). Given ω and B_T as functions of A_ϕ, any A_ϕ satisfying equation (3.14) and appropriate boundary conditions can be used to construct a self-consistent model of a force-free black hole magnetosphere.

At $\theta = \pi/2$, $A_{\phi,\theta}$ will be discontinuous and determined by the toroidal surface current in the disc. As the disc is assumed to be a very good conductor there is no electric field in the comoving frame, and so continuity of the tangential electric field identifies $\omega(A_\phi)$ with the angular velocity of the disc at the point where the field line A_ϕ crosses it.

On the surface of the hole the relevant boundary conditions (Znajek 1977) are that A_ϕ is finite and

$$B_T[A_\phi(r_+,\theta)] = \frac{\sin\theta[\omega(r_+^2 + a^2) - a]}{r_+^2 + a^2\cos^2\theta}A_{\phi,\theta}(r_+,\theta). \tag{3.15}$$

The choice of appropriate boundary conditions at infinity is somewhat more problematical. In the two examples presented in Sections 6 and 7 we match the fields to known flat space solutions at infinity. In so doing we avoid the important question of uniqueness that is as yet unresolved for the purely Newtonian problem. In the next section we demonstrate how these boundary conditions determine the electromagnetic angular velocity ω on field lines crossing the horizon.

4 Energy and angular momentum transport

For any stationary axisymmetric system we can define conserved flux vectors for energy and angular momentum about the axis of symmetry (*cf.* Damour 1975). Let $T^{\mu\nu}$ be the total energy-momentum tensor, and ξ^μ a Killing vector. Then from

$$T^{\mu\nu}_{;\nu} = 0 \tag{4.1}$$

and the Killing equation

$$\xi_{\mu;\nu} + \xi_{\nu;\mu} = 0$$

it follows that

$$(\xi_\mu T^{\mu\nu})_{;\nu} = 0. \tag{4.2}$$

For a force-free field equations (4.1) and (4.2) are satisfied by the electromagnetic part of the energy momentum tensor. Thus we define the conserved electromagnetic energy flux

$$\mathscr{E}^\mu = T^{\mu\nu}\chi_\nu = T^\mu_0$$

and angular momentum flux

$$\mathscr{L}^\mu = -T^{\mu\nu}\eta_\nu = -T^\mu_\phi$$

where χ^ν and η^ν are the timelike and axial Killing vectors with Boyer–Lindquist components $(1, 0, 0, 0)$ and $(0, 0, 0, 1)$ respectively. The poloidal components of $\mathscr{E}^\mu$ and $\mathscr{L}^\mu$ are given by

$$\mathscr{E}^r = -\omega\epsilon_0 A_{\phi,\theta} B_T/\Sigma \sin\theta, \quad \mathscr{E}^\theta = \omega\epsilon_0 A_{\phi,r} B_T/\Sigma \sin\theta \tag{4.3}$$

and

$$\mathscr{E}^r = \omega\mathscr{L}^r, \quad \mathscr{E}^\theta = \omega\mathscr{L}^\theta. \tag{4.4}$$

Equations (4.4) support the interpretation of ω as the electromagnetic angular velocity, and with (4.3) show that energy and angular momentum flow along the poloidal field surfaces. The direction of energy flow cannot reverse on any given field line unless the force-free condition breaks down. Therefore the natural 'radiation condition' at infinity requires energy to flow outwards on all field lines, including lines that cross the event horizon. A physical observer rotating at constant radius close to the horizon will in general see a Poynting flux of energy entering the hole, but he will also see a sufficiently strong flux of angular momentum leaving the hole to ensure that $\mathscr{E}^r \gtrsim 0$. Equations (3.15) and (4.3) show that at the event horizon

$$\mathscr{E}^r = \omega(\Omega_H - \omega)\left(\frac{A_{\phi,\theta}}{r_+^2 + a^2\cos^2\theta}\right)^2 (r_+^2 + a^2)\,\epsilon_0 \tag{4.5}$$

where $\Omega_H \equiv a/(r_+^2 + a^2)$ is the angular velocity of the hole (e.g. Misner *et al.* 1973). Hence $\mathscr{E}^r \gtrsim 0$ implies

$$0 \lesssim \omega \lesssim \Omega_H, \tag{4.6}$$

which with equation (4.4) gives

$$\mathscr{E}^r \lesssim \Omega_H \mathscr{L}^r. \tag{4.7}$$

Inequality (4.7) could have been derived using the classical limit of the Second Law of Black Hole Thermodynamics: the irreducible mass of the hole cannot decrease (e.g. Misner *et al.* 1973). Here it remains constant only when $\omega = \Omega_H$. We can define the efficiency of the energy extraction process to be

$$\epsilon = \text{Actual energy extracted/Maximum extractable energy} \tag{4.8}$$

when unit angular momentum is removed. The numerator is ω, by equation (4.4). (If ω varies from field line to field line we can either consider ϵ to be the efficiency of a given field line or integrate to obtain a mean ϵ, as is done in Section 7.) Let $J = aM$ be the angular momentum of the hole. The irreducible mass of the hole M_{ir} is given by

$$M_{ir} = \tfrac{1}{2}\sqrt{2Mr_+}$$

so that

$$M^2 = M_{ir}^2 + J^2/4M_{ir}^2. \tag{4.9}$$

If the hole gains an amount dJ of angular momentum in a reversible, maximally efficient process, then $M_{\rm ir}$ is constant and (4.9) gives

$$dM = \frac{J\,dJ}{4MM_{\rm ir}^2} = \Omega_{\rm H}\,dJ.$$

Thus the denominator in equation (4.8) is $\Omega_{\rm H}$ and

$$\epsilon = \omega/\Omega_{\rm H}. \tag{4.10}$$

$\epsilon = 1$ can be approached by making ω increase towards $\Omega_{\rm H}$, but perfect efficiency is never achieved because when

$\omega = \Omega_{\rm H}$, $\mathscr{E}^r = \mathscr{L}^r = B_T = 0$*.

