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Abstract

This report contains a detailed analytical and numerical study of the singularities of the electromagnetic field at the tip of a perfectly conducting, elliptic cone. The analytical treatment is concerned with the solution of the Helmholtz equation by separation of variables in sphero-conal coordinates. The solution is in terms of periodic and non-periodic Lamé functions, and the underlying theory of these special functions is amply discussed. The analysis shows that three basic field singularities must be considered at small distance r to the tip of the cone: one electric singularity and two magnetic singularities, in which the electric field or the magnetic field becomes infinite like $r^{\nu-1}$ as $r \to 0$, for some specific singularity exponents $\nu < 1$. Numerical results for the singularity exponents are presented and these are compared with data from the literature.

Key words. Electromagnetic field singularities, Lamé functions, elliptic cone, sphero-conal coordinates, Sturm-Liouville eigenvalue problem.

AMS (MOS) subject classifications. 78A25, 33A55

Dedication

This report is dedicated to our teacher and friend C.J. Bouwkamp, on the occasion of his golden doctor's jubilee on January 23, 1991.

Contents

1. Introduction

It is well known that electromagnetic fields become singular at edges and vertices of perfectly conducting obstacles. A precise knowledge of the field singularities is important, particularly for numerical field calculations. By requiring that the numerical solution behaves in the known singular manner, the convergence and accuracy of the numerical results will improve. This report provides a detailed analytical and numerical study of the singularities of the electromagnetic field at the tip of a perfectly conducting, elliptic cone. The analytical part involves the solution of the Helmholtz equation outside the cone, by separation of variables in sphero-conal coordinates; the solution obtained is in terms of Lamé functions. To make the presentation self-contained, the necessary theory of Lamé functions is developed in Secs. 2-6, whereby most of the material has been adopted from Jansen [14] and Van Vught [24]. The actual determination of the field singularities at the tip of the elliptic cone is discussed in Sec. 7. Here, analytical and numerical results are presented for the so-called singularity exponents, and these are compared with data from the literature. The present report originated from some correspondence with Prof. J. van Bladel (Gent, Belgium) regarding his forthcoming monograph [3]. Singularities at the tip of a cone are treated in [3, Chapter 5] and our numerical data for the elliptic cone will be included there.

We now present a more detailed survey of the contents of this report. In Sec. 2 we introduce the sphero-conal coordinates r, θ, ϕ, represented in trigonometric form. In these coordinates the elliptic cone coincides with a coordinate surface $\theta = \theta_0$, where θ_0 is constant. The coordinate r measures distance to the tip of the cone. The trigonometric form of sphero-conal coordinates which goes back to Kraus [15], is more easy to handle than the classical representation in terms of Jacobian elliptic functions. Sec. 3 deals with the solution of the scalar Helmholtz equation in the space domain bounded by the elliptic cone, subject to either a Dirichlet or a Neumann boundary condition on the cone surface. The pertaining boundary value problems are solved by separation of variables in sphero-conal coordinates. The solution is given by the product of a spherical Bessel function and two Lamé functions. The latter functions are to be determined as solutions of a two-parameter eigenvalue problem, with eigenvalue parameters λ and ν, for the coupled θ-Lamé equation and ϕ-Lamé equation. Both equations are trigonometric forms of Lamé's differential equation. These trigonometric forms which arise due to the trigonometric representation of the sphero-conal coordinates, are less complicated than the traditional Jacobian form of Lamé's differential equation. In the next two sections the solutions of the θ- and ϕ-Lamé equations are determined for a fixed parameter ν.

In Sec. 4 we consider the one-parameter eigenvalue problem for the ϕ-Lamé equation with eigenvalue parameter λ, and fixed ν. By means of the classical Sturm-Liouville theory it is shown that the problem admits an infinite number of eigenvalues, denoted by $\lambda = \lambda^m_{c\nu}$ and $\lambda = \lambda^m_{s\nu}$, $m = (0), 1, 2, \ldots$, with corresponding eigenfunctions denoted by $L^{(m)}_{c\nu}(\phi)$ and $L^{(m)}_{s\nu}(\phi)$, respectively. The eigenfunctions are periodic with period 2π and are therefore called periodic Lamé functions. The Lamé functions $L^{(m)}_{c\nu}(\phi)$ and $L^{(m)}_{s\nu}(\phi)$ are represented by Fourier series involving cosines/sines of even/odd multiples of ϕ. The eigenvalues $\lambda^m_{c\nu}$, $\lambda^m_{s\nu}$, and the coefficients in the Fourier series are to be determined from the eigenvalues and eigenvectors of an infinite tridiagonal matrix. In Sec. 5 we construct the solutions of the θ-Lamé equation with fixed ν, and $\lambda = \lambda^m_{c\nu}$ or $\lambda = \lambda^m_{s\nu}$, as determined in the previous section. These solutions are called non-periodic Lamé functions and are denoted by $L^{(m)}_{cp\nu}(\theta)$ and $L^{(m)}_{sp\nu}(\theta)$, respectively. The latter functions are represented by series involving Legendre functions $P^r_\nu(\cos\theta)$, $r = 0, 1, 2, \ldots$. It is shown that the coefficients in the Legendre-function series are simply related to the coefficients in the Fourier series for the functions $L^{(m)}_{c\nu}(\phi)$ and $L^{(m)}_{s\nu}(\phi)$. In this manner, both the periodic and the non-periodic Lamé functions are completely determined for given parameter ν. Finally, by imposing the Dirichlet or the Neumann boundary condition on the elliptic cone $\theta = \theta_0$, we arrive at the transcendental equations $L^{(m)}_{cp\nu}(\theta_0) = 0$, $L^{(m)}_{sp\nu}(\theta_0) = 0$ or $(d/d\theta_0)L^{(m)}_{cp\nu}(\theta_0) = 0$, $(d/d\theta_0)L^{(m)}_{sp\nu}(\theta_0) = 0$ in the parameter ν. In Sec. 6 it is proved that all four equations have an infinite number of positive ν-roots, which are denoted by νc^m_n, νs^m_n and $\dot{\nu} c^m_n$, $\dot{\nu} s^m_n$, respectively, where $n = 1, 2, 3, \ldots$. Moreover, we establish by analytical means that the only ν-roots less than 1, are given by νc^0_1, $\dot{\nu} c^1_1$ and $\dot{\nu} s^1_1$; the details of the derivation are presented in Appendix B.

In Sec. 7 we discuss the singularities of the electromagnetic field at the tip of a perfectly conducting, elliptic cone. The electromagnetic field in the space domain outside the cone is represented in terms of two Debye potentials. These scalar potentials must satisfy the Helmholtz equation, and are subject to either a Dirichlet or a Neumann boundary condition on the cone surface. These boundary value problems for the potentials were already treated in Secs. 3-5. Thus it is found that the Debye potentials are given by products $j_\nu(k_0 r)L^{(m)}_{cp\nu}(\theta)L^{(m)}_{c\nu}(\phi)$ and $j_\nu(k_0 r)L^{(m)}_{sp\nu}(\theta)L^{(m)}_{s\nu}(\phi)$, with $\nu = \nu c^m_n$ and $\nu = \nu s^m_n$ (in case of a Dirichlet boundary condition), or with $\nu = \dot{\nu} c^m_n$ and $\nu = \dot{\nu} s^m_n$ (in case of a Neumann boundary condition). The associated electric or magnetic field behaves like $r^{\nu-1}$ as $r \to 0$; hence, field singularities at the tip will occur only if $\nu < 1$. We conclude that the following basic singularities must be considered for the electromagnetic field at the tip of the elliptic cone:

(i) one electric singularity, in which the electric field becomes infinite like $r^{\nu-1}$ as $r \to 0$, with singularity exponent $\nu = \nu c^0_1$;

(ii) two magnetic singularities, in which the magnetic field becomes infinite like $r^{\nu-1}$ as $r \to 0$, with singularity exponents $\nu = \dot{\nu} c^1_1$ and $\nu = \dot{\nu} s^1_1$.

Numerical results are presented for the singularity exponents as functions of the cone parameters (maximum semi-opening angle, and axial ratio of the elliptic cross-section), and these are compared with data from the literature. The special cases of a circular cone and of a plane sector (flattened elliptic cone) are included as well.

2. Sphero-conal coordinates

Following Jansen [14], we employ sphero-conal coordinates r, θ, ϕ, represented in trigonometric form. These coordinates are connected with Cartesian coordinates $\mathbf{x} = (x, y, z)$ by means of the relations

$$x = r \sin\theta \cos\phi ,$$

$$y = r\sqrt{1 - k^2\cos^2\theta}\ \sin\phi, \tag{2.1}$$

$$z = r \cos\theta \sqrt{1 - k'^2\sin^2\phi},$$

where

$$r \geq 0, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \phi \leq 2\pi \ ;$$

$$0 \leq k \leq 1, \quad 0 \leq k' \leq 1, \quad k^2 + k'^2 = 1 \ . \tag{2.2}$$

Notice that for $k = 1$ the sphero-conal coordinates reduce to the well-known spherical coordinates. It is easily verified that the sphero-conal coordinates r, θ, ϕ form a right-handed orthogonal coordinate system. The metrical coefficients of this coordinate system are given by

$$h_r = \left|\frac{\partial \mathbf{x}}{\partial r}\right| = 1,$$

$$h_\theta = \left|\frac{\partial \mathbf{x}}{\partial \theta}\right| = r\frac{\sqrt{k^2\sin^2\theta + k'^2\cos^2\phi}}{\sqrt{1 - k^2\cos^2\theta}}, \tag{2.3}$$

$$h_\phi = \left|\frac{\partial \mathbf{x}}{\partial \phi}\right| = r\frac{\sqrt{k^2\sin^2\theta + k'^2\cos^2\phi}}{\sqrt{1 - k'^2\sin^2\phi}} .$$

These coefficients appear in the expressions for grad, div, curl and the Laplacian Δ in sphero-conal coordinates. For example, the Laplacian Δ is found to be

$$\Delta = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\Delta_t \ , \tag{2.4}$$

where the transverse Laplacian Δ_t is given by

$$\Delta_t = \frac{1}{k^2 \sin^2\theta + k'^2 \cos^2\phi} \left[\sqrt{1-k^2\cos^2\theta}\,\frac{\partial}{\partial\theta}\left(\sqrt{1-k^2\cos^2\theta}\,\frac{\partial}{\partial\theta}\right) + \sqrt{1-k'^2\sin^2\phi}\,\frac{\partial}{\partial\phi}\left(\sqrt{1-k'^2\sin^2\phi}\,\frac{\partial}{\partial\phi}\right)\right] . \tag{2.5}$$

Next we examine the coordinate surfaces r, θ or ϕ = constant. As derived from (2.1) by elimination, these coordinate surfaces are described by the following equations:

$$r = r_0 \; : \; x^2 + y^2 + z^2 = r_0^2 \; ; \tag{2.6}$$

$$\theta = \theta_0 \; : \; \frac{x^2}{\sin^2\theta_0} + \frac{k^2 y^2}{1-k^2\cos^2\theta_0} = \frac{z^2}{\cos^2\theta_0}, \quad \mathrm{sgn}(z) = \mathrm{sgn}(\cos\theta_0),\; \theta_0 \neq 0,\; \pi/2, \pi \; ; \tag{2.7}$$

$$\phi = \phi_0 \; : \; \frac{x^2}{\cos^2\phi_0} + \frac{k'^2 z^2}{1-k'^2\sin^2\phi_0} = \frac{y^2}{\sin^2\phi_0}, \; \mathrm{sgn}(x) = \mathrm{sgn}(\cos\phi_0),\; \mathrm{sgn}(y) = \mathrm{sgn}(\sin\phi_0), \; \phi_0 \neq 0,\; \pi/2,\; \pi,\; 3\pi/2,\; 2\pi \; . \tag{2.8}$$

Clearly, the surface $r = r_0$ is a sphere of radius r_0 with center at the origin; the coordinate r measures distance to the origin.

The surface $\theta = \theta_0$, described by (2.7), is a semi-infinite elliptic cone with its tip at the origin and its axis along the positive (negative) z-axis if $\cos\theta_0 > 0$ ($\cos\theta_0 < 0$); see Fig. 2.1. The cross-section of the cone with the plane $z = z_0$, where $\mathrm{sgn}(z_0) = \mathrm{sgn}(\cos\theta_0)$, is an ellipse described by the equation

$$\frac{x^2}{z_0^2\tan^2\theta_0} + \frac{k^2\cos^2\theta_0\; y^2}{z_0^2(1-k^2\cos^2\theta_0)} = 1 \; . \tag{2.9}$$

From (2.9) it follows that the ellipse has a major axis of length $2a$ in the (y, z)-plane, and a minor axis of length $2b$ in the (x, z)-plane, where

$$a = |z_0| \frac{\sqrt{1-k^2\cos^2\theta_0}}{k|\cos\theta_0|} \; , \quad b = |z_0 \tan\theta_0| \; . \tag{2.10}$$

The axial ratio of the ellipse is

$$(2.11)\qquad \varepsilon = \frac{b}{a} = \frac{k\sin\theta_0}{\sqrt{1-k^2\cos^2\theta_0}} \ .$$

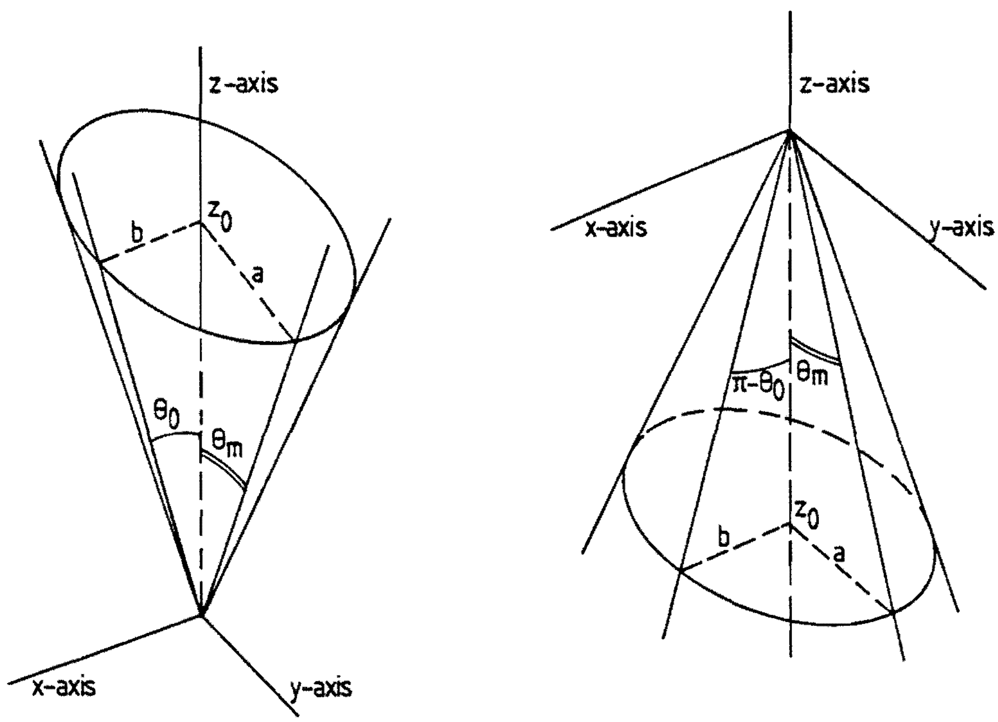

Fig. 2.1. Elliptic cone $\theta = \theta_0$, with $0 < \theta_0 < \pi/2$ (left), $\pi/2 < \theta_0 < \pi$ (right).

For $k = 1$, one has $\varepsilon = 1$ and the elliptic cone becomes a circular cone. The cone $\theta = \theta_0$ has a semi-opening angle equal to $\min(\theta_0, \pi - \theta_0)$ in the (x, z)-plane, whereas in the (y, z)-plane the semi-opening angle is given by (cf. Fig. 2.1)

$$(2.12)\qquad \theta_m = \arcsin(\sqrt{1-k^2\cos^2\theta_0}) \ .$$

For $\theta = \pi/2$ the relations (2.1) simplify to

$$(2.13)\qquad x = r\cos\phi, \quad y = r\sin\phi, \quad z = 0 \ .$$

Hence, the surface $\theta = \pi/2$ coincides with the (x, y)-plane, in which r, ϕ appear as polar coordinates.

For $\theta = 0$ or $\theta = \pi$ the relations (2.1) simplify to

$$(2.14) \quad x = 0, \quad y = k'r \sin\phi, \quad z = \pm r\sqrt{1 - k'^2 \sin^2\phi}\ .$$

The coordinate surfaces $\theta = 0$ and $\theta = \pi$ are sectors in the (y, z)-plane, described by

$$(2.15) \quad \begin{aligned} \theta = 0 &: \ x = 0, \ |y| \le (k'/k)z, \ z > 0\ ; \\ \theta = \pi &: \ x = 0, \ |y| \le -(k'/k)z, \ z < 0\ . \end{aligned}$$

These sectors have their tip at the origin and their axes along the positive (negative) z-axis in case $\theta = 0$ ($\theta = \pi$), while the semi-opening angle is $\theta_m = \arcsin k'$; see Fig. 2.2. The two sides of each sector are to be considered as a degenerate (flattened) elliptic cone.

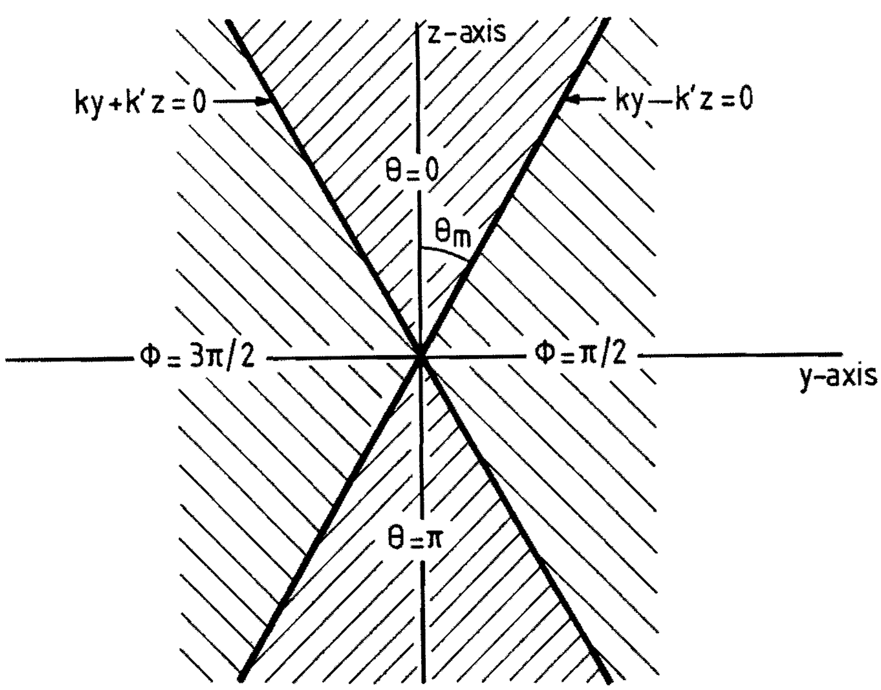

Fig. 2.2. Sectors $\theta = 0$, $\theta = \pi$, and $\phi = \pi/2$, $\phi = 3\pi/2$, in the (y, z)-plane.

Although not used any further, we briefly disuss the coordinate surface $\phi = \phi_0$, described by (2.8). Because of the restrictions $\mathrm{sgn}(x) = \mathrm{sgn}(\cos\phi_0)$, $\mathrm{sgn}(y) = \mathrm{sgn}(\sin\phi_0)$, the surface $\phi = \phi_0$ is one half of a semi-infinite elliptic cone with its tip at the origin and its axis along the positive (negative) y-axis if $\sin\phi_0 > 0$ ($\sin\phi_0 < 0$); see Fig. 2.3.

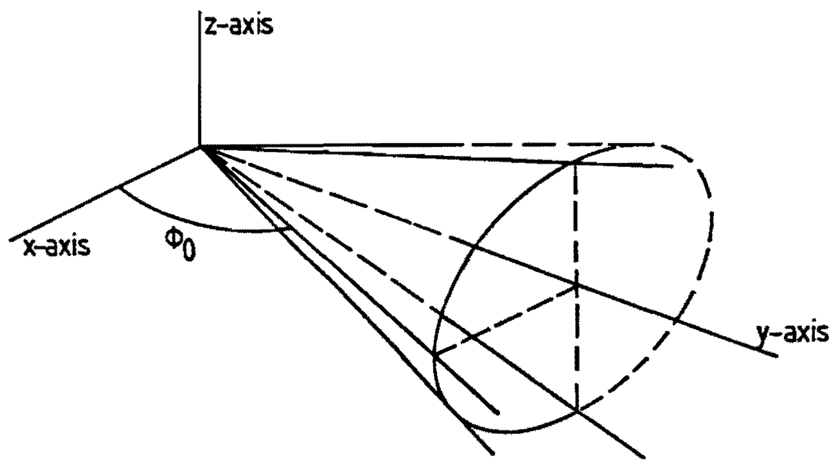

Fig. 2.3. Half elliptic cone $\phi = \phi_0$, with $0 < \phi_0 < \pi/2$.

It is easily seen that the surfaces $\phi = 0$ or $\phi = 2\pi$, and $\phi = \pi$ coincide with the half-planes $x > 0,\ y = 0$ and $x < 0,\ y = 0$, respectively. For $\phi = \pi/2$ or $\phi = 3\pi/2$ the relations (2.1) simplify to

$$(2.16) \qquad x = 0, \quad y = \pm r\sqrt{1 - k^2\cos^2\theta}, \qquad z = kr\cos\theta\ .$$

Hence, the surfaces $\phi = \pi/2$ and $\phi = 3\pi/2$ are sectors in the (y, z)-plane, described by

$$(2.17) \qquad \begin{aligned} \phi = \pi/2 &: \quad x = 0,\ y > 0,\ |z| \leq (k/k')\,y\ ; \\ \phi = 3\pi/2 &: \quad x = 0,\ y < 0,\ |z| \leq -(k/k')\,y\ . \end{aligned}$$

These sectors are complementary to the sectors $\theta = 0$ and $\theta = \pi$ given by (2.15), see also Fig. 2.2.

Finally, we investigate the one-to-one correspondence between the Cartesian coordinates $\mathbf{x} = (x, y, z)$ and the sphero-conal coordinates r, θ, ϕ. The relations (2.1) define a mapping which is shortly written as $\mathbf{x} = \mathbf{x}(r, \theta, \phi)$. This mapping is indeed one-to-one apart from the following exceptions:

$$(2.18) \qquad \mathbf{x}(r, \theta, 0) \ = \ \mathbf{x}(r, \theta, 2\pi)\ , \quad 0 \leq \theta \leq \pi,$$

because the function $\mathbf{x}(r, \theta, \phi)$ is periodic in ϕ with period 2π;

$$
(2.19)\quad
\begin{aligned}
\mathbf{x}(r,0,\phi) &= \mathbf{x}(r,0,\pi-\phi), \quad \mathbf{x}(r,\pi,\phi) = \mathbf{x}(r,\pi,\pi-\phi), \quad 0\le\phi\le\pi,\\
\mathbf{x}(r,0,\phi) &= \mathbf{x}(r,0,3\pi-\phi), \quad \mathbf{x}(r,\pi,\phi) = \mathbf{x}(r,\pi,3\pi-\phi), \quad \pi\le\phi\le 2\pi,
\end{aligned}
$$

because each point $\mathbf{x}$ in the two-sided sectors $\theta = 0$ and $\theta = \pi$ is described by two triples of sphero-conal coordinates.

