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Electromagnetic fields in curved spacetimes

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Abstract

We consider the evolution of electromagnetic fields in curved spacetimes and calculate the exact wave equations for the associated electric and magnetic components. Our analysis is fully covariant, applies to a general spacetime and isolates all the sources that affect the propagation of these waves. Among others, we explicitly show how the different components of the gravitational field act as driving sources of electromagnetic disturbances. When applied to perturbed Friedmann–Robertson–Walker cosmologies, our results argue for a superadiabatic-type amplification of large-scale cosmological magnetic fields in Friedmann models with open spatial curvature.

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1. Introduction

Electromagnetic studies in curved spaces have long established the direct coupling between the Maxwell and the Einstein fields. The interaction emerges from the vector nature of the electromagnetic field and from the geometrical approach to gravity introduced by general relativity and it is interpreted as a sort of scattering of the electromagnetic waves by the spacetime curvature.

In the present paper we study electromagnetic fields in general curved spacetimes by using the covariant approach to general relativity. Our analysis is non-perturbative, in the sense that it does not perturb away from a given metric but provides the full nonlinear equations before linearizing them about a chosen background. In addition, we study the physically measurable electric and magnetic components of the Maxwell field, rather than the Faraday tensor or the electromagnetic 4-potential. This on one hand complements earlier work on the subject, while on the other it allows for a more compact mathematical presentation and for a more transparent physical interpretation of the results. The evolution of the Maxwell field is studied in a general spacetime without imposing any *a priori* symmetries on the latter. The only restriction is on the matter component which is of the perfect fluid form. We derive, from first principles, the electric and magnetic wave equations and identify all the kinematical, dynamical and geometrical sources that drive the propagation of these waves. We demonstrate the effects of the observers' motion and show how the different parts of the gravitational field, namely the Ricci and Weyl fields, affect propagating electromagnetic disturbances. Moreover, the non-perturbative nature of our formalism means that the nonlinear equations apply to a variety of situations of either astrophysical or cosmological interest.

With the full equations at hand, we proceed to consider the evolution of electromagnetic fields in spatially curved Friedmann–Robertson–Walker (FRW) models. Noting that the symmetries of the FRW spacetime are generally incompatible with the generic anisotropy of the electromagnetic field, we consider the evolution of the latter in perturbed Friedmann universes. In particular, we look at the spacetime curvature effects on the linear evolution of the magnetic component of the Maxwell field. Our results show that, when the model is spatially closed, the magnetic field has an oscillatory behaviour with a decreasing amplitude according to the familiar a^{-2}-law (where a is the FRW scale factor). The same is also true for spatially open models with the exception of large-scale magnetic fields. There, we find that the field decays as a^{-1} and therefore that 'magnetic flux' conservation no longer holds at long wavelengths. This result can be seen as an effective superadiabatic-type amplification of large-scale magnetic fields in spatially open FRW universes due to curvature effects alone. Crucially, the amplification is achieved without introducing any new physics and without breaking away from the standard properties of Maxwell's theory.

We start with an outline of the covariant formalism in section 2 and provide a covariant treatment of the electromagnetic and gravitational fields in sections 3 and 4, respectively. The nonlinear electromagnetic wave equations are derived in section 5, and in section 6 they are linearized and solved around curved FRW models. We discuss our results in section 7.

2. The covariant description

The covariant approach to general relativity uses the kinematic quantities of the fluid, its energy density and pressure and the gravito-electromagnetic tensors instead of the metric which in itself does not provide a covariant description. The key equations are the Ricci and Bianchi identities, applied to the fluid 4-velocity vector, while Einstein's equations are incorporated via algebraic relations between the Ricci and the energy–momentum tensors. Here, we will only give a brief description of the approach and direct the reader to a number of review articles for further details and references [1–4].

2.1. The 1 + 3 spacetime splitting

Consider a general spacetime with a Lorentzian metric g_{ab} of signature $(-,+,+,+)$. Then, allow for a family of fundamental observers living along a timelike congruence of worldlines tangent to the 4-velocity vector

$$u_a = \frac{\mathrm{d}x^a}{\mathrm{d}\tau}, \tag{1}$$

where τ is the associated proper time and $u_a u^a = -1$ [4]. This fundamental velocity field introduces a local, 1 + 3 'threading' of the spacetime into time and space. The vector u_a determines the time direction and the tensor $h_{ab} = g_{ab} + u_a u_b$ projects orthogonal to u_a into what is known as the observers' instantaneous rest space. Note that, in the absence of rotation, h_{ab} also acts as the metric of the spatial sections.

Employing u_a and h_{ab} one defines the covariant time derivative and the orthogonally projected gradient of any given tensor field $T_{ab\cdots}{}^{cd\cdots}$ according to

$$\dot{T}_{ab\cdots}{}^{cd\cdots} = u^e\nabla_e T_{ab\cdots}{}^{cd\cdots} \qquad \text{and} \qquad \mathrm{D}_e T_{ab\cdots}{}^{cd\cdots} = h_e{}^s h_a{}^f h_b{}^p h_q{}^c h_r{}^d \cdots \nabla_s T_{fp\cdots}{}^{qr\cdots}, \quad (2)$$

respectively. The former indicates differentiation along the timelike direction and the latter operates on the observers' rest space.

2.2. The matter field

Relative to the aforementioned fundamental observers, the energy–momentum tensor of a general imperfect fluid decomposes into its irreducible parts as [4][1]

$$T_{ab} = \mu u_a u_b + p h_{ab} + 2q_{(a}u_{b)} + \pi_{ab}. \quad (3)$$

Here, $\mu = T_{ab}u^a u^b$ and $p = T_{ab}h^{ab}/3$ are respectively the energy density and the isotropic pressure of the medium, $q_a = -h_a{}^b T_{bc}u^c$ is the energy-flux vector relative to u_a and $\pi_{ab} = h_{\langle a}{}^c h_{b\rangle}{}^d T_{cd}$ is the symmetric and trace-free tensor that describes the anisotropic pressure of the fluid[2]. It follows that $q_a u^a = 0 = \pi_{ab}u^a$. When the fluid is perfect, both q_a and π_{ab} are identically zero and the remaining degrees of freedom are determined by the equation of state. For a barotropic medium the latter reduces to $p = p(\mu)$, with $c_s^2 = \mathrm{d}p/\mathrm{d}\mu$ representing the associated adiabatic sound speed.

When dealing with a multi-component medium, one needs to account for the velocity 'tilt' between the various matter components and the fundamental observers [5]. Here, however, we will consider a single-component fluid and we will assume that the fundamental observers are moving along with it.