Integrating equations (4.3) and (4.4), we find that the total rate of energy extraction from the hole (measured at infinity) is

$$\int_0^{2\pi} d\phi \int_0^{\pi} d\theta\, \mathscr{E}^r \Sigma \sin\theta = -4\pi\epsilon_0 \int_{A_{\rm p}}^{A_{\rm e}} dA_\phi B_T(A\phi)\,\omega(A\phi). \tag{4.11}$$

Similarly the total rate of angular momentum extraction is

$$-4\pi\epsilon_0 \int_{A_{\rm p}}^{A_{\rm e}} B_T(A\phi)\,dA\phi. \tag{4.12}$$

Some physical insight into this process can be obtained by means of the following mechanical analogy. Consider a solid disc of thermally conducting material rotating with angular velocity $\Omega_{\rm H} > 0$ and surrounded by a large annular ring of thermally insulating material with angular velocity ω. Let there be a frictional couple exerted by the disc on the ring proportional to their relative velocity, $C = K(\Omega_{\rm H} - \omega)$. Then unless $\omega = \Omega_{\rm H}$, heat will be generated which will raise the internal energy of the disc. Furthermore, the frictional couple will increase the rotational energy of the ring at a rate $K\omega(\Omega_{\rm H} - \omega)$ and decrease the rotational energy of the disc at a rate $K\Omega_{\rm H}(\Omega_{\rm H} - \omega)$. The surplus energy (which is of course positive) is the rate of heat production $K(\Omega_{\rm H} - \omega)^2$. The efficiency of mechanical energy transfer to the ring – consistent with the definition (4.8) – is clearly $\omega/\Omega_{\rm H}$ as long as $0 \leqslant \omega \leqslant \Omega_{\rm H}$. When $\omega > \Omega_{\rm H}$, the disc gains energy and angular momentum at the expense of the ring just as the hole's mass and angular momentum would be increased by the magnetic stresses if we did not impose a radiation condition at infinity. When $\omega < 0$, both the ring and the disc lose rotational energy, but there is a transfer of angular momentum from the disc to the ring and correspondingly for the case of the hole.

What then acts as an effective frictional force between the hole and the magnetic field?

* It is interesting that a black hole magnetosphere possesses two light surfaces (defined to be the loci where the speed of a particle moving purely toroidally with angular velocity $d\phi/dt = \omega$, satisfying (4.6), equals c). The outer light surface corresponds to the conventional pulsar light cylinder and physical particles must travel radially outwards beyond it. Within the inner light surface, whose existence can be attributed to the dragging of inertial frames and gravitational redshift, particles must travel radially inwards. (The inner surface is by definition the boundary of the ergosphere when $\omega = 0$.) Both surfaces are given by

$$\Delta(1 - a\omega \sin^2\theta)^2 = [\omega(r^2 + a^2) - a]^2 \sin^2\theta$$

(Znajek 1977). The spark gaps discussed in Section 2 must therefore lie between these two surfaces.

As we have assumed force-free conditions for $r > r_+$ this friction must act within the event horizon, and so by the 'no hair' theorems the detailed nature of this interaction cannot influence any exterior observable (e.g. Hawking & Ellis 1973). Nevertheless we can demonstrate that if (as seems likely) the electrical circuit is complete inside the hole, then precisely the correct amount of energy and angular momentum transfer takes place between matter and the electromagnetic field inside the horizon. The actual position of the region (X) where the magnetospheric current meets the disc current is irrelevant.

First there is a slight technical difficulty in that Boyer–Lindquist coordinates become singular at the horizon. This problem is caused by the t and ϕ coordinates, and if these are replaced by two new coordinates, as in the Kerr metric (e.g. Misner *et al.* 1973), we can continue through the horizon using the same coordinates r and θ. The previously defined Killing vectors χ^μ and η^μ can also be continued through the horizon, and so we can give covariant definitions of, for example,

$$A_0(r, \theta) = A_\mu \chi^\mu$$

and

$$A_\phi(r, \theta) = A_\mu \eta^\mu.$$

These scalars are identical to their Boyer–Lindquist namesakes outside the horizon and are well behaved inside.

A particle of rest-mass m and charge e has a generalized 4-momentum $p^\mu = mu^\mu - eA^\mu$. The energy p_0 and angular momentum $-p_\phi$ are constants of the particle's motion. (See e.g. Carter 1973.) These constants can be trivially split into mechanical and electromagnetic components proportional to m and e respectively.

When the field is force-free, the particles travel on surfaces of constant A_0 and A_ϕ. Inside the event horizon these surfaces must cross the equatorial plane at finite r (see Fig. 1). This implies that they must become spacelike and so the particles have to leave them, thus violating the force-free assumption and altering the electromagnetic contributions to the constants of motion. The mechanical contributions to those constants must undergo opposite changes. What is likely to happen is that the particles will be rapidly accelerated just before the field lines become null, with the emission of a large number of photons, but whatever the details, the energy and angular momentum must end up in 'mechanical' form travelling towards the singularity. We assume that at X, A_0 and A_ϕ take their axial values outside the event horizon. (This is done for simplicity and because it is actually true in the limiting case of a Schwarzschild hole when X is the singularity.) Whatever A_0 and A_ϕ really are at X, it can be shown, by inserting an extra closed circuit, that the result obtained below is unaltered. As the total current to X is zero, the total electromagnetic component of the energy and angular momentum flow is also zero. We calculate the rate of transfer of electromagnetic into mechanical energy within the event horizon, using as our time derivative $\chi^\mu(\partial/\partial x^\mu)$, which corresponds to time as measured at infinity. As the $\theta = 0$ axis is a line of constant A_0, the potential difference between a surface of constant A_ϕ in the force-free magnetosphere and X is – using equation (3.13)

$$\Delta A_0 = \int_{A_\mathrm{p}}^{A_\phi} dA'_\phi \, \omega(A'_\phi).$$

Using equations (3.10), (3.11) and (3.12) the rate of increase of mechanical energy inside