Under the mapping (2.1), any single-valued function $f(\mathbf{x}) = f(x,y,z) \in C^1$ passes into the composite function $f(\mathbf{x}(r,\theta,\phi)) =: g(r,\theta,\phi) \in C^1$. In view of (2.18) and (2.19), the function $g(r,\theta,\phi)$ has the following additional properties:

$$
(2.20)\quad g(r,\theta,0)=g(r,\theta,2\pi),\ \ \frac{\partial g}{\partial\phi}(r,\theta,0)=\frac{\partial g}{\partial\phi}(r,\theta,2\pi),\ \ 0\le\theta\le\pi;
$$

$$
(2.21)\quad
\begin{aligned}
&g(r,0,\phi)=g(r,0,\pi-\phi),\ \ \frac{\partial g}{\partial\theta}(r,0,\phi)=-\,\frac{\partial g}{\partial\theta}(r,0,\pi-\phi),\ \ 0\le\phi\le\pi,\\
&g(r,0,\phi)=g(r,0,3\pi-\phi),\ \ \frac{\partial g}{\partial\theta}(r,0,\phi)=-\,\frac{\partial g}{\partial\theta}(r,0,3\pi-\phi),\ \ \pi\le\phi\le 2\pi;
\end{aligned}
$$

$$
(2.22)\quad
\begin{aligned}
&g(r,\pi,\phi)=g(r,\pi,\pi-\phi),\ \ \frac{\partial g}{\partial\theta}(r,\pi,\phi)=-\,\frac{\partial g}{\partial\theta}(r,\pi,\pi-\phi),\ \ 0\le\phi\le\pi,\\
&g(r,\pi,\phi)=g(r,\pi,3\pi-\phi),\ \ \frac{\partial g}{\partial\theta}(r,\pi,\phi)=-\,\frac{\partial g}{\partial\theta}(r,\pi,3\pi-\phi),\ \ \pi\le\phi\le 2\pi.
\end{aligned}
$$

Conversely, let $g(r,\theta,\phi) \in C^1$, then the relations (2.20)-(2.22) are necessary and sufficient conditions in order that $g(r,\theta,\phi) = f(\mathbf{x}(r,\theta,\phi))$ for some single-valued $f(\mathbf{x}) \in C^1$. In the next section the conditions (2.20)-(2.22) come up as regularity conditions on the solution of the Helmholtz equation in sphero-conal coordinates. These regularity conditions should be considered as additional boundary conditions due to the use of sphero-conal coordinates.

3. Solution of the Helmholtz equation in sphero-conal coordinates

In Sec. 7 it is shown that the singularities of the electromagnetic field at the tip of an elliptic cone are determined via solution of the Helmholtz equation in the space domain bounded by the elliptic cone. The required solution of the Helmholtz equation is discussed in the present section. Let D be the domain that is bounded by the elliptic cone ∂D. In sphero-conal coordinates r, θ, ϕ, as introduced in (2.1), the domain D and its boundary ∂D are described by

$$
\begin{aligned}
&D = \{(r, \theta, \phi)|\ r > 0,\ \ 0 \le \theta < \theta_0,\ \ 0 \le \phi \le 2\pi\}, \\
&\partial D = \{(r, \theta_0, \phi)|\ r \ge 0,\ \ 0 \le \phi \le 2\pi\},
\end{aligned}
\tag{3.1}
$$

where $0 < \theta_0 < \pi$. In the domain D we consider the scalar Helmholtz equation

$$\Delta u + k_0^2 u = 0, \quad \mathbf{x} \in D, \tag{3.2}$$

with either the Dirichlet boundary condition

$$u = 0, \quad \mathbf{x} \in \partial D, \tag{3.3}$$

or the Neumann boundary condition

$$\frac{\partial u}{\partial n} = 0, \quad \mathbf{x} \in \partial D\ . \tag{3.4}$$

Here, k_0 is the wave number and $\mathbf{n}$ denotes the outward unit normal to ∂D. The present boundary value problems are solved by separation of variables in two steps. First we split off the r-dependence and search for solutions of the form

$$u(r, \theta, \phi) = R(r)\ w(\theta, \phi)\ . \tag{3.5}$$

Substitute (3.5) into (3.2) and replace the Laplacian Δ by (2.4), then by separation we find

$$\left[\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) + k_0^2 r^2 R(r)\right] \Big/ R(r) = -\frac{\Delta_t w(\theta, \phi)}{w(\theta, \phi)} = \mu, \tag{3.6}$$

where μ is the separation constant and Δ_t denotes the transverse Laplacian. Thus we have for $R(r)$ the differential equation

$$(3.7)\qquad \frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + (k_0^2 r^2 - \mu)R = 0, \quad r > 0 .$$

From (3.6), (3.3) and (3.4) it follows that $w(\theta,\phi)$ is solution of the following problem:

$$(3.8)\qquad \begin{aligned} &\Delta_t w + \mu w = 0, \quad 0 \le \theta < \theta_0, \quad 0 \le \phi \le 2\pi, \\ &w(\theta_0,\phi) = 0 \quad \text{or} \quad \frac{\partial w}{\partial \theta}(\theta_0,\phi) = 0, \quad 0 \le \phi \le 2\pi . \end{aligned}$$

In addition, as a consequence of (2.20) and (2.21), $w(\theta,\phi)$ must satisfy the regularity conditions

$$(3.9)\qquad w(\theta,0) = w(\theta,2\pi), \ \frac{\partial w}{\partial \phi}(\theta,0) = \frac{\partial w}{\partial \phi}(\theta,2\pi), \quad 0 \le \theta \le \theta_0;$$

$$(3.10)\qquad \begin{aligned} &w(0,\phi) = w(0,\pi-\phi), \ \frac{\partial w}{\partial \theta}(0,\phi) = -\frac{\partial w}{\partial \theta}(0,\pi-\phi), \quad 0 \le \phi \le \pi, \\ &w(0,\phi) = w(0,3\pi-\phi), \ \frac{\partial w}{\partial \theta}(0,\phi) = -\frac{\partial w}{\partial \theta}(0,3\pi-\phi), \quad \pi \le \phi \le 2\pi . \end{aligned}$$

Notice that the conditions (2.22) need not be imposed, because the sector $\theta = \pi$ lies outside the domain D of interest.

The problem (3.8) for $w(\theta,\phi)$ is in fact an eigenvalue problem for the transverse Laplacian Δ_t (also called Beltrami operator) on the domain $\Omega = \{(\theta,\phi) | 0 \le \theta < \theta_0, \ 0 \le \phi \le 2\pi\}$ that is cut out by the elliptic cone ∂D in the unit sphere $r = 1$. For specific values of μ, called eigenvalues, the problem (3.8) has a non-trivial solution $w \not\equiv 0$, which is the corresponding eigenfunction. Jansen [14, p. 24, Thm. 2.1] has proved the following theorem on the eigenvalues and eigenfunctions of problem (3.8).

THEOREM 3.1. The eigenvalue problem (3.8) with boundary condition either $w(\theta_0, \phi) = 0$ or $\partial w / \partial\theta(\theta_0, \phi) = 0$, admits an infinite sequence of eigenvalues μ_n with $\mu_n \to \infty$ as $n \to \infty$, and arranged according to

$$0 < \mu_1 \leq \mu_2 \leq \mu_3 \leq \; ..... \; \text{(boundary condition } w(\theta_0, \phi) = 0),$$

$$0 = \mu_0 \leq \mu_1 \leq \mu_2 \leq \; ..... \; \text{(boundary condition } \frac{\partial w}{\partial \theta}(\theta_0, \phi) = 0) \, .$$

The corresponding eigenfunctions $w_n(\theta, \phi)$ form a complete orthogonal system with respect to the inner product

$$(v, w) = \int_0^{\theta_0} \int_0^{2\pi} v(\theta, \phi) w(\theta, \phi) h_\theta h_\phi r^{-2} d\theta d\phi \, .$$

REMARK. In the case of the Neumann boundary condition $\partial w / \partial\theta(\theta_0, \phi) = 0$, the smallest eigenvalue $\mu_0 = 0$ goes with the corresponding eigenfunction $w_0(\theta, \phi) \equiv 1$.

Henceforth we set

$$(3.11) \quad \mu = \nu(\nu + 1), \quad \nu \geq 0 \, ,$$

where $\nu \geq 0$ may be taken because $\mu \geq 0$ by Theorem 3.1. Then eq. (3.7) is recognized as the differential equation for the spherical Bessel functions. As independent solutions for $R(r)$ we use either the spherical Bessel functions of the first and second kinds

$$(3.12) \quad j_\nu(k_0 r) = \sqrt{\pi/2k_0 r}\, J_{\nu+1/2}(k_0 r), \;\; y_\nu(k_0 r) = \sqrt{\pi/2k_0 r}\, Y_{\nu+1/2}(k_0 r),$$

or the spherical Hankel functions

$$(3.13) \quad h_\nu^{(1)}(k_0 r) = \sqrt{\pi/2k_0 r}\, H_{\nu+1/2}^{(1)}(k_0 r), \;\; h_\nu^{(2)}(k_0 r) = \sqrt{\pi/2k_0 r}\, H_{\nu+1/2}^{(2)}(k_0 r) \, .$$

Consider next the eigenvalue problem (3.8) in which $\mu = \nu(\nu+1)$ and the transverse Laplacian Δ_t is replaced by (2.5). This problem is solved by a second application of separation of variables, where we search for solutions of the form

$$(3.14) \quad w(\theta, \phi) = X(\theta)\, Y(\phi) \, .$$

On substitution of (3.14) into (3.8) and separation, we are led to the following differential equations for $X(\theta)$ and $Y(\phi)$:

$$(3.15)\quad \sqrt{1-k^2\cos^2\theta}\,\frac{d}{d\theta}\left(\sqrt{1-k^2\cos^2\theta}\,\frac{dX}{d\theta}\right)+(\nu(\nu+1)\,(1-k^2\cos^2\theta)-\lambda)X=0,\quad 0<\theta<\theta_0;$$

$$(3.16)\quad \sqrt{1-k'^2\sin^2\phi}\,\frac{d}{d\phi}\left(\sqrt{1-k'^2\sin^2\phi}\,\frac{dY}{d\phi}\right)+(\lambda-\nu(\nu+1)\,k'^2\sin^2\phi)Y=0,\quad 0<\phi<2\pi;$$

here, the separation constant has been taken as $\lambda-\nu(\nu+1)k'^2$. Equations (3.15) and (3.16) are called θ-Lamé equation and ϕ-Lamé equation, respectively; both equations are trigonometric forms of Lamé's differential equation.

In addition, $X(\theta)$ and $Y(\phi)$ must satisfy a number of boundary conditions which follow from (3.8)-(3.10) on substitution of (3.14). Thus we find that the ϕ-Lamé equation is accompanied by the boundary conditions

$$(3.17)\quad Y(0)=Y(2\pi),\quad Y'(0)=Y'(2\pi)\,.$$

As for $X(\theta)$, at $\theta=\theta_0$ we have either the Dirichlet boundary condition or the Neumann boundary condition

$$(3.18)\quad X(\theta_0)=0\quad \text{or}\quad X'(\theta_0)=0\,.$$

From the regularity conditions (3.10) it follows that

$$(3.19)\quad \begin{aligned} &X(0)Y(\phi)=X(0)Y(\pi-\phi),\quad X'(0)Y(\phi)=-X'(0)Y(\pi-\phi),\quad 0\le\phi\le\pi,\\ &X(0)Y(\phi)=X(0)Y(3\pi-\phi),\quad X'(0)Y(\phi)=-X'(0)Y(3\pi-\phi),\quad \pi\le\phi\le 2\pi\,. \end{aligned}$$

In the next section it is shown that $Y(\phi)$ is either even symmetric, i.e. $Y(\phi)=Y(\pi-\phi)$, $0\le\phi\le\pi$, or odd symmetric, i.e. $Y(\phi)=-Y(\pi-\phi)$, $0\le\phi\le\pi$. Correspondingly we have for $X(\theta)$ at $\theta=0$ the boundary condition

$$(3.20)\quad X'(0)=0\quad \text{or}\quad X(0)=0\,.$$

The Lamé equations (3.15) and (3.16) together with the boundary conditions (3.17), (3.18) and (3.20), define a two-parameter eigenvalue problem involving pairs of eigenvalues λ,ν.

The further solution of this problem is presented in the sections hereafter. In Sec. 4 we consider, for given ν, the one-parameter eigenvalue problem formed by the ϕ-Lamé equation (3.16) (with eigenvalue parameter λ) and the boundary conditions (3.17). By use of classical Sturm-Liouville theory it is shown that there exists an infinite sequence of eigenvalues $\lambda_n = \lambda_n(\nu)$ which depend on ν. In Sec. 5 we construct the solution $X(\theta)$ of the θ-Lamé equation (3.15) with $\lambda = \lambda_n$, subject to the boundary condition (3.20). By imposing the remaining boundary condition (3.18) on $X(\theta)$, we are led to a transcendental equation for the parameter ν, the solution of which is investigated in Sec. 6.

Finally, for later use we quote the inequality

(3.21) $\quad 0 \leq \lambda < \nu(\nu + 1),$

valid if $\nu > 0,\ 0 \leq k' < 1$, and proved by Jansen [14, p. 31, Lemma 2.6].

4. Analysis of the ϕ-Lamé equation

This section deals with the solution of the ϕ-Lamé equation (3.16) subject to the boundary conditions (3.17). By rewriting the ϕ-Lamé equation in its self-adjoint form, we arrive at the problem

$$
\begin{aligned}
&(\sqrt{1-k'^2\sin^2\phi}\,Y')' + \frac{\lambda-\nu(\nu+1)k'^2\sin^2\phi}{\sqrt{1-k'^2\sin^2\phi}}\,Y = 0, \quad 0<\phi<2\pi, \\
&Y(0)=Y(2\pi), \quad Y'(0)=Y'(2\pi)\,.
\end{aligned} \tag{4.1}
$$

Here the parameter $\nu \geq 0$ is given, while it is understood that $0 \leq k' < 1$, so that $\sqrt{1-k'^2\sin^2\phi} > 0$. Then the problem (4.1) is recognized as a Sturm-Liouville eigenvalue problem with periodic boundary conditions; see Coddington and Levinson [8, Chapter 8], Courant-Hilbert [9, Chapter V, §3,3]. For specific values of λ, called eigenvalues, the problem (4.1) has a non-trivial solution $Y(\phi) \not\equiv 0$, which is the corresponding eigenfunction.

We observe that the ϕ-Lamé equation remains invariant when ϕ is replaced by $\pi+\phi$ or by $\pi-\phi$. On the basis of this property we reduce the original problem (4.1) to four separate Sturm-Liouville problems with four classes of eigenvalues and eigenfunctions. Let $Y(\phi)$ be an eigenfunction of (4.1) corresponding to the eigenvalue λ. Then also the function $Z(\phi)$ defined by

$$
Z(\phi)=Y(\phi+\pi), \quad 0\leq\phi\leq\pi; \quad Z(\phi)=Y(\phi-\pi), \quad \pi\leq\phi\leq 2\pi \tag{4.2}
$$

is an eigenfunction at the eigenvalue λ. We now introduce the combinations

$$
Z_1(\phi)=Y(\phi)+Z(\phi), \quad Z_2(\phi)=Y(\phi)-Z(\phi), \tag{4.3}
$$

which cannot both vanish, since $Y(\phi)\not\equiv 0$. Obviously, if $Z_i(\phi)\not\equiv 0$, then $Z_i(\phi)$ is an eigenfunction of (4.1) corresponding to the eigenvalue λ. Because of the properties

$$
Z_1(\phi)=Z_1(\pi+\phi), \quad Z_2(\phi)=-Z_2(\pi+\phi), \tag{4.4}
$$

the function $Z_1(\phi)$ is periodic with period π, whereas $Z_2(\phi)$ is called half-periodic with half-period π ($Z_2(\phi)$ is also periodic but with period 2π). Thus the eigenfunctions of (4.1) are either periodic or half-periodic.

Next, let $Y(\phi)$ be a periodic or half-periodic eigenfunction of (4.1) corresponding to the eigenvalue λ. Then also the functions

(4.5) $$Z_3(\phi) = Y(\phi) + Y(\pi - \phi), \quad Z_4(\phi) = Y(\phi) - Y(\pi - \phi), \quad 0 \le \phi \le \pi,$$

are eigenfunctions at the eigenvalue λ, provided $Z_i(\phi) \not\equiv 0$. Because of the properties

(4.6) $$Z_3(\phi) = Z_3(\pi - \phi), \quad Z_4(\phi) = -Z_4(\pi - \phi), \quad 0 \le \phi \le \pi,$$

the functions $Z_3(\phi)$ and $Z_4(\phi)$ are called even symmetric and odd symmetric, respectively. Thus the eigenfunctions of (4.1) are either even or odd symmetric.

We may now restrict our search for eigenfunctions to the four classes of periodic/half-periodic, even/odd eigenfunctions. Further, it is sufficient to consider the ϕ-Lamé equation on the interval $0 < \phi < \pi/2$ with boundary conditions at $\phi = 0$ and $\phi = \pi/2$. Then the eigenfunction obtained is extended to the complete interval $[0, 2\pi]$ by even/odd symmetry and periodicity/half-periodicity. Thus the four classes of eigenvalues and eigenfunctions are determined as solutions of the ϕ-Lamé equation

(4.7) $$(\sqrt{1 - k'^2 \sin^2 \phi}\, Y')' + \frac{\lambda - \nu(\nu+1)k'^2 \sin^2 \phi}{\sqrt{1 - k'^2 \sin^2 \phi}}\, Y = 0, \quad 0 < \phi < \pi/2,$$

subject to the following boundary conditions:

Class I. $Y'(0) = Y'(\pi/2) = 0$,
$Y(\phi) = Y(\pi + \phi) = Y(\pi - \phi)$, i.e. $Y(\phi)$ is periodic, even symmetric.

Class II. $Y'(0) = Y(\pi/2) = 0$,
$Y(\phi) = -Y(\pi + \phi) = -Y(\pi - \phi)$, i.e. $Y(\phi)$ is half-periodic, odd symmetric.

Class III. $Y(0) = Y(\pi/2) = 0$,
$Y(\phi) = Y(\pi + \phi) = -Y(\pi - \phi)$, i.e. $Y(\phi)$ is periodic, odd symmetric.

Class IV. $Y(0) = Y'(\pi/2) = 0$,
$Y(\phi) = -Y(\pi + \phi) = Y(\pi - \phi)$, i.e. $Y(\phi)$ is half-periodic, even symmetric.

The eigenvalue problems of classes I-IV are regular Sturm-Liouville eigenvalue problems. From Coddington and Levinson [8, p. 212, Thm. 2.1] we quote the following theorem on the eigenvalues and corresponding eigenfunctions.

THEOREM 4.1. A regular Sturm-Liouville eigenvalue problem defined on the interval $[a,b]$, admits an infinite number of simple eigenvalues $\lambda_0, \lambda_1, \lambda_2, \ldots$ forming a monotonic increasing sequence with $\lambda_n \to \infty$ as $n \to \infty$. Moreover, the eigenfunction corresponding to λ_n has exactly n zeros on (a,b).

Accordingly, the original Sturm-Liouville problem (4.1) admits four infinite sequences of eigenvalues with corresponding eigenfunctions that are called (periodic) Lamé functions. These Lamé functions can be represented by Fourier series which involve either cosines or sines of either even or odd multiples of ϕ, in accordance with the eigenfunction being periodic/half-periodic and even/odd symmetric. We now introduce the following notations adopted from Jansen [14, p. 48].

Class I. $Y(\phi) = Y(\pi+\phi) = Y(\pi-\phi)$, i.e. $Y(\phi)$ is periodic, even symmetric.
The eigenvalues are denoted by $\lambda_{c\nu}^{2m}$, $m = 0,1,2,\ldots$, and the corresponding Lamé function is represented by the Fourier series

$$(4.8) \qquad L_{c\nu}^{(2m)}(\phi) = \sum_{r=0}^{\infty} A_{2r}^{(2m)} \cos(2r\phi), \quad m = 0,1,2,\ldots .$$

The function $L_{c\nu}^{(2m)}(\phi)$ has exactly m zeros on $(0,\pi/2)$.
Class II. $Y(\phi) = -Y(\pi+\phi) = -Y(\pi-\phi)$, i.e. $Y(\phi)$ is half-periodic, odd symmetric.
The eigenvalues are denoted by $\lambda_{c\nu}^{2m+1}$, $m = 0,1,2,\ldots$, and the corresponding Lamé function is represented by the Fourier series

$$(4.9) \qquad L_{c\nu}^{(2m+1)}(\phi) = \sum_{r=0}^{\infty} A_{2r+1}^{(2m+1)} \cos((2r+1)\phi), \quad m = 0,1,2,\ldots .$$

The function $L_{c\nu}^{(2m+1)}(\phi)$ has exactly m zeros on $(0,\pi/2)$.
Class III. $Y(\phi) = Y(\pi+\phi) = -Y(\pi-\phi)$, i.e. $Y(\phi)$ is periodic, odd symmetric.
The eigenvalues are denoted by $\lambda_{s\nu}^{2m}$, $m = 1,2,3,\ldots$, and the corresponding Lamé function is represented by the Fourier series

$$(4.10) \qquad L_{s\nu}^{(2m)}(\phi) = \sum_{r=1}^{\infty} B_{2r}^{(2m)} \sin(2r\phi), \quad m = 1,2,3,\ldots .$$

The function $L_{s\nu}^{(2m)}(\phi)$ has exactly $m-1$ zeros on $(0,\pi/2)$.

Class IV. $Y(\phi) = -Y(\pi + \phi) = Y(\pi - \phi)$, i.e. $Y(\phi)$ is half-periodic, even symmetric.
The eigenvalues are denoted by $\lambda_{s\nu}^{2m+1}$, $m = 0, 1, 2, \ldots$, and the corresponding Lamé function is represented by the Fourier series

$$(4.11) \qquad L_{s\nu}^{(2m+1)}(\phi) = \sum_{r=0}^{\infty} B_{2r+1}^{(2m+1)} \sin((2r+1)\phi), \quad m = 0, 1, 2, \ldots .$$

The function $L_{s\nu}^{(2m+1)}(\phi)$ has exactly m zeros on $(0, \pi/2)$.

REMARK. In view of the even/odd symmetry of the Lamé functions, the results on the number of zeros may be combined to:
the functions $L_{c\nu}^{(m)}(\phi)$ and $L_{s\nu}^{(m)}(\phi)$ have exactly m zeros on the interval $0 \leq \phi < \pi$.

We now determine the mutual ordering of the eigenvalues of classes I-IV by use of the interlacing properties established in Appendix A; see Lemmas A.1 and A.2. The interlacing property holds for the eigenvalues of classes I and II, of classes I and IV, of classes II and III, and of classes III and IV, since the underlying eigenvalue problems differ in just one boundary condition. The interlacing property does not apply to the eigenvalues of classes I and III, and of classes II and IV, because the eigenvalue problems differ in both boundary conditions. Thus we find the following ordering of the eigenvalues $\lambda_{c\nu}^{m}$ and $\lambda_{s\nu}^{m}$:

$$(4.12) \qquad 0 \leq \lambda_{c\nu}^{0} < \begin{matrix} \lambda_{c\nu}^{1} \\ \lambda_{s\nu}^{1} \end{matrix} < \begin{matrix} \lambda_{c\nu}^{2} \\ \lambda_{s\nu}^{2} \end{matrix} < \ldots < \begin{matrix} \lambda_{c\nu}^{m} \\ \lambda_{s\nu}^{m} \end{matrix} < \begin{matrix} \lambda_{c\nu}^{m+1} \\ \lambda_{s\nu}^{m+1} \end{matrix} < \ldots ,$$

where the inequality $0 \leq \lambda_{c\nu}^{0}$ results from (3.21). In (4.12), the writing of two eigenvalues on top of each other indicates that their mutual ordering is not known. We shall come back to this point at the end of this section.
In the special case $k' = 0$ the eigenvalue problems can be explicitly solved and the eigenvalues and eigenfunctions are found to be

$$(4.13) \qquad \begin{aligned} &\lambda_{c\nu}^{0} = 0 \ , \quad L_{c\nu}^{(0)}(\phi) = 1 \ , \\ &\lambda_{c\nu}^{m} = m^2 \ , \quad L_{c\nu}^{(m)}(\phi) = \cos(m\phi), \quad m = 1, 2, 3, \ldots \\ &\lambda_{s\nu}^{m} = m^2 \ , \quad L_{s\nu}^{(m)}(\phi) = \sin(m\phi), \quad m = 1, 2, 3, \ldots . \end{aligned}$$

Notice that $\lambda_{c\nu}^{m} = \lambda_{s\nu}^{m}$ in this case.

