PHILOSOPHICAL TRANSACTIONS

OF THE

ROYAL SOCIETY OF LONDON

Series A. Mathematical and Physical Sciences

No. 784 Vol 237 pp. 509–540 20 January 1939

Electromagnetic Induction in Non-uniform Conductors, and the Determination of the Conductivity of the Earth from Terrestrial Magnetic Variations

By

B. N. Lahiri and A. T. Price

Imperial College, London

LONDON

Printed and published for the Royal Society

By the Cambridge University Press

Bentley House, N.W. 1

Price Four Shillings

ELECTROMAGNETIC INDUCTION IN NON-UNIFORM CONDUCTORS, AND THE DETERMINATION OF THE CONDUCTIVITY OF THE EARTH FROM TERRESTRIAL MAGNETIC VARIATIONS

By B. N. LAHIRI AND A. T. PRICE

Imperial College, London

(*Communicated by S. Chapman, F.R.S.—Received* 28 *March* 1938)

1. Introduction

The possibility of obtaining some knowledge of the distribution of electrical conductivity within the earth, from the observed variations of the earth's magnetic field, was first considered by Schuster (1889), in developing his theory of the daily magnetic variations. He separated these variations into parts of external and internal origin, and then applied the theory of electromagnetic induction in a uniform sphere, due to Lamb (1883), to show that the "internal" part could be attributed to electric currents induced in the earth by the "external" part. Chapman (1919) made a more complete analysis of the diurnal variation field, and showed that it was consistent with the earth having a core of conductivity $\kappa = 3{\cdot}6 \times 10^{-13}$ e.m.u., surrounded by a non-conducting shell of about 250 km. thickness. Chapman and Whitehead (1922) found, however, that the relatively highly conducting oceans probably have an appreciable effect on the internal field, and thus introduce some uncertainty in the estimate of κ.

The theory for a uniformly conducting sphere was extended by Price (1930, 1931) to the case when the field varies aperiodically, and this extension was applied by Chapman and Price (1930) to the storm-time variations. They found that, in order to obtain a theoretical induced field in agreement with the observed internal field, it was necessary to assume a higher value of κ than the above. They also found that the induced currents associated with the storm-time field penetrate deeper than those associated with the daily magnetic variations. They concluded that κ continues to increase with increasing depth to values considerably greater than $3{\cdot}6 \times 10^{-13}$ e.m.u., beyond 250 km. The main object of the present paper is to test and amplify this conclusion, by developing the theory of induction of electric currents in a non-uniform sphere, and applying this theory to the terrestrial magnetic variations.

The consequences of assuming different values for the magnetic permeability μ inside the earth were also considered by Chapman and Whitehead (p. 472), and in further detail by Chapman and Price (p. 439). However, it seems likely on physical grounds—see, for example, the memoir just cited—that μ does not differ appreciably from unity in those layers of the earth which are of main importance in our investigation. We shall therefore treat μ as a constant in the present work, and in the numerical applications its value will be taken as unity.

The general theory for any non-uniform conductor is given in §§ 2, 3, where some special features of the mathematical problem are noted. The formal solution of the problem for any conductor with spherical symmetry is given in § 4. Detailed formulae for the induced field and current distribution, in the special case where $\kappa = k\rho^{-m}$, where k and m are constants, are derived and interpreted in §§ 5, 6. The terrestrial magnetic data to be used are summarized in § 7, and the above theory applied to them in §§ 8, 9. The results obtained support Chapman and Price's view, that there is an increase in κ with increasing depth beyond 250 km., but they further indicate that the increase takes place very rapidly, actually at about 700 km. depth, and that there is also an effective distribution of κ at or near the surface of the earth. It seems likely that the latter represents the influence of the oceans, which is considered in § 10. The distribution and depth of penetration of the induced currents are investigated in § 11, from which it appears that the knowledge of κ afforded by the daily and storm-time variations is not likely to extend beyond a depth of about one-fifth of the earth's radius.

The conclusions as to the distribution of κ are summarized in § 12 and fig. 1. The most important one is that, at about 700 km. from the earth's surface, there is a very rapid rise in κ, from a value of order 10^{-13} e.m.u. or less, to a value at least as high as 10^{-11} e.m.u., and possibly much higher. This suggests that there is some change in the composition of the substance of the earth at that depth, though seismological evidence appears to indicate that the transition to a denser, and therefore presumably, a more metallic, content takes place at a considerably greater depth (Jeffreys 1929, pp. 130, 218; Gutenberg and Richter 1938, p. 363). It may, however, be of interest to note that the greatest observed depth of earthquake foci is also about 600 or 700 km. (Berlage 1937).

2. The field equations for a non-uniform medium

Inside a continuous medium at rest the fundamental relations of the electromagnetic field reduce to

$$\operatorname{curl} \mathbf{E} = -\dot{\mathbf{B}}, \qquad \operatorname{div} \mathbf{B} = 0, \tag{2, 1}$$

$$\operatorname{curl} \mathbf{H} = 4\pi\mathbf{c} + \dot{\mathbf{D}}, \quad \operatorname{div} \mathbf{D} = 4\pi\rho, \tag{2, 2}$$

where $\mathbf{E}$ and $\mathbf{H}$ are respectively the electric and magnetic field intensities, $\mathbf{D}$ and $\mathbf{B}$ the electric and magnetic inductions, $\mathbf{c}$ the conduction current density, and ρ the space-charge density, all being measured in electromagnetic units.

If the medium is isotropic, we have also the constitutive relations

$$\mathbf{D} = \epsilon\mathbf{E}, \quad \mathbf{B} = \mu\mathbf{H}, \quad \mathbf{c} = \kappa\mathbf{E}, \tag{2, 3}$$

where ϵ, μ and κ are, in general, functions of the space variables. We shall assume that the gradients of these functions exist at all points inside the medium considered.

From (2, 2) we have the equation of continuity,

$$4\pi \operatorname{div} \mathbf{c} = -\operatorname{div} \dot{\mathbf{D}} = -4\pi\dot{\rho}, \tag{2, 4}$$

which, together with (2, 3), leads to the relation

$$\dot{\rho} + 4\pi\kappa\rho/\epsilon = (\kappa/\epsilon)\,\mathbf{c}\,.\operatorname{grad}(\epsilon/\kappa). \tag{2, 5}$$

This shows that, if (*a*) ϵ/κ is constant or (*b*) $\mathbf{c}$ is perpendicular to grad (ϵ/κ), ρ is independent of the field vectors and will decay like $\exp(-4\pi\kappa t/\epsilon)$. In these two cases the space-charge distribution is independent of the electromagnetic field, so that if ρ is initially zero it will remain so, and consequently $\mathbf{c}$ will be non-divergent.

If $\mathbf{A}$ is the vector potential of the magnetic field, we have

$$\operatorname{curl} \mathbf{A} = \mathbf{B}, \quad \mathbf{E} = -\dot{\mathbf{A}} - \operatorname{grad}\phi, \tag{2, 6}$$

where ϕ is an arbitrary continuous function. If we identify ϕ with the potential of the instantaneous charge distribution, we have

$$\operatorname{div}(\epsilon \operatorname{grad}\phi) = -4\pi\rho, \tag{2, 7}$$

which, on substituting in (2, 6), shows that $\epsilon\dot{\mathbf{A}}$ will be non-divergent. We can therefore take, without loss of generality,

$$\operatorname{div} \epsilon\mathbf{A} = 0. \tag{2, 8}$$

It is also convenient in the present case to define a new vector $\mathbf{A}_1$ by the relations

$$\mathbf{A}_1 = \mathbf{A} + \operatorname{grad}\psi, \quad \dot{\psi} = \phi, \quad \text{so that } \mathbf{E} = -\dot{\mathbf{A}}_1. \tag{2, 9}$$

It is then readily found that $\mathbf{A}_1$ satisfies the equation

$$\operatorname{curl}\operatorname{curl}\mathbf{A}_1 - \operatorname{grad}\log\mu \wedge \operatorname{curl}\mathbf{A}_1 + 4\pi\kappa\mu\dot{\mathbf{A}}_1 + \mu\epsilon\ddot{\mathbf{A}}_1 = 0. \tag{2, 10}$$

If the field varies sufficiently slowly, the displacement current $\dot{\mathbf{D}}/4\pi$ will be small and $\mathbf{c}$ will be approximately non-divergent. In fact $\dot{\mathbf{D}}$ may be neglected in equation (2, 2) if it is small enough compared with either $\mathbf{c}$ or curl $\mathbf{H}$. When the field is oscillatory with a period of order T, $\dot{\mathbf{D}}$ is of order $\epsilon\mathbf{E}/T$ and $\mathbf{c}$ of order $\kappa\mathbf{E}$, so that

$$\dot{\mathbf{D}} \ll \mathbf{c} \quad \text{if} \quad T \gg \epsilon/\kappa. \tag{2, 11}$$

If this condition is satisfied the medium will behave like an ordinary conductor. In electromagnetic units ϵ is of order c^{-2}, where $c = 3\cdot10^{10}$ cm.sec.$^{-1}$, so that even if κ is as small as 10^{-20} e.m.u., T has only to be greater than a few seconds.

In a dielectric, $\dot{\mathbf{D}}$ may still be ignored if it is small compared with curl $\mathbf{H}$. The latter is of order $\mathbf{H}/L$, where L is a length representing the scale of the field; also $\dot{\mathbf{H}}$ is of order $\mathbf{E}/\mu L$, so that curl $\mathbf{H}$ is of order $\mathbf{E}T/\mu L^2$. Hence

$$\dot{\mathbf{D}} \ll \operatorname{curl}\mathbf{H} \quad \text{if} \quad T \gg \surd(\epsilon\mu)\,L. \tag{2, 12}$$

This is simply the condition that T is large compared with the time taken by electromagnetic waves to travel across the region under consideration. In the case of the earth L is of order 10^9 cm., and $\surd(\epsilon\mu)$ of order c^{-1}, so that this condition is satisfied if $T \gg 0\cdot03$ sec.

3. Electromagnetic induction in a non-uniform conductor

Consider a non-uniform conductor (ϵ, μ, κ) surrounded by a dielectric $(\epsilon', \mu', 0)$ and situated in a varying electromagnetic field. The problem of determining the disturbance produced in the field by the conductor reduces to solving the above equations for $\mathbf{A}$ and ϕ (or $\mathbf{A}_1$ and ψ), subject to the conditions that they contain a part representing the field of external origin and satisfy the proper relations at the boundary of the conductor.