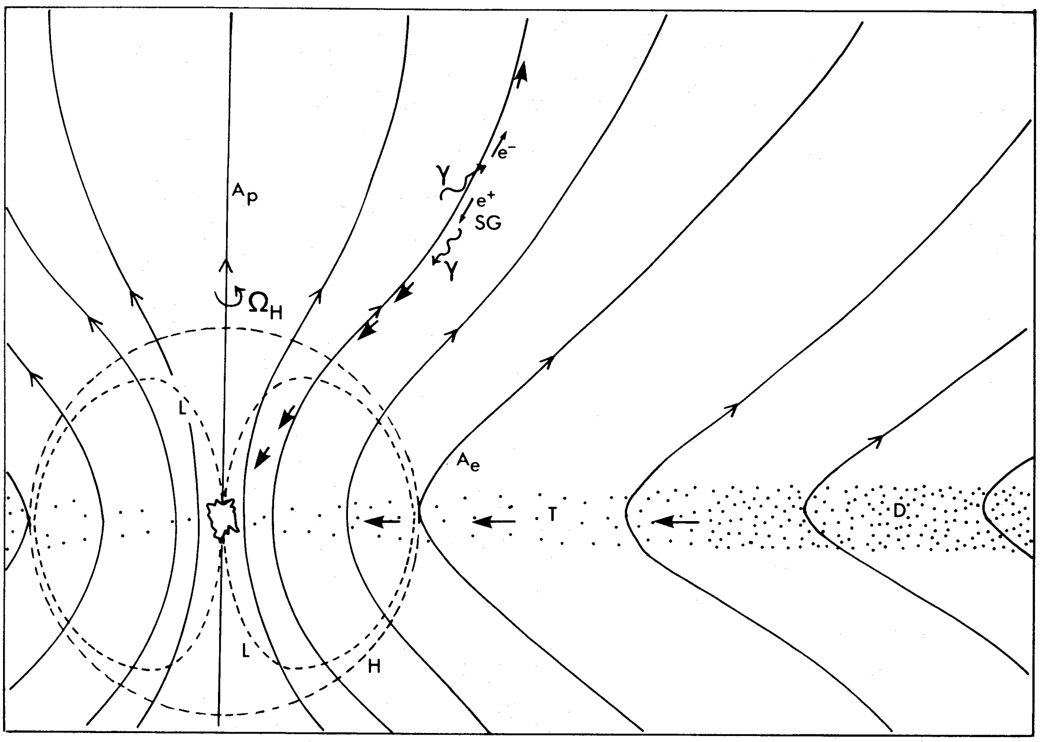

Figure 1. Schematic cross-section of black hole and magnetosphere, using r and θ coordinates in normal way. (Due to axial and time symmetry the diagram is independent of the azimuthal and time coordinates that are being held constant; these can be the Kerr coordinates v and $\tilde{\phi}$, or for $r > r_+$ the Boyer–Lindquist coordinates t and ϕ.) The poloidal field has been chosen so that $\mathbf{\Omega}_H \cdot \mathbf{B} > 0$. H is the event horizon $r = r_+$. The poloidal field surfaces (i.e. surfaces of constant A_ϕ) are shown as solid lines, with the polar and equatorial surfaces $A_\phi = A_p$ and $A_\phi = A_e$ specifically labelled. A current I is flowing from the magnetosphere into the hole, and back out of the hole into the disc D lying in the $\theta = \pi/2$ plane (denoted by heavy stippling). Particles can only cross the event horizon one way, into the hole. In the magnetosphere there are spark gaps like SG creating pairs of positrons e^+ and electrons e^-. Positrons are flowing into the hole along surfaces A_ϕ = const. at a faster rate than electrons, and there is a higher density of electrons (as the space charge has to be negative). Projections of typical particle velocities are shown by arrows. Particles can remain on the hypersurfaces of constant A_ϕ only as long as the normals to these surfaces are space-like. The locus where these surfaces become null is L. Between the disc and the hole there is a transition region T in which the matter is falling from the disc to the hole. This is shown by lighter stippling.

the hole due to the current from the magnetosphere is

$$-\int_{A_p}^{A_e} dA_\phi 4\pi\epsilon_0 \mu(A_\phi) \int_{A_p}^{A_\phi} dA'_\phi \omega(A'_\phi)$$

$$= -I \int_{A_p}^{A_e} dA_\phi \omega(A_\phi) + 4\pi\epsilon_0 \int_{A_p}^{A_e} dA_\phi B_T(A_\phi)\, \omega(A_\phi).$$

The first term is exactly cancelled by the work done on matter in the hole by the disc current. Thus the hole is losing energy electromagnetically at the rate

$$-4\pi\epsilon_0 \int_{A_p}^{A_e} dA_\phi B_T(A_\phi)\, \omega(A_\phi)$$

which is identical to the expression obtained by integrating $\mathscr{E}^r$ over the horizon. A similar result can be obtained for angular momentum.

Note that we have only considered the electromagnetic contributions to the hole's energy and angular momentum budgets. In practice, we also expect significant contributions from the material accreted from the disc.

5 Perturbation method

The only exact solution of equation (3.14) that we have been able to find is a trivial relativistic generalization of Michel's (1973) monopole solution in a Schwarzschild metric. However, it does not satisfy boundary condition (3.15) at the event horizon which is consistent with the notion that it is impossible to extract energy from a non-rotating black hole. (It could, however, be appropriate for a neutron star.) We therefore resort to a perturbative technique in which we expand in powers of the ratio a/M. Such a technique can only be of use when the change in the poloidal field caused by 'spinning up' a non-rotating field configuration (supported by currents in an equatorial disc) can be regarded as small. This cannot be used for a pulsar magnetosphere where there are closed field lines which have to be opened by the rotation. No such problem appears to arise for a slowly rotating hole surrounded by open field lines.