Henceforth the case $k' = 0$ is ignored in this section, so that $0 < k' < 1$. We shall now determine the coefficients in the Fourier series (4.8)-(4.11) for the periodic Lamé functions. To that end we substitute the Fourier series into the ϕ-Lamé equation (3.16). By collecting corresponding terms it is found that the coefficients must satisfy the following recurrence relations quoted from Jansen [14, p. 49]:

(4.14a) $(a_0 - \Lambda)A_0^{(2m)} + c_0 A_2^{(2m)} = 0,$

(4.14b) $b_{2r-2}A_{2r-2}^{(2m)} + (a_{2r} - \Lambda)A_{2r}^{(2m)} + c_{2r}A_{2r+2}^{(2m)} = 0, \quad r = 1,2,3,\dots;$

(4.15a) $(a_1 - \frac{1}{4}\nu(\nu+1)k'^2 - \Lambda)A_1^{(2m+1)} + c_1 A_3^{(2m+1)} = 0,$

(4.15b) $b_{2r-1}A_{2r-1}^{(2m+1)} + (a_{2r+1} - \Lambda)A_{2r+1}^{(2m+1)} + c_{2r+1}A_{2r+3}^{(2m+1)} = 0, \quad r = 1,2,3,\dots;$

(4.16a) $(a_2 - \Lambda)B_2^{(2m)} + c_2 B_4^{(2m)} = 0,$

(4.16b) $b_{2r-2}B_{2r-2}^{(2m)} + (a_{2r} - \Lambda)B_{2r}^{(2m)} + c_{2r}B_{2r+2}^{(2m)} = 0, \quad r = 2,3,4,\dots;$

(4.17a) $(a_1 + \frac{1}{4}\nu(\nu+1)k'^2 - \Lambda)B_1^{(2m+1)} + c_1 B_3^{(2m+1)} = 0,$

(4.17b) $b_{2r-1}B_{2r-1}^{(2m+1)} + (a_{2r+1} - \Lambda)B_{2r+1}^{(2m+1)} + c_{2r+1}B_{2r+3}^{(2m+1)} = 0, \quad r = 1,2,3,\dots;$

here we introduced the notations

$$
(4.18) \qquad
\begin{aligned}
&\Lambda = \lambda - \tfrac{1}{2}\nu(\nu+1)k'^2, \\
&a_r = r^2(1 - \tfrac{1}{2}k'^2), \quad r = 0,1,2,\dots, \\
&b_0 = -\tfrac{1}{2}\nu(\nu+1)k'^2, \quad b_r = -\tfrac{1}{4}(\nu - r)\,(\nu + r + 1)k'^2, \quad r = 1,2,3,\dots, \\
&c_r = -\tfrac{1}{4}(\nu - r - 1)\,(\nu + r + 2)k'^2, \quad r = 0,1,2,\dots.
\end{aligned}
$$

The four sets of equations (4.14)-(4.17) have the same structure. Each set consists of an initial two-term relation (suffix a) and a homogeneous three-term recurrence relation (suffix b) of the form

(4.19a) $(a_0^* - \Lambda)y_0 + c_0^* y_1 = 0,$

(4.19b) $b_r^* y_{r-1} + (a_r^* - \Lambda)y_r + c_r^* y_{r+1} = 0, \quad r = 1,2,3,\dots.$

We recall some results from the asymptotic theory of linear second-order difference equations; see Gautschi [11, Sec. 2]. From (4.18) it is readily found that

$$(4.20) \qquad \lim_{r\to\infty} \frac{a_r^* - \Lambda}{c_r^*} = \frac{2(1+k^2)}{1-k^2}, \quad \lim_{r\to\infty} \frac{b_r^*}{c_r^*} = 1 .$$

These limits enter as coefficients into the characteristic polynomial

$$(4.21) \qquad \Phi(t) = t^2 + \frac{2(1+k^2)}{1-k^2} t + 1$$

associated with (4.19b), and $\Phi(t)$ has zeros $t_1 = -(1-k)/(1+k)$ and $t_2 = -(1+k)/(1-k)$ with $|t_1| < |t_2|$. Then according to Perron's theorem [11, p. 34, Thm. 2.2] the recurrence relation (4.19b) has two linearly independent solutions $\{y_r^{(1)}\}$ and $\{y_r^{(2)}\}$, such that

$$(4.22) \qquad \lim_{r\to\infty} \frac{y_{r+1}^{(1)}}{y_r^{(1)}} = t_1 = -\frac{1-k}{1+k}, \quad \lim_{r\to\infty} \frac{y_{r+1}^{(2)}}{y_r^{(2)}} = t_2 = -\frac{1+k}{1-k} .$$

The sequence $\{y_r^{(1)}\}$ is a minimal solution and the sequence $\{y_r^{(2)}\}$ is a dominant solution of (4.19b). Since $|t_1| < 1$ and $|t_2| > 1$, it is obvious that $y_r^{(1)} \to 0$ and $|y_r^{(2)}| \to \infty$ as $r \to \infty$. The elements y_r stand for coefficients in Fourier series, which implies that $y_r \to 0$ as $r \to \infty$. Consequently, the dominant solution $\{y_r^{(2)}\}$ is not admissible and the solution $\{y_r\}$ of (4.19b) is proportional to the minimal solution $\{y_r^{(1)}\} : y_r = \alpha y_r^{(1)}$. In addition, this solution must satisfy the initial relation (4.19a) which leads to

$$(4.23) \qquad (a_0^* - \Lambda) y_0^{(1)} + c_0^* y_1^{(1)} = 0 .$$

The latter equation is to be considered as a transcendental equation in Λ, which determines the eigenvalues λ of the ϕ-Lamé equation. The sequences of Fourier coefficients, $\{A_{2r}^{(2m)}\}$, $\{A_{2r+1}^{(2m+1)}\}$, $\{B_{2r}^{(2m)}\}$ and $\{B_{2r+1}^{(2m+1)}\}$, are proportional to the minimal solution $\{y_r^{(1)}\}$ of the pertaining recurrence relation. Then it is found from (4.22) that

$$(4.24) \qquad \lim_{r\to\infty} \frac{A_{2r+2}^{(2m)}}{A_{2r}^{(2m)}} = \lim_{r\to\infty} \frac{A_{2r+3}^{(2m+1)}}{A_{2r+1}^{(2m+1)}} = \lim_{r\to\infty} \frac{B_{2r+2}^{(2m)}}{B_{2r}^{(2m)}} = \lim_{r\to\infty} \frac{B_{2r+3}^{(2m+1)}}{B_{2r+1}^{(2m+1)}} = -\frac{1-k}{1+k} .$$

By means of these limit results it follows that the Fourier series (4.8)-(4.11) and their derivatives are uniformly convergent on the interval $[0, 2\pi]$. Thus the periodic Lamé functions $L_{c\nu}^{(m)}(\phi)$ and $L_{s\nu}^{(m)}(\phi)$ have been determined as regular solutions of the ϕ-Lamé equation (3.16).

In practice, the eigenvalues $\lambda_{c\nu}^{m}$, $\lambda_{s\nu}^{m}$ and the Lamé functions $L_{c\nu}^{(m)}(\phi)$, $L_{s\nu}^{(m)}(\phi)$ are calculated by the following procedure. The four sets of equations (4.14)-(4.17) are rewritten in matrix notation as

$$\begin{bmatrix} a_0 & c_0 & & & & \\ b_0 & a_2 & c_2 & & & \\ & \ddots & \ddots & \ddots & & \\ & & b_{2r-2} & a_{2r} & c_{2r} & \\ & & & \ddots & \ddots & \ddots \end{bmatrix} \begin{bmatrix} A_0^{(2m)} \\ A_2^{(2m)} \\ \vdots \\ A_{2r}^{(2m)} \\ \vdots \end{bmatrix} = \Lambda \begin{bmatrix} A_0^{(2m)} \\ A_2^{(2m)} \\ \vdots \\ A_{2r}^{(2m)} \\ \vdots \end{bmatrix}, \tag{4.25}$$

$$\begin{bmatrix} a_1 - \frac{1}{4}\nu(\nu+1)k'^2 & c_1 & & & & \\ b_1 & a_3 & c_3 & & & \\ & \ddots & \ddots & \ddots & & \\ & & b_{2r-1} & a_{2r+1} & c_{2r+1} & \\ & & & \ddots & \ddots & \ddots \end{bmatrix} \begin{bmatrix} A_1^{(2m+1)} \\ A_3^{(2m+1)} \\ \vdots \\ A_{2r+1}^{(2m+1)} \\ \vdots \end{bmatrix} = \Lambda \begin{bmatrix} A_1^{(2m+1)} \\ A_3^{(2m+1)} \\ \vdots \\ A_{2r+1}^{(2m+1)} \\ \vdots \end{bmatrix}, \tag{4.26}$$

$$\begin{bmatrix} a_2 & c_2 & & & & \\ b_2 & a_4 & c_4 & & & \\ & \ddots & \ddots & \ddots & & \\ & & b_{2r-2} & a_{2r} & c_{2r} & \\ & & & \ddots & \ddots & \ddots \end{bmatrix} \begin{bmatrix} B_2^{(2m)} \\ B_4^{(2m)} \\ \vdots \\ B_{2r}^{(2m)} \\ \vdots \end{bmatrix} = \Lambda \begin{bmatrix} B_2^{(2m)} \\ B_4^{(2m)} \\ \vdots \\ B_{2r}^{(2m)} \\ \vdots \end{bmatrix}, \tag{4.27}$$

$$\begin{bmatrix} a_1 + \frac{1}{4}\nu(\nu+1)k'^2 & c_1 & & & & \\ b_1 & a_3 & c_3 & & & \\ & \ddots & \ddots & \ddots & & \\ & & b_{2r-1} & a_{2r+1} & c_{2r+1} & \\ & & & \ddots & \ddots & \ddots \end{bmatrix} \begin{bmatrix} B_1^{(2m+1)} \\ B_3^{(2m+1)} \\ \vdots \\ B_{2r+1}^{(2m+1)} \\ \vdots \end{bmatrix} = \Lambda \begin{bmatrix} B_1^{(2m+1)} \\ B_3^{(2m+1)} \\ \vdots \\ B_{2r+1}^{(2m+1)} \\ \vdots \end{bmatrix}, \tag{4.28}$$

where a_r, b_r, c_r, $r = 0, 1, 2, ...$ are given by (4.18). The problems (4.25)-(4.28) are recognized as eigenvalue problems for infinite tridiagonal matrices, with eigenvalues $\Lambda = \lambda - \frac{1}{2}\nu(\nu+1)k'^2$ and eigenvectors formed by the sequences of Fourier coefficients, $\{A_{2r}^{(2m)}\}$, $\{A_{2r+1}^{(2m+1)}\}$, $\{B_{2r}^{(2m)}\}$ and $\{B_{2r+1}^{(2m+1)}\}$. To numerically determine the eigenvalues and corresponding eigenvectors, we truncate the infinite matrix to a finite $(N \times N)$-matrix with N sufficiently large, whereupon a standard numerical procedure for tridiagonal matrices is employed. It turns out that a choice of $N = 25$ yields accurate results for the first few eigenvalues and eigenvectors.
The periodic Lamé functions $L_{c\nu}^{(m)}(\phi)$ and $L_{s\nu}^{(m)}(\phi)$ introduced in this section are uniquely determined except for a multiplicative constant. Following Jansen [14, p. 52], one may normalize the Lamé functions by the conditions

$$(4.29) \qquad \frac{1}{\pi}\int_0^{2\pi}(L_{c\nu}^{(m)}(\phi))^2 d\phi = \frac{1}{\pi}\int_0^{2\pi}(L_{s\nu}^{(m)}(\phi))^2 d\phi = 1,$$

$$(4.30) \qquad L_{c\nu}^{(m)}(0) > 0, \quad L_{s\nu}^{(m)\prime}(0) > 0 .$$

Furthermore, the Lamé functions $L_{c\nu}^{(m)}(\phi), m = 0, 1, 2, ...,$ and $L_{s\nu}^{(m)}(\phi), m = 1, 2, 3, ...,$ form a complete orthogonal system on $[0, 2\pi]$ with respect to the inner product

$$(4.31) \qquad (u, v) = \int_0^{2\pi} \frac{u(\phi)v(\phi)}{\sqrt{1 - k'^2 \sin^2\phi}} d\phi \; ;$$

cf. Courant-Hilbert [9, Chapter V, §3,3].

Finally we present some additional results on the eigenvalues $\lambda_{c\nu}^m$, $\lambda_{s\nu}^m$, taken from Van Vught [24].

PROPOSITION 4.1. For $0 < k' < 1$ and non-integral $\nu > 0$, one has

$$(4.32) \qquad \lambda_{c\nu}^m \neq \lambda_{s\nu}^m, \quad m = 1, 2, 3, ... \; ;$$

for $0 < k' < 1$ and integral $\nu \geq 0$, one has

$$(4.33) \qquad \lambda_{c\nu}^m \neq \lambda_{s\nu}^m, \quad m = 1, 2, ..., \nu; \quad \lambda_{c\nu}^m = \lambda_{s\nu}^m, \quad m = \nu + 1, \; \nu + 2, ... \; .$$

For the proof of this proposition see Van Vught [24, pp. 33,85, Lemmas 4.1, 6.3]. The same results have been found before by Ince [13, §5] in his discussion of the so-called coexistence question.

PROPOSITION 4.2.

$$\lambda_{c\nu}^{m}-\lambda_{s\nu}^{m}=\frac{(\frac{1}{2}\nu+\frac{1}{2}-\frac{1}{2}m)_m\,(-\frac{1}{2}\nu-\frac{1}{2}m)_m}{2^{2m-3}((m-1)!)^2}(k')^{2m}+O((k')^{2m+2}), \tag{4.34}$$

valid for $k'\downarrow 0,\quad m=1,2,3,\dots$.

In (4.34) we use as a short notation Pochhammer's symbol $(a)_m$, defined by

$$(a)_0:=1,\quad (a)_m:=a(a+1)\cdots(a+m-1),\quad m=1,2,3,\dots . \tag{4.35}$$

The result (4.34) can be derived by a perturbation analysis for small k', based on expansion of the eigenvalues and of the Lamé functions in power-series in powers of k'. The leading terms of these expansions are given by (4.13). For the details of the analysis we refer to Van Vught [24, p. 28-32].

Since $\lambda_{c\nu}^m$ and $\lambda_{s\nu}^m$ are continuous functions of k' for $0<k'<1$, Proposition 4.1 implies that $\mathrm{sgn}(\lambda_{c\nu}^m-\lambda_{s\nu}^m)$ is constant and independent of k'. Thus we may determine $\mathrm{sgn}(\lambda_{c\nu}^m-\lambda_{s\nu}^m)$ by means of Proposition 4.2.

PROPOSITION 4.3. For $0<k'<1$ and $\nu\geq 0$, one has

$$\mathrm{sgn}(\lambda_{c\nu}^m-\lambda_{s\nu}^m)=\begin{cases}(-1)^m, & m=1,2,\dots,N,\\ \\ (-1)^N, & m=N+1,N+2,\dots,\text{ if }\nu\text{ is non-integral,}\end{cases} \tag{4.36}$$

where the integer N is determined by $N-1<\nu\leq N$, i.e. $N=\lceil\nu\rceil$.

By means of this proposition we now complete the ordering of the eigenvalues in (4.12).

PROPOSITION 4.4. For $0 < k' < 1$ and $\nu \geq 0$, we have the following ordering of the eigenvalues $\lambda_{c\nu}^m$ and $\lambda_{s\nu}^m$:

$$
(4.37) \qquad 0 \leq \lambda_{c\nu}^0 < \lambda_{c\nu}^1 < \lambda_{s\nu}^1 < \lambda_{s\nu}^2 < \lambda_{c\nu}^2 < \lambda_{c\nu}^3 < \ldots < \begin{array}{ll} \lambda_{c\nu}^N, & N \text{even}, \\ \lambda_{s\nu}^N, & N \text{odd}, \end{array}
$$

where $N = \lceil \nu \rceil$.

PROPOSITION 4.5. In case $k' = 1$, the eigenvalue problem for the ϕ-Lamé equation with periodic boundary conditions has a finite number of eigenvalues

$$
(4.38) \qquad \begin{array}{l} \lambda_{c\nu}^{2m} = \lambda_{c\nu}^{2m+1} = (4m+1)\nu - 4m^2, \quad m = 0, 1, \ldots, [\frac{1}{2}N - \frac{1}{2}], \\ \lambda_{s\nu}^{2m+1} = \lambda_{s\nu}^{2m+2} = (4m+3)\nu - (2m+1)^2, \quad m = 0, 1, \ldots, [\frac{1}{2}N - 1], \end{array}
$$

where $N = \lceil \nu \rceil$, and a continuous spectrum $\nu(\nu+1) \leq \lambda < \infty$.

In the degenerate case $k' = 1$ the eigenvalue problem (4.1) can be reduced to a singular Sturm-Liouville eigenvalue problem on the interval $[0, \infty)$. Proposition 4.5 has been proved by Van Vught [24, p. 34-37] and by Ince [12, §9], however, Ince did not notice the occurrence of a continuous spectrum.

5. Analysis of the θ-Lamé equation

This section deals with the solution of the θ-Lamé equation

$$(5.1) \qquad \sqrt{1-k^2\cos^2\theta}\,\frac{d}{d\theta}\left(\sqrt{1-k^2\cos^2\theta}\,\frac{dX}{d\theta}\right)+(\nu(\nu+1)(1-k^2\cos^2\theta)-\lambda)X=0, \quad 0<\theta<\theta_0,$$

quoted from (3.15). In this equation, $\nu \geq 0$ is given while $\lambda = \lambda_{c\nu}^m$ or $\lambda = \lambda_{s\nu}^m$, as determined in the previous section. At $\theta = \theta_0$, the function $X(\theta)$ must satisfy either the Dirichlet boundary condition or the Neumann boundary condition (cf. (3.18))

$$(5.2) \qquad X(\theta_0) = 0 \quad \text{or} \quad X'(\theta_0) = 0 \ .$$

At $\theta = 0$, we have for $X(\theta)$ the boundary condition

$$(5.3) \qquad X'(0) = 0 \quad \text{or} \quad X(0) = 0 \ ,$$

dependent on the function $Y(\phi)$ being even symmetric or odd symmetric; see (3.20). Corresponding to the four classes of periodic Lamé functions, we distinguish also four classes of solutions $X(\theta)$ of the θ-Lamé equation. These solutions are called non-periodic Lamé functions and we employ the following notations adopted from Jansen [14, p. 58].

Class I. $Y(\phi) = L_{c\nu}^{(2m)}(\phi)$, $m = 0, 1, 2, \ldots$, is even symmetric;
hence, $X(\theta)$ must satisfy (5.1) with $\lambda = \lambda_{c\nu}^{2m}$ and the boundary condition $X'(0) = 0$.
The solution is denoted by $X(\theta) = L_{cp\nu}^{(2m)}(\theta)$, $m = 0, 1, 2, \ldots$.
Class II. $Y(\phi) = L_{c\nu}^{(2m+1)}(\phi)$, $m = 0, 1, 2, \ldots$, is odd symmetric;
hence, $X(\theta)$ must satisfy (5.1) with $\lambda = \lambda_{c\nu}^{2m+1}$ and the boundary condition $X(0) = 0$.
The solution is denoted by $X(\theta) = L_{cp\nu}^{(2m+1)}(\theta)$, $m = 0, 1, 2, \ldots$.
Class III. $Y(\phi) = L_{s\nu}^{(2m)}(\phi)$, $m = 1, 2, 3, \ldots$, is odd symmetric;
hence, $X(\theta)$ must satisfy (5.1) with $\lambda = \lambda_{s\nu}^{2m}$ and the boundary condition $X(0) = 0$.
The solution is denoted by $X(\theta) = L_{sp\nu}^{(2m)}(\theta)$, $m = 1, 2, 3, \ldots$.
Class IV. $Y(\phi) = L_{s\nu}^{(2m+1)}(\phi)$, $m = 0, 1, 2, \ldots$, is even symmetric;
hence, $X(\theta)$ must satisfy (5.1) with $\lambda = \lambda_{s\nu}^{2m+1}$ and the boundary condition $X'(0) = 0$.
The solution is denoted by $X(\theta) = L_{sp\nu}^{(2m+1)}(\theta)$, $m = 0, 1, 2, \ldots$.

Notice that the non-periodic Lamé functions $L_{cp\nu}^{(m)}(\theta)$, $L_{sp\nu}^{(m)}(\theta)$ are uniquely determined except for a multiplicative constant. Since there is no generally accepted normalization, we leave these constants undetermined and we suppress constant factors in further relations involving non-periodic Lamé functions.

To evaluate the non-periodic Lamé functions, we follow the approach of Jansen [14, p. 54-58] which is briefly described for the Lamé functions of class I. Let $k = 1$, then $\lambda_{c\nu}^{2m} = 4m^2$ by (4.13), and the θ-Lamé equation (5.1) reduces to Legendre's differential equation. The solution of class I is given by the Legendre function of the first kind, $P_\nu^{2m}(\cos\theta)$. Generally, for $0 < k < 1$, the Lamé function of class I is represented by a series of Legendre functions, viz.

$$(5.4) \qquad L_{cp\nu}^{(2m)}(\theta) = \sum_{r=0}^{\infty} C_{2r}^{(2m)} P_\nu^{2r}(\cos\theta), \quad m = 0, 1, 2, ...,$$

which is substituted into (5.1). By collecting corresponding terms one is led to a set of linear equations for the coefficients $C_{2r}^{(2m)}$. These equations transform into the recurrence relations (4.14) for the coefficients $A_{2r}^{(2m)}$, by the substitution

$$(5.5) \qquad C_{2r}^{(2m)} = (-1)^r T(2r) A_{2r}^{(2m)},$$

provided that $T(2r)$ satisfies

$$(5.6) \qquad T(2r+2) = \frac{T(2r)}{(\nu - 2r)(\nu + 2r + 1)}, \quad r = 0, 1, 2, ...$$

The solution of (5.6) is readily found to be

$$(5.7) \qquad T(2r) = 2^{-2r} \frac{\Gamma(\frac{1}{2}\nu + 1 - r)}{\Gamma(\frac{1}{2}\nu + \frac{1}{2} + r)} \frac{\Gamma(\frac{1}{2}\nu + \frac{1}{2})}{\Gamma(\frac{1}{2}\nu + 1)} T(0), \quad r = 0, 1, 2, ...$$

in which the final constant factors may be suppressed. This completes the evaluation of $L_{cp\nu}^{(2m)}(\theta)$ and the remaining Lamé functions are determined in a similar manner.
From Jansen [14, p. 58] we quote the following results for the non-periodic Lamé functions of classes I-IV:

$$(5.8) \qquad L_{cp\nu}^{(2m)}(\theta) = \sum_{r=0}^{\infty} (-1)^r T(2r) A_{2r}^{(2m)} P_\nu^{2r}(\cos\theta), \quad m = 0, 1, 2, ...,$$

$$(5.9)\qquad L_{cp\nu}^{(2m+1)}(\theta)=\sum_{r=0}^{\infty}(-1)^r T(2r+1)A_{2r+1}^{(2m+1)}P_\nu^{2r+1}(\cos\theta),\quad m=0,1,2,...,$$

$$(5.10)\qquad L_{sp\nu}^{(2m)}(\theta)=\frac{\sqrt{1-k^2\cos^2\theta}}{\sin\theta}\sum_{r=1}^{\infty}(-1)^r(2r)T(2r)B_{2r}^{(2m)}P_\nu^{2r}(\cos\theta),\quad m=1,2,3,...,$$

$$(5.11)\qquad L_{sp\nu}^{(2m+1)}(\theta)=\frac{\sqrt{1-k^2\cos^2\theta}}{\sin\theta}\sum_{r=0}^{\infty}(-1)^r(2r+1)T(2r+1)B_{2r+1}^{(2m+1)}P_\nu^{2r+1}(\cos\theta),\quad m=0,1,2,...$$

where

$$(5.12)\qquad T(r)=2^{-r}\frac{\Gamma(\frac{1}{2}\nu+1-\frac{1}{2}r)}{\Gamma(\frac{1}{2}\nu+\frac{1}{2}+\frac{1}{2}r)},\quad r=0,1,2,\dots.$$

Notice the slight change in notation: Jansen's $T(2r), T(2r+1)$ [14, p. 58] have been replaced by $(-1)^r\ T(2r)$, $(-1)^r\ T(2r+1)$. It is observed that the same coefficients $A_{2r}^{(2m)}$, $A_{2r+1}^{(2m+1)}$, $B_{2r}^{(2m)}$ and $B_{2r+1}^{(2m+1)}$, appear in the expansions (5.8)-(5.11) and in the Fourier series (4.8)-(4.11). Thus, both the non-periodic Lamé functions and the periodic Lamé functions of classes I-IV are expressible in terms of four sequences of coefficients, one sequence for each class. It is recalled that these sequences of coefficients can be numerically determined as eigenvectors of the tridiagonal matrices in (4.25)-(4.28). The expression (5.12) for $T(r)$ becomes singular if $\nu\geq 0$ is an integer and $r=\nu+2j$, $j=1,2,3,\dots$. Fortunately, this singularity is not serious because for integral $\nu\geq 0$ the series-expansions (5.8)-(5.11) terminate at $r=[\frac{1}{2}\nu]$ or $r=[\frac{1}{2}\nu]-1$, and the Lamé functions reduce to Lamé polynomials; for details see Van Vught [14, p. 89-91].