2.3. The covariant kinematics

The observers' motion is characterized by the irreducible kinematical quantities of the u_a-congruence, which emerge from the following covariant decomposition of the 4-velocity gradient,

$$\nabla_b u_a = \sigma_{ab} + \omega_{ab} + \tfrac{1}{3}\Theta h_{ab} - \dot{u}_a u_b, \quad (4)$$

where $\sigma_{ab} = \mathrm{D}_{\langle b}u_{a\rangle}$, $\omega_{ab} = \mathrm{D}_{[b}u_{a]}$, $\Theta = \nabla^a u_a = \mathrm{D}^a u_a$ and $\dot{u}_a = u^b\nabla_b u_a$ are respectively the shear and the vorticity tensors, the expansion (or contraction) scalar and the 4-acceleration vector [4]. Then, $\sigma_{ab}u^a = 0 = \omega_{ab}u^a = \dot{u}_a u^a$ by definition. Also, on using the orthogonally projected alternating tensor ϵ_{abc} (with $\dot{\epsilon}_{abc} = 3u_{[a}\epsilon_{bc]d}\dot{u}^d$), one defines the vorticity vector $\omega_a = \epsilon_{abc}\omega^{bc}/2$.

The nonlinear covariant kinematics are determined by a set of three propagation equations complemented by an equal number of constraints [4]. The former contains Raychaudhuri's formula

$$\dot{\Theta} = -\tfrac{1}{3}\Theta^2 - \tfrac{1}{2}(\mu + 3p) - 2(\sigma^2 - \omega^2) + \mathrm{D}^a\dot{u}_a + \dot{u}_a\dot{u}^a, \quad (5)$$

for the time evolution of Θ, the shear propagation equation

$$\dot{\sigma}_{\langle ab\rangle} = -\tfrac{2}{3}\Theta\sigma_{ab} - \sigma_{c\langle a}\sigma^c{}_{b\rangle} - \omega_{\langle a}\omega_{b\rangle} + \mathrm{D}_{\langle a}\dot{u}_{b\rangle} + \dot{u}_{\langle a}\dot{u}_{b\rangle} - E_{ab} + \tfrac{1}{2}\pi_{ab}, \quad (6)$$

which describes kinematical anisotropies, and the evolution equation of the vorticity

$$\dot{\omega}_{\langle a\rangle} = -\tfrac{2}{3}\Theta\omega_a - \tfrac{1}{2}\,\mathrm{curl}\,\dot{u}_a + \sigma_{ab}\omega^b. \quad (7)$$

[1] Throughout this paper we use geometrized units with $c = 1 = 8\pi G$. Consequently, all geometrical variables have physical dimensions that are integer powers of length.

[2] Angle brackets denote the symmetric and trace-free part of projected second-rank tensors and the orthogonally projected component of vectors.

Note that $\sigma^2 = \sigma_{ab}\sigma^{ab}/2$ and $\omega^2 = \omega_{ab}\omega^{ab}/2 = \omega_a\omega^a$ are respectively the magnitudes of the shear and the vorticity, while E_{ab} is the electric component of the Weyl tensor (see section 3.2). Also, $\operatorname{curl} v_a = \epsilon_{abc}\mathrm{D}^b v^c$ for any orthogonally projected vector v_a by definition.

Equations (5)–(7) are complemented by a set of three nonlinear constraints. These are the shear

$$\mathrm{D}^b\sigma_{ab} = \tfrac{2}{3}\mathrm{D}_a\Theta + \operatorname{curl}\omega_a + 2\epsilon_{abc}\dot{u}^b\omega^c - q_a, \tag{8}$$

the vorticity

$$\mathrm{D}^a\omega_a = \dot{u}_a\omega^a, \tag{9}$$

and the magnetic Weyl constraint

$$H_{ab} = \operatorname{curl}\sigma_{ab} + \mathrm{D}_{\langle a}\omega_{b\rangle} + 2\dot{u}_{\langle a}\omega_{b\rangle}, \tag{10}$$

where $\operatorname{curl} T_{ab} = \epsilon_{cd\langle a}\mathrm{D}^c T_{b\rangle}{}^d$ for any orthogonally projected tensor T_{ab}.

3. The electromagnetic field

Covariant studies of electromagnetic fields date back to the work of Ehlers [1] and Ellis [3]. In addition to its inherent mathematical compactness and clarity, the covariant formalism facilitates a physically intuitive fluid description of the Maxwell field. In particular, the latter is represented as an imperfect fluid with properties specified by its electric and magnetic components.

3.1. The electric and magnetic components

The Maxwell field is covariantly characterized by the antisymmetric electromagnetic (Faraday) tensor F_{ab}, which relative to a fundamental observer decomposes into an electric and a magnetic component as [3, 6]

$$F_{ab} = 2u_{[a}E_{b]} + \epsilon_{abc}H^c. \tag{11}$$

In the above $E_a = F_{ab}u^b$ and $H_a = \epsilon_{abc}F^{bc}/2$ are respectively the electric and magnetic fields experienced by the observer. Note that $E_a u^a = 0 = H_a u^a$, ensuring that both E_a and H_a are spacelike vectors living in the observer's three-dimensional rest space. Also, the expression $H_a = \epsilon_{abc}F^{bc}/2$ guarantees that H_a is the dual of the antisymmetric (pseudo) tensor F_{ab}.

The Faraday tensor also determines the energy–momentum tensor of the Maxwell field. In particular, we have

$$T_{ab}^{(\mathrm{em})} = -F_{ac}F^c{}_b - \tfrac{1}{4}F_{cd}F^{cd}g_{ab}, \tag{12}$$

which, on using (11), provides an irreducible decomposition for $T_{ab}^{(\mathrm{em})}$. More precisely, relative to a fundamental observer, the latter splits into [3, 6]

$$T_{ab}^{(\mathrm{em})} = \tfrac{1}{2}(E^2 + H^2)u_a u_b + \tfrac{1}{6}(E^2 + H^2)h_{ab} + 2\mathcal{Q}_{(a}u_{b)} + \mathcal{P}_{ab}. \tag{13}$$

Here $E^2 = E_a E^a$ and $H^2 = H_a H^a$ are the magnitudes of the two fields, $\mathcal{Q}_a = \epsilon_{abc}E^b H^c$ is the electromagnetic Poynting vector and $\mathcal{P}_{ab}$ is a symmetric, trace-free tensor given by

$$\mathcal{P}_{ab} = \mathcal{P}_{\langle ab\rangle} = \tfrac{1}{3}(E^2 + H^2)h_{ab} - E_a E_b - H_a H_b. \tag{14}$$