We require an exact axisymmetric vacuum solution for the magnetic field in a Schwarzschild metric given by $A_\phi(r, \theta) = X(r, \theta)$ and, in order that we can impose boundary conditions at infinity, an exact rotating force-free Newtonian solution. We anticipate that the electromagnetic angular frequency will be comparable to the angular frequency of the hole, and so write

$$\omega(r, \theta) = \frac{a}{M^2} W(r, \theta). \tag{5.1}$$

From equation (3.15)

$$B_T(r, \theta) = \frac{a}{M^2} Y(r, \theta) + O\left(\frac{a}{M}\right)^3. \tag{5.2}$$

Changing the sign of a does not alter the shape of the field lines and so we put

$$A_\phi(r, \theta) = X(r, \theta) + \frac{a^2}{M^2} x(r, \theta) + O\left(\frac{a}{M}\right)^4. \tag{5.3}$$

To the order we require ω, μ and B_T can be regarded as functions of X only. Substituting equations (5.1)–(5.3) into the fundamental differential equation (3.14) and expanding in powers of a, the terms $O(1)$ give the equation for the unperturbed potential $X(r, \theta)$ in the Schwarzschild metric

$$LX = 0, \tag{5.4}$$

where L is the self-adjoint partial differential operator

$$L = \frac{1}{\sin\theta} \frac{\partial}{\partial r} \left(1 - \frac{2M}{r}\right) \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial}{\partial \theta} \frac{1}{\sin\theta} \frac{\partial}{\partial \theta}.$$

There are no terms $O(a/M)$ and the terms $O(a/M)^2$ give the equation

$$Lx = S(r, \theta) \tag{5.5}$$

where the source function is given in terms of the known potential $X(r, \theta)$ and the functions $W(r,\theta)$, $Y(r,\theta)$ by

$$\begin{aligned}\sin\theta S(r,\theta) = &-\frac{M^2}{r^2} X_{,r,r} + \frac{2M^3}{r^4}(3\cos^2\theta - 1) X_{,r} - \frac{4M^3}{r^5}\frac{\sin\theta}{(1-2M/r)}(M\sin\theta X_{,r} - \cos\theta X_{,\theta})\\ &- 4MW\sin\theta\left[\sin\theta\left(\frac{X_{,r}}{r}\right)_{,r} + \frac{(\sin\theta X_{,\theta})_{,\theta}}{r^3(1-2M/r)}\right]\\ &+\frac{W^2\sin\theta}{M^2}\left[\sin\theta(r^2X_{,r})_{,r} + \frac{(\sin\theta X_{,\theta})_{,\theta}}{(1-2M/r)}\right]\\ &+\frac{W'}{M^2}\sin^2\theta\left[W - \frac{2M^3}{r}\right]\left[r^2(X_{,r})^2 + \frac{(X_{,\theta})^2}{(1-2M/r)}\right] - \frac{YY'}{[M^2(1-2M/r)]}.\end{aligned} \tag{5.6}$$

The prime denotes differentiation with respect to X. In deriving equation (5.6) use was made of (5.4). For $\theta \ll 1$, A_ϕ = const. + $O(\theta^2)$ and $Y = O(\theta^2)$. Hence $S = O(\theta)$. If the toroidal disc current in the perturbed situation is different from the unperturbed current that generates X, an appropriate term $\propto \delta(\cos\theta)$ should be added t0 S.

Discussion of the solution of equation (5.5) is deferred to the Appendix. Here it is sufficient to note that the solution exists if, and only if, the integral

$$\int_0^\pi \int_{2M}^\infty \frac{|S(r,\theta)|}{r}\, dr\, d\theta$$

converges. This requires that S be $o[1/(1-2M/r)]$ at the horizon $r = 2M$. This in turn imposes the condition

$$Y(2M, \theta) = (W - ¼) \sin\theta X_{,\theta} \tag{5.7}$$

which can be seen to be identical to the boundary condition (3.15) expanded to $O(a/M^2)$.

Equation (5.7) effectively fixes W as a function of Y, but Y itself must be determined as a function of X before the source function can be completely specified. We cannot guarantee that the solution of equation (5.6) with any particular choice of Y will correspond to a solution of the full non-linear equations. In the two examples that we present below, we are in possession of exact analytic solutions of the non-linear flat space equations. We believe it to be likely (but cannot prove) that whenever there exists a suitable flat space solution there also exists a Kerr metric solution approaching it at large distances. (The existence and uniqueness problem seems to be qualitatively unchanged from that in flat space (Scharlemann & Wagoner 1973) by the introduction of general relativity.) If we assume that this is true, then we can specify a second relation between W and Y, thus determining the efficiency, ϵ, and the source function, S. This second relation effectively fixes the current flowing through the hole for a given potential difference. Alternative electromagnetic conditions at infinity would yield an alternative relation.

6 Split monopole magnetic field

Our first example of the use of this perturbation technique is in calculating the effect of spinning up a radial magnetic field of opposite polarity in the two hemispheres. The unperturbed vector potential,

$$X(r,\theta) = -C\cos\theta, \quad 0 \leqslant \theta \leqslant \pi/2 \tag{6.1}$$

with C constant, is an exact solution of the vacuum Maxwell equations in a Schwarzschild metric, except on an equatorial disc containing a toroidal surface current density (as measured by a static observer)

$$I = \frac{2C\epsilon_0}{r^2}. \tag{6.2}$$

This is admittedly a somewhat artificial application as none of the field lines, defined by (6.1) actually cross the disc and furthermore the current given by equation (6.2) must extend right up to the horizon at $r = 2M$ despite the fact that stable circular orbits only exist up to $r = 6M$. However, there is no objection in principle to maintaining this current distribution, e.g. by using non-gravitational forces.

We now let the hole rotate slowly. The boundary condition at the horizon, (5.7), gives one equation relating Y and W

$$Y = C(W - \tfrac{1}{4})\sin^2\theta. \tag{6.3}$$

The exact solution for a rotating radial field in a flat spacetime has been given by Michel (1973) and it is to this that we choose to match our complete solution in the far field, well beyond the gravitational influence of the hole. A trivial generalization of Michel's solution to include the possibility that the angular velocity be variable gives a second equation

$$Y = -CW\sin^2\theta, \tag{6.4}$$

where we have used the boundary condition $Y \to 0$ as $\theta \to 0$. As θ, W and Y are constant along the unperturbed field lines, we can equate (6.3) and (6.4) to obtain

$$W = \tfrac{1}{8}$$

$$Y = -\tfrac{1}{8}C\sin^2\theta. \tag{6.5}$$

This means that the electromagnetic angular velocity ω is constant and equal to half the hole angular velocity Ω_H (to $O(a/M)$). The value of ω is effectively determined by the shape of the poloidal field lines rather than the geometry of spacetime in the vicinity of the hole. The efficiency of electromagnetic energy extraction, as defined by (4.10) is then 50 per cent.