By means of (4.24) and the asymptotic expansion [14, p. 66, Lemma 3.18]

$$(5.13)\qquad P_\nu^r(x)=\frac{(-1)^r}{r!}\,\frac{\Gamma(\nu+r+1)}{\Gamma(\nu-r+1)}\left(\frac{1-x}{1+x}\right)^{r/2}\left[1+O\left(\frac{1}{r}\right)\right],\quad r\to\infty,\quad -1<x\leq 1,$$

we establish the following limit result for the ratio of successive terms of the series (5.8):

$$(5.14)\qquad \lim_{r\to\infty}\frac{(-1)^{r+1}T(2r+2)A_{2r+2}^{(2m)}P_\nu^{2r+2}(\cos\theta)}{(-1)^r T(2r)A_{2r}^{(2m)}P_\nu^{2r}(\cos\theta)}=-\frac{1-k}{1+k}\,\frac{1-\cos\theta}{1+\cos\theta}\,.$$

The same limit is obtained for the ratio of successive terms of the series (5.9)-(5.11). By means of (5.14) it follows that the series (5.8)-(5.11) are convergent if $\cos\theta > -k$. Furthermore, the four series and their derivatives are uniformly convergent on any closed interval $[0,\theta_1]$ with $\theta_1 < \arccos(-k)$. Thus the non-periodic Lamé functions $L^{(m)}_{cp\nu}(\theta)$ and $L^{(m)}_{sp\nu}(\theta)$ have been determined as regular solutions of the θ-Lamé equation (5.1) on the interval $[0,\arccos(-k))$.

Alternative series-expansions for non-periodic Lamé functions, in terms of Legendre functions of argument $k\cos\theta$, can be derived by use of suitable integral transformations which transform periodic Lamé functions into non-periodic Lamé functions. Referring to Jansen [14, Chapter 7] and Van Vught [24, p. 49-56] for the details of the derivation, we present the following results for the non-periodic Lamé functions of classes I-IV:

$$(5.15)\qquad L^{(2m)}_{cp\nu}(\theta) = \sum_{r=0}^{\infty} T(2r)A^{(2m)}_{2r}P^{2r}_{\nu}(k\cos\theta),\quad m = 0,1,2,...,$$

$$(5.16)\qquad L^{(2m+1)}_{cp\nu}(\theta) = \frac{\sin\theta}{\sqrt{1-k^2\cos^2\theta}}\sum_{r=0}^{\infty}(2r+1)\,T(2r+1)A^{(2m+1)}_{2r+1}P^{2r+1}_{\nu}(k\cos\theta),\quad m = 0,1,2,...,$$

$$(5.17)\qquad L^{(2m)}_{sp\nu}(\theta) = \frac{\sin\theta}{\sqrt{1-k^2\cos^2\theta}}\sum_{r=1}^{\infty} 2r\,T(2r)B^{(2m)}_{2r}P^{2r}_{\nu}(k\cos\theta),\quad m = 1,2,3,...,$$

$$(5.18)\qquad L^{(2m+1)}_{sp\nu}(\theta) = \sum_{r=0}^{\infty} T(2r+1)B^{(2m+1)}_{2r+1}P^{2r+1}_{\nu}(k\cos\theta),\quad m = 0,1,2,...,$$

where $T(r)$ is given by (5.12). Notice again the occurrence of the same coefficients $A^{(2m)}_{2r}$, $A^{(2m+1)}_{2r+1}$, $B^{(2m)}_{2r}$ and $B^{(2m+1)}_{2r+1}$, in the expansions (5.15)-(5.18), in the Fourier series (4.8)-(4.11), and in the previous expansions (5.8)-(5.11). By means of (4.24) and the asymptotic expansion (5.13), we establish the following limit result for the ratio of successive terms of the series (5.15):

$$(5.19)\qquad \lim_{r\to\infty}\frac{T(2r+2)A^{(2m)}_{2r+2}P^{2r+2}_{\nu}(k\cos\theta)}{T(2r)A^{(2m)}_{2r}P^{2r}_{\nu}(k\cos\theta)} = \frac{(1-k)\,(1-k\cos\theta)}{(1+k)\,(1+k\cos\theta)}\,.$$

The same limit is obtained in the cases of the series (5.16)-(5.18). From (5.19) it readily follows that the series (5.15)-(5.18) and their derivatives are uniformly convergent on any closed interval $[0,\theta_1]$ with $\theta_1 < \pi$. Thus the non-periodic Lamé functions have now been determined as regular solutions of the θ-Lamé equation on the interval $[0,\pi)$, which encloses the previous interval $[0, \arccos(-k))$.
Finally, it is recalled that in the special case $k = 1$, all non-periodic Lamé functions reduce to Legendre functions of the first kind, viz.

(5.20) $$L^{(0)}_{cp\nu}(\theta) = P_\nu(\cos\theta), \quad L^{(m)}_{cp\nu}(\theta) = L^{(m)}_{sp\nu}(\theta) = P^m_\nu(\cos\theta), \quad m = 1,2,3,\dots\,, \ (k=1)\,.$$

The non-periodic Lamé functions as introduced in this section must satisfy the Dirichlet or Neumann boundary conditions (5.2). When imposed on the functions $L^{(m)}_{cp\nu}(\theta)$, $L^{(m)}_{sp\nu}(\theta)$, $m = (0), 1, 2, \dots$, we are led to the equations

(5.21) $$L^{(m)}_{cp\nu}(\theta_0) = 0, \quad L^{(m)}_{sp\nu}(\theta_0) = 0, \qquad \text{(Dirichlet boundary condition)}$$

or

(5.22) $$\frac{d}{d\theta}L^{(m)}_{cp\nu}(\theta)\Big|_{\theta=\theta_0} = 0, \quad \frac{d}{d\theta}L^{(m)}_{sp\nu}(\theta)\Big|_{\theta=\theta_0} = 0, \qquad \text{(Neumann boundary condition)}$$

which should be considered as transcendental equations in the parameter ν. In the next section it is shown that the equations (5.21) and (5.22) have an infinite number of positive roots. The ν-roots of the equations (5.21) are denoted by νc^m_n, νs^m_n, $n = 1,2,3,\dots$, ordered by ascending magnitude, i.e.

(5.23) $$0 < \nu c^m_1 < \nu c^m_2 < \nu c^m_3 < \dots, \quad 0 < \nu s^m_1 < \nu s^m_2 < \nu s^m_3 < \dots\,.$$

The ν-roots of the equations (5.22) are denoted by $\dot\nu c^m_n$, $\dot\nu s^m_n$, $n = 1,2,3,\dots$, ordered by ascending magnitude, i.e.

(5.24) $$0 \le \dot\nu c^m_1 < \dot\nu c^m_2 < \dot\nu c^m_3 < \dots, \quad 0 < \dot\nu s^m_1 < \dot\nu s^m_2 < \dot\nu s^m_3 < \dots\,.$$

Here, $\dot\nu c^m_1 > 0$ for $m = 1,2,3,\dots$, while $\dot\nu c^0_1 = 0$ because of $L^{(0)}_{cp\nu}(\theta) = \text{constant}$ for $\nu = 0$.
In the special case $k = 1$, we derive from (5.20) the transcendental equations

(5.25) $$P^m_\nu(\cos\theta_0) = 0, \qquad \text{(Dirichlet boundary condition)}$$

(5.26) $$\frac{d}{d\theta}\,P^m_\nu(\cos\theta)\Big|_{\theta=\theta_0} = 0, \qquad \text{(Neumann boundary condition)}$$

which are known to have an infinite number of positive roots; see Robin [19, Chap. VIII, §§131-137]. The ν-roots of the equations (5.25), (5.26) are denoted by ν_n^m, $\dot{\nu}_n^m$, $n = 1,2,3,...,$ ordered by ascending magnitude, i.e.

$$(5.27)\qquad 0 < \nu_1^m < \nu_2^m < \nu_3^m < ..., \quad 0 \le \dot{\nu}_1^m < \dot{\nu}_2^m < \dot{\nu}_3^m < ... .$$

Here, $\dot{\nu}_1^m > 0$ for $m = 1,2,3,...,$ while $\dot{\nu}_1^0 = 0$ because of $P_\nu(\cos\theta) = 1$ for $\nu = 0$. It is clear that for $k = 1$ one has

$$(5.28)\qquad \nu c_n^m = \nu s_n^m = \nu_n^m, \quad \dot{\nu} c_n^m = \dot{\nu} s_n^m = \dot{\nu}_n^m, \quad (k = 1)\ .$$

With the parameter ν determined from the equations (5.21) or (5.22), we return to the eigenvalue problem (3.8) which can be completely solved now. In the case of the Dirichlet boundary condition ($w = 0$ at the elliptic cone ∂D), the eigenvalues μ and eigenfunctions $w(\theta, \phi)$ are found to be

$$(5.29)\qquad \begin{cases} \mu = \nu(\nu+1), \\ \\ w(\theta,\phi) = L_{cp\nu}^{(m)}(\theta) L_{c\nu}^{(m)}(\phi), \end{cases} \qquad \nu = \nu c_n^m, \quad m = 0,1,2,..., \quad n = 1,2,3,...,$$

$$(5.30)\qquad \begin{cases} \mu = \nu(\nu+1), \\ \\ w(\theta,\phi) = L_{sp\nu}^{(m)}(\theta) L_{s\nu}^{(m)}(\phi), \end{cases} \qquad \nu = \nu s_n^m, \quad m = 1,2,3,..., \quad n = 1,2,3,... .$$

In the case of the Neumann boundary condition ($\partial w/\partial\theta = 0$ at the elliptic cone ∂D), the eigenvalues μ and eigenfunctions $w(\theta, \phi)$ are found to be

$$(5.31)\qquad \begin{cases} \mu = \nu(\nu+1), \\ \\ w(\theta,\phi) = L_{cp\nu}^{(m)}(\theta) L_{c\nu}^{(m)}(\phi), \end{cases} \qquad \nu = \dot{\nu} c_n^m, \quad m = 0,1,2,..., \quad n = 1,2,3,...,$$

$$(5.32)\qquad \begin{cases} \mu = \nu(\nu+1), \\ \\ w(\theta,\phi) = L_{sp\nu}^{(m)}(\theta) L_{s\nu}^{(m)}(\phi), \end{cases} \qquad \nu = \dot{\nu} s_n^m, \quad m = 1,2,3,..., \quad n = 1,2,3,... .$$

To $\dot{\nu}c_1^0 = 0$, there corresponds the smallest eigenvalue $\mu_0 = 0$ with associated eigenfunction $w_0(\theta, \phi) = \text{constant}$; see the Remark following Theorem 3.1.

In the special case $k = 1$, the conical boundary ∂D as described by (3.1), becomes a circular cone. Then by means of (4.13) and (5.20) the results for the eigenvalues μ and eigenfunctions $w(\theta, \phi)$ specialize to

$$(5.33)\quad \begin{cases} \mu = \nu(\nu+1), & \nu = \nu_n^m, \quad m = 0,1,2,..., \quad n = 1,2,3,\dots, \\ \\ w(\theta,\phi) = P_\nu^m(\cos\theta) \begin{cases} \cos(m\phi) \\ \sin(m\phi) \end{cases}, & \text{(Dirichlet boundary condition)}; \end{cases}$$

$$(5.34)\quad \begin{cases} \mu = \nu(\nu+1), & \nu = \dot{\nu}_n^m, \quad m = 0,1,2,..., \quad n = 1,2,3,\dots, \\ \\ w(\theta,\phi) = P_\nu^m(\cos\theta) \begin{cases} \cos(m\phi) \\ \sin(m\phi) \end{cases}, & \text{(Neumann boundary condition)}. \end{cases}$$

Notice that for $m = 1,2,3,...$, each eigenvalue is associated with two linearly independent eigenfunctions. To $\dot{\nu}_1^0 = 0$, there corresponds the smallest eigenvalue $\mu_0 = 0$ with associated eigenfunction $w_0(\theta, \phi) = \text{constant}$.

For integral $\nu \geq 0$, both the periodic and the non-periodic Lamé functions reduce to Lamé polynomials. Referring to Van Vught [24, p. 89-92] for the details of the derivation, we present the following results for the non-periodic Lamé functions:

(i) $\nu = 0$,

$$(5.35)\quad L_{cp,0}^{(0)}(\theta) = \text{constant};$$

(ii) $\nu = 2n+1, \quad n = 0,1,2,...,$

$$(5.36)\quad \begin{aligned} &L_{cp,2n+1}^{(2m)}(\theta) = \cos\theta \ \mathrm{Pol}_n(\cos^2\theta), \quad m = 0,1,...,n, \\ &L_{cp,2n+1}^{(2m+1)}(\theta) = \sin\theta \ \mathrm{Pol}_n(\cos^2\theta), \quad m = 0,1,...,n, \\ &L_{sp,2n+1}^{(2m)}(\theta) = \sin\theta \ \cos\theta \sqrt{1-k^2\cos^2\theta} \ \mathrm{Pol}_{n-1}(\cos^2\theta), \quad m = 1,2,...,n, \\ &L_{sp,2n+1}^{(2m+1)}(\theta) = \sqrt{1-k^2\cos^2\theta} \ \mathrm{Pol}_n(\cos^2\theta), \quad m = 0,1,...,n; \end{aligned}$$

(iii) $\nu = 2n+2, \quad n = 0,1,2,...,$

(5.37)
$$L^{(2m)}_{cp,2n+2}(\theta) = \mathrm{Pol}_{n+1}(\cos^2\theta), \quad m = 0,1,...,n+1,$$
$$L^{(2m+1)}_{cp,2n+2}(\theta) = \sin\theta\ \cos\theta\ \mathrm{Pol}_n(\cos^2\theta), \quad m = 0,1,...,n,$$
$$L^{(2m)}_{sp,2n+2}(\theta) = \sin\theta\ \sqrt{1-k^2\cos^2\theta}\ \mathrm{Pol}_n(\cos^2\theta), \quad m = 1,2,...,n+1,$$
$$L^{(2m+1)}_{sp,2n+2}(\theta) = \cos\theta\ \sqrt{1-k^2\cos^2\theta}\ \mathrm{Pol}_n(\cos^2\theta), \quad m = 0,1,...,n\ .$$

Here the notation $\mathrm{Pol}_n(x)$ stands for a polynomial in x of degree n. It can easily be verified that the Lamé functions (5.36), (5.37) agree with the eight types of Lamé polynomials introduced by Arscott [1, §§9.2, 9.3].

By means of the representations (5.36), (5.37), the ν-roots of equations (5.21) and (5.22), which of course depend on θ_0, can be explicitly determined in the two cases $\theta_0 = \pi/2$ and $\theta_0 = \pi$. First we observe that the functions $L^{(m)}_{cp\nu}(\theta)$ and $L^{(m)}_{sp\nu}(\theta)$, as given by (5.36) and (5.37), are even (odd) with respect to $\theta = \pi/2$, if the integers ν and m have the same (different) parities. From this symmetry it follows that

(5.38)
$$L^{(m)}_{cp\nu}(\pi/2) = L^{(m)}_{sp\nu}(\pi/2) = 0 \quad \text{if} \quad \nu = m+2n-1, \quad n = 1,2,3,...\ ;$$
$$\frac{d}{d\theta}\ L^{(m)}_{cp\nu}(\theta)\Big|_{\theta=\pi/2} = \frac{d}{d\theta}\ L^{(m)}_{sp\nu}(\theta)\Big|_{\theta=\pi/2} = 0 \quad \text{if} \quad \nu = m+2n-2, \quad n = 1,2,3,....$$

Consequently, for $\theta_0 = \pi/2$ the ν-roots of (5.21) and (5.22) are given by

(5.39)
$$\nu c^m_n = \nu s^m_n = m+2n-1, \qquad m = 0,1,2,..., \quad n = 1,2,3,..., \quad (\theta_0 = \pi/2)\ .$$
$$\dot{\nu} c^m_n = \dot{\nu} s^m_n = m+2n-2,$$

By setting $\theta_0 = \pi/2$ in (3.1), the domain D is recognized to be the half-space $z > 0$. Then the results (5.39) may also be found by solution of the Helmholtz equation in D, using separation of variables in spherical coordinates.

Secondly, it is obvious from (5.36) and (5.37) that the Lamé functions of classes II and III and the derivatives of the Lamé functions of classes I and IV vanish at $\theta = \pi$, that is

$$(5.40)\quad \begin{aligned} &L_{cp\nu}^{(2m+1)}(\pi)=0 \quad \text{if} \quad \nu=2m+n; && L_{sp\nu}^{(2m)}(\pi)=0 \quad \text{if} \quad \nu=2m+n-1; \\ &\frac{d}{d\theta}\, L_{cp\nu}^{(2m)}(\theta)\Big|_{\theta=\pi}=0 \quad \text{if} \quad \nu=2m+n-1; && \frac{d}{d\theta}\, L_{sp\nu}^{(2m+1)}(\theta)\Big|_{\theta=\pi}=0 \quad \text{if} \quad \nu=2m+n; \end{aligned}$$

where $n=1,2,3,\ldots$. Thus for $\theta_0=\pi$ we have the following partial results for the ν-roots of (5.21) and (5.22):

$$(5.41)\quad \begin{aligned} &\nu c_n^{2m+1}=2m+n, \quad \nu s_n^{2m}=2m+n-1, \\ &\dot{\nu} c_n^{2m}=2m+n-1, \quad \dot{\nu} s_n^{2m+1}=2m+n, \end{aligned} \qquad m=0,1,2,\ldots,\quad n=1,2,3,\ldots,\qquad (\theta_0=\pi)\,.$$

In the special case $k=1$, the ν-roots coalesce by (5.28) and the explicit values in (5.39) and (5.41) pass into

$$(5.42)\quad \begin{aligned} &\nu_n^m=m+2n-1, \quad \dot{\nu}_n^m=m+2n-2, \quad (\theta_0=\pi/2) \\ &\nu_n^m=m+n-1, \quad \dot{\nu}_n^m=m+n-1, \quad (\theta_0=\pi) \end{aligned} \qquad m=0,1,2,\ldots,\quad n=1,2,3,\ldots\,.$$

6. Transcendental equations for the parameter ν

In this section it is shown that the transcendental equations (5.21) and (5.22) have an infinite number of positive ν-roots. The main idea in the analysis is to treat the eigenvalue problem (4.1) for the ϕ-Lamé equation and the eigenvalue problem for the θ-Lamé equation (5.1) with boundary conditions (5.2)-(5.3), as two separate Sturm-Liouville eigenvalue problems. Both problems admit an infinite number of eigenvalues which depend on the parameter ν. Next we require the mth eigenvalue of the ϕ-Lamé equation to be equal to the nth eigenvalue of the θ-Lamé equation, where $m, n = 1, 2, 3, \dots$. The transcendental equations for the parameter ν, thus obtained, are equivalent to the equations (5.21) and (5.22).

In the Lamé equations we replace the eigenvalue parameter λ by μ, where λ and μ [1] are related by

$$(6.1) \qquad \lambda = \mu + \nu(\nu+1)k'^2, \quad \mu = \lambda - \nu(\nu+1)k'^2.$$

As a result, the ϕ-Lamé equation (4.7) and the θ-Lamé equation (5.1) change into

$$(6.2) \qquad \frac{d}{d\phi}\left(\sqrt{1-k'^2\sin^2\phi}\,\frac{dY}{d\phi}\right) + \frac{\nu(\nu+1)k'^2\cos^2\phi + \mu}{\sqrt{1-k'^2\sin^2\phi}}\,Y = 0, \quad 0 < \phi < \pi/2,$$

$$(6.3) \qquad \frac{d}{d\theta}\left(\sqrt{1-k^2\cos^2\theta}\,\frac{dX}{d\theta}\right) + \frac{\nu(\nu+1)k^2\sin^2\theta - \mu}{\sqrt{1-k^2\cos^2\theta}}\,X = 0, \quad 0 < \theta < \theta_0.$$

Furthermore, it is found from (6.1) and the inequality (3.21) that the relevant eigenvalues μ lie in the interval

$$(6.4) \qquad -\nu(\nu+1)k'^2 \le \mu < \nu(\nu+1)k^2 \quad \text{if } \nu > 0\ ,\ 0 \le k' < 1\ .$$

Consider first the eigenvalue problem for the ϕ-Lamé equation (6.2). As shown in Sec. 4, there exist four classes of eigenvalues and eigenfunctions which are determined as solutions of equation (6.2), subject to the following boundary conditions:

---

[1]) The present parameter μ should not be confused with the separation constant μ introduced in (3.6).

Class I. $\quad Y'(0) = Y'(\pi/2) = 0, \quad (Y(\phi)$ periodic, even symmetric).

Class II. $\quad Y'(0) = Y(\pi/2) = 0, \quad (Y(\phi)$ half-periodic, odd symmectric).

Class III. $\quad Y(0) = Y(\pi/2) = 0, \quad (Y(\phi)$ periodic, odd symmetric).

Class IV. $\quad Y(0) = Y'(\pi/2) = 0, \quad (Y(\phi)$ half-periodic, even symmetric).

The corresponding μ-eigenvalues are denoted by μ_I^m, μ_{II}^m, μ_{III}^m, μ_{IV}^m, $m = 1,2,3,\ldots$, where the subscript refers to the pertaining class. In view of (6.1), the μ-eigenvalues are related to the previous eigenvalues $\lambda_{c\nu}^m$, $\lambda_{s\nu}^m$ by

$$
\begin{aligned}
&\mu_I^m = \lambda_{c\nu}^{2m-2} - \nu(\nu+1)k'^2, \quad \mu_{II}^m = \lambda_{c\nu}^{2m-1} - \nu(\nu+1)k'^2,\\
(6.5)\qquad &\\
&\mu_{III}^m = \lambda_{s\nu}^{2m} - \nu(\nu+1)k'^2, \quad \mu_{IV}^m = \lambda_{s\nu}^{2m-1} - \nu(\nu+1)k'^2.
\end{aligned}
$$

The ordering of the μ-eigenvalues is now readily found from Proposition 4.4, cf. (4.37); for $0 < k' < 1$ and $\nu \geq 0$, one has

$$
\begin{aligned}
&-\nu(\nu+1)k'^2 \leq \mu_I^1 < \mu_{II}^1 < \mu_{IV}^1 < \mu_{III}^1\\
(6.6)\qquad &< \mu_I^2 < \mu_{II}^2 < \mu_{IV}^2 < \mu_{III}^2 < \ldots < \begin{cases} \mu_I^{N/2+1}, & N \text{ even},\\ \mu_{IV}^{(N+1)/2}, & N \text{ odd},\end{cases}
\end{aligned}
$$

where $N = \lceil \nu \rceil$.

