Electromagnetic Field in Lyra Manifold: A First Order Approach

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(Received on 23 October, 2005)

We discuss the coupling of the electromagnetic field with a curved and torsioned Lyra manifold using the Duffin-Kemmer-Petiau theory. We will show how to obtain the equations of motion and energy-momentum and spin density tensors by means of the Schwinger Variational Principle.

I. INTRODUCTION

First order Lagrangians are one of the most profitable tools in Field Theory. By means of first order approach, Hamiltonian dynamics becomes more transparent, constrained systems can be dealt with a wide range of methods [1], and CPT and spin-statistics theorems can be proved by variational statements [2].

Otherwise, the coupling between electromagnetism and the torsion content of spacetime has been an intringuing puzzle for many years. Minimal coupling of the Einstein-Cartan gravity with eletromagnetism breaks local gauge covariance by the presence of the torsion interaction [3–5].

Here, we want to add another piece to the puzzle, showing that the torsion coupling problem is related to scale invariance which we will model together with the gravitational field by means of the Lyra geometry. Electromagnetic field will be described by the first order approach of Duffin-Kemmer-Petiau (DKP).

II. THE LYRA GEOMETRY

The Lyra manifold [6] is defined giving a tensor metric $g_{\mu\nu}$ and a positive definite scalar function ϕ which we call the scale function. In Lyra geometry, one can change scale and coordinate system in an independent way, to compose what is called a *reference system* transformation: let $M \subseteq \mathbb{R}^N$ and U an open ball in $\mathbb{R}^n$, $(N \geq n)$ and let $\chi : U \curvearrowright M$. The pair (χ, U) defines a *coordinate system*. Now, we define a reference system by (χ, U, ϕ) where ϕ transforms like

$$\overline{\phi}(\overline{x}) = \overline{\phi}(x(\overline{x}); \phi(x(\overline{x}))) \quad , \quad \frac{\partial\overline{\phi}}{\partial\phi} \neq 0$$

under a reference system transformation.

In the Lyra's manifold, vectors transform as

$$\overline{A}^{\nu} = \frac{\overline{\phi}}{\phi}\frac{\partial\overline{x}^{\nu}}{\partial x^{\mu}}A^{\mu}$$

In this geometry, the affine connection is

$$\tilde{\Gamma}^{\rho}{}_{\mu\nu} = \frac{1}{\phi}\mathring{\Gamma}^{\rho}{}_{\mu\nu} + \frac{1}{\phi}\left[\delta^{\rho}_{\mu}\partial_{\nu}\ln\left(\frac{\phi}{\phi}\right) - g_{\mu\nu}g^{\rho\sigma}\partial_{\sigma}\ln\left(\frac{\phi}{\phi}\right)\right]$$

whose transformation law is given by

$$\tilde{\Gamma}^{\rho}{}_{\mu\nu} = \frac{\overline{\phi}}{\phi}\overline{\Gamma}^{\sigma}{}_{\lambda\varepsilon}\frac{\partial x^{\rho}}{\partial\overline{x}^{\sigma}}\frac{\partial\overline{x}^{\lambda}}{\partial x^{\mu}}\frac{\partial\overline{x}^{\varepsilon}}{\partial x^{\nu}} + \frac{1}{\phi}\frac{\partial x^{\rho}}{\partial\overline{x}^{\sigma}}\frac{\partial^2\overline{x}^{\sigma}}{\partial x^{\mu}\partial x^{\nu}} + \\ + \frac{1}{\phi}\delta^{\rho}_{\nu}\frac{\partial}{\partial x^{\mu}}\ln\left(\frac{\overline{\phi}}{\phi}\right).$$

One can define the covariant derivative for a vector field as

$$\nabla_{\mu}A^{\nu} = \frac{1}{\phi}\partial_{\mu}A^{\nu} + \tilde{\Gamma}^{\nu}{}_{\mu\alpha}A^{\alpha}, \quad \nabla_{\mu}A_{\nu} = \frac{1}{\phi}\partial_{\mu}A_{\nu} - \tilde{\Gamma}^{\alpha}{}_{\mu\nu}A_{\alpha}.$$

We use the notation $\Gamma^{\nu}{}_{[\alpha\mu]} = \frac{1}{2}\left(\Gamma^{\nu}{}_{\alpha\mu} - \Gamma^{\nu}{}_{\mu\alpha}\right)$ for the antisymmetric part of the connection and $\mathring{\Gamma}^{\rho}_{\mu\nu} \equiv \frac{1}{2}g^{\rho\sigma}\left(\partial_{\mu}g_{\nu\sigma} + \partial_{\nu}g_{\sigma\mu} - \partial_{\sigma}g_{\mu\nu}\right)$ for the analogous of the Levi-Civita connection.