If we substitute (6.1) and (6.5) into the source function, we obtain

$$S(r,\theta) = -\frac{CM}{r^3}\left(1 + \frac{2M}{r}\right)\sin\theta\cos\theta. \tag{6.6}$$

Only the $l = 1$ term of the summation in the Greens function (A2) contributes to the solution of equation (4.5), i.e. the equation is separable in r and θ. Straightforward but lengthy calculation yields the following expression for the perturbation $x(r,\theta)$.

$$x(r,\theta) = \frac{C}{4}\left(\frac{r}{M}\right)^2 \left[\left(2 - 6\left(\frac{M}{r}\right) + \frac{7}{6}\left(\frac{M}{r}\right)^2 + \frac{2}{9}\left(\frac{M}{r}\right)^3 + \left(\frac{1}{2} - \frac{M}{r}\right)\left(\frac{2r}{M} - 3\right)\ln\left(1 - \frac{2M}{r}\right)\right.\right.$$
$$+ \frac{1}{2}I\left(\frac{r}{2M}\right)\left(2\left(\frac{r}{M}\right) - 3\right) - \left(\left(\frac{r}{M}\right) - \frac{1}{2} - 3\left(\frac{M}{r}\right) + \frac{1}{2}\ln\left(\frac{r}{2M}\right)\right)$$
$$\left.\times\left(4 - 2\left(\frac{M}{r}\right) - \frac{2}{3}\left(\frac{M}{r}\right)^2 + \left(\frac{2r}{M} - 3\right)\ln\left(1 - 2\left(\frac{M}{r}\right)\right)\right)\right]\sin^2\theta\cos\theta$$

where

$$I(x) = \int_x^\infty \frac{dt}{t} \ln\left(\frac{t}{t-1}\right); \quad x \geqslant 1. \tag{6.7}$$

We have chosen that solution of the differential equation which is finite at the horizon $[x(2M, \theta) = C(\pi^2/12 - 49/72)\sin^2\theta\cos\theta]$ and falls off faster than $X(r, \theta)$ as $r \to \infty$:

$$x(r \gg M, \theta) \sim \frac{CM}{4r} \sin^2\theta \cos\theta.$$

We have found, correct to $O(a/M)^2$, a force-free electromagnetic solution in which electromagnetic energy is being extracted continuously from the rotating hole. Further electromagnetic properties of this solution can be derived using the results of Section 3. At distances well beyond the light cylinder $(r \gg M^2/a)$, the magnetic field is predominantly toroidal and becomes equal in magnitude to the (poloidal) electric field – see Michel (1973) and Fig. 2(a).

7 Paraboloidal magnetic field

In our second application of the perturbation equations, we generalize the paraboloidal magnetic field solution discussed in Blandford (1976). This is an exact Newtonian solution for a force-free magnetosphere in which the magnetic field lines lie on paraboloidal surfaces cutting an equatorial disc, rotating with angular velocity $\omega(r)$ and with surface current density $I = C\epsilon_0/r(1 + \omega^2 r^2)^{1/2}$.

Fortunately, we can also satisfy our second requirement as there exists a simple analytic solution for the magnetic field in a Schwarzschild metric when the surface current distribution $I = C\epsilon_0/r$, with C constant, extends down to the horizon. By direct substitution in the source-free Maxwell relations (5.4) we can confirm that the required solution is given by

$$X(r, \theta) = \frac{C}{2}\,[r(1 - \cos\theta) + 2M(1 + \cos\theta)(1 - \ln(1 + \cos\theta))]. \tag{7.1}$$

Imposing boundary condition (5.7) at the horizon, we obtain

$$Y(2M, \theta) = CM(W - \tfrac{1}{4})\sin^2\theta(1 + \ln(1 + \cos\theta)). \tag{7.2}$$

The vector potential for the far field Newtonian solution is

$$X(r, \theta) = \frac{C}{2} r(1 - \cos\theta) + 2CM(1 - \ln 2); \quad r \gg M \tag{7.3}$$

where the second term is a constant chosen so that the solutions match when $r \gg M$ and $\theta \to 0$. From Blandford (1976) we have for the second relation between Y and W,

$$Y = -CWr(1 - \cos\theta), \quad r \gg M. \tag{7.4}$$

Now Y and W are conserved on the field lines, given to sufficient accuracy by $X(r, \theta)$ = constant. Treating the value of θ at the horizon, θ_H as a parameter labelling the field lines, we obtain from (7.1)–(7.4)

$$W(\theta_H) = \frac{\tfrac{1}{4}\sin^2\theta_H[1 + \ln(1 + \cos\theta_H)]}{[4\ln 2 + \sin^2\theta_H + [\sin^2\theta_H - 2(1 + \cos\theta_H)]\ln(1 + \cos\theta_H)]} \tag{7.5}$$

which varies monotonically over the horizon from $W(0) = 0.125$ to $W(\pi/2) = 0.0663$.