Consider next the eigenvalue problem for the θ-Lamé equation (6.3) with boundary conditions (5.2)-(5.3). There exist again four classes of eigenvalues and eigenfunctions which are determined as solutions of equation (6.3), subject to the following boundary conditions chosen from (5.2) and (5.3):

Class A. $\quad X(0) = X(\theta_0) = 0, \quad$ (odd Dirichlet problem).

Class B. $\quad X'(0) = X(\theta_0) = 0, \quad$ (even Dirichlet problem).

Class C. $\quad X(0) = X'(\theta_0) = 0, \quad$ (odd Neumann problem).

Class D. $\quad X'(0) = X'(\theta_0) = 0, \quad$ (even Neumann problem).

Here, the terminology odd/even Dirichlet/Neumann problem refers to the boundary conditions imposed at $\theta = 0$ and $\theta = \theta_0$. More specific, the problem is called odd (even) in the case of a boundary condition $X(0) = 0$ ($X'(0) = 0$) , which applies if the function $Y(\phi)$ is odd (even) symmetric; see (3.20). Thus the eigenvalue problems of classes A and C (B and D) for the θ-Lamé equation are coupled to the eigenvalue problems of classes II and III (I and IV) for the ϕ-Lamé equation.

The eigenvalue problems of classes A-D are regular Sturm-Liouville eigenvalue problems for the θ-Lamé equation (6.3) with eigenvalue parameter $-\mu$. According to Theorem 4.1, the eigenvalue problems admit an infinite number of simple eigenvalues denoted by μ_A^n, μ_B^n, μ_C^n, μ_D^n, $n = 1, 2, 3, \dots$, where the subscript refers to the pertaining class. The eigenvalues μ_j^n, $j = A, B, C, D$, form a monotonic decreasing sequence with $\mu_j^n \to -\infty$ as $n \to \infty$.

The mutual ordering of the eigenvalues of classes A-D is determined by use of the interlacing properties established in Appendix A; see Lemmas A.1 and A.2. The interlacing property holds for the eigenvalues of classes A and B, of classes A and C, of classes B and D, and of classes C and D, since the underlying eigenvalue problems differ in just one boundary condition. Thus we are led to the following inequalities for the eigenvalues μ_j^n:

$$\mu_B^n > \mu_A^n > \mu_B^{n+1},\ \mu_C^n > \mu_A^n > \mu_C^{n+1},\ \mu_D^n > \mu_B^n > \mu_D^{n+1},\ \mu_D^n > \mu_C^n > \mu_D^{n+1}, \tag{6.7}$$

where $n = 1, 2, 3, \dots$.

We now examine the dependence of the eigenvalues μ_i^m, $i = I, II, III, IV$, and μ_j^n, $j = A, B, C, D$, on the parameter ν. It will be shown that μ_i^m is a decreasing function of ν with $\mu_i^m \to -\infty$ as $\nu \to \infty$, whereas μ_j^n is an increasing function of ν with $\mu_j^n \to \infty$ as $\nu \to \infty$. The proof uses the maximum-minimum property of eigenvalues, also called Courant's maximum-minimum principle; see Courant-Hilbert [9, Chapter VI, §1,4].

Consider first the eigenvalue problems of classes I–IV for the ϕ-Lamé equation (6.2). In view of the boundary conditions of the four classes we introduce the function spaces

$$(6.8)\qquad \begin{aligned} &S_I = PC^1[0,\pi/2], && S_{II} = \{Y \in PC^1[0,\pi/2] \mid Y(\pi/2) = 0\}, \\ &S_{III} = \{Y \in PC^1[0,\pi/2] \mid Y(0) = Y(\pi/2) = 0\}, && S_{IV} = \{Y \in PC^1[0,\pi/2] \mid Y(0) = 0\}, \end{aligned}$$

of functions $Y(\phi)$ defined for $0 \le \phi \le \pi/2$. Here, $PC^1[0,\pi/2]$ denotes the space of continuous functions on $[0,\pi/2]$ which possess a piecewise continuous first derivative. Associated with the ϕ-Lamé equation (6.2), we introduce the quadratic functional $D_\phi[Y]$ and the inner product $H_\phi[v,w]$, defined by

$$(6.9)\qquad D_\phi[Y] = \int_0^{\pi/2} \sqrt{1-k'^2\sin^2\phi}\,\left(\frac{dY}{d\phi}\right)^2 d\phi \;-\nu(\nu+1)k'^2 \int_0^{\pi/2} \frac{\cos^2\phi}{\sqrt{1-k'^2\sin^2\phi}} Y^2(\phi)d\phi\,,$$

$$(6.10)\qquad H_\phi[v,w] = \int_0^{\pi/2} \frac{v(\phi)w(\phi)}{\sqrt{1-k'^2\sin^2\phi}} d\phi\,,$$

similar to Courant-Hilbert [9, Chapter V, pp. 398,402]. Then the eigenvalues μ_i^m are characterized by the maximum-minimum property

$$(6.11)\qquad \mu_i^m = \max_{V_{m-1}} \min_{Y\in S_i, Y\perp V_{m-1}} \frac{D_\phi[Y]}{H_\phi[Y,Y]},\quad i = I, II, III, IV,\quad m = 1,2,3,\ldots\,.$$

In (6.11) the maximum should be taken over all $(m-1)$-dimensional subspaces V_{m-1} of the space $PC[0,\pi/2]$ of piecewise continuous functions on $[0,\pi/2]$; the notation $Y \perp V_{m-1}$ means that $H_\phi[Y,v] = 0$ for any $v \in V_{m-1}$.

Consider next the eigenvalue problems of classes A-D for the θ-Lamé equation (6.3). Starting from the boundary conditions of the four classes we introduce the function spaces

$$(6.12)\qquad \begin{aligned} &S_A = \{X \in PC^1[0,\theta_0] \mid X(0) = X(\theta_0) = 0\}, && S_B = \{X \in PC^1[0,\theta_0] \mid X(\theta_0) = 0\}, \\ &S_C = \{X \in PC^1[0,\theta_0] \mid X(0) = 0\}, && S_D = PC^1[0,\theta_0], \end{aligned}$$

of functions $X(\theta)$ defined for $0 \le \theta \le \theta_0$. As before, $PC^1[0,\theta_0]$ denotes the space of continuous functions on $[0,\theta_0]$ which possess a piecewise continuous first derivative. Associated with the θ-Lamé equation (6.3), we introduce the quadratic functional $D_\theta[X]$ and the inner product $H_\theta[v,w]$, defined by

$$(6.13)\qquad D_\theta[X] = \int_0^{\theta_0} \sqrt{1-k^2\cos^2\theta}\,\left(\frac{dX}{d\theta}\right)^2 d\theta - \nu(\nu+1)k^2 \int_0^{\theta_0} \frac{\sin^2\theta}{\sqrt{1-k^2\cos^2\theta}}\, X^2(\theta)d\theta\ ,$$

$$(6.14)\qquad H_\theta[v,w] = \int_0^{\theta_0} \frac{v(\theta)w(\theta)}{\sqrt{1-k^2\cos^2\theta}}\, d\theta\ ,$$

similar to Courant-Hilbert [9, Chapter V, pp. 398, 402]. Then the eigenvalues μ_j^n are characterized by the maximum-minimum property

$$(6.15)\qquad -\mu_j^n = \max_{V_{n-1}} \min_{X\in S_j, X\perp V_{n-1}} \frac{D_\theta[X]}{H_\theta[X,X]},\ \ j=A,B,C,D,\ \ n=1,2,3,\dots\ ,$$

or equivalently

$$(6.16)\qquad \mu_j^n = \min_{V_{n-1}} \max_{X\in S_j, X\perp V_{n-1}} \frac{-D_\theta[X]}{H_\theta[X,X]},\ \ j=A,B,C,D,\ n=1,2,3,\dots\ .$$

In (6.16) the minimum should be taken over all $(n-1)$-dimensional subspaces V_{n-1} of the space $PC[0,\theta_0]$ of piecewise continuous functions on $[0,\theta_0]$; the notation $X\perp V_{n-1}$ means that $H_\theta[X,v]=0$ for any $v\in V_{n-1}$.

We establish some further inequalities for the eigenvalues μ_i^m and μ_j^n. By replacing $\cos^2\phi$ by $1-\sin^2\phi$ in the second integral of (6.9), it is easily seen that

$$(6.17)\qquad D_\phi[Y] \ge -\nu(\nu+1)k'^2 \int_0^{\pi/2} \frac{Y^2(\phi)}{\sqrt{1-k'^2\sin^2\phi}} d\phi = -\nu(\nu+1)k'^2 H_\phi[Y,Y]$$

with equality iff $\nu=0$ and $Y(\phi)=$ constant $\neq 0$. Since the latter function is element of the space S_I only, we find by use of (6.11),

$$(6.18)\qquad \mu_i^m > -\nu(\nu+1)k'^2,\ \ i=I,II,III,IV,\ \ m=1,2,3,\dots\ ,$$

except that $\mu_I^1 = 0$ at $\nu=0$. Likewise, by replacing $\sin^2\theta$ by $1-\cos^2\theta$ in the second integral of (6.13), it follows that

$$(6.19)\qquad -D_\theta[X] \le \nu(\nu+1)k^2 \int_0^{\theta_0} \frac{X^2(\theta)}{\sqrt{1-k^2\cos^2\theta}}d\theta = \nu(\nu+1)k^2 H_\theta[X,X]$$

with equality iff $\nu = 0$ and $X(\theta) = \text{constant} \neq 0$. Since the latter function is element of the space S_D only, we find by use of (6.16),

$$(6.20)\qquad \mu_j^n < \nu(\nu+1)k^2, \quad j = A,B,C,D, \quad n = 1,2,3,\ldots,$$

except that $\mu_D^1 = 0$ at $\nu = 0$. Notice that the inequalities (6.18) and (6.20) agree with (6.4). From (6.9) and (6.13) it is obvious that, for fixed $Y \in S_i$ and $X \in S_j$, the functional $D_\phi[Y]$ decreases and the functional $-D_\theta[X]$ increases with increasing ν, whereas $H_\phi[Y,Y]$ and $H_\theta[X,X]$ are independent of ν. Then it follows from (6.11) that the eigenvalues $\mu_i^m = \mu_i^m(\nu)$, $i = I,II,III,IV$, $m = 1,2,3,\ldots$, are decreasing functions of ν, with $\mu_i^m > 0$ at $\nu = 0$ (except that $\mu_I^1 = 0$ at $\nu = 0$) and $\mu_i^m \to -\infty$ as $\nu \to \infty$. Likewise we infer from (6.16) that the eigenvalues $\mu_j^n = \mu_j^n(\nu)$, $j = A,B,C,D$, $n = 1,2,3,\ldots$, are increasing functions of ν, with $\mu_j^n < 0$ at $\nu = 0$ (except that $\mu_D^1 = 0$ at $\nu = 0$) and $\mu_j^n \to \infty$ as $\nu \to \infty$. These monotony properties and the inequalities (6.6), (6.7), (6.18) and (6.20) underlie the qualitative sketch of the eigenvalue curves $\mu = \mu_i^m(\nu)$ and $\mu = \mu_j^n(\nu)$ in Figs. 6.1 and 6.2. Because of the monotony of the eigenvalues as functions of ν, the transcendental equation $\mu_i^m(\nu) = \mu_j^n(\nu)$ has precisely one positive ν-root, or equivalently, the corresponding eigenvalue curves have precisely one intersection point. By varying m and n over the positive integers, we have proved in principle that the transcendental equations (5.21) and (5.22) have an infinite number of positive ν-roots, as it will be detailed below.

We first investigate the transcendental equations (5.21) obtained by imposing the Dirichlet boundary condition at $\theta = \theta_0$. It is shown that both equations (5.21) have an infinite number of positive ν-roots which are denoted by $\nu = \nu c_n^m$ and $\nu = \nu s_n^m$, $m = (0),1,2,\ldots$, $n = 1,2,3,\ldots$. Because of the Dirichlet boundary condition $X(\theta_0) = 0$, the eigenvalue problem for the θ-Lamé equation (6.3) is either of class A or of class B, with eigenvalues μ_A^n or μ_B^n, $n = 1,2,3,\ldots$. Let the eigenfunction $X(\theta)$ be of class A, then the boundary condition $X(0) = 0$ implies that the solution $Y(\phi)$ of (6.2) should be odd symmetric.

Hence, the eigenvalue problem for the ϕ-Lamé equation (6.2) is either of class II or of class III. The eigenvalues of classes II and III are given by μ_{II}^m and μ_{III}^m, $m = 1,2,3,\dots$, and the corresponding eigenfunctions are $Y(\phi) = L_{c\nu}^{(2m-1)}(\phi)$ and $Y(\phi) = L_{s\nu}^{(2m)}(\phi)$, respectively. The eigenvalue problems of class A and of classes II or III are now coupled by setting up the transcendental equations (in the parameter ν)

(6.21) $\quad \mu_{II}^m = \mu_A^n \;$ or $\; \mu_{III}^m = \mu_A^n \;.$

Both equations have precisely one positive ν-root and these roots are given by $\nu = \nu c_n^{2m-1}$ and $\nu = \nu s_n^{2m}$, respectively. Indeed, the coupling equations (6.21) imply that $X(\theta) = L_{cp\nu}^{(2m-1)}(\theta)$ or $X(\theta) = L_{sp\nu}^{(2m)}(\theta)$. Hence, we have $L_{cp\nu}^{(2m-1)}(\theta_0) = 0$ at $\nu = \nu c_n^{2m-1}$ and $L_{sp\nu}^{(2m)}(\theta_0) = 0$ at $\nu = \nu s_n^{2m}$, in accordance with the original equations (5.21). In the same manner, the eigenvalue problem of class B for the θ-Lamé equation (6.3) is coupled to the eigenvalue problems of classes I or IV for the ϕ-Lamé equation (6.2). The eigenvalues of classes I and IV are given by μ_I^m and μ_{IV}^m, $m = 1,2,3,\dots$, and the corresponding eigenfunctions are $Y(\phi) = L_{c\nu}^{(2m-2)}(\phi)$ and $Y(\phi) = L_{s\nu}^{(2m-1)}(\phi)$, respectively. The coupling of the eigenvalue problems is effected by setting up the transcendental equations

(6.22) $\quad \mu_I^m = \mu_B^n \;$ or $\; \mu_{IV}^m = \mu_B^n .$

Both equations have precisely one positive ν-root and it is readily seen that these roots are given by $\nu = \nu c_n^{2m-2}$ and $\nu = \nu s_n^{2m-1}$, respectively. It is now clear that the transcendental equations (5.21) are equivalent to the equations (6.21), (6.22) with $m, n = 1,2,3,\dots$. Thus it has been shown that the equations (5.21) have an infinite number of positive roots $\nu = \nu c_n^m$ and $\nu = \nu s_n^m$, where $m = (0),1,2,\dots$ and $n = 1,2,3,\dots$. The present procedure is illustrated in Fig. 6.1: the roots νc_n^m and νs_n^m may be determined graphically from the intersections of the eigenvalue curves $\mu = \mu_i^m(\nu)$, $i = I, II, III, IV$, $m = 1,2,3,\dots$, and the eigenvalue curves $\mu = \mu_j^n(\nu)$, $j = A, B$, $n = 1,2,3,\dots$.

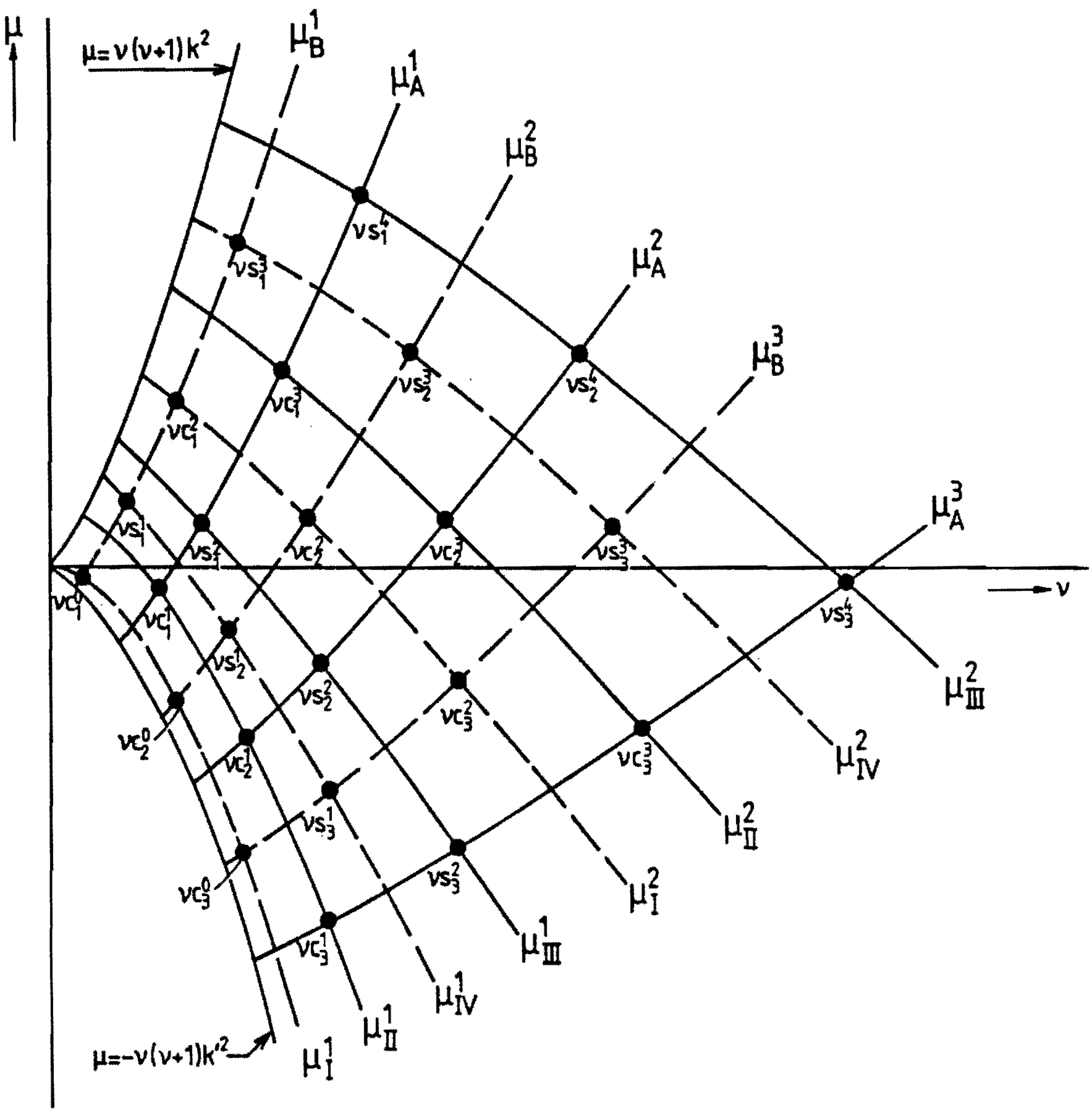

Fig. 6.1. Intersections of the eigenvalue curves $\mu = \mu_i^m(\nu)$, $i = I, II, III, IV$, $m = 1,2$, and the eigenvalue curves $\mu = \mu_j^n(\nu)$, $j = A, B$, $n = 1,2,3$.

From Fig. 6.1 we deduce the following orderings of the roots νc_n^m and νs_n^m:

$$\begin{aligned} &0 < \nu c_1^m < \nu c_2^m < \nu c_3^m < \ldots.. , \quad m = 0,1,2,\ldots , \\ (6.23)\qquad & \\ &0 < \nu s_1^m < \nu s_2^m < \nu s_3^m < \ldots.. , \quad m = 1,2,3,\ldots ; \end{aligned}$$

$$
(6.24)\qquad
\begin{aligned}
&0 < \nu c_n^0 < \nu s_n^1 < \nu c_n^2 < \nu s_n^3 < \ldots.. ,\ n = 1,2,3,\ldots ,\\
&0 < \nu c_n^1 < \nu s_n^2 < \nu c_n^3 < \nu s_n^4 < \ldots.. ,\ n = 1,2,3,\ldots ;
\end{aligned}
$$

$$
(6.25)\qquad
\begin{aligned}
&\nu c_n^{2m} < \nu c_n^{2m+1} < \nu s_{n+1}^{2m+1},\ m = 0,1,2,\ldots ,\ n = 1,2,3,\ldots. ,\\
&\nu s_n^{2m+1} < \nu s_n^{2m+2} < \nu c_{n+1}^{2m+2},\ m = 0,1,2,\ldots ,\ n = 1,2,3,\ldots .
\end{aligned}
$$

Obviously the smallest root is $\nu = \nu c_1^0$.

Next we investigate the transcendental equations (5.22) obtained by imposing the Neumann boundary condition at $\theta = \theta_0$. It is shown that both equations (5.22) have an infinite number of ν-roots which are denoted by $\nu = \dot{\nu} c_n^m$ and $\nu = \dot{\nu} s_n^m$, $m = (0),1,2,\ldots$, $n = 1,2,3,\ldots$; all roots are positive except that $\dot{\nu} c_1^0 = 0$. Because of the Neumann boundary condition $X'(\theta_0) = 0$, the eigenvalue problem for the θ-Lamé equation (6.3) is either of class C or of class D, with eigenvalues μ_C^n or μ_D^n, $n = 1,2,3,\ldots$. Let the eigenfunction $X(\theta)$ be of class C (class D), then the boundary condition $X(0) = 0$ ($X'(0) = 0$) implies that the solution $Y(\phi)$ of (6.2) should be odd (even) symmetric. Hence, the eigenvalue problem of class C (class D) is coupled to the eigenvalue problems of classes II or III (classes I or IV) for the ϕ-Lamé equation (6.2). The coupling of the eigenvalue problems is effected by setting up the transcendental equations

$$(6.26)\qquad \mu_{II}^m = \mu_C^n \quad \text{or} \quad \mu_{III}^m = \mu_C^n,$$

and

$$(6.27)\qquad \mu_I^m = \mu_D^n \quad \text{or} \quad \mu_{IV}^m = \mu_D^n.$$

Both equations (6.26) have precisely one positive ν-root and it is readily seen that these roots are given by $\nu = \dot{\nu} c_n^{2m-1}$ and $\nu = \dot{\nu} s_n^{2m}$, respectively. Likewise the first/second equation (6.27) has the single root $\nu = \dot{\nu} c_n^{2m-2}$ and $\nu = \dot{\nu} s_n^{2m-1}$, respectively. The latter roots are positive with one exception: for $m = n = 1$ the equation $\mu_I^1 = \mu_D^1$ has the single root $\nu = 0$, in accordance with $\dot{\nu} c_1^0 = 0$. It is again clear that the transcendental equations (5.22) are equivalent to the equations (6.26), (6.27) with $m, n = 1,2,3,\ldots$. Thus it has been shown that the equations (5.22) have an infinite number of positive roots $\nu = \dot{\nu} c_n^m$ and $\nu = \dot{\nu} s_n^m$, where

$m = (0), 1, 2, \ldots$, $n = 1, 2, 3, \ldots$, except that $\dot{\nu}c_1^0 = 0$. The present procedure is illustrated in Fig. 6.2: the roots $\dot{\nu}c_n^m$ and $\dot{\nu}s_n^m$ may be determined graphically from the intersections of the eigenvalue curves $\mu = \mu_i^m(\nu)$, $i = I, II, III, IV$, $m = 1, 2, 3, \ldots$, and the eigenvalue curves $\mu = \mu_j^n(\nu)$, $j = C, D$, $n = 1, 2, 3, \ldots$.