In all the cases to be considered, only the conductivity κ is variable*, ϵ and μ being constants; also the above conditions (2, 11) and (2, 12) are satisfied, so that $\dot{\mathbf{D}}$ may be neglected. With these simplifications, we have

$$\operatorname{div} \mathbf{A}_1 = -\nabla^2\psi, \quad \nabla^2\dot{\psi} = -\mathbf{c}.\operatorname{grad}(1/\kappa) = -4\pi\rho/\epsilon, \tag{3, 1}$$

$$\nabla^2\mathbf{A}_1 + \operatorname{grad}\nabla^2\dot{\psi} = 4\pi\kappa\mu\dot{\mathbf{A}}_1. \tag{3, 2}$$

At the boundary of the conductor, the tangential components of $\mathbf{E}$ and $\mathbf{H}$ and the normal component of $\mathbf{B}$ are continuous. Also the function ϕ, which is the potential of the space-charge distribution ρ together with a charge distribution σ on the surface of the conductor, is continuous. To determine σ we have $4\pi\sigma = D'_n - D_n = \epsilon' E'_n - \epsilon E_n$, where the suffix $(_n)$ denotes the outward normal component of any vector.

Also $\kappa E_n = c_n = \dot{\sigma}$, and therefore

$$4\pi\sigma + \epsilon\dot{\sigma}/\kappa = \epsilon' E'_n, \tag{3, 3}$$

which corresponds to equation (2, 5) for ρ. But since (2, 11) is satisfied, the second term on the left is negligible compared with the first, so that σ is very nearly equal to $\epsilon' E'_n/4\pi$ e.m.u. Further, $E_n = \dot{\sigma}/\kappa \ll 4\pi\sigma/\epsilon$, i.e. $E_n \ll (\epsilon'/\epsilon) E'_n$, so that E_n is in general negligible compared with E'_n. We thus see that a minute charge distribution (of order $c^{-2} E'_n$ e.m.u.) is set up on the surface of the conductor, which practically reduces the normal component of $\mathbf{E}$ just inside to zero. It is, in fact, the electrostatic field of this surface distribution which deflects any currents approaching the surface so that they flow parallel to it. Since $\mathbf{E} = -\dot{\mathbf{A}}_1$, we see also that the normal component of $\mathbf{A}_1$ just inside the surface must be zero. The other boundary conditions given above imply the continuity of $\dot{\psi}$ and of the tangential components of $\mathbf{A}_1$ and $\mu^{-1}\operatorname{curl}\dot{\mathbf{A}}_1$.

It will be observed that $\dot{\psi}$ enters into the differential equations only in the form $\nabla^2\dot{\psi}$. For certain distributions of κ these equations will have a solution in which $\nabla^2\dot{\psi}$ is zero. This indicates that there will be no charge distribution inside the conductor, but only a surface charge, as is the case when the conductivity is uniform. Also the equation (3, 1) indicates that, for these special distributions of κ, the currents will flow in the surfaces for which κ is constant. Moreover, in such cases, the induced currents can be determined by solving the above equations for $\mathbf{A}_1$, without further considering $\dot{\psi}$. It will, however, be noted that this does not imply that electrostatic effects are ignored;

* I.e. κ is a function of $\mathbf{r}$, but not, of course, of t.

the new vector potential $\mathbf{A}_1$ in fact contains grad ϕ, and its normal component is zero just inside the conductor simply because of this term, which represents the effect of the electrostatic field of the surface distribution.

4. Conductors with spherical symmetry

We shall now show that there is a solution of the above special form when the conductor is a sphere, with κ a function of the distance r from the centre. If there is such a solution corresponding to a given inducing field, it will be unique, because, if two different solutions were found, their difference would, since the equations are linear, correspond to a zero inducing field. It could therefore represent only the decay of some initial distribution of current in the conductor, which we take as zero if we are concerned only with the effect of the inducing field.

Let the radius of the sphere be a, and let $\kappa = \kappa(\rho)$, where $\rho = r/a$. When $\nabla^2\psi = 0$, we find the elementary solutions of (3, 2) for $\mathbf{A}_1$, appropriate to spherical boundaries, to be of the forms $\mathbf{r}\wedge\operatorname{grad} u$ and $v \operatorname{grad} w$, where u, v and w are functions depending on $\kappa(\rho)$. Since the radial component of $\mathbf{A}_1$ is to be zero just inside the surface, we consider the solutions of the form $\mathbf{A}_1 = \mathbf{r}\wedge\operatorname{grad} u$. This makes div $\mathbf{A}_1$ zero and therefore satisfies (3, 1). Also it satisfies (3, 2) provided $\nabla^2 u = 4\pi\kappa\mu\dot{u}$, and the solution of this equation, when $\kappa\mu$ is a function of ρ only, is the sum of terms of the form

$$u_n = a_0 f_n(t, \rho)\, S_n,$$

where S_n is any *surface* spherical harmonic of degree n, a_0 is the radius of the sphere of reference of the spherical harmonics, and $f_n(t, \rho)$ satisfies the equation

$$\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial f_n}{\partial\rho}\right) = \left\{n(n+1) + 4\pi\mu a^2\rho^2\kappa(\rho)\frac{\partial}{\partial\rho}\right\} f_n. \tag{4, 1}$$

We thus have a solution of the equations for $\mathbf{A}_1$ in the form of a series, of which a typical term is

$$\mathbf{A}_{1n} = a_0\mathbf{r}\wedge\operatorname{grad}\{f_n(t,\rho)\, S_n\} = a_0(\mathbf{r}\wedge\operatorname{grad} S_n) f_n(t,\rho). \tag{4, 2}$$

Outside the conductor, where $\kappa = 0$ and $\mu = 1$, both $\nabla^2 \mathbf{A}_1$ and div $\mathbf{A}_1$ are zero, and therefore curl $\mathbf{H}$ = curl curl $\mathbf{A}_1 = 0$, so that

$$\mathbf{H} = -\operatorname{grad}\Omega, \tag{4, 3}$$

where Ω satisfies Laplace's equation, and is therefore the sum of spherical harmonic terms of the form

$$\Omega_n = a_0\{e_n(t)\,(q\rho)^n + i_n(t)\,(q\rho)^{-n-1}\}\, S_n, \tag{4, 4}$$

where $q = a/a_0 \leqslant 1$. The function $e_n(t)$ represents the inducing field of external origin and will be a known function of t, while $i_n(t)$, corresponding to the induced field, has yet to be determined.

The term in $\mathbf{A}_1$, corresponding to Ω_n, is found by solving the equation

$$\operatorname{curl} \mathbf{A}_{1n} = -\operatorname{grad} \Omega_n,$$

which gives
$$\mathbf{A}_{1n} = a_0(\mathbf{r} \wedge \operatorname{grad} S_n) \left\{ \frac{(q\rho)^n}{n+1} e_n(t) - \frac{(q\rho)^{-n-1}}{n} i_n(t) \right\}, \tag{4, 5}$$

which is, of course, in agreement with (4, 1) and (4, 2) when $\kappa = 0$.

At the boundary $\rho = 1$, the continuity of the tangential components of $\mathbf{A}_1$ and of the tangential components of $\mu^{-1} \operatorname{curl} \mathbf{A}_1$ lead to the relations

$$f_n(t, 1) = \frac{q^n}{n+1} e_n(t) - \frac{q^{-n-1}}{n} i_n(t), \tag{4, 6}$$

$$f_n(t, 1) + \left[\frac{\partial}{\partial \rho} f_n(t, \rho) \right]_{\rho=1} = \mu\{q^n e_n(t) + q^{-n-1} i_n(t)\}. \tag{4, 7}$$

All the boundary conditions considered in § 3 are then satisfied.

The problem is thus reduced to the determination of the unknown functions* $i(t)$ and $f(t, \rho)$ in terms of the known function $e(t)$, from the equations (4, 1), (4, 6) and (4, 7). The induced field is then given by the relevant part of (4, 4), and the induced current is given by

$$\mathbf{c} = \kappa \mathbf{E} = -\kappa \dot{\mathbf{A}}_1 = -a_0(\mathbf{r} \wedge \operatorname{grad} S_n)\, \kappa(\rho) \frac{\partial}{\partial t} f(t, \rho). \tag{4, 8}$$

It will be observed that the current is in the same direction at all points on the same radius vector.

There are two cases to be considered, (i) when the inducing field is periodic and has existed long enough for the transient effects of the initial circumstances to be negligible, so that the induced field is also periodic with the same period, (ii) when the inducing field is aperiodic and the effects of the initial conditions have to be determined.

In the first case, it is sufficient to consider a single harmonic constituent of the field, of period $2\pi/\alpha$ say, and we may then treat e, i and f as the real parts of $Ee^{i\alpha t}$, $Ie^{i\alpha t}$ and $F(\rho)\, e^{i\alpha t}$ respectively, so that equation (4, 1) becomes an ordinary differential equation with $i\alpha$ in place of $\partial/\partial t$.

In case (ii) the most convenient method of solution is afforded by the Heaviside operational calculus, in which the relations between the above time functions are expressed by means of operators in the forms

$$i(t) = I(p)\, e(t), \tag{4, 9}$$

$$f(t, \rho) = F(p, \rho)\, e(t), \tag{4, 10}$$

the functions $I(p)$ and $F(p, \rho)$ being obtained by solving the equations (4, 1), (4, 6) and (4, 7), with the operator $\partial/\partial t$ replaced by a constant p. The values of $i(t)$ and $f(t, \rho)$ can then be determined from (4, 9) and (4, 10) by known theorems in the operational calculus.

* The suffix n will now be omitted when no confusion can arise.

Thus, whether the field is periodic or aperiodic, the equations are first solved with $\partial/\partial t$ replaced by a constant, and we can therefore treat both cases together, taking p, $I(p)$ and $F(p,\rho)$ to represent either the complex quantities $i\alpha$, $I(i\alpha)$ and $F(i\alpha,\rho)$ in the first case, or the operators in the second.