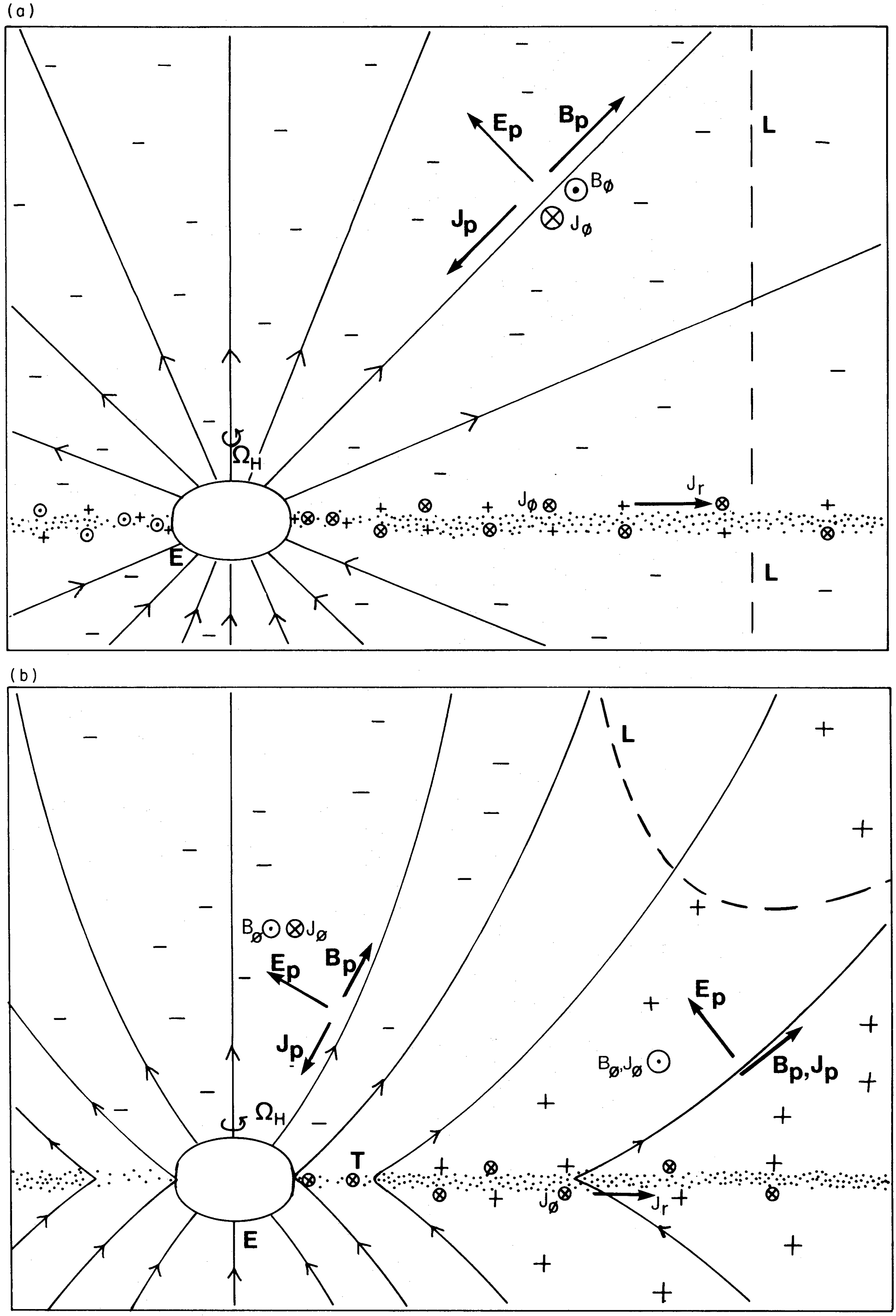

Figure 2. Electromagnetic structure of force-free magnetosphere with (a) radial and (b) paraboloidal magnetic fields. $\Omega_H \cdot \mathbf{B}$ is taken to be positive. Space charge, currents and non-zero field components shown are as seen by a static observer outside the ergosphere E. Physical observers travelling round the hole at constant r and θ and angular velocity $d\phi/dt$ will see the electric field reverse direction on the surface $d\phi/dt = \omega$. Inside this surface they see a Poynting flux of energy going towards the hole. (For a system of observers with time-like worldlines $d\phi/dt \rightarrow \Omega_H$ on the event horizon and $d\phi/dt \rightarrow 0$ at infinity. Hence when $0 < \omega < \Omega_H$, i.e. when the hole is losing energy electromagnetically, this surface always exists.) The discs are assumed to be Keplerian; the electromagnetic structure of the magnetosphere of the transition region, T, in (b) cannot be determined without additional assumptions. Outside this region Blandford's (1976) Newtonian solution applies. Note the difference in the shape of the light surface, L, in the two cases. For a paraboloidal field the energy appears to be focussed along the rotation axis.

We now know how W varies in the far field on those field lines that emerge from the hole. We can therefore calculate an average efficiency of energy extraction appropriate to this field geometry. If **S** is the Poynting flux well beyond the light surface and $d\Sigma$ the element of area, then the overall efficiency is given by

$$\bar{\epsilon} = \frac{\int d\Sigma \cdot \mathbf{S}}{\int d\Sigma \cdot (\mathbf{S}/4W)} \tag{7.6}$$

where the integrals are over all field lines crossing the horizon. A straightforward calculation using (4.11) gives $\bar{\epsilon} = 38$ per cent, somewhat less than the efficiency for the radial field. Further electromagnetic properties of this approximate solution can be calculated following Blandford (1976) – see also Fig. 2(b).

Unfortunately, this second example is also rather artificial because there is no natural way to match the black hole solution at $\theta_H = \pi/2$ to an equatorial disc solution. Unless there is a current sheet in the magnetosphere, W must be continuous. At large radii, W is presumably determined by the angular velocity of the disc which increases to $\sim (6\sqrt{6M})^{-1}$ at $r \sim 6M$. For $6M \gtrsim r \gtrsim 2M$, there would have to be a transition region within which the angular velocity changes smoothly to $0.27\,\Omega_H$. This could in principle be achieved using non-gravitational (especially electromagnetic) forces, although it would involve dynamical considerations totally absent from the present treatment. In particular any fluid velocity would have to have a substantial radial component at the horizon, because the angular velocity of a purely circular orbit must approach Ω_H there. As most of the gravitational energy of the infalling material is actually liberated within this transition region we cannot obtain a reliable estimate of the efficiency of electromagnetic energy extraction from the disc. However, there is no reason to suppose that given a similar field geometry to that just described, the efficiency of removal of the hole's rotational energy should be seriously different from 38 per cent and that the total efficiency for the disc cannot exceed that associated with the last stable circular orbit (6–42 per cent as a increases from 0 to M – Bardeen (1970)).