The richness of the Lyra's geometry is demonstrated by the *curvature* [7]

$$\tilde{R}^{\rho}_{\beta\alpha\sigma} \equiv \frac{1}{\phi^2}\left[\partial_{\beta}\left(\phi\tilde{\Gamma}^{\rho}_{\alpha\sigma}\right) - \partial_{\alpha}\left(\phi\tilde{\Gamma}^{\rho}_{\beta\sigma}\right)\right] + \\ + \frac{1}{\phi^2}\left[\phi\tilde{\Gamma}^{\rho}_{\beta\lambda}\phi\tilde{\Gamma}^{\lambda}_{\alpha\sigma} - \phi\tilde{\Gamma}^{\rho}_{\alpha\lambda}\phi\tilde{\Gamma}^{\lambda}_{\beta\sigma}\right]$$

and the *torsion*

$$\tilde{\tau}_{\mu\nu}{}^{\rho} = -\frac{2}{\phi}\delta^{\rho}_{[\mu}\partial_{\nu]}\ln\overline{\phi} \tag{1}$$

which has intrinsic link with the scale functions and whose trace is given by

$$\tilde{\tau}_{\mu\rho}{}^{\rho} \equiv \tilde{\tau}_{\mu} = \frac{3}{\phi}\partial_{\mu}\ln\overline{\phi}. \tag{2}$$

In the next section we introduce the behavior of massless DKP field in the Lyra geometry.

III. THE MASSLESS DKP FIELD IN LYRA MANIFOLD

DKP theory describes in a unified way the spin 0 and 1 fields [8–10]. The massless DKP theory can not be obtained as a zero mass limit of the massive DKP case, so we consider the Harish-Chandra Lagrangian density for the massless DKP theory in the Minkowski space-time $\mathcal{M}^4$, given by [11]

$$\mathcal{L}_{\mathcal{M}} = i\overline{\psi}\gamma\beta^a\partial_a\psi - i\partial_a\overline{\psi}\beta^a\gamma\psi - \overline{\psi}\gamma\psi, \tag{3}$$

where the β^a matrices satisfy the usual DKP algebra

$$\beta^a\beta^b\beta^c+\beta^c\beta^b\beta^a=\beta^a\eta^{bc}+\beta^c\eta^{ba}$$

and γ is a *singular* matrix satisfying [16]

$$\beta^a\gamma+\gamma\beta^a=\beta^a \qquad , \qquad \gamma^2=\gamma.$$

From the above lagrangian it follows the massless DKP wave equation

$$i\beta^a\partial_a\psi-\gamma\psi=0\ .$$

As it was known, the Minkowskian Lagrangian density (3) in its massless spin 1 sector reproduces the electromagnetic or Maxwell theory with its respective $U(1)$ local gauge symmetry.

To construct the covariant derivative of massless DKP field in Lyra geometry, we follow the standard procedure of analyzing the behavior of the field under local Lorentz transformations,

$$\psi(x)\rightarrow\psi\prime(x)=U(x)\psi(x) \tag{4}$$

where U is a spin representation of Lorentz group characterizing the DKP field. Now we define a *spin connection* S_μ in a such way that the object

$$\nabla_\mu\psi\equiv\frac{1}{\phi}\partial_\mu\psi+S_\mu\psi \tag{5}$$

transforms like a DKP field in (4), thus, we set

$$\nabla_\mu\psi\rightarrow(\nabla_\mu\psi)'=U(x)\nabla_\mu\psi$$

and therefore S transforms like

$$S'_\mu=U(x)S_\mu U^{-1}(x)-\frac{1}{\phi}(\partial_\mu U)U^{-1}(x) \tag{6}$$

From the covariant derivative of the DKP field (5) and remembering that $\bar{\psi}\psi$ must be a scalar under the transformation (4), it follows that $\nabla_\mu\bar{\psi}=\frac{1}{\phi}\partial_\mu\bar{\psi}-\bar{\psi}S_\mu$. Using the covariant derivative of the DKP current

$$\begin{aligned}\nabla_\mu(\bar{\psi}\beta^\nu\psi)&=\frac{1}{\phi}\partial_\mu(\bar{\psi}\beta^\nu\psi)+\Gamma^\nu{}_{\mu\lambda}\left(\bar{\psi}\beta^\lambda\psi\right)=\\&=(\nabla_\mu\bar{\psi})\beta^\nu\psi+\bar{\psi}(\nabla_\mu\beta^\nu)\psi+\bar{\psi}\beta^\nu(\nabla_\mu\psi)\end{aligned}$$

one gets the following expression for the covariant derivative of β^ν

$$\nabla_\mu\beta^\nu=\frac{1}{\phi}\partial_\mu\beta^\nu+\Gamma^\nu{}_{\mu\lambda}\beta^\lambda+S_\mu\beta^\nu-\beta^\nu S_\mu$$