The equation (4, 1), with the condition that the field remains finite at $\rho = 0$, determines $F(p,\rho)$ completely, except for an arbitrary coefficient independent of ρ, so that we can write

$$F(p,\rho) = C(p)\,R(p,\rho), \tag{4, 11}$$

where $R(p,\rho)$ will be a known function. On substituting in equations (4, 6) and (4, 7) we then easily find

$$C(p) = \frac{q^n}{n+1}\frac{(2n+1)\,\mu}{(n\mu+1)\,R+R'}, \tag{4, 12}$$

$$I(p) = \frac{nq^{2n+1}}{n+1}\left\{1-\frac{(2n+1)\,\mu R}{(n\mu+1)\,R+R'}\right\}, \tag{4, 13}$$

where R and R' denote the values of $R(p,\rho)$ and $\frac{d}{d\rho}R(p,\rho)$ at $\rho = 1$. Thus when $R(p,\rho)$ has been found from (4, 1), $C(p)$ and $I(p)$ will be known, and $f(t,\rho)$ and $i(t)$ can be determined.

When the field is periodic, the time factor $i(t)$ of the induced field is equal, by (4, 9), to the real part of $I(i\alpha)\,Ee^{i\alpha t}$. Hence the amplitude ratio, and the phase difference at $r = a_0$, of the induced and inducing fields, are equal, respectively, to the modulus and argument of $I(i\alpha)$, which is given by (4, 13). In a similar way the amplitude and phase of the induced current can be obtained from (4, 8) and (4, 10) on replacing $e(t)$ by $Ee^{i\alpha t}$.

When the field is aperiodic, the time factors $i(t)$ and $f(t,\rho)$ are found by solving the operational equations (4, 9), (4, 10), the operators $I(p)$ and $F(p,\rho)$ being given by (4, 11)–(4, 13). The solution of any operational equation, say $g(t) = G(p)\,e(t)$, where $G(p)$ is a known operator, is given by

$$g(t) = \frac{d}{dt}\int_0^t e(t-u)\,h(u)\,du, \tag{4, 14}$$

where

$$h(u) = \frac{1}{2\pi i}\int_L e^{\lambda u}\,G(\lambda)\frac{d\lambda}{\lambda}, \tag{4, 15}$$

the path of integration L being a curve from $c-i\infty$ to $c+i\infty$, where c is positive and finite, and the singularities of the integrand are on the left side of L, with $|\arg\lambda| \leqslant \pi$. This contour integral (4, 15) is due to Bromwich (1916); the function $h(t)$, which it determines, is the value of $g(t)$ when $e(t)$ is equal to Heaviside's unit function $H(t)$, defined by $H(t) = 0$ when $t \leqslant 0$, $H(t) = 1$ when $t > 0$. Hence $h(t)$ gives the result for a sudden increase in the inducing field. When $h(t)$ has been determined, the formula* (4, 14) gives the result for any other variation of the inducing field, beginning at time $t=0$.

Thus to determine $i(t)$ or $f(t,\rho)$, we have simply to substitute $I(\lambda)$ or $F(\lambda,\rho)$ respectively for $G(\lambda)$ in the above formulae. Alternately, having found $f(t,\rho)$, we may obtain $i(t)$ from (4, 6), which gives

$$i(t)=\frac{nq^{2n+1}}{n+1}e(t)-nq^{n+1}f(t,1). \tag{4, 16}$$

The above general formulae hold good for any spherical conductor in which κ is a (differentiable) function of ρ only. It will be observed that $\kappa(\rho)$ affects the result entirely through the function $R(p,\rho)$, which, from (4, 11) and (4, 1) is the solution of

$$\frac{d}{d\rho}\left(\rho^2\frac{dR}{d\rho}\right)=\{n(n+1)+4\pi\mu a^2\rho^2\kappa(\rho)\,p\}\,R, \tag{4, 17}$$

which does not become infinite as ρ tends to zero. The arbitrary constant factor remaining in R can be given any value, since it will obviously cancel out in the expressions for $I(p)$ and $F(p,\rho)$.

5. The function $R(p,\rho)$, when $\kappa=k\rho^{-m}$

The particular case when κ is a constant has been dealt with in the memoirs already cited, and the solution is expressible in terms of Bessel functions of half an odd integer. Another case of interest is when $\kappa=k\rho^{-2}$, where k is a constant, for in this case the above equation for R reduces to a homogeneous linear one, and is solved by

$$R=A\rho^{-\frac{1}{2}-s}+B\rho^{-\frac{1}{2}+s},\quad s=\sqrt{\{n(n+1)+\tfrac{1}{4}+\zeta^2\}}, \tag{5, 1}$$

where

$$\zeta^2=4\pi\mu a^2kp. \tag{5, 2}$$

In the more general case where $\kappa=k\rho^{-m}$, where m is any real constant except $+2$, the equation for R, on writing

$$R=w\rho^{-\frac{1}{2}},\quad \rho^{1-\frac{1}{2}m}=\frac{|m-2|\,z}{2\zeta},\quad \nu=\frac{2n+1}{|m-2|}, \tag{5, 3}$$

reduces to the Bessel equation

$$z^2\frac{d^2w}{dz^2}+z\frac{dw}{dz}=(\nu^2+z^2)\,w, \tag{5, 4}$$

of which the general solution is

$$w=A'I_\nu(z)+B'K_\nu(z), \tag{5, 5}$$

where I_ν and K_ν are the modified Bessel functions of the first and second kinds and of order ν.

* This formula expresses a superposition principle discovered independently in many branches of physics; it is true when the differential equations involved are linear and hence whenever the methods of the operational calculus are applicable. References to various proofs are given by Goldstein (1932, p. 104).

Since $R(p,\rho)$ is to remain finite as ρ tends to zero, we find that (i) for $m>2$, A' is zero, (ii) for $m=2$, B is zero, and (iii) for $m<2$, B' is zero. Hence, taking the remaining arbitrary constant in each case to be unity, we have

$$\begin{array}{lll} m>2 & m=2 & m<2 \\ R(p,\rho)=\rho^{-\frac{1}{2}}K_\nu(z), & \rho^{-\frac{1}{2}+s}, & \rho^{-\frac{1}{2}}I_\nu(z). \end{array} \tag{5, 6}$$

6. The induced field and current-distribution when $\kappa = k\rho^{-m}$, $\mu=1$

On substituting the above expression for $R(p,\rho)$ in (4, 12) and (4, 13) and taking $\mu=1$, we obtain, after some transformations using the recurrence relations of Bessel functions,

$$\begin{array}{llll} & m>2 & m=2 & m<2 \\ C(p)= & \dfrac{(2n+1)\,q^n}{(n+1)\,\zeta K_{\nu+1}(z_1)}, & \dfrac{(2n+1)\,q^n}{(n+1)\,(s+n+\frac{1}{2})}, & \dfrac{(2n+1)\,q^n}{(n+1)\,\zeta I_{\nu-1}(z_1)}, \end{array} \tag{6, 1}$$

$$I(p)=\frac{nq^{2n+1}\,K_{\nu-1}(z_1)}{(n+1)\,K_{\nu+1}(z_1)},\ \frac{nq^{2n+1}(s-n-\frac{1}{2})}{(n+1)\,(s+n+\frac{1}{2})},\ \frac{nq^{2n+1}\,I_{\nu+1}(z_1)}{(n+1)\,I_{\nu-1}(z_1)}, \tag{6, 2}$$

where z_1 is the value of z when $\rho=1$. From these we can now derive expressions for the induced field and current when the inducing field is either periodic or aperiodic.

(i) *Periodic field.* The amplitude ratio, i/e say, and the phase difference, $\iota-\epsilon$ say, of the field at $r=a_0$ are given by

$$i/e=\operatorname{mod} I(i\alpha), \quad \iota-\epsilon=\arg I(i\alpha), \tag{6, 3}$$

where $I(i\alpha)$ is obtained from (6, 2) on writing $i\alpha$ for p, so that $z_1=x\surd i$, where

$$x=4a\surd(\pi k\alpha)/|\,m-2\,|. \tag{6, 4}$$

The numerical evaluation of $I(i\alpha)$ is easy when $m=2$. It is also easy when $m>2$ and ν is half an odd integer, because then the Bessel functions in (6, 2) are of specially simple form, and $I(i\alpha)$ consequently reduces to a rational function of $\surd(i\alpha)$. In a few other cases, e.g., when ν is a small integer, it is possible to make use of tables of ker and kei functions when $m>2$, or ber and bei functions when $m<2$; but in general ν is fractional, and it is necessary to use the ascending power series expressions for the Bessel functions when x is small, or their asymptotic expansions when x is large.

When $m>2$, we find, on expressing $K_\nu(x\surd i)$ in terms of $I_\nu(x\surd i)$ and $I_{-\nu}(x\surd i)$, and expanding these in ascending (fractional) powers of x,

$$K_\nu(x\surd i)=\tfrac{1}{2}\pi\operatorname{cosec}\nu\pi\{(\Phi_{-\nu}+i\Psi_{-\nu})\,e^{-i\nu\pi/4}-(\Phi_\nu+i\Psi_\nu)\,e^{i\nu\pi/4}\}, \tag{6, 5}$$

where
$$\Phi_\nu=\sum_{r=0}^{\infty}\frac{(-)^r\,(\frac{1}{2}x)^{\nu+4r}}{(2r)!\,\Gamma(\nu+2r+1)}, \quad \Psi_\nu=\sum_{r=0}^{\infty}\frac{(-)^r\,(\frac{1}{2}x)^{\nu+4r+2}}{(2r+1)!\,\Gamma(\nu+2r+2)}. \tag{6, 6}$$

This is useful for evaluating $I(i\alpha)$ when x is sufficiently small, say $x \leqslant 2$. It leads when $\nu < 1$ to the approximation

$$I(i\alpha) = \frac{nq^{2n+1}}{n+1}\frac{\Gamma(1-\nu)}{\Gamma(1+\nu)}\left(\frac{x}{2}\right)^{2\nu}\left(\cos\frac{\nu\pi}{2}+i\sin\frac{\nu\pi}{2}\right), \quad \nu<1,\ x \text{ small.} \tag{6, 7}$$

If, however, $\nu > 1$, the approximation becomes

$$I(i\alpha) = \frac{nq^{2n+1}}{n+1}\nu(\nu-1)\left(\frac{x}{2}\right)^{2} i, \quad \nu>1,\ x \text{ small.} \tag{6, 8}$$

It follows from (6, 7) that, when $\nu < 1$ and $a^2k\alpha \to 0$, $i/e \to 0$ and $\iota - \epsilon \to \frac{1}{2}\nu\pi$; while from (6, 8) when $\nu > 1$ and $a^2k\alpha \to 0$, $i/e \to 0$ and $\iota - \epsilon \to \frac{1}{2}\pi$. Hence, as the surface conductivity k is decreased, the phase difference $\iota - \epsilon$ will tend to $90°$ only if $\nu > 1$, i.e. if $m < 2n+3$. If m is greater than this, the phase difference will tend to $\nu \times 90°$.