8 Black holes in active galactic nuclei

We have demonstrated the existence of an astrophysically efficient mechanism for extracting rotational energy from a Kerr hole: electromagnetic braking resulting from fields supported by external currents. This process has been exhibited in the slow rotation limit for two basic field geometries that occur when the magnetic fields and Poynting fluxes are either radial or parallel to the rotation axis at infinity. The simplest application of these ideas is to a model of an active galactic nucleus containing a massive black hole surrounded by an accretion disc (e.g. Lynden-Bell 1969; Lynden-Bell & Rees 1971). As the length scales associated with the observations of nuclear activity get shorter (e.g. Epstein *et al.* 1972; Kellermann 1974; Martin, Angel & Maza 1976) the idea that the central gravitational machine be a black hole becomes increasingly attractive. One light day is only 100 Schwarzschild radii for a $10^8 M_\odot$ black hole, which is more or less the minimum mass necessary to supply the energy requirements of quasars and double radio sources. The argument against the alternative type of model involving a cluster of stellar mass objects (e.g. Arons, Kulsrud & Ostriker 1975) which associates each outburst with a supernovae-like event is that many outbursts are either too rapid and individually too feeble, or too energetic (*cf.* Fabian *et al.* 1976). No characteristic 'quantum' of energy $\sim 10^{52}$ erg appears to be involved. Nevertheless the above theory can also be applied to a cluster of stellar mass holes.

There are two well-known difficulties associated with the black hole hypothesis. Firstly in spite of the gravitational potential well being as deep as possible, a hole may turn out to be a rather inefficient emitter of non-thermal and high-frequency radiation. It is quite possible that most of the liberated potential energy of accreted matter may be swallowed by the hole as thermal energy, kinetic energy or trapped radiation. Alternatively, in the case of a viscous disc, the effective temperature may be too low for the production of X-rays or relativistic particles. This is discussed in Fabian *et al.* (1976) where some methods for improving the efficiency are described.

The second problem involves inverse Compton losses. If, as is generally believed, the non-thermal radio and optical emission from quasars is synchrotron radiation by relativistic electrons in an ordered magnetic field, then as originally pointed out by Hoyle, Burbidge & Sargent (1966), the radiation energy density must not greatly exceed the field energy density. This imposes a serious lower limit on both the field strength and, for the radio emission, on the size of the source. The implication is that the synchrotron emitting electrons cannot be accelerated in the vicinity of the event horizon as their radiative losses would then be far too large.

Both these objections can be met in the present model by the device of liberating the gravitational energy from the region near the hole using a large-scale Poynting flux. As discussed in Blandford (1976), inertial effects may dominate beyond the light surface, leading to a relativistic electromagnetic wind. The energy in this form is usable in the sense that it can, for example, accelerate relativistic particles behind a distant shock located where the momentum flux in the wind balances the ambient pressure in the nucleus. This can avoid serious Compton losses because the total energies of particles in the wind can be much less than those inferred to be responsible for the non-thermal emission. (In this respect the present model is similar to a pulsar radiating strong waves, for example as in Arons *et al.* (1975).) We might expect that if there were a fairly abrupt change in the field arrangement close to the horizon (e.g. arising from some field reconnection in the transition region) the electromagnetic luminosity would increase very abruptly leading to the possibility of rapid variability (*cf.* Pringle, Rees & Pacholczyk 1973; Shields & Wheeler 1976). The final conversion to non-thermal radiation can be very efficient if the cooling time of the emitting relativistic electrons is sufficiently short (*cf.* Blandford & McKee 1976).

Variability and VLB observations of nuclear radio sources have in many cases to be interpreted in terms of apparent 'super-luminal' expansion speeds if the radio emission is incoherent electron synchrotron radiation (e.g. Burbidge, Jones & O'Dell 1974; Cohen *et al.* 1976). Such rapid expansion can be explained in terms of light travel time effects (Rees 1967) involving either relativistic material motion of the emitting plasma or emission by a stationary plasma in response to a signal travelling at the speed of light. Both possibilities can occur with a time-varying relativistic wind. In addition the double structure characteristic of many VLB radio observations arises naturally in this interpretation as the poloidal magnetic field lines which determine the direction of the Poynting flux are probably curved towards the direction of the rotation axis. Alternatively, we can further exploit the analogy with pulsars. As the electromagnetic conditions around the hole and inner disc only differ by scale from those believed to be present in a pulsar magnetosphere, it would not be surprising if coherent curvature radiation were emitted (*cf.* Cocke & Pacholczyk 1975). It may be possible to discriminate between these alternatives by means of polarization observations.

The overall efficiency of electromagnetic energy extraction from a disc around a black hole is difficult to calculate with any precision. Neither of the electromagnetic solutions presented in the last two sections have been matched satisfactorily to solutions in regions where the magnetic field lines are attached to the disc. Within the transition region, separating the horizon from the disc, the efficiency of energy extraction depends critically on the dynamical behaviour of the accreting material and this is where the greatest uncertainty in the physics lies. Attempts have been made (e.g. Novikov & Thorne 1974 and Stoeger 1976) to understand these problems, but such investigations may be irrelevant in the present context because most of the radial stress on fluid elements in this region could be magnetic (Znajek 1976).

As discussed by Blandford (1976), if the magnetic field strength in the magnetosphere is to be stationary, then there must be an effective finite conductivity that enables the accreting material to cross the field lines. Perhaps the most likely way of achieving this is through magnetic reconnection occurring on sufficiently small scales for it to appear as an effective electrical resistance rather than as a global instability. The 'ohmic' dissipation associated with this reconnection is a fraction $\sim |V_r c/V_\phi^2|$ of the rate of liberation of gravitational energy, V_r and V_ϕ being the average radial and tangential fluid velocities of the disc. It is quite likely that this fraction, small in the outer parts of the disc, becomes of order unity in the transition region. Furthermore, even in the outer parts of the disc viscous transport of angular momentum could be dominant (as is conventionally assumed). The ratio of electromgnetic to viscous torques cannot be determined prior to obtaining an adequate understanding of various dissipative processes that occur in the disc.