Expression (13) provides a fluid description of the Maxwell field and manifests its generically anisotropic nature. In particular, the electromagnetic field corresponds to an imperfect fluid with energy density $(E^2+H^2)/2$, isotropic pressure $(E^2+H^2)/6$, anisotropic stresses given by $\mathcal{P}_{ab}$ and an energy-flux vector represented by $\mathcal{Q}_a$. Equation (13) also ensures that $T_a^{(\mathrm{em})\,a} = 0$, in agreement with the trace-free nature of the radiation stress–energy tensor. Finally, we note that by putting the isotropic and anisotropic pressure together, one arrives at the familiar Maxwell tensor, which assumes the covariant form

$$\mathcal{M}_{ab} = \tfrac{1}{2}(E^2 + H^2)h_{ab} - E_a E_b - H_a H_b. \tag{15}$$

3.2. Maxwell's equations

We follow the evolution of the electromagnetic field by means of Maxwell's equations. In their standard tensor form the latter read

$$\nabla_{[c}F_{ab]} = 0, \tag{16a}$$

and

$$\nabla^b F_{ab} = J_a, \tag{16b}$$

where (16*a*) manifests the existence of a 4-potential and J_a is the 4-current that sources the electromagnetic field. With respect to the u_a-congruence, the 4-current splits into its irreducible parts according to

$$J_a = \rho_e u_a + \mathcal{J}_a, \tag{17}$$

with $\rho_e = -J_a u^a$ representing the charge density and $\mathcal{J}_a = h_a{}^b J_b$ the orthogonally projected current (i.e. $\mathcal{J}_a u^a = 0$).

Relative to a fundamental observer, each of Maxwell's equations decomposes into a timelike and a spacelike component. Thus, by projecting (16*a*) and (16*b*) along and orthogonal to the 4-velocity vector u_a, we obtain a set of two propagation equations [3, 6]

$$\dot{E}_{\langle a\rangle} = \left(\sigma_{ab} + \varepsilon_{abc}\omega^c - \tfrac{2}{3}\Theta h_{ab}\right) E^b + \varepsilon_{abc}\dot{u}^b H^c + \text{curl}\, H_a - \mathcal{J}_a, \tag{18}$$

$$\dot{H}_{\langle a\rangle} = \left(\sigma_{ab} + \varepsilon_{abc}\omega^c - \tfrac{2}{3}\Theta h_{ab}\right) H^b - \varepsilon_{abc}\dot{u}^b E^c - \text{curl}\, E_a, \tag{19}$$

and the following pair of constraints

$$\mathrm{D}^a E_a + 2\omega^a H_a = \rho_{\mathrm{e}}, \tag{20}$$

$$\mathrm{D}^a H_a - 2\omega^a E_a = 0. \tag{21}$$

Note that in addition to the usual 'curl' and 'divergence' terms, there are terms due to the observer's motion. According to equation (20), in the absence of an electric field the observed charge density is $\rho_{\mathrm{e}} = 2\omega^a H_a$. This means non-zero charge density unless $\omega^a H_a = 0$ (see [7] for a discussion on the charge asymmetry of the universe). Also, following (21), the magnetic vector is not solenoidal unless $\omega^a E_a = 0$.

3.3. The conservation laws

The antisymmetry of the Faraday tensor (see equation (11)) and the second of Maxwell's formulae (see equation (16*b*)) imply the conservation law

$$\nabla^a J_a = 0, \tag{22}$$

for the 4-current density. Then, on using decomposition (17), expression (22) provides the covariant form of the charge density conservation law [3, 8]

$$\dot{\rho}_{\mathrm{e}} = -\Theta\rho_{\mathrm{e}} - \mathrm{D}^a\mathcal{J}_a - \dot{u}^a\mathcal{J}_a. \tag{23}$$

Thus, in the absence of spatial currents, the charge density evolution depends entirely on the average volume expansion (or contraction) of the fluid element.

3.4. Ohm's law

The electrical conductivity of the medium determines the relation between the 4-current and the associated electric field via Ohm's law. In covariant form the latter reads

$$J_a - \rho_{\rm e} u_a = \sigma E_a, \tag{24}$$

where σ is the scalar conductivity of the medium [9]. Projecting the above into the observer's rest space one arrives at

$$\mathcal{J} = \sigma E_a. \tag{25}$$

Thus, non-zero spatial currents are compatible with a vanishing electric field as long as the conductivity of the medium is infinite (i.e. for $\sigma \to \infty$). Alternatively, one can say that at the infinite conductivity limit, which defines the well-known MHD approximation, the electric field vanishes in the frame of the fluid. On the other hand, zero electrical conductivity implies that the spatial currents vanish even when the electric field is non-zero.

4. The gravitational field

Covariantly, the local gravitational field is monitored by a set of algebraic relations between the Ricci curvature tensor and the energy–momentum tensor of the matter. The free gravitational field, on the other hand, is described by the electric and magnetic components of the conformal curvature (Weyl) tensor.

4.1. The local Ricci curvature

In the general relativistic geometrical interpretation of gravity, matter determines the spacetime curvature which in turn dictates the motion of the matter. This interaction is manifested in the Einstein field equations, which in the absence of a cosmological constant take the form

$$R_{ab} = T_{ab} - \tfrac{1}{2} T g_{ab}, \tag{26}$$

where $R_{ab} = R_{acb}{}^{c}$ is the spacetime Ricci tensor and T_{ab} is the energy–momentum tensor of the matter fields, with $T = T_a{}^a$ being the trace. For our purposes the total energy–momentum tensor has the form $T_{ab} = T_{ab}^{(f)} + T_{ab}^{(\rm em)}$, where $T_{ab}^{(f)}$ is given by equation (3) and $T_{ab}^{(\rm em)}$ by equation (13). Thus,

$$T_{ab} = \left[\mu + \tfrac{1}{2}(H^2 + E^2)\right] u_a u_b + \left[p + \tfrac{1}{6}(H^2 + E^2)\right] h_{ab} + 2(q_{(a} + \mathcal{Q}_{(a})u_{b)} + \pi_{ab} + \mathcal{P}_{ab}, \tag{27}$$

ensuring that $\mu + (H^2 + E^2)/2$ is the total energy density of the system, $p + (H^2 + E^2)/6$ is the total isotropic pressure, $q_a + \mathcal{Q}_a$ is the total heat flux vector and $\pi_{ab} + \mathcal{P}_{ab}$ is the total anisotropic pressure. The inclusion of electromagnetic terms in the energy–momentum tensor of the matter guarantees that the contribution of the Maxwell field to the spacetime geometry is fully incorporated.