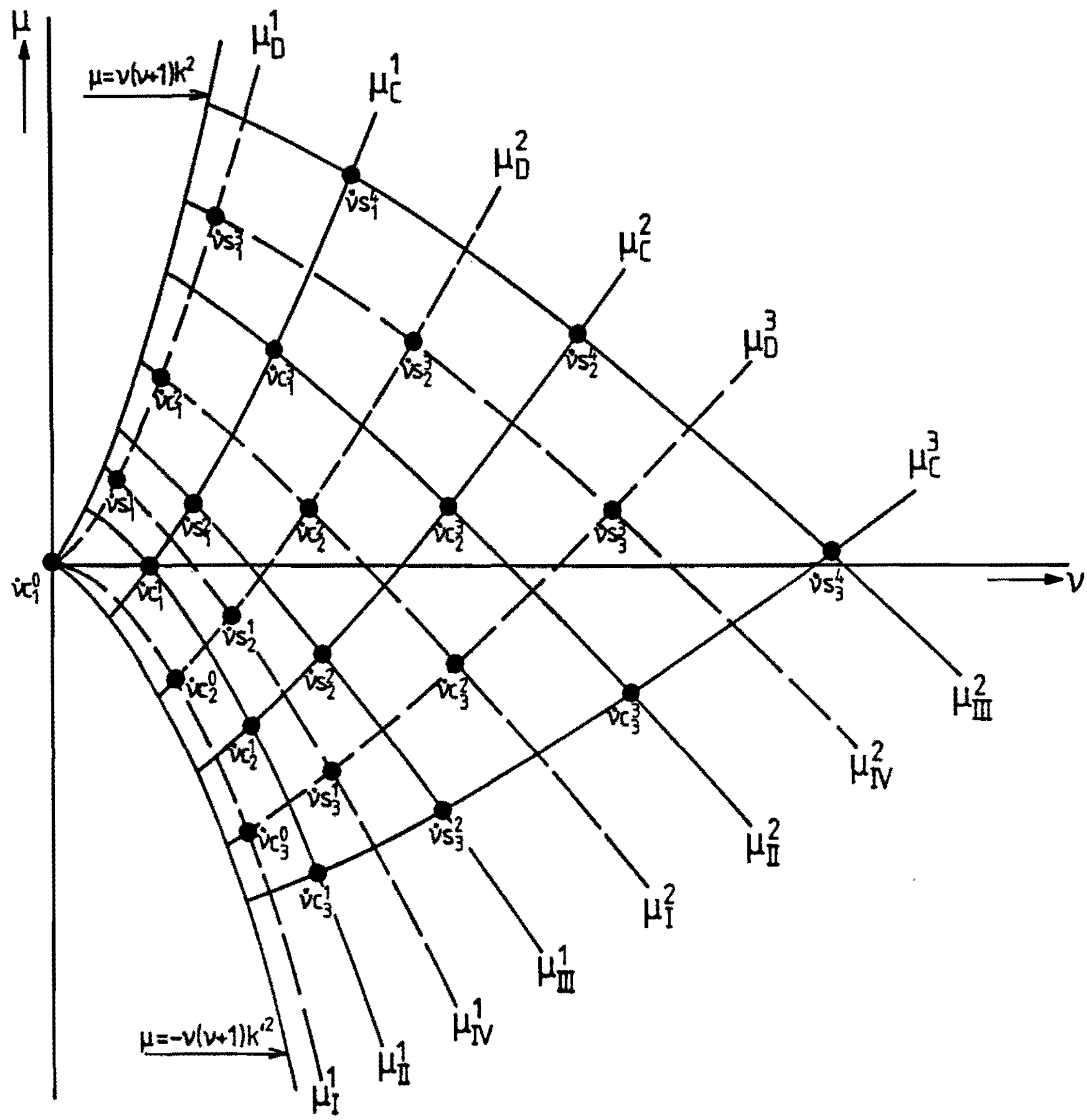

Fig. 6.2. Intersections of the eigenvalue curves $\mu = \mu_i^m(\nu)$, $i = I, II, III, IV$, $m = 1, 2$, and the eigenvalue curves $\mu_j^n(\nu)$, $j = C, D$, $n = 1, 2, 3$,

From Fig. 6.2 we deduce the following orderings of the roots $\dot\nu c_n^m$ and $\dot\nu s_n^m$:

$$
(6.28) \qquad \begin{aligned}
&0 \le \dot\nu c_1^m < \dot\nu c_2^m < \dot\nu c_3^m < \dots\dots, \quad m = 0,1,2,\dots,\\
&0 < \dot\nu s_1^m < \dot\nu s_2^m < \dot\nu s_3^m < \dots\dots, \quad m = 1,2,3,\dots;
\end{aligned}
$$

$$
(6.29) \qquad \begin{aligned}
&0 \le \dot\nu c_n^0 < \dot\nu s_n^1 < \dot\nu c_n^2 < \dot\nu s_n^3 < \dots\dots, \quad n = 1,2,3,\dots,\\
&0 < \dot\nu c_n^1 < \dot\nu s_n^2 < \dot\nu c_n^3 < \dot\nu s_n^4 < \dots\dots, \quad n = 1,2,3,\dots;
\end{aligned}
$$

$$
(6.30) \qquad \begin{aligned}
&\dot\nu c_n^{2m} < \dot\nu c_n^{2m+1} < \dot\nu s_{n+1}^{2m+1}, \quad m = 0,1,2,\dots, \; n = 1,2,3,\dots,\\
&\dot\nu s_n^{2m+1} < \dot\nu s_n^{2m+2} < \dot\nu c_{n+1}^{2m+2}, \quad m = 0,1,2,\dots, \; n = 1,2,3,\dots.
\end{aligned}
$$

Apart from $\dot\nu c_1^0 = 0$, the smallest root is given by $\nu = \min(\dot\nu c_2^0,\ \dot\nu c_1^1,\ \dot\nu s_1^1)$.

In Sec. 7 the ν-roots νc_n^m, νs_n^m, $\dot\nu c_n^m$, $\dot\nu s_n^m$, if less than 1, appear as singularity exponents of the electromagnetic field singularities at the tip of an elliptic cone. The question which of the ν-roots are < 1 and which are ≥ 1, is answered in Appendix B; see Theorems B.3 and B.4. It is shown that all ν-roots are ≥ 1, except for $\dot\nu c_1^0 = 0$ and

$$(6.31) \qquad 0 < \nu c_1^0,\ \dot\nu c_1^1 < 1, \text{ when } \pi/2 < \theta_0 \le \pi; \quad 0 < \dot\nu s_1^1 < 1, \text{ when } \pi/2 < \theta_0 < \pi\,.$$

Notice that $\dot\nu s_1^1 = 1$ at $\theta_0 = \pi$ by (5.41).

We briefly discuss the numerical calculation of νc_1^0, $\dot\nu c_1^1$ and $\dot\nu s_1^1$, which are the only ν-roots that are relevant in Sec. 7. As an example we consider νc_1^0 which is the smallest positive root of the transcendental equation $L_{cp\nu}^{(0)}(\theta_0) = 0$; see (5.21). This root is numerically evaluated by a standard zero-searching procedure applied to $L_{cp\nu}^{(0)}(\theta_0)$; the procedure is based on a combination of the regula falsi and the bisection method. For the necessary calculation of $L_{cp\nu}^{(0)}(\theta_0)$ at given ν, we proceed as follows. First, we determine the smallest eigenvalue $\lambda_{c\nu}^0$ of the corresponding ϕ-Lamé equation (4.7) with boundary conditions of class I. In practice, $\lambda_{c\nu}^0$ is calculated as the smallest eigenvalue of the tridiagonal matrix in (4.25), which is truncated to order 25, say. Next, the function $X(\theta) = L_{cp\nu}^{(0)}(\theta)$ is determined as the solution of the θ-Lamé equation (cf. (5.1))

$$\frac{d}{d\theta}\left(\sqrt{1-k^2\cos^2\theta}\ \frac{dX}{d\theta}\right)+\frac{\nu(\nu+1)(1-k^2\cos^2\theta)-\lambda^0_{c\nu}}{\sqrt{1-k^2\cos^2\theta}}X=0, \tag{6.32}$$

subject to the initial conditions $X(0) = 1$, $X'(0) = 0$. The numerical value of $X(\theta_0) = L^0_{cp\nu}(\theta_0)$ is obtained by a standard numerical integration procedure based on the Runge-Kutta method. The calculation of $\dot{\nu}c^1_1$ and $\dot{\nu}s^1_1$ runs along similar lines.

7. Electromagnetic field singularities at the tip of an elliptic cone

In this section we determine the singularities of the electromagnetic field at the tip of a perfectly conducting, elliptic cone. The geometry of the semi-infinite elliptic cone ∂D is specified by the maximum semi-opening angle θ_m and the axial ratio $\varepsilon = b/a$, where $0 \leq \theta_m \leq \pi/2$ and $0 \leq \varepsilon \leq 1$; see Fig. 7.1. In Cartesian coordinates $\mathbf{x} = (x, y, z)$, the tip of the cone is at the origin and its axis lies along the negative z-axis. The cross-section of the cone with the plane $z = z_0 < 0$ is an ellipse with a major axis of length $2a$ in the (y, z)-plane, and a minor axis of length $2b$ in the (x, z)-plane. The semi-opening angle θ_m is in the (y, z)-plane and faces the major axis of the ellipse.

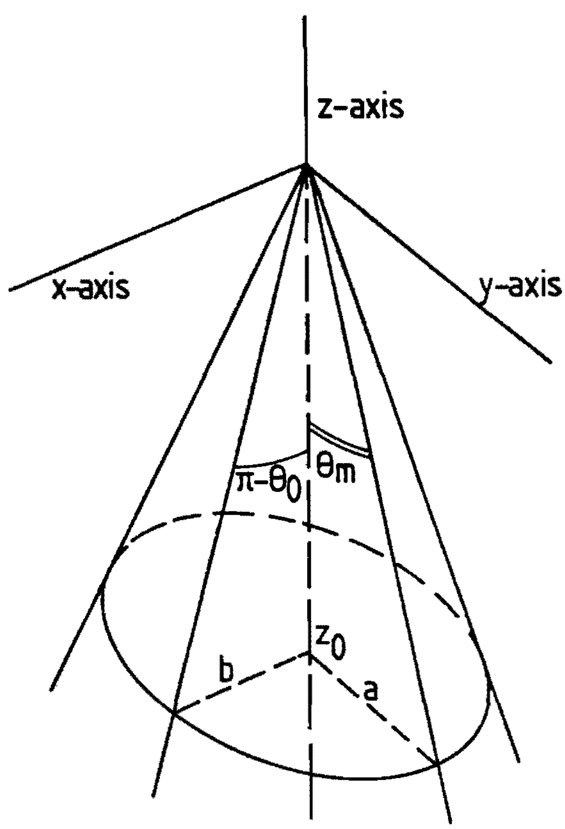

Fig. 7.1. Geometry of the semi-infinite elliptic cone ∂D.

We now employ sphero-conal coordinates r, θ, ϕ, as introduced in (2.1), whereby the parameter k is chosen such that the cone ∂D coincides with a coordinate surface $\theta = \theta_0$, $\pi/2 \le \theta_0 \le \pi$. From (2.11) and (2.12) it is readily found that

$$(7.1) \qquad k = \sqrt{\cos^2\theta_m + \varepsilon^2 \sin^2\theta_m}, \ \tan\theta_0 = -\varepsilon \tan\theta_m, \ \pi/2 \le \theta_0 \le \pi,$$

expressed in terms of the primary parameters θ_m and ε. In the (x, z)-plane the cone has the semi-opening angle $\pi - \theta_0$, which faces the minor axis of the ellipse.

Two special cases are of interest. If $\varepsilon = 1$, the elliptic cone becomes a circular cone of semi-opening angle $\theta_m = \pi - \theta_0$; from (7.1) we have $k = 1$, $\theta_0 = \pi - \theta_m$. If $\varepsilon = 0$, the elliptic cone degenerates into a plane sector in the (y, z)-plane, of total opening angle $2\theta_m$; see Fig. 2.2. In this case, $k = \cos\theta_m$, $\theta_0 = \pi$ by (7.1).

In sphero-conal coordinates the space domain D exterior to the cone ∂D is described by

$$(7.2) \qquad D = \{(r, \theta, \phi) \mid r > 0,\ 0 \le \theta < \theta_0,\ 0 \le \phi \le 2\pi\}.$$

Let the medium in D be free space with permittivity ε_0 and permeability μ_0. To determine the electromagnetic field singularities at the tip of the cone ∂D, we start from the source-free Maxwell's equations

$$(7.3) \qquad \begin{aligned} &\nabla \times \mathbf{E} - i\omega\mu_0\mathbf{H} = \mathbf{0}, \quad \nabla\cdot\mathbf{E} = 0, \\ &\nabla \times \mathbf{H} + i\omega\varepsilon_0\mathbf{E} = \mathbf{0}, \quad \nabla\cdot\mathbf{H} = 0, \end{aligned}$$

for the time-harmonic electromagnetic field $\mathbf{E}(\mathbf{x})e^{-i\omega t}$, $\mathbf{H}(\mathbf{x})e^{-i\omega t}$ in D. In sphero-conal coordinates the electromagnetic field has components $\mathbf{E} = (E_r, E_\theta, E_\phi)$, $\mathbf{H} = (H_r, H_\theta, H_\phi)$. The elliptic cone ∂D is perfectly conducting which implies that the tangential electric field components should vanish on ∂D:

$$(7.4.) \qquad E_r = E_\phi = 0, \ \ r \ge 0,\ \theta = \theta_0,\ 0 \le \phi \le 2\pi\ .$$

In addition, the electromagnetic field must satisfy the so-called tip condition. This condition requires that the electromagnetic energy contained in any finite volume V about the tip of the cone ∂D must be finite, i.e.

$$\iiint\limits_V [\varepsilon_0 \mid \mathbf{E} \mid^2 + \mu_0 \mid \mathbf{H} \mid^2] \, dV < \infty \, . \tag{7.5}$$

With reference to [7, Sec. I.2.6], the electromagnetic field $\mathbf{E}, \mathbf{H}$ in D is represented in terms of two Debye potentials Π_e and Π_m:

$$\begin{aligned} \mathbf{E} &= \nabla \times \nabla \times (r \Pi_e \mathbf{e}_r) + i\omega\mu_0 \nabla \times (r \Pi_m \mathbf{e}_r), \\ \mathbf{H} &= -i\omega\varepsilon_0 \nabla \times (r \Pi_e \mathbf{e}_r) + \nabla \times \nabla \times (r \Pi_m \mathbf{e}_r), \end{aligned} \tag{7.6}$$

where $\mathbf{e}_r$ is the unit vector in the radial direction. The scalar potentials Π_e and Π_m must satisfy the Helmholtz equation

$$\Delta \Pi_{e,m} + k_0^2 \Pi_{e,m} = 0, \tag{7.7}$$

where $k_0 = \omega\sqrt{\varepsilon_0 \mu_0}$. By use of the well-known rules of vector analysis, we elaborate the representation (7.6) in components in sphero-conal coordinates. As a result it is found that the electromagnetic field components due to the potential Π_e are given by

$$\begin{aligned} E_r &= -\frac{1}{r} \Delta_t \Pi_e, & H_r &= 0, \\ E_\theta &= \frac{1}{h_\theta} \frac{\partial^2}{\partial r \partial \theta} (r \Pi_e), & H_\theta &= -\frac{i\omega\varepsilon_0 r}{h_\phi} \frac{\partial \Pi_e}{\partial \phi}, \\ E_\phi &= \frac{1}{h_\phi} \frac{\partial^2}{\partial r \partial \phi} (r \Pi_e), & H_\phi &= \frac{i\omega\varepsilon_0 r}{h_\theta} \frac{\partial \Pi_e}{\partial \theta}, \end{aligned} \tag{7.8}$$

while the field components due to the potential Π_m are given by

$$\begin{aligned} E_r &= 0, & H_r &= -\frac{1}{r} \Delta_t \Pi_m, \\ E_\theta &= \frac{i\omega\mu_0 r}{h_\phi} \frac{\partial \Pi_m}{\partial \phi}, & H_\theta &= \frac{1}{h_\theta} \frac{\partial^2}{\partial r \partial \theta} (r \Pi_m), \\ E_\phi &= -\frac{i\omega\mu_0 r}{h_\theta} \frac{\partial \Pi_m}{\partial \theta}, & H_\phi &= \frac{1}{h_\phi} \frac{\partial^2}{\partial r \partial \phi} (r \Pi_m). \end{aligned} \tag{7.9}$$

Here, h_θ and h_ϕ are the metrical coefficients of the sphero-conal coordinate system as presented in (2.3); Δ_t stands for the transverse Laplacian given by (2.5).

Because of $H_r = 0$ in (7.8) and $E_r = 0$ in (7.9), the corresponding electromagnetic fields are recognized to be transverse magnetic (TM-field) and transverse electric (TE-field), respectively. Thus the representation (7.6) involves a decomposition of the original electromagnetic field into a TM-field represented by the potential Π_e, and a TE-field represented by the potential Π_m.

Next we establish the boundary conditions for Π_e and Π_m on the cone ∂D. In the case of a TE-field, it readily follows from (7.4) and (7.9) that Π_m must satisfy the Neumann boundary condition

$$\text{(7.10)} \quad \frac{\partial \Pi_m}{\partial \theta} = 0, \quad r \geq 0, \ \theta = \theta_0, \ 0 \leq \phi \leq 2\pi.$$

In the case of a TM-field, we are led to the boundary conditions

$$\text{(7.11)} \quad \Delta_t \Pi_e = 0, \quad \frac{\partial^2}{\partial r \partial \phi}(r\Pi_e) = 0, \ r \geq 0, \quad \theta = \theta_0, \ 0 \leq \phi \leq 2\pi,$$

derived from (7.4) and (7.8). By means of (2.4) and the Helmholtz equation (7.7) for Π_e, the first condition in (7.11) is reduced to

$$\text{(7.12)} \quad \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \frac{\partial \Pi_e}{\partial r}) + k_0^2 \Pi_e = 0, \quad r \geq 0, \ \theta = \theta_0, \ 0 \leq \phi \leq 2\pi.$$

By integration of (7.12) and of the second condition in (7.11) we obtain

$$\text{(7.13)} \quad \Pi_e(r, \theta_0, \phi) = \alpha \frac{e^{ik_0 r}}{r} + \beta \frac{e^{-ik_0 r}}{r}, \quad r \geq 0, \ 0 \leq \phi \leq 2\pi,$$

where α and β are arbitrary constants. Consider now the potential

$$\text{(7.14)} \quad \tilde{\Pi}_e(r, \theta, \phi) = \alpha \frac{e^{ik_0 r}}{r} + \beta \frac{e^{-ik_0 r}}{r},$$

defined in the domain D. Obviously, $\tilde{\Pi}_e$ satisfies the Helmholtz equation (7.7) in D and assumes the boundary value (7.13) on ∂D. Furthermore, the TM-field (7.8) associated with $\tilde{\Pi}_e$, vanishes identically. Thus the potentials Π_e and $\Pi_e - \tilde{\Pi}_e$ give rise to the same TM-field.

Therefore, without loss of generality we may require that Π_e satisfies the Dirichlet boundary condition

$$(7.15)\quad \Pi_e = 0,\;\; r \geq 0,\; \theta = \theta_0,\;\; 0 \leq \phi \leq 2\pi.$$

The latter condition replaces (7.13).
Summarizing, we have shown that the potentials Π_e and Π_m of the TM- and TE-fields are solutions of the following boundary value problems for the Helmholtz equation:

$$(7.16)\quad \Delta\Pi_e + k_0^2\Pi_e = 0,\;\; \mathbf{x} \in D;\;\; \Pi_e = 0,\;\; \mathbf{x} \in \partial D,$$

and

$$(7.17)\quad \Delta\Pi_m + k_0^2\Pi_m = 0,\;\; \mathbf{x} \in D;\;\; \frac{\partial \Pi_m}{\partial n} = 0,\;\; \mathbf{x} \in \partial D,$$

where $\mathbf{n}$ denotes the outward unit normal to ∂D.

The problems (7.16) and (7.17) were already announced in (3.2)-(3.4), and were subsequently solved by separation of variables. In Secs. 3-5 it was shown that each of the problems (7.16) and (7.17) has an infinite number of solutions. Referring tot (3.5) and (5.29)-(5.32), we present the following solutions for the potentials Π_e and Π_m:

$$(7.18)\; \Pi_e(r,\theta,\phi) = \begin{cases} R(r)L_{cp\nu}^{(m)}(\theta)L_{c\nu}^{(m)}(\phi), & \nu = \nu c_n^m,\; m = 0,1,2,\dots,\; n = 1,2,3,\dots, \\ \\ R(r)L_{sp\nu}^{(m)}(\theta)L_{s\nu}^{(m)}(\phi), & \nu = \nu s_n^m,\; m = 1,2,3,\dots,\; n = 1,2,3,\dots; \end{cases}$$

$$(7.19)\; \Pi_m(r,\theta,\phi) = \begin{cases} R(r)L_{cp\nu}^{(m)}(\theta)L_{c\nu}^{(m)}(\phi), & \nu = \dot{\nu} c_n^m,\; m = 0,1,2,\dots,\; n = 1,2,3,\dots, \\ \\ R(r)L_{sp\nu}^{(m)}(\theta)L_{s\nu}^{(m)}(\phi), & \nu = \dot{\nu} s_n^m,\; m = 1,2,3,\dots,\; n = 1,2,3,\dots. \end{cases}$$

Here, the values $\nu = \nu c_n^m$, νs_n^m, $\dot{\nu} c_n^m$, $\dot{\nu} s_n^m$ are the ν-roots of the transcendental equations (5.21) and (5.22); these ν-roots are amply dicussed in Sec. 6 and Appendix B. The function $R(r)$ in (7.18) and (7.19) is a solution of the differential equation (3.7) for the spherical Bessel functions. From (3.12) we choose

$$(7.20)\quad R(r) = j_\nu(k_0 r) = \sqrt{\pi/2k_0 r}\, J_{\nu+1/2}(k_0 r)$$

in compliance with the tip condition (7.5). Indeed, the alternative $R(r) = y_\nu(k_0 r)$ would give rise to the field behaviour $\mathbf{E} = O(r^{-\nu-2})$ or $\mathbf{H} = O(r^{-\nu-2})$ as $r \to 0$, for TM- and TE-fields, respectively; since $\nu > 0$, the condition (7.5) would be violated. The potentials Π_e and Π_m give rise to TM- and TE-fields with components given by (7.8) and (7.9), respectively. By use of (3.8) and (5.29)-(5.32), we may simplify the expressions for the field components E_r in (7.8) and H_r in (7.9), viz.

$$(7.21) \quad E_r = -\frac{1}{r}\Delta_t \Pi_e = \frac{\nu(\nu+1)}{r}\Pi_e, \quad \text{(TM–field)}$$

$$(7.22) \quad H_r = -\frac{1}{r}\Delta_t \Pi_m = \frac{\nu(\nu+1)}{r}\Pi_m, \quad \text{(TE–field)} \ .$$

Below (5.24) it was found that $\dot{\nu}c_1^0 = 0$. For $\nu = 0$ the potential Π_m of (7.19) reduces to

$$(7.23) \quad \Pi_m(r, \theta, \phi) = \text{const.}\ j_0(k_0 r) \ .$$

It is easily seen that the associated TE-field (7.9) vanishes identically. Therefore, the solution (7.19) with $\nu = \dot{\nu}c_1^0$ is henceforth discarded.