A particular solution to this equation is given by

$$S_\mu=\frac{1}{2}\omega_{\mu ab}S^{ab}\ ,\ S^{ab}=\left[\beta^a,\beta^b\right].$$

With a covariant derivative of the DKP field well-defined we can consider the Lagrangian density (3) of the massless DKP field minimally coupled [3, 12, 13] to the Lyra manifold, introducing the tetrad field,

$$g^{\mu\nu}(x)=\eta^{ab}\,e^\mu{}_a(x)e^\nu{}_b(x)\ ,\ g_{\mu\nu}(x)=\eta_{ab}e_\mu{}^a(x)e_\nu{}^b(x)\ .$$

$$S=\int_\Omega d^4x\,\phi^4e\,\left(i\bar{\psi}\gamma e^\mu_a\beta^a\nabla_\mu\psi-i\nabla_\mu\bar{\psi}\beta^a e^\mu_a\gamma\psi-\bar{\psi}\gamma\psi\right)\ . \tag{7}$$

where ∇_μ is the Lyra covariant derivative of DKP field defined above.

IV. EQUATIONS OF MOTION AND THE DESCRIPTION OF MATTER CONTENT

In following we use a classical version of the Schwinger Action Principle such as it was treated in the context of Classical Mechanics by Sudarshan and Mukunda [14]. The Schwinger Action Principle is the most general version of the usual variational principles. It was proposed originally at the scope of the Quantum Field Theory [2], but its application goes beyond this area. Here, we will apply the Action Principle to derive equations of motion of the Dirac field in an external Lyra background and expression for the energy-momentum and spin density tensors.

Thus, making the variation of the action integral (7) we get

$$\begin{aligned}\delta S=&\int_\Omega dx\,e\phi^4\left[4\mathcal{L}-\frac{i}{\phi}\bar{\psi}\gamma\beta^\mu\partial_\mu\psi+\frac{i}{\phi}\partial_\mu\bar{\psi}\beta^\mu\gamma\psi\right]\left(\frac{\delta\phi}{\phi}\right)+\\&+\int_\Omega dx\,\phi^4e\left(\frac{\delta e}{e}\right)\mathcal{L}+\\&+\int_\Omega dx\,e\phi^4\left[i\bar{\psi}\gamma(\delta\beta^\mu)\nabla_\mu\psi-i\nabla_\mu\bar{\psi}(\delta\beta^\mu)\gamma\psi\right]+\\&+\int_\Omega dx\,e\phi^4\left[i\bar{\psi}\gamma\beta^\mu(\delta S_\mu)\psi+i\bar{\psi}(\delta S_\mu)\beta^\mu\gamma\psi\right]+\\&+\int_\Omega dx\,e\phi^4\,\delta\bar{\psi}(i\gamma\beta^\mu\nabla_\mu\psi-\gamma\psi+iS_\mu\beta^\mu\gamma\psi)+\\&+\int_\Omega dx\,e\phi^4\left[\frac{i}{\phi}\bar{\psi}\gamma\beta^\mu(\delta\partial_\mu\psi)-\frac{i}{\phi}(\delta\partial_\mu\bar{\psi})\beta^\mu\gamma\psi\right]\\&-\int_\Omega dx\,e\phi^4\,(i\nabla_\mu\bar{\psi}\beta^\mu\gamma+\bar{\psi}\gamma-i\bar{\psi}\gamma\beta^\mu S_\mu)\delta\psi\end{aligned} \tag{8}$$

Choosing different specializations of the variations, one can easily obtain the equations of motion and the energy-momentum and spin density tensor.