Starting from the Einstein field equations and assuming that T_{ab} is given by equation (27), we arrive at the following algebraic relations [8]:

$$R_{ab} u^a u^b = \tfrac{1}{2}(\mu + 3p + E^2 + H^2), \tag{28}$$

$$h_a{}^b R_{bc} u^c = -(q_a + \mathcal{Q}_a), \tag{29}$$

$$h_a{}^c h_b{}^d R_{cd} = \left[\tfrac{1}{2}\left(\mu - p + \tfrac{1}{3}E^2 + \tfrac{1}{3}H^2\right)\right] h_{ab} + \pi_{ab} + \mathcal{P}_{ab}. \tag{30}$$

In addition, the trace of (26) gives $R = -T$, with $R = R_a{}^a$ and $T = T_a{}^a = 3p - \mu$, where the latter result is guaranteed by the trace-free nature of $T_{ab}^{(\rm em)}$. Note that the above expressions are valid irrespective of the strength of the electromagnetic components. When the Maxwell field is weak relative to the matter, namely for E^2, $H^2 \ll \mu$, one might treat the electromagnetic contribution to the spacetime curvature as a first-order perturbation. Finally, recall that $q_a = 0 = \pi_{ab}$ when dealing with a perfect fluid.

4.2. The long-range Weyl curvature

The Ricci tensor describes the local gravitational field of the nearby matter. The long-range gravitational field, namely gravitational waves and tidal forces, propagates through the Weyl conformal curvature tensor. The splitting of the gravitational field into its local and non-local components is demonstrated in the following decomposition of the Riemann tensor,

$$R_{abcd} = C_{abcd} + \tfrac{1}{2}\left(g_{ac}R_{bd} + g_{bd}R_{ac} - g_{bc}R_{ad} - g_{ad}R_{bc}\right) - \tfrac{1}{6}R\left(g_{ac}g_{bd} - g_{ad}g_{bc}\right), \tag{31}$$

where C_{abcd} is the Weyl tensor. The latter shares all the symmetries of the Riemann tensor and is also trace-free (i.e. $C^c{}_{acb} = 0$). Relative to the fundamental observers, the Weyl tensor decomposes further into its irreducible parts according to

$$C_{abcd} = (g_{abqp}g_{cdsr} - \eta_{abqp}\eta_{cdsr})u^q u^s E^{pr} - (\eta_{abqp}g_{cdsr} + g_{abqp}\eta_{cdsr})u^q u^s H^{pr}, \tag{32}$$

where $g_{abcd} = g_{ac}g_{bd} - g_{ad}g_{bc}$ (e.g. see [10, 11]). The symmetric and trace-free tensors E_{ab} and H_{ab} are known as the electric and magnetic Weyl components and they are given by

$$E_{ab} = C_{acbd}u^c u^d \qquad \text{and} \qquad H_{ab} = \tfrac{1}{2}\epsilon_a{}^{cd}C_{cdbe}u^e, \tag{33}$$

with $E_{ab}u^b = 0 = H_{ab}u^b$. Given that E_{ab} has a Newtonian counterpart, the electric part of the Weyl tensor is associated with the tidal field. The magnetic component, on the other hand, has no Newtonian analogue and therefore is primarily associated with gravitational waves [2]. Of course, both tensors are required if gravitational waves are to exist. For a comparison with the non-covariant metric-based treatments of gravitational waves, we note that in perturbed FRW models the harmonically decomposed, pure-tensor metric perturbation is $H_T = 2E + \sigma'/n$ [16]. Here, E and σ represent the harmonic parts of the transverse traceless electric Weyl and shear tensors, respectively. Also, n is the associated wavenumber and a prime denotes derivatives with respect to conformal time.

The Weyl tensor represents the part of the curvature that is not determined locally by matter. However, the dynamics of the Weyl field is not entirely arbitrary because the Riemann tensor satisfies the Bianchi identities. When contracted the latter take the form [10]

$$\nabla^d C_{abcd} = \nabla_{[b}R_{a]c} + \tfrac{1}{6}g_{c[b}\nabla_{a]}R, \tag{34}$$

by means of decomposition (31). In a sense the contracted Bianchi identities act as the field equations for the Weyl tensor, determining the part of the spacetime curvature that depends on the matter distribution at other points [10]. The form of the contracted Bianchi identities guarantees that once the electromagnetic contribution to the Ricci curvature has been incorporated, through the Einstein field equations, the effect of the Maxwell field on the Weyl curvature has also been fully accounted for.

Expression (34) splits into a set of two propagation and two constraint equations, which monitor the evolution of the electric and magnetic Weyl components [2–4]. These formulae are not used to derive the electromagnetic wave equations of section 5.3 and are therefore not essential for our purposes. Here we simply note that the aforementioned set of equations is remarkably similar to Maxwell's formulae, which in turn explains the names of E_{ab} and H_{ab}. This Maxwell-like form of the free gravitational field underlines the rich correspondence between electromagnetism and general relativity, which has been the subject of theoretical debate for many decades (see [12–15] for a representative list).

5. The electromagnetic wave equations

Studies of electromagnetic waves in curved spacetimes have long established that, while propagating similar to any other travelling wave, electromagnetic disturbances also interact with the spacetime curvature. As a result, electromagnetic signals propagate inside as well as on the future light cone of an event, indicating the failure of Huygens' principle in curved spaces [17–19].

5.1. The wave equation for the electromagnetic field tensor

Maxwell's equations immediately provide a wave equation for the electromagnetic field tensor. In particular, taking the covariant derivative of (16*a*) and using (16*b*) we arrive at

$$\nabla^2 F_{ab} = -2R_{acbd}F^{cd} + R_a{}^c F_{cb} + F_a{}^c R_{cb} + \nabla_b J_a - \nabla_a J_b, \tag{35}$$

where $\nabla^2 = \nabla^a\nabla_a$ is the generalized covariant Laplacian operator (e.g. see [20, 21]). The above, which holds in a general spacetime, reveals the role of the curvature as a driving source of electromagnetic disturbances. Note that the Riemann and Ricci curvature terms on the left-hand side of equation (35) emerge after using the Ricci identity

$$2\nabla_{[a}\nabla_{b]}F_{cd} = R_{abce}F^e{}_d + R_{abde}F_c{}^e, \tag{36}$$

which here monitors the commutation between the covariant derivatives of F_{ab}. Expression (35) can also provide the individual wave equations for the electric and magnetic components of F_{ab}. For example, contracting equation (35) along u_a eventually leads to the wave equation of E_a, while its dual provides the magnetic wave equation. Here, we will follow an alternative route and obtain these expressions directly from the decomposed Maxwell formulae (18) and (19).