We now examine the behaviour of the electromagnetic field as $r \to 0$, i.e. at the tip of the elliptic cone. Since $R(r) = j_\nu(k_0 r) = O(r^\nu)$ as $r \to 0$, we infer from (7.18) and (7.19) that

$$(7.24) \quad \Pi_e = O(r^\nu),\ \nu = \nu c_n^m, \nu s_n^m; \quad \Pi_m = O(r^\nu),\ \nu = \dot{\nu}c_n^m, \dot{\nu}s_n^m \ .$$

These results are used in (7.8), (7.9), (7.21) and (7.22), to establish the behaviour of the TM- and TE-field components as $r \to 0$:

$$(7.25) \quad \mathbf{E} = O(r^{\nu-1}), \ \ \mathbf{H} = O(r^\nu), \ \ \nu = \nu c_n^m, \nu s_n^m, \ \text{(TM–field)} \ ;$$

$$(7.26) \quad \mathbf{E} = O(r^\nu), \ \ \mathbf{H} = O(r^{\nu-1}), \ \ \nu = \dot{\nu}c_n^m, \dot{\nu}s_n^m, \ \text{(TE–field)} \ .$$

Since $\nu > 0$, the magnetic field in (7.25) and the electric field in (7.26) remain finite and even vanish at the tip of the elliptic cone. Singularities of the electric field in (7.25) and of the magnetic field in (7.26) will arise if any of the ν-roots νc_n^m, νs_n^m and $\dot{\nu}c_n^m$, $\dot{\nu}s_n^m$, is less than 1. According to (6.31), the only ν-roots less than 1 are given by νc_1^0, $\dot{\nu}c_1^1$ and $\dot{\nu}s_1^1$, provided that $\pi/2 < \theta_0 \le \pi$. The latter restriction implies that electromagnetic field singularities at the tip of the cone ∂D only occur in the exterior domain D and not in the interior of the cone.

Accordingly, two basic singularities must be considered for the electromagnetic field at the tip of the elliptic cone, as it is now detailed.

(i) Consider the TM-field with components (7.8), (7.21), and represented by the potential

$$(7.27) \qquad \Pi_e = j_\nu(k_0 r) L^{(0)}_{cp\nu}(\theta)\, L^{(0)}_{c\nu}(\phi), \quad \nu = \nu c^0_1 \ .$$

According to (7.25), the electric field becomes infinite like $r^{\nu-1}$ as $r \to 0$, whereas the magnetic field remains finite. The present field singularity is called the electric singularity with singularity exponent $\nu = \nu c^0_1$. The Lamé functions in (7.27) are of class I which implies an even/even symmetry of Π_e with respect to the major/minor axis of the elliptic cross-section of the cone.

(ii) Consider the TE-field with components (7.9), (7.22), and represented by either of the potentials

$$(7.28) \qquad \Pi^{(1)}_m = j_\nu(k_0 r) L^{(1)}_{cp\nu}(\theta)\, L^{(1)}_{c\nu}(\phi), \quad \nu = \dot{\nu} c^1_1 \ ,$$

$$(7.29) \qquad \Pi^{(2)}_m = j_\nu(k_0 r) L^{(1)}_{sp\nu}(\theta)\, L^{(1)}_{s\nu}(\phi), \quad \nu = \dot{\nu} s^1_1 \ .$$

According to (7.26), the magnetic field becomes infinite like $r^{\nu-1}$ as $r \to 0$, whereas the electric field remains finite. The present field singularities are called the first and second magnetic singularities with singularity exponents $\nu = \dot{\nu} c^1_1$ and $\nu = \dot{\nu} s^1_1$, respectively. The Lamé functions in (7.28) and (7.29) are of classes II and IV, respectively. This implies an odd/even symmetry of $\Pi^{(1)}_m$ and an even/odd symmetry of $\Pi^{(2)}_m$ with respect to the major/minor axis of the elliptic cross-section of the cone.

The present results for the singularity exponents and for the symmetries of the potentials agree with those of Blume and Kahl [4], De Smedt and Van Bladel [22]. In these references, the question which of the ν-roots are less than 1, was settled by numerical calculations, whereas our result (6.31) is proved in Appendix B by analytical means.

Numerical results for the singularity exponents νc^0_1, $\dot{\nu} c^1_1$ and $\dot{\nu} s^1_1$, as functions of the maximum semi-opening angle θ_m and the axial ratio ε, are presented in Tables 7.1, 7.2 and 7.3. The numerical values of νc^0_1, $\dot{\nu} c^1_1$ and $\dot{\nu} s^1_1$, have been calculated correct to five decimal places by the procedure outlined at the end of Sec. 6.

TABLE 7.1. Singularity exponent νc_1^0 of the electric singularity.

θ_m	$\varepsilon = 0$	$\varepsilon = 0.1$	$\varepsilon = 0.2$	$\varepsilon = 0.4$	$\varepsilon = 0.6$	$\varepsilon = 0.8$	$\varepsilon = 1$
5°	0.13004	0.13331	0.13644	0.14236	0.14790	0.15315	0.15814
10	0.15822	0.16308	0.16777	0.17674	0.18525	0.19339	0.20122
20	0.20168	0.20969	0.21752	0.23266	0.24719	0.26114	0.27450
30	0.24010	0.25181	0.26333	0.28573	0.30712	0.32732	0.34618
40	0.27759	0.29410	0.31042	0.34203	0.37155	0.39838	0.42228
45	0.29658	0.31615	0.33553	0.37283	0.40695	0.43709	0.46310
50	0.31595	0.33924	0.36232	0.40628	0.44535	0.47862	0.50629
60	0.35635	0.39035	0.42392	0.48504	0.53447	0.57250	0.60151
70	0.39982	0.45393	0.50602	0.59043	0.64747	0.68555	0.71180
80	0.44736	0.55612	0.64486	0.74709	0.79697	0.82489	0.84225
90	0.50000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000

TABLE 7.2. Singularity exponent νc_1^1 of the first magnetic singularity.

θ_m	$\varepsilon = 0$	$\varepsilon = 0.1$	$\varepsilon = 0.2$	$\varepsilon = 0.4$	$\varepsilon = 0.6$	$\varepsilon = 0.8$	$\varepsilon = 1$
0°	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
5	0.99807	0.99788	0.99769	0.99730	0.99692	0.99655	0.99618
10	0.99209	0.99131	0.99053	0.98902	0.98756	0.98617	0.98486
20	0.96615	0.96308	0.96020	0.95509	0.95091	0.94767	0.94533
30	0.91904	0.91339	0.90872	0.90222	0.89913	0.89879	0.90051
40	0.85257	0.84680	0.84345	0.84287	0.84817	0.85690	0.86719
45	0.81466	0.81047	0.80955	0.81526	0.82729	0.84202	0.85717
50	0.77543	0.77409	0.77686	0.79105	0.81109	0.83220	0.85196
60	0.69748	0.70604	0.72062	0.75847	0.79676	0.82964	0.85631
70	0.62461	0.65243	0.68844	0.75956	0.81416	0.85304	0.88093
80	0.55880	0.64037	0.72058	0.82258	0.87608	0.90741	0.92757
90	0.50000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000

TABLE 7.3. Singularity exponent $\dot{\nu}s_1^1$ of the second magnetic singularity.

θ_m	$\varepsilon = 0$	$\varepsilon = 0.1$	$\varepsilon = 0.2$	$\varepsilon = 0.4$	$\varepsilon = 0.6$	$\varepsilon = 0.8$	$\varepsilon = 1$
0°	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
5	1.00000	0.99979	0.99954	0.99893	0.99817	0.99725	0.99618
10	1.00000	0.99917	0.99819	0.99577	0.99274	0.98910	0.98486
20	1.00000	0.99683	0.99309	0.98393	0.97272	0.95973	0.94533
30	1.00000	0.99340	0.98570	0.96740	0.94624	0.92353	0.90051
40	1.00000	0.98955	0.97757	0.95041	0.92144	0.89318	0.86719
45	1.00000	0.98771	0.97378	0.94303	0.91172	0.88267	0.85717
50	1.00000	0.98606	0.97045	0.93698	0.90457	0.87599	0.85196
60	1.00000	0.98379	0.96602	0.93048	0.89974	0.87530	0.85631
70	1.00000	0.98361	0.96607	0.93434	0.91055	0.89343	0.88093
80	1.00000	0.98683	0.97365	0.95406	0.94167	0.93340	0.92757
90	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000

We will now discuss our numerical results in comparison with results from the literature. In Tables 7.2 and 7.3, the entries $\dot{\nu}c_1^1 = 1$ and $\dot{\nu}s_1^1 = 1$ at $\theta_m = 0$, can be shown to be exact. For $\theta_m = \pi/2$, $0 < \varepsilon \leq 1$, one has $k = \varepsilon$, $\theta_0 = \pi/2$ by (7.1), and the elliptic cone degenerates into the (x, y)-plane; the corresponding values $\nu c_1^0 = 1$, $\dot{\nu}c_1^1 = 1$ and $\dot{\nu}s_1^1 = 1$ follow from (5.39). For $\theta_m = \pi/2$, $\varepsilon = 0$, the elliptic cone degenerates into the half-plane $x = 0$, $z \leq 0$, with its edge along the y-axis. By a direct calculation of the edge singularity, we find $\nu c_1^0 = 0.5$, $\dot{\nu}c_1^1 = 0.5$ and $\dot{\nu}s_1^1 = 1$, in accordance with the entries in Tables 7.1-7.3. Graphical results for the singularity exponents, as functions of the maximum and minimum semi-opening angles of the elliptic cone, were presented by Blume and Kahl [4]. Their analysis is very similar to ours, and their eigenvalues $\mu_{\min}$ and $\nu_{\min}$ (with eigenfunctions of type 2 or 4) agree with our νc_1^0, $\dot{\nu}c_1^1$ and $\dot{\nu}s_1^1$, respectively. De Smedt and Van Bladel [22] established a general variational principle for the singularity exponents pertaining to a perfectly conducting cone of arbitrary cross-section. This variational principle was implemented in a numerical procedure for the approximate calculation of the singularity exponents for an elliptic cone.

These singularity exponents are denoted by ν, τ_{oe} and τ_{eo} (identical to our νc_1^0, $\dot{\nu} c_1^1$ and $\dot{\nu} s_1^1$, respectively), and the cone is specified by the maximum semi-opening angle θ_m and the axial ratio ε. The data in [22, Tables I, III, IV] turn out to be correct to two or three decimal places, as it follows from a direct comparison with our Tables 7.1, 7.2, 7.3.

In the special case $\varepsilon = 1$, one has $k = 1$, $\theta_0 = \pi - \theta_m$ by (7.1), and the elliptic cone becomes a circular cone of semi-opening angle $\theta_m = \pi - \theta_0$. For $k = 1$, the ν-roots coalesce by (5.28), so that $\nu c_1^0 = \nu_1^0$, $\dot{\nu} c_1^1 = \dot{\nu} s_1^1 = \dot{\nu}_1^1$, where ν_1^0 and $\dot{\nu}_1^1$ are the smallest positive ν-zeros of $P_\nu(\cos\theta_0)$ and $(d/d\theta_0)P_\nu^1(\cos\theta_0)$, respectively; see (5.25) and (5.26). Notice that the $(\varepsilon = 1)$-columns of Tables 7.2 and 7.3 are indeed identical. The numerical values in [22, Tables I, III, IV] for $\varepsilon = 1$ do agree, up to four or five decimal places, with the corresponding data in Tables 7.1, 7.2, 7.3. Previous numerical results for the singularity exponents ν and τ (identical to our ν_1^0 and $\dot{\nu}_1^1$, respectively) for a circular cone were presented by Van Bladel [2]. Recently, Marchetti and Rozzi [16] determined the electric field behaviour at the tip of various circular conical structures. The electric singularity exponent for a circular cone, correct to seven decimal places, can be obtained from [16, Table I, column 5a]. When rounded to five decimal places, the pertaining values reduce precisely to the data in the $(\varepsilon = 1)$-column of Table 7.1.

In the special case $\varepsilon = 0$, the elliptic cone degenerates into a plane sector in the (y, z)-plane, of total opening angle $2\theta_m$; from (7.1) we have $k = \cos\theta_m$, $\theta_0 = \pi$. Notice that $\dot{\nu} s_1^1 = 1$ at $\theta_0 = \pi$ by (5.41), which is confirmed by the data in the $(\varepsilon = 0)$-column of Table 7.3. Hence, the second magnetic singularity disappears. Following De Smedt and Van Bladel [22], [23], we change the notations νc_1^0 and $\dot{\nu} c_1^1$ for the remaining singularity exponents into $\nu = \nu(\alpha)$ and $\tau = \tau(\alpha)$, respectively, where $\alpha = 2\theta_m$ is the total opening angle of the sector. The data for $\nu(\alpha)$ and $\tau(\alpha)$ as presented in the $(\varepsilon = 0)$-column of Tables 7.1 and 7.2, apply over the range $0 \leq \alpha \leq \pi$, corresponding to a sector with a salient corner. These data are extended to the case of a reentrant corner, where $\pi \leq \alpha \leq 2\pi$, by means of the symmetry relation

$$\nu(\alpha) = \tau(2\pi - \alpha) \ . \tag{7.30}$$

The latter relation expresses that the electric singularity exponent for a conducting plane sector is identical to the magnetic singularity exponent for the complementary sector. The equality (7.30), which was observed more or less casually in [5], [22], [23], is a consequence of Babinet's principle, as it has been proved by Boersma [6]. The data for $\nu(\alpha)$ and $\tau(\alpha)$, thus extended, have been collected in Table 7.4.

TABLE 7.4. Singularity exponents ν and τ for a sector of opening angle α.

salient corner			reentrant corner		
α	ν	τ	α	ν	τ
10°	0.13004	0.99807	180°	0.50000	0.50000
20	0.15822	0.99209	200	0.55880	0.44736
40	0.20168	0.96615	220	0.62461	0.39982
60	0.24010	0.91904	240	0.69748	0.35635
80	0.27759	0.85257	260	0.77543	0.31595
90	0.29658	0.81466	270	0.81466	0.29658
100	0.31595	0.77543	280	0.85257	0.27759
120	0.35635	0.69748	300	0.91904	0.24010
140	0.39982	0.62461	320	0.96615	0.20168
160	0.44736	0.55880	340	0.99209	0.15822
180	0.50000	0.50000	350	0.99807	0.13004

De Smedt and Van Bladel [22], [23] employed a variational method for the calculation of the singularity exponents for a sector. Their numerical data in [22, Tables I, II, III for $\varepsilon = 0$], [23, Table 1] show an agreement up to three decimal places with the values in Table 7.4. Kraus [15] was the first to analyse the field singularities at the tip of a plane sector. In [15, pp. 47-49], some preliminary calculations of the singularity exponent n_0 (identical to our ν or τ) have been made, however, the results are inaccurate. Crucial in Kraus's analysis is the introduction of sphero-conal coordinates represented in trigonometric form (as in (2.1)), rather than in terms of Jacobian elliptic functions. Separation of variables in these coordinates leads to trigonometric forms of Lamé's differential equation (as in (3.15) and (3.16)), which are much easier to handle than the Jacobian form. By use of the same sphero-conal coordinates, the work of Kraus has been continued and improved by Satterwhite and Kouyoumjian [21], Sahalos and Thiele [20]. In [21, p. 124], there is a short table of the singularity exponents ν_{e1} and ν_{o2} (identical to our ν and τ, respectively) for a sector of opening angle $\alpha = 0.205\pi$, $\alpha = \pi/2$ and $\alpha = 0.795\pi$. As an example, we quote the values $\nu = 0.296$ and $\tau = 0.814$ of the singularity exponents for the quarter plane with $\alpha = \pi/2$; these values agree very well with the data in Table 7.4. It is interesting to mention that in [21, Appendix B] the singularity exponents are determined graphically from the intersections of certain eigenvalue curves; see [21, Figs. 29-32] which should be compared to our Figs. 6.1 and 6.2. Graphical results for the eigenvalues ν_{1e1} and ν_{1o2} (identical to our ν and τ, respectively), as functions of the parameter $k^2 = \cos^2(\alpha/2)$, were presented in [20, Fig. 3]. The electric singularity exponent ν also determines the singularity in surface charge density at the tip of a sectorial conducting plate placed in an electrostatic field. The strength of the charge singularity was calculated numerically by Morrison and Lewis [17], again by use of sphero-conal coordinates represented in trigonometric form. In addition, for a sector of opening angle close to 0, π or 2π, analytical expressions for the singularity exponent were found by singular perturbation techniques. It has been verified that the numerical values in [17, Table 1], correct to four decimal places, do agree with the data in Table 7.4.

Sphero-conal coordinates represented in terms of Jacobian elliptic functions were employed by Blume and Kirchner [5], Marchetti and Rozzi [16]. Their analysis of the field singularities at the tip of a plane sector is more complicated, because separation of variables now leads to the Jacobian form of Lamé's differential equation. The numerical values of the eigenvalue $\nu_{\min}$ (identical to our ν or τ), presented in [5, Tabelle 3], differ from the data in Table 7.4 by at most one unit of the fifth decimal. The electric singularity exponent for a plane sector, correct to seven decimal places, can be obtained from [16, Table II, column 6a]. When rounded to five decimal places, the pertaining values precisely reduce to the data in the ν-columns of Table 7.4.

Appendix A. Interlacing property of eigenvalues of Sturm-Liouville eigenvalue problems

In this appendix we shall prove an interlacing property of the eigenvalues of two regular Sturm-Liouville eigenvalue problems that differ in just one boundary condition. Consider the two eigenvalue problems for the Sturm-Liouville differential equation

(A.1) $\quad (p(x)u')' + (\lambda r(x) - q(x))u = 0, \quad a < x < b,$

with boundary conditions either

(A.2) $\quad u'(a) = 0, \quad u'(b) = 0,$

or

(A.3) $\quad u(a) = 0, \quad u'(b) = 0 .$

Here the functions $p(x)$, $p'(x)$, $q(x)$ and $r(x)$ are continuous on $[a,b]$, while $p(x) > 0$ and $r(x) > 0$ on $[a,b]$. It is well known (see Coddington and Levinson [8, p. 212, Thm. 2.1]) that both problems admit an infinite number of simple eigenvalues which can be arranged into two monotonic increasing, unbounded sequences. The eigenvalues of the problem with boundary conditions (A.2) and (A.3) are denoted by λ_n and $\tilde{\lambda}_n$, $n = 0, 1, 2, \ldots$, respectively, ordered by ascending magnitude with $\lambda_n \to \infty$, $\tilde{\lambda}_n \to \infty$ as $n \to \infty$. The corresponding eigenfunctions are denoted by $u_n(x)$ and $\tilde{u}_n(x)$, respectively; both functions have exactly n zeros on (a,b). We now establish the following lemma.

LEMMA A.1. The eigenvalues $\lambda_n, \tilde{\lambda}_n$ have the interlacing property

(A.4) $\quad \lambda_n < \tilde{\lambda}_n < \lambda_{n+1}, \quad n = 0, 1, 2, \ldots .$

PROOF. The proof is indirect and proceeds in three steps.
(i) Suppose $\lambda_m = \tilde{\lambda}_n$ for a certain pair m, n. Then the corresponding eigenfunctions $u_m(x)$ and $\tilde{u}_n(x)$ are solutions of the same Sturm-Liouville equation (A.1) with $\lambda = \lambda_m = \tilde{\lambda}_n$. Because of the common boundary condition $u'_m(b) = \tilde{u}'_n(b) = 0$, the Wronskian $W(u_m, \tilde{u}_n)$ vanishes identically which implies that $u_m(a) = 0$ and/or $\tilde{u}'_n(a) = 0$. Combined with (A.2) and (A.3) we then find that $u_m(x) \equiv 0$ and/or $\tilde{u}_n(x) \equiv 0$, in contradiction with the fact that

$u_m(x)$ and $\tilde{u}_n(x)$ are eigenfunctions. Thus we conclude that $\lambda_m \neq \tilde{\lambda}_n$ for any pair m, n.
(ii) Suppose $\lambda_n > \tilde{\lambda}_n$. According to Sturm's comparison theorem [8, p. 208, Thm. 1.1] the function $u_n(x)$ has at least one zero between any two successive zeros of $\tilde{u}_n(x)$. Let the zeros of $u_n(x)$ and $\tilde{u}_n(x)$ on (a,b) be denoted by x_i and $\tilde{x}_i$, $i = 1,2,...,n$, respectively. Then we are led to the following ordering of zeros:

$$a < x_1 < \tilde{x}_1 < x_2 < \tilde{x}_2 < ... < x_n < \tilde{x}_n < b,$$

where it is recalled that $\tilde{u}_n(a) = 0$. From the differential equations for $u_n(x)$ and $\tilde{u}_n(x)$ we deduce the relation

$$\text{(A.5)} \qquad \frac{d}{dx}[p(x)(u_n'(x)\tilde{u}_n(x) - u_n(x)\tilde{u}_n'(x))] = -(\lambda_n - \tilde{\lambda}_n)r(x)u_n(x)\tilde{u}_n(x),$$

which is integrated over the interval $[\tilde{x}_n, b]$, with $\tilde{x}_n = a$ if $n = 0$. As a result we find

$$\text{(A.6)} \qquad p(\tilde{x}_n)u_n(\tilde{x}_n)\,\tilde{u}_n'(\tilde{x}_n) = -(\lambda_n - \tilde{\lambda}_n)\int_{\tilde{x}_n}^{b} r(x)u_n(x)\tilde{u}_n(x)dx$$

by use of the boundary conditions $u_n'(b) = \tilde{u}_n'(b) = 0$ and $\tilde{u}_n(\tilde{x}_n) = 0$. Without loss of generality we may assume that $u_n(x) > 0$, $\tilde{u}_n(x) > 0$ on $(\tilde{x}_n, b)$, so that the right-hand side of (A.6) is negative. The left-hand side of (A.6) is positive because of $u_n(\tilde{x}_n) > 0$, $\tilde{u}_n'(\tilde{x}_n) > 0$. Thus we arrive at a contradiction and we conclude that $\lambda_n < \tilde{\lambda}_n$, $n = 0,1,2,....$
(iii) Suppose $\tilde{\lambda}_n > \lambda_{n+1}$. Again by Sturm's comparison theorem the function $\tilde{u}_n(x)$ has at least one zero between any two successive zeros of $u_{n+1}(x)$. Let the zeros of $u_{n+1}(x)$ and $\tilde{u}_n(x)$ be denoted by $x_1, x_2, ..., x_{n+1}$ and $\tilde{x}_1, \tilde{x}_2, ..., \tilde{x}_n$, respectively, then we have the following ordering of zeros:

$$a < x_1 < \tilde{x}_1 < x_2 < \tilde{x}_2 < ... < \tilde{x}_n < x_{n+1} < b\ .$$

By integration of (A.5) with λ_n, u_n replaced by λ_{n+1}, u_{n+1}, over the interval $[x_{n+1}, b]$ we find

$$\text{(A.7)} \qquad -p(x_{n+1})u_{n+1}'(x_{n+1})\tilde{u}_n(x_{n+1}) = -(\lambda_{n+1} - \tilde{\lambda}_n)\int_{x_{n+1}}^{b} r(x)u_{n+1}(x)\tilde{u}_n(x)dx$$

by use of the boundary conditions $u_{n+1}'(b) = \tilde{u}_n'(b) = 0$ and $u_{n+1}(x_{n+1}) = 0$. Without loss of generality we may assume that $u_{n+1}(x) > 0$, $\tilde{u}_n(x) > 0$ on (x_{n+1}, b), so that the right-hand side of (A.7) is positive. The left-hand side of (A.7) is negative because of $u_{n+1}'(x_{n+1}) > 0$, $\tilde{u}_n(x_{n+1}) > 0$. Thus we arrive at a contradiction and we conclude that

$\tilde{\lambda}_n < \lambda_{n+1},\ n = 0, 1, 2, ...$ $\square$

Consider next the two eigenvalue problems for the Sturm-Liouville equation (A.1) with boundary conditions either

$$\text{(A.8)} \quad u'(a) = 0, \quad u(b) = 0,$$

or

$$\text{(A.9)} \quad u(a) = 0, \quad u(b) = 0 \ .$$

Let the corresponding eigenvalues be denoted by μ_n and $\tilde{\mu}_n$, $n = 0, 1, 2, ...$, respectively, ordered by ascending magnitude with $\mu_n \to \infty$, $\tilde{\mu}_n \to \infty$ as $n \to \infty$. Then we have the interlacing property

$$\text{(A.10)} \quad \mu_n < \tilde{\mu}_n < \mu_{n+1}, \quad n = 0, 1, 2, ...,$$

the proof of which is almost identical to that of Lemma A.1.