If we assume that the disc is electromagnetic then we can attempt to compare the Poynting fluxes radiated by the hole and the disc beyond the last stable circular orbit. For $a \ll M$ and a paraboloidal field, the power radiated by the hole, L_{H}, satisfies,

$$L_{\mathrm{H}} \sim 0.3(a/M)^2 L_{\mathrm{D}} \sim 10^{38}(a/M)^2 \dot{M}\ \mathrm{W}$$

where L_{D} is the power radiated by the disc and $\dot{M}$ the mass accretion rate measured in $M_\odot$/yr. (We have estimated the ratio $L_{\mathrm{H}}/L_{\mathrm{D}}$ using the results of Section 7.) Even when $a \sim M$ we do not expect L_{H} to exceed L_{D} and so if the disc is completely electromagnetic we can probably ignore the hole's contribution to the power. For self-consistency, we must confirm that the field strength is large enough to break down the vacuum near the event horizon. The field strength, B is related to L_{H} by

$$B \sim 0.2 \left(\frac{L_{\mathrm{H}}}{10^{38}\ \mathrm{W}}\right)^{1/2} \left(\frac{M}{a}\right) \left(\frac{M}{10^9 M_\odot}\right)^{-1} \mathrm{T}$$

using equation (4.11). For a hole of mass in the range 10^8–$10^{10} M_\odot$ supplying $L_{\mathrm{H}} \gtrsim 10^{38}$ W of electromagnetic power to a galactic nucleus, the field strength must exceed 0.01 T and from (2.8) we see that this is certainly adequate for breakdown.

If, alternatively, the torque in the disc is viscous then the gravitational energy of the accreting material in the disc will be mainly radiated away. (In the transition region most of this energy will probably be swallowed by the hole.) Energy liberated in this way is unlikely to be transformable into the non-thermal emission characteristic of active nuclei and we must rely on the electromagnetic power from the hole. If a fraction $0.1 f_1$ of the rest mass of the accreting material is radiated from the disc, then the rate of mass accretion is restricted to the Salpeter (1964) limit:

$$\frac{M}{\dot{M}} \gtrsim 4.5 \times 10^7 f_{-1}\ \mathrm{yr}$$

assuming electron scattering.

In conclusion, we have discussed a mechanism for extracting rotational energy from a Kerr hole. The presence of the three principle ingredients: angular momentum, a magnetic field and a massive black hole, seems difficult to avoid within a galactic nucleus. It appears that this could be an effective agency for transforming gravitational energy into non-thermal radiation.

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Appendix

Green's function for Schwarzschild equation

Petterson (1974) obtained an expression for the vector potential of a current loop around a Schwarzschild black hole at $r = r_0$, $\theta = \pi/2$. One can generalize his result to the case of a current loop at $r = r_0$, $\theta = \theta_0$. This tells us that the solution of the differential equation

$$LG = \delta(r - r_0)\,\delta(\theta - \theta_0) \tag{A1}$$

where L is the self-adjoint operator

$$L \equiv \frac{1}{\sin\theta}\frac{\partial}{\partial r}\left(1 - \frac{2M}{r}\right)\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}$$

with boundary conditions such that G is finite at $r = 2M$ and tends to zero as $r \to \infty$ is

$$G(r, \theta; r_0, \theta_0) = \sum_{l=0}^{\infty} \frac{(l + 3/2)}{(l+1)(l+2)} \sin\theta \sin\theta_0\, P'_{l+1}(\cos\theta)\, P'_{l+1}(\cos\theta_0)\, R_l(r, r_0). \tag{A2}$$

Here

$$R_l(r, r_0) = \begin{cases} U_l(r)\, V_l(r_0), & 2M \leqslant r \leqslant r_0 \\ V_l(r)\, U_l(r_0), & r \geqslant r_0 \geqslant 2M \end{cases}$$

$$U_l(r) = (2M)^l\, r^2 P_l^{(2,0)}(1 - r/M) \tag{A3}$$

$P_l^{(2,0)}$ being a Jacobi polynomial, and

$$V_l(r) = U_l(r) \int_r^{\infty} \frac{r'\, dr'}{(2M - r')[U_l(r')]^2}. \tag{A4}$$

From (A1) we can write down the solution to $Lx = S$ that falls to zero as $r \to \infty$ and is non-singular at the event horizon:

$$x(r, \theta) = \int_{2M}^{\infty} dr_0 \int_0^{\pi} d\theta_0 G(r, \theta; r_0, \theta_0)\, S(r_0, \theta_0). \tag{A5}$$

G is logarithmically singular at $(r, \theta) = (r_0, \theta_0)$, so we cannot use uniform convergence of the series for G to prove that (A5) converges. However, using a property of the roots of Jacobi polynomials (e.g. Abramowitz & Stegun 1964) equation (A3) can be rewritten as

$$U_l(r) = (-1)^l \binom{2l+2}{l} r^2 (r - \alpha_1) \ldots (r - \alpha_l)$$

where $0 < \alpha_i < 2M$. It can then be shown that

$$|R_l(r, r_0)| = |R_l(r_0, r)| < \frac{r^2}{r_0 - 2M}\left(\frac{r}{r_0}\right)^l \frac{1}{2l+2}, \quad (r \leqslant r_0),$$

and we then have sufficient information about R_l to be able to use the dominated convergence theorem (e.g. Kingman & Taylor 1966) to prove that if $S(r, \theta)$ is continuous in $r > 2M$, then (A5) converges – and one can interchange the summation and integration in (A5) – if, and only if,

$$\int_{2M}^{\infty} dr \int_0^{\pi} d\theta \, \frac{|S(r, \theta)|}{r}$$

converges. In the two cases treated in Sections 6 and 7, convergence at the horizon is ensured by the boundary condition (3.15).