5.2. The electro/magneto-curvature coupling

In addition to the Einstein field equations, vector sources, like the electromagnetic field, obey an extra set of equations, known as the Ricci identities, which manifest the direct interaction between electromagnetism and spacetime geometry. This coupling emerges naturally from the vector nature of the Maxwell field and from the geometrical approach to gravity of general relativity. When applied to the magnetic field vector the Ricci identity reads

$$2\nabla_{[c}\nabla_{b]}H_a = R_{dabc}H^d; \tag{37}$$

with an exactly analogous expression for the electric component. Clearly, on using decomposition (31), the Ricci identity couples the electromagnetic field explicitly with both the local and the long-range gravitational field. Also, by projecting the above into the observer's rest space one arrives at what is known as the 3-Ricci identity

$$2\mathrm{D}_{[c}\mathrm{D}_{b]}H_a = -2\varepsilon_{cbd}\omega^d\dot{H}_{\langle a\rangle} + \mathcal{R}_{dabc}H^d, \tag{38}$$

describing the interaction between the magnetic field and the local spatial geometry [22, 23]. Clearly an exactly analogous relation holds for E_a as well. Note that $\mathcal{R}_{abcd}$ is the orthogonally projected part of R_{abcd}, namely the Riemann tensor of the observer's local 3-space. Finally, we should emphasize that the validity of both (37) and (38) extends to any arbitrary spacetime (e.g. see [2, 10]).

5.3. The wave equations for the electric and magnetic fields

Equations (18) and (19) monitor the propagation of electromagnetic fields in a general spacetime either in vacuum (i.e. for source-free fields with $\rho_{\rm e} = 0 = \mathcal{J}_a$) or in the presence of matter. Starting from these formulae, one can work out the wave equations for propagating electromagnetic radiation in a general spacetime. In particular, taking the time derivative of equation (18), one obtains the wave-like evolution equation of the electric field. Similarly, the time derivative of equation (19) leads to the corresponding wave equation of the magnetic field. In the Minkowski space of special relativity these calculations are relatively straightforward since the geometry of the space is trivial. In the context of general relativity, however, this is no longer true and one has to account for the coupling between the electromagnetic fields and the spacetime geometry discussed earlier. Technically speaking, this requires using the Ricci identities and leads to spatial curvature terms every time the projected derivatives of E_a or H_a commute. In addition, the Ricci identities guarantee a Weyl field contribution whenever a time derivative and a projected gradient of either the electric or the magnetic field commute.

Assuming that the matter component has a perfect fluid form with a barotropic equation of state, we take the time derivative of equation (18) and project it orthogonal to u_a. Then, using the kinematical propagation and constraint equations of section 2.3, expression (14), relations (19)–(32) and the commutation laws (37), (38) we arrive at the following wave equation for the electric field vector:

$$\begin{aligned} \ddot{E}_{\langle a\rangle} - {\rm D}^2 E_a = {} & \tfrac{1}{3}\mu(1+3w)E_a + \left(\sigma_{ab} - \varepsilon_{abc}\omega^c - \tfrac{5}{3}\Theta h_{ab}\right)\dot{E}^b \\ & + \tfrac{1}{3}\Theta\left(\sigma_{ab} + \varepsilon_{abc}\omega^c - \tfrac{4}{3}\Theta h_{ab}\right)E^b - \sigma_{\langle a}{}^c\sigma_{b\rangle c}E^b + \varepsilon_{abc}E^b\sigma^{cd}\omega_d \\ & + \tfrac{4}{3}\left(\sigma^2 - \tfrac{2}{3}\omega^2\right)E_a + \tfrac{1}{3}\omega_{\langle a}\omega_{b\rangle}E^b + \dot{u}^b\dot{u}_bE_a - \tfrac{5}{2}\varepsilon_{abc}\dot{u}^b\,{\rm curl}\,E^c + {\rm D}_{\langle a}E_{b\rangle}\dot{u}^b \\ & + \tfrac{2}{3}\varepsilon_{abc}H^b{\rm D}^c\Theta + \varepsilon_{abc}H_d{\rm D}^b\sigma^{cd} + {\rm D}_{\langle a}\omega_{b\rangle}H^b + \tfrac{3}{2}\varepsilon_{abc}H^b\,{\rm curl}\,\omega^c + 2{\rm D}_{\langle a}H_{b\rangle}\omega^b \\ & - 2\varepsilon_{abc}\sigma^b{}_d{\rm D}^{\langle c}H^{d\rangle} + \varepsilon_{abc}\ddot{u}^bH^c + \tfrac{7}{3}\dot{u}^b\omega_bH_a + \tfrac{4}{3}H^b\omega_b\dot{u}_a - 3\dot{u}^bH_b\omega_a \\ & + 3\varepsilon_{abc}\dot{u}^b\sigma^{cd}H_d + \tfrac{1}{3}\rho_{\rm e}\dot{u}_a - {\rm D}_a\rho_{\rm e} - \Theta\mathcal{J}_a - \dot{\mathcal{J}}_a - \mathcal{R}_{ab}E^b - E_{ab}E^b + H_{ab}H^b. \end{aligned} \tag{39}$$

Similarly, one may start from equation (19) and proceed in an analogous way to obtain the wave equation of the magnetic field vector

$$\begin{aligned} \ddot{H}_{\langle a\rangle} - {\rm D}^2 H_a = {} & \tfrac{1}{3}\mu(1+3w)H_a + \left(\sigma_{ab} - \varepsilon_{abc}\omega^c - \tfrac{5}{3}\Theta h_{ab}\right)\dot{H}^b \\ & + \tfrac{1}{3}\Theta\left(\sigma_{ab} + \varepsilon_{abc}\omega^c - \tfrac{4}{3}\Theta h_{ab}\right)H^b - \sigma_{\langle a}{}^c\sigma_{b\rangle c}H^b + \varepsilon_{abc}H^b\sigma^{cd}\omega_d \\ & + \tfrac{4}{3}\left(\sigma^2 - \tfrac{2}{3}\omega^2\right)H_a + \tfrac{1}{3}\omega_{\langle a}\omega_{b\rangle}H^b + \dot{u}^b\dot{u}_bH_a - \tfrac{5}{2}\varepsilon_{abc}\dot{u}^b\,{\rm curl}\,H^c + {\rm D}_{\langle a}H_{b\rangle}\dot{u}^b \\ & - \tfrac{2}{3}\varepsilon_{abc}E^b{\rm D}^c\Theta - \varepsilon_{abc}E_d{\rm D}^b\sigma^{cd} - {\rm D}_{\langle a}\omega_{b\rangle}E^b - \tfrac{3}{2}\varepsilon_{abc}E^b\,{\rm curl}\,\omega^c - 2{\rm D}_{\langle a}E_{b\rangle}\omega^b \\ & + 2\varepsilon_{abc}\sigma^b{}_d{\rm D}^{\langle c}E^{d\rangle} - \varepsilon_{abc}\ddot{u}^bE^c - \tfrac{7}{3}\dot{u}^b\omega_bE_a - \tfrac{4}{3}E^b\omega_b\dot{u}_a + 3\dot{u}^bE_b\omega_a \\ & - 3\varepsilon_{abc}\dot{u}^b\sigma^{cd}E_d - \tfrac{2}{3}\rho_{\rm e}\omega_a + 2\varepsilon_{abc}\dot{u}^b\mathcal{J}^c + {\rm curl}\,\mathcal{J}_a - \mathcal{R}_{ab}H^b - E_{ab}H^b - H_{ab}E^b. \end{aligned} \tag{40}$$