When x is much greater than 2, the series in (6, 5) do not converge very rapidly, but in this case we can use the asymptotic expansions of the Bessel functions to obtain fairly accurate results. We find

$$\operatorname{mod} K_\nu(x\sqrt{i}) = \sqrt{\{\tfrac{1}{2}\pi x^{-1} e^{-x\sqrt{2}} (\lambda^2+\mu^2)\}}, \tag{6, 9}$$

$$\arg K_\nu(x\sqrt{i}) = \tan^{-1}\frac{\mu}{\lambda} - \frac{x}{\sqrt{2}} - \frac{\pi}{8}, \tag{6, 10}$$

where
$$\lambda + i\mu \sim \sum_{r=0} \frac{(4\nu^2-1)(4\nu^2-3^2)\ldots\{4\nu^2-(2r-1)^2\}}{r!\,(8x)^r}\left(\cos\frac{r\pi}{4} - i\sin\frac{r\pi}{4}\right). \tag{6, 11}$$

This leads to the approximation

$$I(i\alpha) = \frac{nq^{2n+1}}{n+1}\left(1-\frac{\nu\sqrt{2}}{x}+i\frac{\nu\sqrt{2}}{x}\right), \quad (m-2)x \text{ large.} \tag{6, 12}$$

If m is made to tend to infinity in (6, 7) or k is made to tend to infinity in (6, 12), we get the correct results for an infinitely conducting sphere, viz. $i/e = nq^{2n+1}/(n+1)$ and $\iota - \epsilon = 0$.

The corresponding results for $m < 2$ can be obtained by similar methods, but are not required at present.

The induced current density is given by

$$\mathbf{c} = -(\mathbf{r} \wedge \operatorname{grad} S_n)\, a_0\, k\rho^{-m} i\alpha C(i\alpha)\, R(i\alpha, \rho)\, Ee^{i\alpha t}, \tag{6, 13}$$

and if we write
$$\mathbf{c} = \mathbf{c}_s v, \tag{6, 14}$$

where $\mathbf{c}_s$ is the value of $\mathbf{c}$ at the surface of the conductor, we have

$$\mathbf{c}_s = -(\mathbf{r} \wedge \operatorname{grad} S_n)\, a_0 k i\alpha\left(\frac{q^n}{n+1} - \frac{q^{-n-1}}{n} I(i\alpha)\right) Ee^{i\alpha t}, \tag{6, 15}$$

$$v = \rho^{-m}\frac{R(i\alpha, \rho)}{R(i\alpha, 1)}. \tag{6, 16}$$

Thus the surface current $\mathbf{c}_s$ can be obtained from the induced field already considered, and for the distribution of $\mathbf{c}$ below the surface we have only to consider the complex function v, whose modulus and argument will give the amplitude and phase of $\mathbf{c}$ as compared with $\mathbf{c}_s$.

When $m>2$, we have $\quad v=\rho^{-m-\frac{1}{2}}\dfrac{K_\nu(\gamma x\sqrt{i})}{K_\nu(x\sqrt{i})},\quad \gamma=\rho^{-\frac{1}{2}m+1}.$ (6, 17)

This expression can be evaluated immediately by means of the various formulae already given for $K_\nu(x\sqrt{i})$. When γx is sufficiently small, we obtain from (6, 5) and (6, 6) the approximation

$$v=\rho^{n-m}\left\{1-\frac{\Gamma(1-\nu)}{\Gamma(1+\nu)}(\tfrac{1}{2}x\gamma)^{2\nu}e^{i\nu\pi/2}\right\}, \tag{6, 18}$$

which shows that, if the surface conductivity is sufficiently small, the induced currents, down to a certain depth, will increase or decrease in intensity according as m is greater or less than n. Also, throughout this depth, the phase of the induced currents will remain practically the same. But, in any case, γx will become large at a certain depth, depending on m and x. When this large value is reached, the asymptotic expansions (6, 9)–(6, 11) may be used to determine v. From these we obtain the approximations

$$\operatorname{mod} v=\rho^{-\frac{3}{4}m-1}\exp\{(1-\gamma)\,x/\sqrt{2}\},\quad \arg v=(1-\gamma)\,x/\sqrt{2}, \tag{6, 19}$$

when γx is very large. This shows that the intensity of the induced currents will decrease very rapidly with increasing depth after a certain depth has been reached, since the exponential factor in mod r decreases, and becomes much more important than the other (increasing) factor when ρ reaches a sufficiently small value. Also, when this depth is reached, arg v will be large, and therefore the phase of the induced current will change rapidly with increasing depth.

(ii) *Aperiodic field.* In this case we first obtain, as explained in § 4, the special forms assumed by the time functions $i(t)$ and $f(t,\rho)$, when $e(t)$ is taken equal to $H(t)$. These special forms will be denoted by $\phi(t)$ and $\psi(t,\rho)$ respectively.

When $m>2$, we find, on evaluating the contour integral (4, 15), with $G(\lambda)$ replaced by $I(\lambda)$ and $F(\lambda,\rho)$, respectively,

$$\phi(t)=\frac{nq^{2n+1}}{n+1}\frac{8\nu}{\pi^2}\int_0^\infty e^{-u^2t/\tau}\frac{1}{J_{\nu+1}^2(u)+Y_{\nu+1}^2(u)}\frac{du}{u^3}, \tag{6, 20}$$

$$\psi(t,\rho)=\frac{q^n\rho^{-\frac{1}{2}}}{n+1}\left\{\gamma^\nu+\frac{4\nu}{\pi}\int_0^\infty e^{-u^2t/\tau}\frac{J_\nu(u\gamma)\,Y_{\nu+1}(u)-Y_\nu(u\gamma)\,J_{\nu+1}(u)}{J_{\nu+1}^2(u)+Y_{\nu+1}^2(u)}\frac{du}{u^2}\right\}, \tag{6, 21}$$

where $\quad \tau=16\pi a^2k/(m-2)^2,\quad \gamma=\rho^{-\frac{1}{2}m+1}.$ (6, 22)

Alternative expressions for $\phi(t)$ and $\psi(t,\rho)$, which are more convenient for numerical calculations, can be obtained by expanding the operators $I(p)$ and $F(p,\rho)$ in descending or ascending (fractional) powers of p, and interpreting term by term, using the known result $p^\nu H(t)=t^{-\nu}/\Gamma(1-\nu)$, which satisfies (4, 15) (Heaviside 1899, p. 69; Jeffreys

1931, ch. 5). The conditions under which the formal expansion of any operator such as $I(p)$ in ascending (fractional) powers of p will lead to a series of (fractional) powers of $(1/t)$, which is asymptotic to $\phi(t)$, have been investigated by Sutton (1934). Thus, by using the asymptotic expansions of the $K_\nu(z)$ functions, we obtain an expansion for $I(p)$ in descending powers of p, which leads to the following ascending power series for $\phi(t)$,

$$\phi(t)=\frac{nq^{2n+1}}{n+1}\left\{1+\frac{c_2 t}{\tau}+\frac{c_4 t^2}{2!\,\tau^2}+\ldots+\sqrt{\left(\frac{t}{\pi\tau}\right)}\left(2c_1+\frac{2^2c_3 t}{3\tau}+\frac{2^3c_5 t^2}{3\,.\,5\tau^2}+\ldots\right)\right\}, \tag{6, 23}$$

where

$$c_s=a_s(\nu-1)-\sum_{r=0}^{s-1}c_r a_{s-r}(\nu+1),\quad c_0=1;\quad a_s(\nu)=\frac{(4\nu^2-1)\,(4\nu^2-3^2)\ldots\{4\nu^2-(2s-1)^2\}}{8^s\,.\,s!}. \tag{6, 24}$$

This expression is convenient when t/τ is small. For larger values of t, we find the asymptotic expansion

$$\phi(t)\sim\frac{nq^{2n+1}}{(n+1)\,\Gamma(1+\nu)}\left\{\left(\frac{4t}{\tau}\right)^{-\nu}-2\left(\frac{4t}{\tau}\right)^{-1-\nu}+\frac{(3-2\nu)\,(1+\nu)}{\nu(1-\nu)}\left(\frac{4t}{\tau}\right)^{-2-\nu}+\ldots+\frac{2\{\Gamma(1-\nu)\}^2}{\Gamma(2+\nu)\,\Gamma(1-2\nu)}\left(\frac{4t}{\tau}\right)^{-1-2\nu}+\ldots\right\}. \tag{6, 25}$$

When $\phi(t)$ has been obtained in the above way, the value of $i(t)$ corresponding to any given $e(t)$, can be found by substituting $\phi(u)$ for $h(u)$ in (4, 14); this formula can be evaluated by numerical integration when $e(t)$ is an empirical function. When $e(t)$ is an analytical function, it is sometimes more convenient to express $e(t)$ in the operational form $E(p)\,H(t)$ by means of the theorem*

$$E(p)=p\int_0^\infty e^{-pt}e(t)\,dt, \tag{6, 26}$$

and then to evaluate $i(t)$ by suitably expanding the combined operator $I(p)\,E(p)$. An important case is

$$e(t)=e^{-\lambda t}H(t),\quad E(p)=p/(p+\lambda), \tag{6, 27}$$

which leads to various expressions for $i(t)$ depending on the values of λ and τ. Thus when λt is small but t/τ is large, we find (cf. 6, 25)

$$i(t)\sim\frac{nq^{2n+1}}{(n+1)\,\Gamma(1+\nu)}\left\{\left(\frac{4t}{\tau}\right)^{-\nu}M_{-\nu}(\lambda t)-2\left(\frac{4t}{\tau}\right)^{-1-\nu}M_{-1-\nu}(\lambda t)+\ldots+\frac{2\{\Gamma(1-\nu)\}^2}{\Gamma(2+\nu)\,\Gamma(1-2\nu)}\left(\frac{4t}{\tau}\right)^{-1-2\nu}M_{-1-2\nu}(\lambda t)\ldots\right\}, \tag{6, 28}$$

where $M_{-s}(z)$ is a confluent hypergeometric function, defined by

$$M_{-s}(z)=1-\frac{z}{1-s}+\frac{z^2}{(1-s)\,(2-s)}-\frac{z^3}{(1-s)\,(2-s)\,(3-s)}+\ldots. \tag{6, 29}$$

* Carson (1926, p. 16). The theorem was stated by Heaviside (1912, p. 327), but not explicitly proved.