Finally we compare the two eigenvalue problems for the Sturm-Liouville equation (A.1) with boundary conditions (A.2) and (A.8), and the two problems with boundary conditions (A.3) and (A.9). The pertaining eigenvalue problems differ in just one boundary condition, i.e. $u'(b) = 0$ versus $u(b) = 0$. Similar to Lemma A.1 we have the interlacing properties

$$\text{(A.11)} \quad \lambda_n < \mu_n < \lambda_{n+1}, \quad n = 0, 1, 2, ...,$$

$$\text{(A.12)} \quad \tilde{\lambda}_n < \tilde{\mu}_n < \tilde{\lambda}_{n+1}, \quad n = 0, 1, 2, ... \ .$$

The results (A.4), (A.10), (A.11) and (A.12) are combined into the following lemma.

LEMMA A.2. The eigenvalues $\lambda_n, \tilde{\lambda}_n, \mu_n, \tilde{\mu}_n$ have the interlacing property

$$\text{(A.13)} \quad \lambda_0 < \begin{matrix} \tilde{\lambda}_0 \\ \mu_0 \end{matrix} < \begin{matrix} \lambda_1 \\ \tilde{\mu}_0 \end{matrix} < ... < \begin{matrix} \lambda_n \\ \tilde{\mu}_{n-1} \end{matrix} < \begin{matrix} \tilde{\lambda}_n \\ \mu_n \end{matrix} < \begin{matrix} \lambda_{n+1} \\ \tilde{\mu}_n \end{matrix} < ... \ .$$

The writing of two eigenvalues on top of each other indicates that their mutual ordering is not known.

The present lemmas are probably known. However, we have not been able to find a suitable reference to the vast literature on Sturm-Liouville eigenvalue problems. The interlacing property does appear as a problem in Dunford and Schwartz [10, p. 1547] and Reid [18, p. 165-166]. According to Reid, "the inequalities are some of the most classic of Sturmian theory".

Appendix B. Location of the ν-roots νc_n^m, νs_n^m, $\dot{\nu} c_n^m$, $\dot{\nu} s_n^m$ with respect to $\nu = 1$

This appendix deals with the question which of the ν-roots νc_n^m, νs_n^m, $\dot{\nu} c_n^m$, $\dot{\nu} s_n^m$ are < 1 and which are ≥ 1. It is shown that all roots are ≥ 1, except for $\dot{\nu} c_1^0 = 0$ and

(B.1) $\quad 0 < \nu c_1^0,\ \dot{\nu} c_1^1 < 1,$ when $\pi/2 < \theta_0 \leq \pi$; $\ 0 < \dot{\nu} s_1^1 < 1,$ when $\pi/2 < \theta_0 < \pi$.

Notice that $\dot{\nu} s_1^1 = 1$ at $\theta_0 = \pi$ by (5.41).
Consider first the roots νc_n^m and νs_n^m of the transcendental equations (5.21), which obviously depend on θ_0.

LEMMA B.1. The roots $\nu c_n^m = \nu c_n^m(\theta_0)$ and $\nu s_n^m = \nu s_n^m(\theta_0)$ are decreasing functions of θ_0 for $0 \leq \theta_0 \leq \pi$.
PROOF. The roots νc_n^m, νs_n^m can be determined as solutions of the equivalent transcendental equations (6.21), (6.22); see also Fig. 6.1. In the latter equations the left-hand sides μ_i^m, $i = I, II, III, IV$, are independent of θ_0, whereas the right-hand sides μ_A^n and μ_B^n are functions of θ_0 as it is readily seen from the variational expression (6.16). We now change θ_0 into $\theta_0' > \theta_0$ and, similar tot (6.12), we introduce the spaces S_A' and S_B' of functions $X(\theta)$ defined for $0 \leq \theta \leq \theta_0'$. Each function $X(\theta) \in S_A$ or S_B can be extended to a function $X(\theta) \in S_A'$ or S_B' , by setting $X(\theta) = 0$ for $\theta_0 \leq \theta \leq \theta_0'$; indeed, the extended function is continuous at $\theta = \theta_0$ and belongs to $PC^1[0, \theta_0']$. Thus we have the space inclusions $S_A \subset S_A'$ and $S_B \subset S_B'$. Furthermore, the functionals $D_\theta[X]$ and $H_\theta[X, X]$, as given by (6.13) and (6.14), remain the same under the extension of $X(\theta)$. Therefore, when changing θ_0 into $\theta_0' > \theta_0$, the min-max in (6.16) is to be taken over a "larger" set which implies that $\mu_A^n |_{\theta_0} \leq \mu_A^n |_{\theta_0'}$ and $\mu_B^n |_{\theta_0} \leq \mu_B^n |_{\theta_0'}$. Hence, μ_A^n and μ_B^n are increasing functions of θ_0. Then the eigenvalue curves $\mu = \mu_A^n(\nu)$ and $\mu = \mu_B^n(\nu)$ in Fig. 6.1 will move upwards with increasing θ_0 . As a result, the intersections with the (fixed) eigenvalue curves $\mu = \mu_i^m(\nu)$, at $\nu = \nu c_n^m$ and $\nu = \nu s_n^m$, will move to the left. This proves that νc_n^m and νs_n^m are decreasing functions of θ_0 . $\square$

NOTE. Numerical results have shown that there is no general monotony of the roots νc_n^m and νs_n^m as functions of θ_0 . This can be understood from the variational expression (6.16) for μ_n^C and μ_n^D, where the min-max is to be taken over the spaces S_C, S_D (see (6.12)) or S'_C, S'_D, if θ_0 is changed into θ'_0 . A function $X(\theta) \in S_C$ or S_D might again be extended by setting $X(\theta) = 0$ for $\theta_0 < \theta \leq \theta'_0$, however, the extended function is in general discontinuous at $\theta = \theta_0$ and does not belong to $PC^1[0, \theta'_0]$. Therefore, there exists no ordering of the spaces S_C, S_D and S'_C, S'_D, and the above proof fails.

LEMMA B.2. $\nu c_n^m,\ \nu s_n^m \geq 1,\ m = (0), 1, 2, \dots,\ n = 1, 2, 3, \dots,$ when $0 \leq \theta_0 \leq \pi/2$.
PROOF. The result follows from Lemma B.1 and the values $\nu c_n^m = \nu s_n^m = m + 2n - 1 \geq 1$ at $\theta_0 = \pi/2$, taken from (5.39).

LEMMA B.3. $0 < \nu c_1^0 < 1,\ 1 \leq \nu c_1^1 < 2$, when $\pi/2 < \theta_0 \leq \pi$.
PROOF. From (5.39) and (5.41) we have $\nu c_1^0 = 1,\ \nu c_1^1 = 2$ at $\theta_0 = \pi/2$, and $\nu c_1^1 = 1$ at $\theta_0 = \pi$. Next the results are obvious by Lemma B.1. Notice that $0 < \nu c_1^0$ by (6.23). □

From Fig. 6.1 it is recognized that, apart from νc_1^0 and νc_1^1, the smallest ν-roots of the transcendental equations (5.21) are given by νc_2^0 and νs_1^1 . In Lemma B.4 we shall prove that $\nu c_2^0,\ \nu s_1^1 \geq 1$, when $\pi/2 < \theta_0 \leq \pi$. The proof uses Sturm's comparison theorem [8, p. 208, Thm. 1.1] which is quoted here for convenience.

THEOREM B.1. Let $y_1(x)$ and $y_2(x)$ be real solutions of the differential equations

$$(p(x)y_i')' + g_i(x)y_i = 0, \quad a < x < b, \quad i = 1, 2,$$

where $p(x) > 0$ on (a, b). Let $g_1(x) < g_2(x)$ on (a, b). If x_1 and x_2 are consecutive zeros of $y_1(x)$ on $[a, b]$, then $y_2(x)$ has at least one zero beteen x_1 and x_2.

As a further preliminary we establish another Sturm-type comparison theorem:

THEOREM B.2. Let $y_1(x)$ and $y_2(x)$ be the solutions of the initial-value problems

$$(p(x)y_i')' + g_i(x)y_i = 0, \quad a < x < b; \quad y_i(a) = 1, \quad y_i'(a) = 0; \quad i = 1,2,$$

where $p(x) > 0$ on (a,b). Let $g_1(x) < g_2(x)$ on (a,b). If $y_1(x)$ has a smallest zero $x_1 \in (a,b]$, then $y_2(x)$ has at least one zero between a and x_1.

PROOF. Multiply the differential equations for y_1 and y_2 by y_2 and y_1, respectively, and subtract, then we obtain

$$\text{(B.2)} \qquad \frac{d}{dx}[p(x)(y_1'(x)y_2(x) - y_1(x)y_2'(x))] = [g_2(x) - g_1(x)]y_1(x)y_2(x)\ .$$

Integration of (B.2) over the interval $[\alpha,\beta] \subset [a,b]$ yields the Green's formula

$$\text{(B.3)} \qquad p(x)(y_1'(x)y_2(x) - y_1(x)y_2'(x))\Big|_{\alpha}^{\beta} = \int_{\alpha}^{\beta}[g_2(x) - g_1(x)]y_1(x)y_2(x)dx\ .$$

For $\alpha = a$ and $\beta = x_1$, the formula (B.3) specializes into

$$\text{(B.4)} \qquad p(x_1)y_1'(x_1)y_2(x_1) = \int_{a}^{x_1}[g_2(x) - g_1(x)]y_1(x)y_2(x)dx\ ,$$

where it has been used that $y_1'(a) = y_2'(a) = 0, \quad y_1(x_1) = 0$.

Obviously $y_1(x) > 0$ on $[a, x_1)$ and $y_1'(x_1) < 0$. Suppose now that $y_2(x)$ has no zeros between a and x_1 , so that $y_2(x) > 0$ on $[a, x_1)$. Then the right-hand side of (B.4) is positive, whereas the left-hand side of (B.4) is non-positive. Thus we arrive at a contradiction and we conclude that $y_2(x)$ has at least one zero between a and x_1 . □

Consider now the θ-Lamé equation (6.3) which is rewritten in the generic form

$$\text{(B.5)} \qquad (p(\theta)X')' + g(\nu,\theta)X = 0\ ,$$

where

$$\text{(B.6)} \qquad p(\theta) = \sqrt{1 - k^2\cos^2\theta}, \quad g(\nu,\theta) = \frac{\nu(\nu+1)k^2\sin^2\theta - \mu}{\sqrt{1-k^2\cos^2\theta}}\ .$$

By setting $\mu = \mu_i^m$, $i = I, II, III, IV$, $m = 1, 2, 3, \ldots$, as given by (6.5), the solution $X(\theta)$ is a non-periodic Lamé function of class I, II, III, IV, respectively. By use of the variational expression (6.11) it has been shown below (6.20) that $\mu_i^m = \mu_i^m(\nu)$ is a decreasing function of ν; see also Figs. 6.1 and 6.2. Hence, the function $g(\nu, \theta)$ in (B.6) is an increasing function of ν, which is the only property that is needed in the sequel.

The comparison theorems B.1 and B.2 will be applied to the solutions $X_1(\theta)$ and $X_2(\theta)$ of the differential equations

(B.7) $\quad (p(\theta)X_i')' + g(\nu_i, \theta)X_i = 0, \quad i = 1, 2$,

obtained from (B.5) by setting $\nu = \nu_1$ and $\nu = \nu_2$, respectively. Similar to (B.3), the associated Green's formula over some interval $[\alpha, \beta]$ is given by

(B.8) $\quad p(\theta)(X_1'(\theta)X_2(\theta) - X_1(\theta)X_2'(\theta))\Big|_\alpha^\beta = \int\limits_\alpha^\beta [g(\nu_2, \theta) - g(\nu_1, \theta)]X_1(\theta)X_2(\theta)d\theta$.

In the sequel we usually set $\nu_1 = 1$. According tot (5.36), the non-periodic Lamé functions with $\nu = 1$ take the form

(B.9) $\quad L_{cp,1}^{(0)}(\theta) = \cos\theta, \quad L_{cp,1}^{(1)}(\theta) = \sin\theta, \quad L_{sp,1}^{(1)}(\theta) = \sqrt{1 - k^2\cos^2\theta}$,

in which any multiplicative constants have been suppressed.

LEMMA B.4. νc_2^0, $\nu s_1^1 \geq 1$, when $\pi/2 < \theta_0 \leq \pi$.

PROOF. (i) The Lamé functions $X_1(\theta) = L_{cp,1}^{(0)}(\theta) = \cos\theta$ and $X_2(\theta) = L_{cp\nu}^{(0)}(\theta)$ with $\nu = \nu c_2^0$, are solutions of the differential equations (B.7) with $\nu_1 = 1$ and $\nu_2 = \nu c_2^0$. Both functions satisfy the boundary condition $X_i'(0) = 0$, $i = 1, 2$. Furthermore, $X_2(\theta_0) = 0$ and $X_2(\theta)$ has exactly one zero θ^* between 0 and θ_0 by Theorem 4.1.

Suppose $\nu_2 < \nu_1$, then $g(\nu_2, \theta) < g(\nu_1, \theta)$. Next it follows from Theorems B.1 and B.2 that $X_1(\theta)$ has at least two zeros at α_1 and α_2 with $0 < \alpha_1 < \theta^* < \alpha_2 < \theta_0 \leq \pi$. Since $X_1(\theta) = \cos\theta$ has only one zero between 0 and π, we arrive at a contradiction and we conclude that $\nu c_2^0 \geq 1$.

(ii) The Lamé functions $X_1(\theta) = L_{sp,1}^{(1)}(\theta) = \sqrt{1 - k^2\cos^2\theta}$ and $X_2(\theta) = L_{sp\nu}^{(1)}(\theta)$ with $\nu = \nu s_1^1$, are solutions of the differential equations (B.7) with $\nu_1 = 1$ and $\nu_2 = \nu s_1^1$. Both functions satisfy the boundary condition $X_i'(0) = 0$, $i = 1, 2$, while $X_2(\theta_0) = 0$.

Suppose $\nu_2 < \nu_1$, then $g(\nu_2,\theta) < g(\nu_1,\theta)$. From Theorem B.2 it follows that $X_1(\theta)$ has at least one zero between 0 and θ_0 . Since $X_1(\theta) = \sqrt{1-k^2\cos^2\theta}$ has no zeros, we arrive at a contradiction and we conclude that $\nu s_1^1 \geq 1$. □

The results of Lemmas B.2-4 are combined into the following theorem.

THEOREM B.3. νc_n^m, $\nu s_n^m \geq 1$, $m = (0),1,2,\ldots$, $n = 1,2,3,\ldots$, when $0 \leq \theta_0 \leq \pi$, except that $0 < \nu c_1^0 < 1$, when $\pi/2 < \theta_0 \leq \pi$.

Secondly, we consider the roots $\dot{\nu}c_n^m$ and $\dot{\nu}s_n^m$ of the transcendental equations (5.22). Notice that $\dot{\nu}c_1^0 = 0$, as observed below (5.24). From Fig. 6.2 it is recognized that, apart from $\dot{\nu}c_1^0 = 0$, the smallest ν-roots of equations (5.22) are given by $\dot{\nu}c_2^0$, $\dot{\nu}c_1^1$ and $\dot{\nu}s_1^1$.

LEMMA B.5. $\dot{\nu}c_2^0 \geq 1$, when $0 \leq \theta_0 \leq \pi$.

PROOF. The Lamé functions $X_1(\theta) = L_{cp,1}^{(0)}(\theta) = \cos\theta$ and $X_2(\theta) = L_{cp\nu}^{(0)}(\theta)$ with $\nu = \dot{\nu}c_2^0$, are solutions of the differential equations (B.7) with $\nu_1 = 1$ and $\nu_2 = \dot{\nu}c_2^0$. Both functions satisfy the boundary condition $X_i'(0) = 0,\ i = 1,2$. Furthermore, $X_2'(\theta_0) = 0$ and $X_2(\theta)$ has exactly one zero θ^* between 0 and θ_0 by Theorem 4.1. Without loss of generality we may assume that $X_2(\theta) > 0$ for $0 \leq \theta < \theta^*$ and $X_2(\theta) < 0$ for $\theta^* < \theta \leq \theta_0$.

Let $0 < \theta_0 \leq \pi/2$ and suppose $\nu_2 < \nu_1$, then $g(\nu_2,\theta) < g(\nu_1,\theta)$. From Theorem B.2 it follows that $X_1(\theta)$ has at least one zero between 0 and $\theta^* < \pi/2$. On the other hand, $X_1(\theta) = \cos\theta$ has no zeros between 0 and $\pi/2$, and we arrive at a contradiction.

Let $\pi/2 < \theta_0 \leq \pi$ and suppose $\nu_2 < \nu_1$, then $g(\nu_2,\theta) < g(\nu_1,\theta)$. From Theorem B.2 it follows that $X_1(\theta)$ has at least one zero between 0 and θ^*. Since $X_1(\theta) = \cos\theta$ vanishes at $\theta = \pi/2$, we conclude that $\theta^* > \pi/2$. Next we employ the Green's formula (B.8) over the interval $[\theta^*,\theta_0]$, viz.

$$(B.10)\quad p(\theta_0)X_1'(\theta_0)X_2(\theta_0) + p(\theta^*)X_1(\theta^*)X_2'(\theta^*) = \int_{\theta^*}^{\theta_0} [g(\nu_2,\theta) - g(\nu_1,\theta]X_1(\theta)(X_2(\theta)d\theta ,$$

where it has been used that $X_2'(\theta_0) = 0$ and $X_2(\theta^*) = 0$. It is easily seen that the left-hand side of (B.10) is positive, whereas the right-hand side is negative, and we arrive at a contradiction.

Thus we conclude that $\dot{\nu}c_2^0 \geq 1$, when $0 \leq \theta_0 \leq \pi$. □

LEMMA B.6. $\dot{\nu}c_1^1$, $\dot{\nu}s_1^1 \geq 1$, when $0 \leq \theta_0 \leq \pi/2$; $0 < \dot{\nu}c_1^1 < 1$, when $\pi/2 < \theta_0 \leq \pi$; $0 < \dot{\nu}s_1^1 < 1$, when $\pi/2 < \theta_0 < \pi$; $\dot{\nu}s_1^1 = 1$ at $\theta_0 = \pi$.

PROOF. (i) The Lamé functions $X_1(\theta) = L_{cp,1}^{(1)}(\theta) = \sin\theta$ and $X_2(\theta) = L_{cp\nu}^{(1)}(\theta)$ with $\nu = \dot{\nu}c_1^1$, are solutions of the differential equations (B.7) with $\nu_1 = 1$ and $\nu_2 = \dot{\nu}c_1^1$. Both functions satisfy the boundary condition $X_i(0) = 0,\ i = 1,2$, while $X_2(\theta) > 0$ for $0 < \theta \leq \theta_0$ and $X_2'(\theta_0) = 0$. Hence, the Green's formula (B.8) over the interval $[0,\theta_0]$ specializes into

$$\text{(B.11)} \quad p(\theta_0)X_1'(\theta_0)X_2(\theta_0) = \int_0^{\theta_0} [g(\nu_2,\theta) - g(\nu_1,\theta)]X_1(\theta)X_2(\theta)d\theta \,.$$

Let $0 < \theta_0 \leq \pi/2$ and suppose $\nu_2 < \nu_1$, then $g(\nu_2,\theta) < g(\nu_1,\theta)$. Obviously the right-hand side of (B.11) is negative, whereas the left-hand side is non-negative. From this contradiction we conclude that $\dot{\nu}c_1^1 \geq 1$, when $0 \leq \theta_0 \leq \pi/2$.

Let $\pi/2 < \theta_0 \leq \pi$ and suppose $\nu_2 \geq \nu_1$, then $g(\nu_2,\theta) \geq g(\nu_1,\theta)$. The right-hand side of (B.11) is now non-negative, whereas the left-hand side is negative. Because of this contradiction we have $\dot{\nu}c_1^1 < 1$, when $\pi/2 < \theta_0 \leq \pi$. Notice that $0 < \dot{\nu}c_1^1$ by (6.29).

(ii) The Lamé functions $X_1(\theta) = L_{sp,1}^{(1)}(\theta) = \sqrt{1-k^2\cos^2\theta}$ and $X_2(\theta) = L_{sp\nu}^{(1)}(\theta)$ with $\nu = \dot{\nu}s_1^1$, are solutions of the differential equations (B.7) with $\nu_1 = 1$ and $\nu_2 = \dot{\nu}s_1^1$. Both functions satisfy the boundary condition $X_i'(0) = 0,\ i = 1,2$, while $X_2(\theta) > 0$ for $0 \leq \theta \leq \theta_0$ and $X_2'(\theta_0) = 0$. Hence, the Green's formula (B.8) over the interval $[0,\theta_0]$ specializes into

$$\text{(B.12)} \quad p(\theta_0)X_1'(\theta_0)X_2(\theta_0) = \int_0^{\theta_0} [g(\nu_2,\theta) - g(\nu_1,\theta)]X_1(\theta)X_2(\theta)d\theta \,.$$

Let $0 < \theta_0 \leq \pi/2$ and suppose $\nu_2 < \nu_1$, then $g(\nu_2,\theta) < g(\nu_1,\theta)$. Obviously the right-hand side of (B.12) is negative, whereas the left-hand side is non-negative. From this contradiction we conclude that $\dot{\nu}s_1^1 \geq 1$, when $0 \leq \theta_0 \leq \pi/2$.

Let $\pi/2 < \theta_0 < \pi$ and suppose $\nu_2 \geq \nu_1$, then $g(\nu_2, \theta) \geq g(\nu_1, \theta)$. The right-hand side of (B.12) is now non-negative, whereas the left-hand side is negative. Because of this contradiction we have $\dot{\nu}s_1^1 < 1$, when $\pi/2 < \theta_0 < \pi$. Notice that $0 < \dot{\nu}s_1^1$ by (6.28) and $\dot{\nu}s_1^1 = 1$ at $\theta_0 = \pi$ by (5.41). □

From Lemmas B.5 and B.6 and Fig. 6.2 we infer that $\dot{\nu}c_n^m$, $\dot{\nu}s_n^m \geq 1$ in the following two cases: (i) $m = (0), 1, 2, \ldots$, $n = 1, 2, 3, \ldots$, when $0 \leq \theta_0 \leq \pi/2$ (except that $\dot{\nu}c_1^0 = 0$) ; (ii) $m = (0), 1, 2, \ldots$, $n = 2, 3, 4, \ldots$, when $\pi/2 < \theta_0 \leq \pi$. It only remains to determine the location with respect to $\nu = 1$ of the ν-roots $\dot{\nu}c_1^m$ and $\dot{\nu}s_1^m$, $m = 2, 3, 4, \ldots$, when $\pi/2 < \theta_0 \leq \pi$. The smallest of these roots are given by $\dot{\nu}c_1^2$ and $\dot{\nu}s_1^2$.

LEMMA B.7. $\dot{\nu}c_1^2$, $\dot{\nu}s_1^2 \geq 1$, when $\pi/2 < \theta_0 \leq \pi$.
Since we have not been able to find an analytical proof, the lemma has been verified by some representative numerical calculations of $\dot{\nu}c_1^2$ and $\dot{\nu}s_1^2$ over the range $\pi/2 < \theta_0 \leq \pi$ for various values of the parameter k .

Finally, the results of Lemmas B.5-7 are combined into the following theorem.
THEOREM B.4. $\dot{\nu}c_n^m$, $\dot{\nu}s_n^m \geq 1$, $m = (0), 1, 2, \ldots$, $n = 1, 2, 3, \ldots$, when $0 \leq \theta_0 \leq \pi$, except that $\dot{\nu}c_1^0 = 0$; $0 < \dot{\nu}c_1^1 < 1$, when $\pi/2 < \theta_0 \leq \pi$; $0 < \dot{\nu}s_1^1 < 1$, when $\pi/2 < \theta_0 < \pi$.

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