As expected, when there are no charges and currents, one recovers equation (40) from (39) by simply replacing E_a with H_a and H_a with $-E_a$. Similarly, we obtain (39) from (40) after replacing H_a with E_a and E_a with $-H_a$. In the presence of charges and currents, however, this symmetry no longer holds and the apparent breakdown reflects the absence of magnetic monopoles.

The above expressions provide a covariant description of propagating electromagnetic waves in a general spacetime and incorporate the electromagnetic input to the curvature of the latter[3]. So far the only restrictions are those imposed on the fluid, which has a barotropic equation of state. That aside, equations (39) and (40) are fully nonlinear in perturbative terms. Once the background is specified, these equations can describe the evolution of the electromagnetic field at any perturbative level. In general, of course, one needs to couple these formulae with the appropriate propagation equations of the various kinematical, dynamical and geometrical variables that appear on the right-hand sides of (39) and (40). Clearly, the more complicated the background the more equations are necessary for the system to close.

Among others, the above given wave equations show how the kinematical quantities, namely the expansion, the shear, the vorticity and the acceleration, drive the propagation of electromagnetic waves. Here, the barotropic nature of the matter component means that the 4-acceleration takes the form

$$\mu(1+w)\dot{u}_a = -\mathrm{D}_a p + \rho_{\mathrm{e}} E_a + \varepsilon_{abc}\mathcal{J}^b H^c, \tag{41}$$

with contributions from gradients in the fluid pressure and from the electromagnetic Lorentz force only. The input from the spacetime geometry to equations (39) and (40) is through the spatial and the Weyl curvature components. The former is represented by $\mathcal{R}_{ab}$, the orthogonally projected 3-Ricci tensor, defined by

$$\mathcal{R}_{ab} = \mathcal{R}^c{}_{acb} = h_a{}^c h_b{}^d R_{cd} + R_{acbd}u^c u^d + v_{ac}v^c{}_b - \Theta v_{ab}, \tag{42}$$

where $v_{ab} = \mathrm{D}_b u_a$ is the second fundamental form describing the extrinsic curvature of the space (e.g. see [10, 23]). Note that the tidal part of the Weyl field contributes to the evolution of either E_a or H_a via its direct coupling with the aforementioned fields. The effect of the magnetic Weyl tensor, on the other hand, is indirect and requires the presence of both the electromagnetic field components.

The non-perturbative nature of our analysis, namely the fact that we have not yet specified our background spacetime, means that equations (39) and (40) apply to a range of physical situations (e.g. see [24–26]). For example, in the absence of matter sources one can always set the observer's acceleration to zero (see equation (41)). If, in addition, the spacetime is stationary and non-rotating (i.e. set $\Theta = 0 = \omega_a$), expression (39) and reduces to

$$\begin{aligned}\ddot{E}_a - \mathrm{D}^2 E_a &= \sigma_{ab}\dot{E}^b - \sigma_{\langle a}{}^c\sigma_{b\rangle c}E^b + \tfrac{4}{3}\sigma^2 E_a + \varepsilon_{abc}H_d\mathrm{D}^b\sigma^{cd} - 2\varepsilon_{abc}\sigma^b{}_d\mathrm{D}^{\langle c}H^{d\rangle} \\ &\quad - \mathcal{R}_{ab}E^b - E_{ab}E^b + H_{ab}H^b,\end{aligned} \tag{43}$$

with an exactly analogous wave equation for H_a. When the shear and the Weyl components are divergence-free (i.e. for $\mathrm{D}^b\sigma_{ab} = 0 = \mathrm{D}^b E_{ab} = \mathrm{D}^b H_{ab}$), the above describes the propagation of electromagnetic radiation in the presence of gravitational waves alone. Thus, using equation (43) one can revisit the age old problem of the interaction between electromagnetic and gravitational waves in isolated astrophysical environments away from the gravitational field of massive compact stars (e.g. see [27–30] and references therein). In what follows, however, we will consider a cosmological application of (39) and (40).

6. Electromagnetic fields in curved FRW models

The generic anisotropy of the electromagnetic energy–momentum tensor makes the Maxwell field incompatible with the high symmetry of the FRW spacetime. The implication is that the simplest models where one can study cosmological electromagnetic fields are the perturbed Friedmann universes.

[3] By including the Maxwell field in the Einstein field equations (see equations (26)–(30)), the electromagnetic contribution to the spacetime geometry has been fully accounted for. In practice this means ensuring that μ has been replaced with $\mu + (E^2 + H^2)/2$, p with $p + (E^2 + H^2)/6$, q_a with $\mathcal{Q}_a$ and π_{ab} with $\mathcal{P}_{ab}$ in every formula used to derive equations (39) and (40). For example, by implementing the aforementioned substitution into the kinematical expressions of section 2.3, we incorporate fully the electromagnetic impact on the model's kinematics.

6.1. The linear wave equations

Consider a FRW background cosmology with curved spatial sections. In covariant terms, the isotropy of the FRW model translates into $\omega_a = 0 = \sigma_{ab} = \dot{u}_a$ and $E_{ab} = 0 = H_{ab}$, while their spatial homogeneity ensures that all orthogonally projected gradients vanish identically (i.e. $\mathrm{D}_a\mu = 0 = \mathrm{D}_a p = \mathrm{D}_a\Theta$). This means that μ, p, Θ, $\mathcal{R}_{ab} = \mathcal{R}h_{ab}/3$ and their time derivatives are the only non-vanishing background quantities.