The series (6, 28) is asymptotic to $i(t)$ for *large* t/τ, though each term of this series is expressed in (6, 29) by means of an *ascending* series in λt. When λt is itself large, $M_{-s}(\lambda t)$ has the asymptotic expansion

$$M_{-s}(\lambda t) \sim -\frac{s}{\lambda t}-\frac{s(s+1)}{\lambda^2 t^2}-\frac{s(s+1)(s+2)}{\lambda^3 t^3}-\dots. \tag{6, 30}$$

The corresponding results for $m<2$ will not be given in detail, but it may be noted that in this case $\phi(t)$ and $\psi(t,\rho)$ are expressible as series of exponential terms of the form $C_s \exp(-\alpha_s t/\tau)$, where the α's are the (real and positive) zeros of $J_{\nu-1}(z)$ (see, for example, the result for $m=0$ given by Price 1931). This is because $I(\lambda)$ and $F(\lambda,\rho)$, regarded as functions of a complex variable λ, possess Mittag-Leffler expansions, so that $i(t)$ and $\psi(t,\rho)$ can be determined from an extension of the Heaviside partial fraction rule.

It is of interest to note how the form of the solution changes as m passes through the value 2. As m approaches 2 from below, ν tends to infinity and the simple poles of $I(\lambda)$ (cf. (6, 2)) converge together at the origin, which becomes an essential singularity for $m \geqslant 2$. The influence of this on $\phi(t)$ is to replace the above-mentioned exponential series by an exponential integral of the form (6, 20).

In the transitional case $m=2$, we find

$$\phi(t)=\frac{nq^{2n+1}}{n+1}\sqrt{\left(\frac{8}{\pi}\right)}e^{-t/(2\tau_1)}D_{-3}\{\sqrt{(2t/\tau_1)}\}, \tag{6, 31}$$

where

$$\tau_1=16\pi a^2 k/(2n+1)^2, \tag{6, 32}$$

and $D_{-3}(z)$ is a Weber parabolic function (Whittaker and Watson (1920), ch. 16), which can be conveniently expressed in terms of the error function.

The distribution of induced currents at time t, due to a sudden change in the inducing field, is obtained by substituting $\psi(t,\rho)$ for $f(t,\rho)$ in (4, 8). An explicit expression for $\psi(t,\rho)$ has been obtained in (6, 21), but this is not very convenient for numerical calculations. Alternative expressions for the current density can be obtained by substituting the operational form for $\psi(t,\rho)$ in (4, 8), and suitably transforming the resultant operator in the manner already described. Denoting the value of **c** when $e(t)=H(t)$ by **C**, we find

$$\mathbf{C}=-\frac{(m-2)(2n+1)q^{n-2}}{8\pi a_0(n+1)}(\mathbf{r}\wedge \operatorname{grad} S_n)\,\chi(t,\rho), \tag{6, 33}$$

where

$$\chi(t,\rho)=\rho^{-m-\frac{1}{2}}\sqrt{(\tau p)}\frac{K_\nu\{\gamma\sqrt{(\tau p)}\}}{K_{\nu+1}\{\sqrt{(\tau p)}\}}H(t). \tag{6, 34}$$

Using the asymptotic expansion

$$\frac{K_\nu(\gamma z)}{K_{\nu+1}(z)}\sim\frac{e^{-(\gamma-1)z}}{\sqrt{\gamma}}\left\{1+\frac{b_1}{z}+\frac{b_2}{z^2}+\dots\right\}, \tag{6, 35}$$

where the coefficients $b_1, b_2, \ldots$ are readily found by dividing the asymptotic series for $K_\nu(\gamma z)$ by that for $K_\nu(z)$, we obtain

$$\chi(t,\rho) = \rho^{-1-\frac{3}{2}m}\{E_0(t/\tau)+b_1E_1(t/\tau)+b_2E_2(t/\tau)+\ldots\}, \tag{6, 36}$$

where

$$E_n(t/\tau) = \int_0^{t/\tau} E_{n-2}(u)\,du; \quad E_0(t/\tau) = \sqrt{\left(\frac{\tau}{\pi t}\right)}e^{-\xi^2}, \quad E_1(t/\tau) = 1-\operatorname{erf}\xi = \frac{2}{\sqrt{\pi}}\int_\xi^\infty e^{-u^2}\,du, \tag{6, 37}$$

$$\xi^2 = (\gamma-1)^2\tau/(4t). \tag{6, 38}$$

This form of the result is useful for all values of ρ (including unity) provided t is sufficiently small. If, however, ρ differs sufficiently from unity to make $\gamma-1$ large, the above expression may be reduced, by means of the asymptotic series for the error function, to the much simpler form

$$\chi(t,\rho) \sim \rho^{-1-\frac{3}{2}m}\sqrt{\left(\frac{\tau}{4t}\right)}e^{-\xi^2}\left[1+b_1\left\{\frac{2t}{(\gamma-1)\tau}\right\}+\frac{2!}{2}b_2\left\{\frac{2t}{(\gamma-1)\tau}\right\}^2+\ldots\right], \tag{6, 39}$$

which is useful for all values of t and ρ which make $2t/(\gamma-1)\tau$ small. For larger values of t, we find, by expanding $1/K_{\nu+1}\{\sqrt{(\tau p)}\}$ in ascending powers of p,

$$\chi(t,\rho) \sim \frac{2}{\Gamma(1+\nu)}\rho^{-\frac{3}{2}-\frac{1}{2}m}\eta^{1+\nu}e^{-\gamma\eta}\left\{1+\frac{1}{\nu}\eta^2w_2(\gamma\eta)+\frac{2-\nu}{2\nu^2(1-\nu)}\eta^4w_4(\gamma\eta)+\ldots\right.$$
$$\left.+\frac{\Gamma(-\nu)}{\Gamma(2+\nu)}\eta^{2+2\nu}w_{2+2\nu}(\gamma\eta)+\ldots\right\}, \tag{6, 40}$$

where

$$\eta = \gamma\tau/(4t), \tag{6, 41}$$

and the function $w_s(z)$ is given in terms of the confluent hypergeometric function $W_{k,m}(z)$ (Whittaker and Watson's notation) by

$$w_s(z) = e^{\frac{1}{2}z}z^{-\frac{1}{2}(s+1+\nu)}W_{\frac{1}{2}(s+1+\nu),\frac{1}{2}\nu}(z), \tag{6, 42}$$

$$\sim 1+\frac{\nu^2-(s+\nu)^2}{4z}+\frac{\{\nu^2+(s+\nu)^2\}\{\nu^2+(s+\nu-2)^2\}}{2!\,(4z)^2}+\ldots. \tag{6, 43}$$

The series (6, 40) is asymptotic to $\chi(t,\rho)$ for large t, and is useful when η $(=\gamma\tau/(4t))$ is fairly small. The values of ρ and t which are of most interest (e.g. the value of ρ at which the maximum current density occurs for a given value of t) are frequently those for which η is small but $\gamma\eta$ is large. In such cases it is convenient to use the series (6, 40), which is asymptotic to $\chi(t,\rho)$ for *small* η, but to evaluate each term of this series by means of (6, 42) which is asymptotic to $w_s(\gamma\eta)$ for *large* $\gamma\eta$.

The general character of the induced current distribution can be readily deduced from the above expressions for χ. The current density is proportional to χ, and (6, 39) shows that, as $t\to 0$, $\chi\to 0$ for $\rho<1$ but $\to\infty$ for $\rho=1$. Also, for the period immediately following, in which $t \ll \tau$, χ is proportional to $\sqrt{(\tau/t)}$ at the surface (i.e. at $\rho = 1$), while for points inside the conductor $(\rho < 1)$ we have the approximation

$$\chi = \rho^{-1-\frac{3}{4}m}\sqrt{\left(\frac{\tau}{4t}\right)}\exp\left\{-\frac{\tau(1-\gamma)^2}{4t}\right\} \text{ when either } t \ll \tau \text{ or } t \ll (1-\gamma)\,\tau. \qquad (6,\ 44)$$

This increases as t increases, and also as ρ increases, in the range of values of t for which the approximation holds. Hence the first effect of a sudden change in the inducing field is to produce a sheet-current distribution on the surface of the conductor, which then gradually penetrates inwards. This is, of course, in agreement with physical considerations.

For larger values of t we obtain, from (6, 40), the approximation

$$\chi = \frac{2}{\Gamma(1+\nu)}\left(\frac{\tau}{4t}\right)^{1+\nu}\rho^{-m-n-1}\exp\left(-\frac{\tau\gamma^2}{4t}\right), \quad \text{when } t \gg \frac{\tau\gamma}{4}. \qquad (6,\ 45)$$

In this case $$\frac{\partial\chi}{\partial\rho} = 0, \quad \text{when } t = \frac{m-2}{m+n+1}\frac{\tau\gamma^2}{4}.$$

This value of t is sufficiently large for the above approximation to hold good if

$$\gamma\ (= \rho^{-\frac{1}{2}m+1}) \gg \frac{m+n+1}{m-2},$$

and this condition can always be satisfied by taking ρ small enough; actually, for large values of m, it is satisfied even when ρ differs only slightly from unity. Also for the same value of t, $\partial\chi/\partial\rho$ will become negative if ρ is increased, and positive if ρ is decreased. Hence at sufficiently great depths for the above conditions to be satisfied, the current distribution at time t has its maximum intensity at ρ_0 where

$$\rho_0^{2-m} = \frac{4t_0}{\tau}\frac{m+n+1}{m-2}, \qquad (6,\ 46)$$

and the above approximations (6, 45) will hold good for a portion of the forward part, and for the peak and the whole of the rear part of the wave of current density which is gradually penetrating inwards. The peak of this wave, i.e. the point of maximum current density, travels inwards with a velocity

$$-\frac{\partial\rho_0}{\partial t_0} = \frac{4(m+n+1)}{\tau}\rho_0^{m-1}, \qquad (6,\ 47)$$

which thus continually decreases with increasing depth. The magnitude of this maximum current density at ρ_0 is proportional to χ_0, where, from (6, 45) and (6, 46),

$$\chi_0 = \frac{2}{\Gamma(1+\nu)}\rho_0^{-2+n}\frac{m+n+1}{m-2}\exp\left(-\frac{m+n+1}{m-2}\right). \qquad (6,\ 48)$$

This shows that, if κ increases downwards at a rate greater than that corresponding to $\kappa = k\rho^{-2}$, and a sudden change occurs in the inducing field, represented by the first spherical harmonic $n = 1$, then the pulse of induced current will actually *increase* in intensity, as it penetrates (more and more slowly) into the interior of the conductor. If $n = 2$ it will remain constant in intensity, while if $n > 2$ it will decrease. The *total* amount of induced current will, of course, continually decrease, in spite of the fact that, for $n = 1$, the maximum intensity will continually increase. This implies that the maximum becomes more and more sharp as ρ_0 is decreased (see, for example, fig. 3); thus (6, 45) shows that when ρ_0 differs sufficiently from unity, the current density will fall rapidly from the maximum as ρ increases from ρ_0, because of the factor ρ^{-m-n-1}, and also as ρ decreases, because of the factor $\exp\{-\tau\rho^{2-m}/(4t)\}$, and this falling away from the maximum becomes more and more pronounced as ρ_0 is decreased.