A. Equations of Motion

We choose to make functional variations only in the massless DKP field thus we set $\delta\phi=\delta e^\mu{}_b=\delta\omega_{\mu ab}=0$ and considering $[\delta,\partial_\mu]=0$, from (8) we get

$$\begin{aligned}\delta S=&\int_{\partial\Omega}d\sigma_\mu\,e\phi^3\,i\left[\bar{\psi}\gamma\beta^\mu(\delta\psi)-(\delta\bar{\psi})\beta^\mu\gamma\psi\right]+\\&+i\int_\Omega dx\,e\phi^4(\delta\bar{\psi})\left[i\beta^\mu\nabla_\mu\psi+i\tilde{\tau}_\mu\beta^\mu\gamma\psi-\gamma\psi\right]+\\&-\int_\Omega dx\,e\phi^4\left[i\nabla_\mu\bar{\psi}\beta^\mu+i\tilde{\tau}_\mu\bar{\psi}\gamma\beta^\mu+\bar{\psi}\gamma\right]\delta\psi\end{aligned}$$

Following the action principle we get the generator of the variations of the massless DKP field

$$G_{\delta\psi} = \int_{\partial\Omega} d\sigma_\mu \, e\phi^3 \, i\left[\bar{\psi}\gamma\beta^\mu(\delta\psi) - (\delta\bar{\psi})\beta^\mu\gamma\psi\right]$$

and its equations of motion in the Lyra's manifold are

$$\begin{cases} i\beta^\mu(\nabla_\mu + \tilde{\tau}_\mu\gamma)\psi - \gamma\psi = 0 \\ i\nabla_\mu\bar{\psi}\beta^\mu + i\tilde{\tau}_\mu\bar{\psi}\gamma\beta^\mu + \bar{\psi}\gamma = 0 \end{cases}$$

The spin 1 projectors $R^\mu(= e^\mu{}_a R^a\ )$ and $R^{\mu\nu}(= e^\mu{}_a e^\nu{}_b R^{ab})$ [10, 15] are such that $R^\mu\psi$ and $R^{\mu\nu}\psi$ transform respectively as a vector and a second rank tensor under general coordinate transformation. Thus, using the projectors we have

$$R^\mu \quad \rightarrow \quad i\nabla_\nu(R^{\mu\nu}\psi) + i\tilde{\tau}_\nu(R^{\mu\nu}\gamma\psi) - R^\mu\gamma\psi = 0$$

multiplying by $(1-\gamma)$ we get

$$i(\nabla_\nu + \tilde{\tau}_\nu)(R^{\mu\nu}\gamma\psi) = 0$$

and

$$R^{\mu\nu} \quad \rightarrow \quad i\nabla_\rho(R^{\mu\nu}\beta^\rho\psi) + i\tilde{\tau}_\rho(R^{\mu\nu}\beta^\rho\gamma\psi) - R^{\mu\nu}\gamma\psi = 0$$

$$R^{\mu\nu}\gamma\psi = i\left(\nabla_\rho + \tilde{\tau}_\rho\gamma\right)\left[g^{\rho\nu}(R^\mu\psi) - g^{\rho\mu}(R^\nu\psi)\right]$$

from the above equations we get the equation of motion for the massless vector field $R^\mu\psi$

$$(\nabla_\nu + \tilde{\tau}_\nu)(\nabla_\rho + \tilde{\tau}_\rho\gamma)\left[g^{\rho\nu}(R^\mu\psi) - g^{\rho\mu}(R^\nu\psi)\right] = 0\,,$$

We use a specific representation of the DKP algebra in which the singular γ matrix is

$$\gamma = \mathrm{diag}(0,0,0,0,1,1,1,1,1,1)\ .$$

Then in this representation the DKP field ψ is now a 10-component column vector

$$\psi = \left(\ \psi^0, \psi^1, \psi^2, \psi^3, \psi^{23}, \psi^{31}, \psi^{12}, \psi^{10}, \psi^{20}, \psi^{30}\ \right)^T\ ,$$

where ψ^a $(a = 0,1,2,3)$ and ψ^{ab} behave, respectively, as a 4-vector and an antisymmetric tensor under *Lorentz* transformations on the Minkowski tangent space. And we also get

$$\gamma\psi = \left(\ 0,0,0,0,\psi^{23},\psi^{31},\psi^{12},\psi^{10},\psi^{20},\psi^{30}\ \right)^T$$
$$R^\mu\psi = \left(\ \psi^\mu,0,0,0,0,0,0,0,0,0\ \right)^T$$
$$R^{\mu\nu}\psi = \left(\ \psi^{\mu\nu},0,0,0,0,0,0,0,0,0\ \right)^T$$

due to $R^\mu\gamma = \gamma R^\mu$ and $R^{\mu\nu}\gamma = (1-\gamma)R^{\mu\nu}$. Then, we get the following relations among ψ components

$$i\,\psi_{\mu\nu} = \nabla_\mu\psi_\nu - \nabla_\nu\psi_\mu$$

which leads to the equation of motion for the spin 1 sector of the massless DKP field in Lyra space-time

$$(\nabla_\mu + \tilde{\tau}_\mu)(\nabla^\mu\psi^\nu - \nabla^\nu\psi^\mu) = 0\,.$$