When studying cosmological electromagnetic fields there is a widespread perception that, given the conformal invariance of the Maxwell field and the conformal flatness of the FRW spacetimes, flat spaces provide an adequate background (e.g. see [31, 32]). This is only approximately true however, since the FRW symmetries are generally incompatible with the presence of electric or magnetic fields. As is clearly stated in [33], adopting the conformal triviality of Maxwell's equations on FRW backgrounds means ignoring the electromagnetic impact on the FRW symmetries. This is a good approximation when dealing with weak electromagnetic fields but only on small scales in models with nontrivial spatial geometry. In the latter case, the approximation becomes progressively less accurate as one moves on to larger scales and the 3-curvature effects start kicking in. Putting it another way, with the possible exception of incoherent radiation, one must study cosmological electromagnetic fields in perturbed Friedmann universes. The latter, however, are no longer conformally flat.

On these grounds, we consider a perturbed Friedmann universe with non-Euclidean spatial sections and allow for the presence of a weak electromagnetic field. The latter vanishes in the background, thus guaranteeing that both the electric and the magnetic field vectors are first-order, gauge-invariant perturbations [34]. Then, the source-free components of the nonlinear wave equations (39) and (40) linearize to

$$\ddot{E}_a - \mathrm{D}^2 E_a = -\tfrac{5}{3}\Theta\dot{E}_a - \tfrac{4}{9}\Theta^2 E_a + \tfrac{1}{3}\mu(1+3w)E_a - \mathcal{R}_{ab}E^b, \tag{44}$$

and

$$\ddot{H}_a - \mathrm{D}^2 H_a = -\tfrac{5}{3}\Theta\dot{H}_a - \tfrac{4}{9}\Theta^2 H_a + \tfrac{1}{3}\mu(1+3w)H_a - \mathcal{R}_{ab}H^b, \tag{45}$$

respectively. During linearization quantities with non-zero background value have zero perturbative order, while those that vanish in the background are first-order perturbations and higher-order terms are neglected. For example, the Weyl-free nature of the FRW metric guarantees that the Weyl effects are nonlinear. The 3-Ricci curvature, on the other hand, contributes to both (44) and (45). Recall that $\mathcal{R}_{ab} = (2k/a^2)h_{ab}$ to zero order, where $k = 0, \pm 1$ is the curvature index and a represents the scale factor of the unperturbed model. In other words, the symmetries of the FRW metrics ensure that, to linear order, the electromagnetic field interacts only with the 3-Ricci part of the spacetime curvature. The curvature terms in (44) and (45) reflect the earlier mentioned coupling between electromagnetism and spacetime geometry. Unless the background model is spatially flat, these are clearly first-order perturbative terms and should be taken into account in any complete linear study of cosmological electromagnetic fields. These linear curvature terms clearly show why large-scale electromagnetic fields are not adequately treated on flat FRW backgrounds.

Given that the source-free E_a and H_a fields satisfy identical linear wave equations, we will only consider the magnetic component and proceed by introducing the following harmonic decomposition for H_a,

$$H_a = \sum_n H_{(n)}Q_a^{(n)}, \tag{46}$$

where n is the comoving eigenvalue of the nth harmonic component and $Q_a^{(n)}$ are the associated vector harmonics. As usual $\mathrm{D}_a H^{(n)} = 0 = \dot{Q}_a^{(n)}$ and $Q_a^{(n)}$ are eigenfunctions of the

Laplace–Beltrami operator so that $D^2Q_a^{(n)} = -(n^2/a^2)Q_a^{(n)}$. Employing decomposition (46) and introducing the conformal time variable η (with $\dot{\eta} = 1/a$) we recast equation (45) as

$$H''_{(n)} + n^2 H_{(n)} = -4\left(\frac{a'}{a}\right)H'_{(n)} - 2\left(\frac{a'}{a}\right)^2 H_{(n)} - 2\left(\frac{a''}{a}\right)H_{(n)} - 2kH_{(n)}, \tag{47}$$

where a prime indicates differentiation with respect to η. Then, on introducing the 'magnetic flux' variable $\mathcal{H}_{(n)} = a^2 H_{(n)}$, the above reduces to

$$\mathcal{H}''_{(n)} + n^2\mathcal{H}_{(n)} = -2k\mathcal{H}_{(n)}. \tag{48}$$

This is a wave equation for $\mathcal{H}_{(n)}$ with a driving term on the right-hand side which depends on the background spatial curvature and vanishes only when the background is spatially flat. Note that in a model with closed spatial sections the Laplacian eigenvalue is given by $n^2 = \nu(\nu+1)$, where ν takes the discrete values $\nu = 1, 2, 3, \ldots$. Alternatively, $n^2 = \nu^2 + 1$ when $k = -1$ and $n^2 = \nu^2$ for $k = 0$ (with $\nu^2 \geqslant 0$ in both cases).

6.2. The linear solutions

The driving term on the right-hand side of equation (48) is clearly sensitive to the sign of the background spatial curvature. Let us consider first a FRW model with closed spatial sections. When $k = +1$, equation (48) takes the form

$$\mathcal{H}''_{(\nu)} + [2 + \nu(\nu+1)]\,\mathcal{H}_{(\nu)} = 0, \tag{49}$$

with $\nu = 1, 2, 3, \ldots$. The above leads to the following oscillatory solution for the νth magnetic mode,

$$H_{(\nu)} = \frac{1}{a^2}\{\mathcal{C}_1\cos[\sqrt{2+\nu(\nu+1)}\eta] + \mathcal{C}_2\sin[\sqrt{2+\nu(\nu+1)}\eta]\}, \tag{50}$$

where $\mathcal{C}_1$ and $\mathcal{C}_2$ are constants. In other words, for $k = +1$, the magneto-curvature term on the right-hand side of (48) does not have any significant effect on the evolution of the field, which oscillates in time with an amplitude that decays according to the a^{-2}-law. The only difference relative to the $k = 0$ case is a change in the oscillation frequency near the long wavelength limit. Note that the oscillatory behaviour of the field is ensured on all scales by the compactness of the closed space.