7. The terrestrial magnetic variations

The preceding theory will now be used to show that the observed relations between the external and internal parts of certain magnetic variations can be explained consistently on the hypothesis that the internal part is due to electric currents induced in the earth by the primary external part, and that these relations indicate the distribution of conductivity down to a certain depth within the earth.

Any time variation of the earth's magnetic field, of a sufficiently world-wide character, may be analysed by spherical harmonic analysis into terms of the form

$$\Omega_n^p = (A_1 \cos p\lambda + A_2 \sin p\lambda)\, P_n^p(\cos\theta), \tag{7, 1}$$

where θ is the north polar distance, λ the east longitude, and A_1, A_2 are each of the form

$$A_s = a_0(e_{ns}^p \rho_0^n + i_{ns}^p \rho_0^{-n-1}), \tag{7, 2}$$

in which a_0 is the radius of the earth and $\rho_0 = r/a$. The coefficients e_{ns}^p and i_{ns}^p will be functions of the time, and will relate to the fields of external and internal origin respectively.

The particular variations which will be considered are the solar diurnal variations, denoted by S, and the storm-time variations, denoted by D_{st}. The actual observational data will be the same as those used by Chapman and Price (1930): they are collected and tabulated here for convenience, but for a full description and explanation, reference should be made to their paper.

The solar diurnal variations depend only on local solar time, and therefore a typical term in their potential can be expressed in the form

$$\Omega_n^p = a_0\{e_n^p \rho_0^n \cos(p\lambda + \alpha t + \epsilon_n^p) + i_n^p \rho_0^{-n-1} \cos(p\lambda + \alpha t + \iota_n^p)\}\, P_n^p(\cos\theta), \tag{7, 3}$$

where

$$\alpha = 2\pi p/86400, \tag{7, 4}$$

if t is measured in seconds. The principal harmonics are those for which $p = 1, 2, 3, 4$, and $n = p$ or $p+1$. The amplitude ratios e/i and the phase differences $\epsilon - \iota$ of these harmonics, as determined by Chapman (1919), are given in Table I.

Table I. Amplitude ratios and phase differences of harmonics in solar diurnal variations as found by Chapman

	1905 (sunspot maximum)		1902 (sunspot minimum)		
P_n^p	Mean equinox	Mean solstice	Mean equinox	Mean solstice	Mean
P_2^1	(2·9, − 5°)	2·8, − 3°	2·7, −23°	3·0, −20°	2·8, −13°
P_3^2	2·4, −18°	2·3, −19°	2·0, −17°	2·2, −18°	2·2, −18°
P_4^3	2·4, −21°	2·7, −20°	2·5, −21°	2·4, −21°	2·5, −21°
P_5^4	2·2, −23°	2·3, −15°	2·9, −30°	3·2, −24°	2·7, −23°

Seasonal harmonics (mean of 1902 and 1905)

	½ (summer-winter)	½ (spring-autumn)	
P_1^1	2·5, − 7°	2·1, − 1°	2·3, − 4°
P_2^2	2·3, − 8°	2·6, + 2°	2·45, − 3°
P_3^3	2·0, −32°	2·2, −13°	2·1, −22°
P_4^4	1·7, −30°	1·7, −19°	1·7, −24°

The regular part of the changes of the earth's magnetic field during a typical magnetic storm were separated by Chapman (1918) into the disturbance diurnal variations and the storm-time variations. The field of the storm-time variations is symmetrical about the earth's axis and it varies with the time measured from the commencement of the storm. This field for middle and lower latitudes was separated into parts of external and internal origin by Chapman and Price. Its potential is expressed in terms of zonal harmonics, i.e. $p = 0$, and (in lower and middle latitudes) only the first harmonic P_1 is found to be of importance. The values of e_1 and i_1, as found by Chapman and Price, are given for a series of times after the commencement of the storm in the first column of Table III, and are shown graphically in fig. 2.

8. Distributions of conductivity within the earth which are consistent with the solar diurnal variations

It was shown by Chapman that the internal part of S could be explained by electromagnetic induction, if the earth were assumed non-conducting down to a depth of about 250 km., and uniformly conducting below, with a conductivity of $3{\cdot}6\times10^{-13}$ e.m.u., which is considerably higher than the measured value for most surface rocks (about 10^{-15} e.m.u.). But Chapman and Price found that this distribution would not lead, on the induction theory, to sufficiently large values for $i_1(t)$ in the later stages of the storm-time variations, though a distribution in which κ is zero down to about 375 km., and equal to $4{\cdot}4\times10^{-12}$ e.m.u. below this depth, would do so. In view of this they considered that a distribution in which κ continued to increase with increasing depth for some distance below the earth's surface might lead to a consistent explanation of the internal parts of both S and D_{st}.

To investigate this possibility, distributions of κ will now be considered which are given, as in § 5 above, by

$$\kappa = 0 \text{ for } \rho > 1, \quad \kappa = k\rho^{-m} \text{ for } \rho < 1, \tag{8, 1}$$

where $\rho = r/qa_0$ and $q < 1$. This will allow for the possibility of a (relatively) non-conducting layer just below the surface, together with a conducting core of radius qa_0, in which the conductivity increases (assuming m positive) with increasing depth. It should be observed, however, that, since the induced currents flow in an outer shell of the earth whose thickness is probably less than $0{\cdot}2a_0$ (cf. § 11), the value of κ at greater depths than this will not influence the induced fields. Hence results derived for the distribution (8, 1) will still hold good if (8, 1) is true only for the outer shell of thickness $0{\cdot}2a_0$, and any inference as to the actual distribution of κ within the earth will apply only to this outer shell.

By choosing suitable values of m, k and q, various distributions of κ will first be found which are compatible with the observed relations between the external and internal parts of S. We shall then investigate whether any of these distributions will also account satisfactorily for the internal field of the storm-time variations. The influence which electric currents induced in the oceans may have on the internal fields will be ignored for the present, but will be considered later in § 10.

On the assumption that the internal field is due to currents induced in the earth in which the distribution of κ is given by (8, 1), the values of e/i and $\epsilon - \iota$ for any harmonic in S are given by (6, 3) and (6, 2)*, where α is given by (7, 4). Hence if one of the three disposable constants (m, k, q) is chosen, the observed values of e/i and $\epsilon - \iota$ for a single harmonic in S will determine the other two constants.

There is, of course, no a priori reason for supposing that a set of values of (m, k, q) thus found for one harmonic will necessarily fit the other harmonics, though Chapman found that with $m = 0$, i.e with a uniformly conducting core, the values $k = 3{\cdot}6 \times 10^{-13}$ e.m.u., $q = 0{\cdot}96$ gave a satisfactory fit for those harmonics for which the analysis of the observations gave consistent and reliable results. The most satisfactory results are those for the harmonics P_3^2, P_4^3, P_5^4, and it is of interest to note that the values of e/i and $\epsilon - \iota$ are nearly the same for all these harmonics (cf. Table I). Indeed, Chapman obtained the above values of k and q by comparing the calculated results for P_2^1 with the mean of the observed results for all four harmonics P_2^1, P_3^2, P_4^3, and P_5^4, instead of the observed results for P_2^1 only.

The present calculations are based mainly on the observational results for P_3^2, as these seem to be the most reliable. Some preliminary calculations for $m > 0$ showed that the corresponding values of k and q, deduced from the values of e/i and $\epsilon - \iota$ for this harmonic, gave results for the other harmonics, which are actually rather more satisfactory than in the case $m = 0$ above. The question of fitting all the harmonics satisfactorily is dealt with further in § 12.

* Actually the theory leads more directly to the values of i/e and $\iota - \epsilon$, as in these equations, but it is convenient to convert to the forms e/i and $\epsilon - \iota$ for comparison with previous memoirs.