B. Energy-momentum tensor and spin tensor density

Now, we only vary the background manifold and we assume that $\delta\omega_{\mu ab}$ and $\delta e^\mu{}_a$ are independent variations,

$$\delta S = \int_\Omega dx e\phi^4 \left[i\left(\bar{\psi}\gamma\beta^a\nabla_\mu\psi - \nabla_\mu\bar{\psi}\beta^a\gamma\psi\right)\delta e^\mu{}_a + \right.$$
$$\left. + \left(\frac{1}{e}\delta e\right)\mathcal{L} + i\left(\bar{\psi}\gamma\beta^\mu S^{ab}\psi + \bar{\psi}S^{ab}\beta^\mu\gamma\psi\right)\frac{1}{2}\delta\omega_{\mu ab}\right].$$

First, holding only the variations in the tetrad field, $\delta\omega_{\mu ab} = 0$, we found for the variation of the action

$$\delta S = \int_\Omega dx\, e\phi^4 \left[\, i\left(\bar{\psi}\gamma\beta^a\nabla_\mu\psi - \nabla_\mu\bar{\psi}\beta^a\gamma\psi\right) - e_\mu{}^a\mathcal{L}\right]\delta e^\mu{}_a$$

Defining the energy-momentum density tensor as

$$T_\mu{}^a \equiv \frac{1}{\phi^4 e}\frac{\delta S}{\delta e^\mu{}_a} = i\bar{\psi}\gamma\beta^a\nabla_\mu\psi - i\nabla_\mu\bar{\psi}\beta^a\gamma\psi - e_\mu{}^a\mathcal{L}$$

which can be written in coordinates as

$$T_\mu{}^\nu \equiv e^\nu{}_a T_\mu{}^a = i\bar{\psi}\gamma\beta^\nu\nabla_\mu\psi - i\nabla_\mu\bar{\psi}\beta^\nu\gamma\psi - \delta_\mu{}^\nu\mathcal{L}$$

On the mass shell,

$$T_\mu{}^\nu = i\bar{\psi}\gamma\beta^\nu\nabla_\mu\psi - i\nabla_\mu\bar{\psi}\beta^\nu\gamma\psi - \delta_\mu{}^\nu\bar{\psi}\gamma\psi$$

Now, making functional variations only in the components of the spin connection, $\delta e^\mu{}_a = 0$, we found for the action variation

$$\delta S = \int_\Omega dx\, e\phi^4\, \frac{1}{2}\left(\delta\omega_{\mu ab}\right) i\bar{\psi}\left(\gamma\beta^\mu S^{ab} + S^{ab}\beta^\mu\gamma\right)\psi,$$

we define the spin tensor density as being

$$S^{\mu ab} \equiv \frac{2}{\phi^4 e}\frac{\delta S}{\delta\omega_{\mu ab}} = i\bar{\psi}\left(\gamma\beta^\mu S^{ab} + S^{ab}\beta^\mu\gamma\right)\psi$$

The spin 1 component of DKP energy momentum tensor is

$$\begin{aligned} T_\mu{}^\nu \;=\; & \frac{i}{2}\psi^{*\nu\alpha}\left(\nabla_\mu\psi_\alpha - \nabla_\alpha\psi_\mu\right) + \\ & -\frac{i}{2}\psi^{\nu\beta}\left(\nabla_\mu\psi^*_\beta - \nabla_\beta\psi^*_\mu\right) + \\ & -\delta_\mu{}^\nu\left(\psi^{*\alpha\beta}\psi_{\alpha\beta}\right) \end{aligned}$$

which coincides with the first order energy momentum tensor of the electromagnetic field in the real case.

V. FINAL REMARKS

The coupling between torsion and massless vectorial field was showed to be related to scale transformations in Lyra background. Since this scale transformations are governed by an arbitrary function ϕ, it seems plausible that the problem of breaking the local gauge invariance associated with this coupling could be removed from the theory if we had chosen an gauge transformations to be linked to scale invariance in Lyra manifold. A deeper study of this line is under construction.

Acknowledgments

This work is supported by FAPESP grants 01/12611-7 (RC), 01/12584-0 (CAMM) and 02/00222-9 (BMP). BMP also thanks CNPq for partial support.

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