When dealing with the hyperbolic geometry of the spatially open FRW model, however, the oscillatory behaviour of $\mathcal{H}_{(n)}$ is not always guaranteed. Indeed, for $k = -1$ equation (48) takes the form

$$\mathcal{H}''_{(\nu)} + (\nu^2 - 1)\mathcal{H}_{(\nu)} = 0, \tag{51}$$

with $\nu^2 \geqslant 0$. Clearly, when $\nu^2 > 1$ the harmonic mode $\mathcal{H}_{(\nu)}$ oscillates just like in a perturbed closed FRW model. On these scales, the background geometry makes no real difference to the evolution of the field. This agrees with our perception that curvature effects become progressively less important as we move towards smaller scales. At sufficiently long wavelengths (i.e. for $\nu^2 < 1$), the geometrical effects take over and equation (51) no longer accepts an oscillatory solution. In particular, as $\nu^2 \to 0$ we have

$$\mathcal{H}_{(\nu)} = \mathcal{C}_1\cosh\eta + \mathcal{C}_2\sinh\eta = \mathcal{C}_3 e^{\eta} + \mathcal{C}_4\,\mathrm{e}^{-\eta}, \tag{52}$$

where $\mathcal{C}_1$ and $\mathcal{C}_2$ are constants and $\mathcal{C}_{3,4} = (\mathcal{C}_1 \pm \mathcal{C}_2)/2$. Note that, since $n^2 = \nu^2 + 1 > 1$ always, these long wavelength solutions still correspond to subcurvature modes [35]. To have a closer look at the effect of geometry on the linear evolution of the field, we note that the evolution of a spatially open FRW model is monitored by

$$a\Theta = 3\coth(\beta\eta), \tag{53}$$

with $\beta = (1+3w)/2$ by definition and $\beta\eta > 0$ (e.g. see [36]). The above holds throughout the various periods in the lifetime of an open FRW universe, provided the barotropic index w remains constant during each epoch. Then, the relation between the scale factor and the conformal time variable is

$$a = a_0 \left(\frac{1 - \mathrm{e}^{-2\beta\eta}}{1 - \mathrm{e}^{-2\beta\eta_0}} \right)^{1/\beta} \mathrm{e}^{\eta - \eta_0}, \tag{54}$$

where η_0, a_0 depend on the initial conditions. Throughout the dust era $w = 0$ and $\beta = 1/2$, while $w = 1/3$ and $\beta = 1$ when radiation dominates. Finally, during a period of inflationary expansion with $p = -\rho$ we have $\beta = -1$. Note that in the latter case the conformal time variable takes negative values. According to expression (54), there are extensive periods in the lifetime of the universe (i.e. as long as $\eta \ll 0$ or $\eta \gg 0$) when the relation between the cosmological scale factor and the conformal time variable is (see also [37])

$$a \propto \mathrm{e}^{\eta}. \tag{55}$$

Substituting this result into the right-hand side of equation (52), and taking into account that $\mathcal{H}_{(\nu)} = a^2 H_{(\nu)}$ by definition, we arrive at

$$H_{(\nu)} = \mathcal{C}_3 a^{-1} + \mathcal{C}_4 a^{-3}. \tag{56}$$

Therefore, large-scale magnetic fields in perturbed spatially open FRW models decay as a^{-1}, a rate considerably slower than the standard 'adiabatic' a^{-2}-law. The immediate consequence is that, at the long wavelength limit, the cosmological magnetic flux is no longer conserved. Instead, the product $a^2 H_{(\nu)}$ increases with time. This opens the possibility of an effective superadiabatic amplification of the field on large scales similar to that found in [38]. Even if the universe is only marginally open today, this effect could have important implications for the present strength of primordial large-scale magnetic fields, particularly for those fields that survived an epoch of inflation, since they would be much stronger than previously anticipated. Note that during inflation the conductivity of the cosmic medium is effectively zero, which in turn ensures the absence of spatial currents (see section 3.4). In this paper, we have focused primarily on the mathematics of the magneto-geometrical interaction and provided a qualitative measure of its implications for large-scale magnetic fields. A discussion of the physics, together with a detailed quantitative study of the amplification effect, will be given in a subsequent paper.

So far, similar modifications in the evolution of cosmological magnetic fields have been obtained at the expense of standard electromagnetic properties, and in particular of the conformal invariance of Maxwell's equations (e.g. see [38–42] for a representative, though incomplete, list). Moreover, in some cases this effect is achieved by introducing *ad hoc* new physics. Our analysis shows that one can still arrive at the same result by taking into account the natural, general relativistic coupling between the electromagnetic field and the spacetime curvature. In other words, contrary to the widespread perception, superadiabatic magnetic amplification is possible within conventional electromagnetic theory. Here, this has been done through the field's coupling to the intrinsic curvature of spatially open FRW models. Interestingly, however, analogous effects can also occur in perturbed flat FRW cosmologies by coupling the magnetic field to the Weyl curvature of the model, namely to the gravitational waves [44]. All these cast new light on the role and the potential implications of spacetime geometry for the evolution of large-scale cosmic magnetic fields.

7. Discussion

The general relativistic coupling between the electromagnetic and the gravitational fields has long been known in the literature. So far, this interaction has been primarily studied in terms of the Faraday tensor and of the electromagnetic 4-potential [17, 21]. Here, we have taken an alternative approach by looking at the evolution of the individual electromagnetic field components in a general curved spacetime. Assuming that the matter field is of the perfect fluid form, we have derived from first principles the nonlinear wave equations of the electric and the magnetic parts of the Maxwell field. This complements earlier studies which have provided a differential/integral formulation of Maxwell's formulae in terms of the physically measurable components of the electromagnetic field (e.g. see [45–47]). Our approach identifies and isolates all the sources that drive the propagation of electromagnetic fields by keeping the separate aspects of the problem quite distinct. Also, by being manifestly covariant at every step, our calculation avoids undue complexity without introducing any specific coordinate frame. We show explicitly how the electric and magnetic fields are affected by the various kinematical and dynamical quantities and particularly by the different parts of the gravitational field.

Given that large-scale electromagnetic fields are generally incompatible with the FRW symmetries, we consider perturbed models and concentrate on the evolution of large-scale magnetic fields. In particular, we linearize our equations about spatially curved FRW spacetimes and investigate the implications of the background curvature for the evolution of cosmological electromagnetic fields. The gauge invariance of our linear equations ensures that our results are free from any gauge-related problems and ambiguities. We show that when the zero-order spacetime has open spatial sections, the magnetic flux is not always conserved. More specifically, magnetic fields coherent on the largest subcurvature scales are found to decay as a^{-1}, instead of following the familiar a^{-2}-law, where a is the cosmological scale factor. The reason for this deviation is the general relativistic coupling between the magnetic field and the intrinsic curvature of a perturbed spatially open FRW universe. This magneto-geometrical interaction can change the evolution of the field on large scales, where curvature effects become important. The result is a natural superadiabatic-type amplification of cosmological magnetic fields, without the need for new physics and without breaking away from standard electromagnetism.

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