While it would be possible, having chosen one of the constants (m, k, q), to equate the observed values of i/e and $\iota - \epsilon$ to mod $I(i\alpha)$ and arg $I(i\alpha)$ respectively (cf. equations (6, 3)), and, using (6, 2), to solve for the other two constants, the method would in general be laborious, since (6, 2) in general involves Bessel functions of small fractional order and complex argument. Hence the procedure adopted was to start with the values found for k and q by Chapman when $m = 0$, and to find the changes in e/i and $\epsilon - \iota$ produced by taking larger values of m, but the same k and q. These changes indicated how k and q should be modified to obtain the observed values of e/i and $\epsilon - \iota$, and new calculations were then made with these modified k's and q's. Thus, when $m = 8$, it will be seen from Table II A that the original values of k and q give a value

Table II A. Calculated amplitude ratios and phase differences for the harmonic P_3^2 in S, corresponding to various assumed distributions of conductivity within the earth

Distribution of conductivity, $\kappa = k\rho^{-m}$ for $\rho < 1$, $\kappa = 0$ for $\rho > 1$ where $\rho = r/qa_0$, a_0 = radius of earth			P_3^2	
			e/i	$\epsilon - \iota$
			Observed values	
			2·2	−18°
q	$10^{13}k$	m	Calculated values	
0·96	3·6	0	2·41	−18·9°
0·96	3·6	2	2·41	−17·9°
0·96	3·6	8	2·43	−15·5°
0·96	1·8	8	2·70	−18·3°
1·0	1·8	8	1·96	−18·1°
0·992	1·45	16	2·20	−18·0°
0·99	0·36	30	2·70	−17·3°
1·0	0·36	30	2·55	−17·3°

Table II B. Calculated amplitude ratios and phase differences for the harmonic P_3^2 in S, assuming that the oceans have an effect equivalent to a uniform shell of total conductivity 2×10^{-6} e.m.u. $\times$ cm., and corresponding to various assumed distributions of conductivity within the earth

Distribution of conductivity, $\kappa = k\rho^{-m}$ for $\rho < 1$, $\kappa = 0$ for $\rho > 1$ where $\rho = r/qa_0$, a_0 = radius of earth			P_3^2	
			e/i	$\epsilon - \iota$
			Values just inside surface shell	
			2·4	−15°
q	$10^{13}k$	m	Calculated values	
0·96	3·6	8	2·43	−15·5°
0·96	3·6	12	2·42	−14·8°
0·964	2·73	16	2·40	−15·0°
1·0	0·36	37	2·37	−14·8°
1·0	0·1	51	—	−12·9°

for $\epsilon - \iota$ which is (numerically) too small and a value of e/i which is too large (but note that this value of e/i is very nearly the same as that for $m = 0$, which differs somewhat from the observed value, owing to the fact mentioned already that Chapman took the average of the results for several different harmonics in his calculations). The result for $\epsilon - \iota$ is improved by halving the value of k, as in the next entry in the table, but this makes e/i too large; this can be remedied by increasing q, which scarcely affects $\epsilon-\iota$. The next result shows that $q = 1$ is too large; evidently a value about 0·98 would be satisfactory. No attempt was made to determine *exact* values of k and q corresponding to every m, as our object in these preliminary calculations was to explore the general range of permissible values of k, q and m.

There is, however, just one case, viz. when $m = 16$, when the corresponding exact values of k and q can be found very simply; this forms a useful guide when other values of m are considered. When $m = 16$ and $n = 3$, the value of ν, given by (5, 3), is $\frac{1}{2}$, so that the Bessel functions in (6, 2) are of a specially simple type and $I(i\alpha)$ reduces to the rational function

$$I(i\alpha) = \tfrac{3}{4}q^7 \frac{x\sqrt{i}}{1+x\sqrt{i}}, \tag{8, 2}$$

where x is given by (6, 6) and is therefore proportional to $q\sqrt{k}$.

On equating $\arg I(i\alpha)$ to the observed value of $\iota-\epsilon$ (18°), we find $x = 1·47$. Substituting this value of x in $\mathrm{mod}\, I(i\alpha)$, and equating $\mathrm{mod}\, I(i\alpha)$ to the observed value of i/e (0·45), we find $q = 0·992$. Substituting this value for q, together with the known values of a_0 and α, in the expression (6, 4) for x, we find $k = 1·45\times 10^{-13}$ e.m.u. It will be observed that q is now approaching unity, while k has rather less than half the value found when $m = 0$.

In the calculations for which $m<16$ (some further results appear in Table II B), it was found permissible to use the asymptotic expression for the Bessel function appearing in $I(i\alpha)$, since x was then sufficiently large. When $m>16$, the ascending series for the Bessel functions must be used, and these do not converge very rapidly when m is near 16. Hence larger values of m were first considered and calculations were made for $m = 51$, 44, 37, 30, these numbers being chosen because they make $\nu = \frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}$, respectively and slightly simplify the computations. For $m = 51$, it was found that the approximation (6, 7) could be used. This showed that the phase difference would be equal to $\frac{1}{7}\times 90°$, which is too small, and *would be practically unaffected by any change in k*. This implies that, for $m\geqslant 51$, the conductivity increases downwards so rapidly that the outer layer of low conductivity is too thin to shield the inner and more highly conducting region sufficiently, no matter how small κ is taken at the surface, provided, of course, it is not actually zero. Thus the observed value of $\epsilon-\iota$ is of itself sufficient to show that m cannot be greater than about 50. The results for $m = 30$ are shown in the last two lines of Table II A. Here $\epsilon-\iota$ is still slightly too small (numerically), and e/i is too large, even when q is given its largest possible value. The discrepancy in the values of e/i could be removed by taking a value of k a little larger than the chosen value $3·6\times 10^{-14}$; this would, however, further decrease (but only slightly) the value of $\epsilon-\iota$. We conclude that $m = 30$, with $q = 1$ and k about 4×10^{-14} e.m.u., would give results which are possibly just within the margin of error in the observational results. But m cannot be greater than about 30, because q has to be increased as m is increased and it reaches unity, its maximum possible value, when m is about 30. Thus the values of (m, k, q), with m positive, which are consistent with the observed results for P_3^2 in S, range from $(0, 3\cdot6\times10^{-13}, 0\cdot96)$ to $(30, 4\times10^{-14}, 1)$. The distributions of κ which correspond to these values, when $m = 2$, 8 and 30 respectively, are shown graphically as the curves a, b and c in fig. 1.

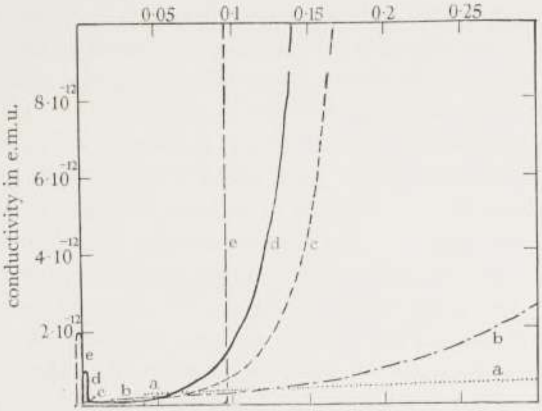

FIG. 1. Distributions of conductivity within the earth compatible with the magnetic variations. All the distributions represented are compatible with the observed relation between the external and internal parts of the harmonic P_3^2 in S, but only d and e are compatible also with the D_{st} observations. In the case of d and e there is a thin shell of relatively high conductivity at the surface. This is represented by a thick shell, for which the product of the thickness δ and the conductivity κ_0 has the correct value for the thin shell. Thus the area under curve d (or e) near the surface gives a correct indication of the total conductivity of the surface shell, but the values chosen for δ and κ_0 have no separate significance.

9. DISTRIBUTION OF κ CONSISTENT WITH THE ZONAL ("STORM-TIME") PART OF FIELD OF MAGNETIC STORMS

The field now to be considered is essentially of an aperiodic character. The time factors $e_1(t)$ and $i_1(t)$, associated with the first zonal harmonic in the field potential (cf. § 7), are given for a series of values of t in Table III, and are shown graphically in fig. 2.

The function $e_1(t)$, during the later part of the storm (which is the part of main interest in the present investigation), can be well represented by $Ae^{-\lambda(t-t_0)}$, where $A = 28$, $\lambda = 3\cdot2\times10^{-6}$ and $t_0 = 18\times3600$, so that $t-t_0$ is the time in seconds, measured from 18 hr. after the commencement of the storm. This is shown by the following table:

Time in hours	18	24	30	36	42	48
Observed $e_1(t)$	28	26	24	23	21	20
$Ae^{-\lambda(t-t_0)}$	28·0	26·1	24·4	22·7	21·2	19·7

In the preliminary calculations we take, for simplicity, not the actual external field as represented by $e_1(t)$, but the field represented by $Ae^{-\lambda(t-t_0)}\,H(t-t_0)$, where $H(t-t_0) = 0$

for $t < t_0$ and $= 1$ for $t > t_0$. This assumed field ignores the initial phase of the storm in which $e_1(t)$ first rapidly assumes a negative value but recovers its initial value again in about 4 hr. It also replaces the gradual (though rapid) increase of $e_1(t)$ between 4 hr. and 18 hr. by an instantaneous increase at 18 hr.

As regards the initial phase, no serious error can arise from ignoring this, because the current systems induced by the first rapid decrease of $e_1(t)$ will be almost exactly annulled by the equal and opposite current system induced by the rapid recovery of $e(t)$. This is, in fact, proved by the observations, which show that $i_1(t)$ recovers its initial value at almost exactly the same time as $e_1(t)$.

An instantaneous increase of $e_1(t)$, such as that assumed at $t = t_0$, would produce the maximum possible value of $i_1(t_0)$, viz. $\frac{1}{2}q^3A$, which is independent of κ. This value of $i_1(t_0)$ would correspond to a distribution of induced currents on the surface of the conductor, which would rapidly diffuse into its interior. Therefore, for a short period after time $t = t_0$, the induced field would rapidly decrease. Hence the effect of replacing the gradual rise of the observed $e(t)$ by the assumed instantaneous rise at $t = t_0$ will be to give too high a value of $i_1(t)$ at time $t = t_0$, and too rapid a decrease of $i_1(t)$ for a short period following this time. If, however, the calculated and observed values of $i_1(t)$ agree at some instant shortly afterwards (say about an hour later), a comparison of the subsequent changes in the two values will indicate whether a suitable distribution of κ has been adopted.

Using the formulae (6, 23)–(6, 32), calculations of $i_1(t)$ were first made, corresponding to the above approximation for $e_1(t)$, and taking $m = 2, 8, 12, 16$, with the corresponding values of k and q already found suitable for P_3^2. The results for $m = 2$ and $m = 8$ are shown as curves a' and b' in fig. 2. It will be seen that the calculated $i_1(t)$ decays too rapidly in these cases; increasing m was found to improve the result, though the decay was still too rapid when $m = 16$.

The largest permissible value of m consistent with the P_3^2 observations has been seen to be about 30, the corresponding distribution of κ being represented by curve c of fig. 1. The time factor $i_1(t)$ was therefore next calculated for this limiting distribution. In this case, in order to obtain greater accuracy, the actually observed $e_1(t)$, as given in Table III, was used; the function $\phi(t)$ was first calculated from (6, 23) and (6, 25), and $\phi(u)$ substituted for $h(u)$ in (4, 14), the integral being evaluated graphically for various values of t. The result obtained is shown as curve c' in fig. 2. This curve is much closer to the observed values of $i_1(t)$ than is the curve a', but it is still consistently below those values, and the discrepancy can hardly be accounted for by the possible errors in the observational material. Moreover, it seems unlikely that the actual distribution of conductivity should just happen to coincide with the limiting case ($m = 30$) of a series of distributions *of special form* which are compatible with the P_3^2 observations. It is more probable that the *form* of the distribution differs in some respect from the (m, k, q)-model, since the observations appear to require in some region a higher value of κ than this model can allow, if it is to remain consistent with the observed results for P_3^2. We conclude that, though the curve c of fig. 1 is a much better approximation to the actual distribution of κ within the earth than is the curve a (which is practically the same as the uniform core distribution previously considered by Chapman and Price), a still better approximation would be obtained if the model could be modified so as to allow for a greater conductivity of the inner core.

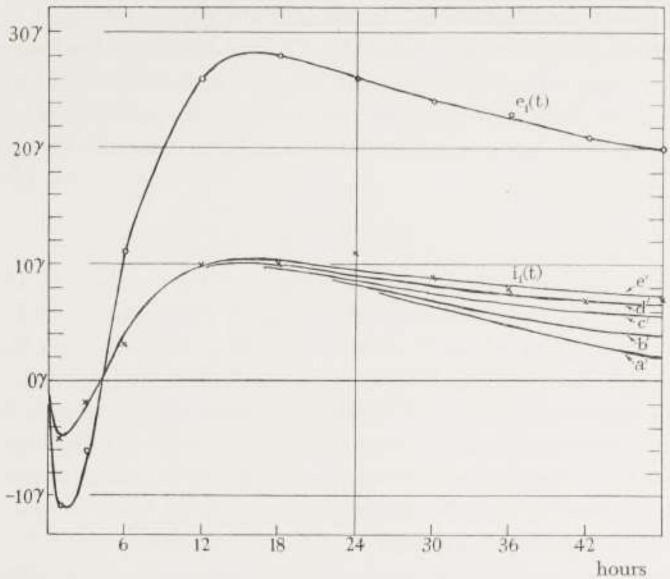

FIG. 2. Values of $i_1(t)$ for storm-time variations calculated for the distributions of conductivity shown in fig. 1.

TABLE III. OBSERVED VALUES OF e_1 AND i_1 FOR THE STORM-TIME VARIATIONS, AND VALUES OF i_1 CALCULATED ON THE INDUCTION HYPOTHESIS, FOR VARIOUS DISTRIBUTIONS OF κ. UNIT = 1γ. TIME MEASURED FROM $\frac{1}{2}$ HR. BEFORE COMMENCEMENT OF STORM

Time in hours	Observed		Calculated values of i_1 for various distributions of κ				
	e_1	i_1	a'	b'	c'	d'	e'
0	0	0	—	—	0	0	0
1	−11	− 5	—	—	—	—	− 4
3	− 6	− 2	—	—	—	—	− 2·2
6	11	3	—	—	3·9	4·0	4·0
12	26	10	—	—	9·0	9·3	9·6
18	28	10	(12)	(12)	9·6	10·1	10·3
24	26	11	7·8	8·1	8·7	9·1	9·4
30	24	9	6·2	7·0	7·8	8·0	8·8
36	23	8	5·0	5·6	7·0	7·4	8·4
42	21	7	3·3	4·5	6·5	7·2	7·7
48	20	7	1·8	4·1	5·7	7·1	7·4

The conductivity of the inner core could be increased if it were screened more effectively, in the case of the periodic variations, by the surrounding layers. This could be done by increasing (within limits) the conductivity near the surface of the earth. This additional conductivity might either be concentrated in a thin shell at the surface, or spread throughout the thick shell surrounding the core. The available *direct* evidence appears to favour the first of these alternatives, for the thin shell might be accounted for as representing the influence of the relatively highly conducting oceans, which cover the greater part of the earth's surface. The conductivity of sea-water is, in fact, about 4×10^{-11} e.m.u., which is considerably higher than the surface value (4×10^{-14} e.m.u.), of κ, given by the distribution *c* of fig. 1; on the other hand, the measured conductivity of surface rocks is only about 10^{-16} or 10^{-15} e.m.u.

10. The influence of the oceans

The possibility that the electric currents induced in the oceans have an appreciable effect on the magnetic variations was pointed out by Chapman and Whitehead, who showed that the internal field of diurnal variations could arise from induction in a uniform core, surrounded by a thin uniformly conducting surface shell, of which the conductivity did not exceed that of a uniform ocean about half a mile deep. The actual oceans, if uniformly distributed, would cover the earth to a depth of over 2 miles, but they are so broken up by land masses that their shielding effect is greatly reduced.

If the total induced field corresponding to any external magnetic variation is due partly to currents in a conducting core, and partly to currents in a thin non-uniform oceanic shell, we may expect the two parts to differ considerably in distribution. Under the probable assumption that the conducting core has spherical symmetry, the first part would have a distribution similar to that of the inducing field, since, as shown in § 4, any particular spherical harmonic in the inducing field will give rise to the same harmonic, and to no others, in the induced field. The second part would, however, be largely influenced by the geographical distribution of the oceans, and its distribution might differ greatly from that of the inducing field.

The fairly constant values of the amplitude ratios for the different harmonics in Table I indicate a close similarity between the actual distributions of the external and internal parts of S. This does not necessarily imply that the induced field is due entirely to currents flowing in the spherically symmetrical core; it does, however, show that, *so far as the local time variations S are concerned*, any influence which the oceans have can be represented approximately by a uniformly conducting shell. The non-uniformity of distribution of the actual oceans would introduce harmonics into the induced field, which do not depend solely on local time. These, however, would be eliminated by the analysis of the observations by which S is determined.

In the case of the storm-time variations, the method of analysis eliminates all harmonics other than zonal harmonics, and the observed internal and external fields are again very similar, being represented almost entirely by P_1 in middle and lower latitudes. We infer that, for these variations also, the oceans can be represented by a uniformly conducting shell. The conductivity of this shell may, however, be different from that required in the case of S; probably it will be rather less because the induced current circuits in the case of D_{st} coincide with the circles of latitude, and therefore have always to pass through considerable land areas.

When the equivalent uniform shell is taken into account, it is necessary to determine the field just inside the shell in terms of that outside, and to use this inside field, instead of the observed field, in any calculations involving κ at greater depths. For a shell of conductivity κ_0 and thickness δ (small), the relations between the time functions e and i for the field outside the shell, and the corresponding functions e' and i' for the field just inside the shell, are found to be

$$e' = e - \frac{c_0}{n}\frac{d}{dt}\{ne-(n+1)\,i\}, \quad i' = i - \frac{c_0}{n+1}\frac{d}{dt}\{ne-(n+1)\,i\}, \tag{10, 1}$$

where $$c_0 = \frac{4\pi\kappa_0 a_0^2 \delta}{2n+1}. \tag{10, 2}$$

When the field is periodic with period $2\pi/\alpha$, we replace d/dt by $i\alpha$ and e by $Ee^{i\alpha t}$ etc. in (10, 1); then the modulus and argument of I'/E' give, respectively, the amplitude ratio and phase difference of the field just inside the shell in terms of the amplitude ratio (mod I/E) and phase difference (arg I/E) of the field outside the shell. As $\kappa_0\delta$ is increased from zero, the amplitude ratio and phase difference of the field just inside the shell, corresponding to a *given* field outside, will both decrease. Now, if the part of internal origin of the field just inside is due, as we suppose, to induction in a spherically symmetrical core, the amplitude ratio and phase difference of the field will be equal to the modulus and argument of the function $I(i\alpha)$, given by (4, 13); i.e. $I'/E' = I(i\alpha)$. But, if $\mu = 1$, arg $I(i\alpha)$ lies between 0 and 90°; it tends to zero if the conductivity of the core, *near its surface*, tends to infinity, and to 90° if the conductivity tends everywhere to zero. Hence the maximum permissible value of $\kappa_0\delta$, consistent with a given field outside the shell, will be obtained when arg $(I'/E') = 0$, and the surface value of κ for the corresponding core will then be infinite.

In the case of the harmonic P_3^2 in S, we find that the maximum value of $\kappa_0\delta$, consistent with the observed phase difference of the outside field, is

$$K = 5{\cdot}1 \times 10^{-6}\ \text{e.m.u.} \times \text{cm}. \tag{10, 3}$$

which is roughly equivalent to a uniform ocean of depth 1 km. If we now suppose that the conducting core has a distribution of κ of the form (8, 1), we find that, corresponding to any value of $\kappa_0\delta$ less than K, there is a certain range of values of m consistent with the observed results for P_3^2; the upper limit of this range tends to infinity as $\kappa_0\delta \to K$. Also, if m is fixed and $\kappa\delta$ gradually increased from zero, the corresponding value of k required to fit the P_3^2 observations increases, and that of q decreases; as $k\delta \to K$, $k \to \infty$

and $q \to 0{\cdot}903$. This limiting distribution, which is the same as that obtained when $m \to \infty$, consists of a thin surface shell of total conductivity K surrounding an infinitely conducting core of radius $0{\cdot}903a_0$. It is represented diagrammatically by the curve e of fig. 1.

In the case of the storm-time variations, the only important harmonic (in low and middle latitudes) is P_1, and it is probable, as indicated above, that the conductivity of the equivalent uniform oceanic shell is now rather less than in the case of any harmonic in S, and therefore less than the maximum value K found above for P_3^2. To estimate the effect of a shell of this conductivity on the storm-time variations, the corresponding values of $c_0 \frac{d}{dt}\{ne_1-(n+1)\,i_1\}$ were found graphically from the values e_1 and i_1, determined from the observations and shown in Table III. These values were found to be less than 1γ, except near the beginning of the storm. It follows from (10, 1) that e_1-e_1' and i_1-i_1' are of the same sign and numerically less than $0{\cdot}5\gamma$ and $0{\cdot}33\gamma$, respectively, so that the influence of the oceans on the later storm-time variations is practically negligible, and is, in any case, considerably less than in the case of the diurnal variations.

It has been seen above that the introduction of the surface shell enables us to take, consistently with the P_3^2 observations, a greater value of m or k for the core than was previously possible. This increase in either m or k will produce a general rise in the calculated values of $i_1(t)$ for the storm-time variations, and the maximum possible $i_1(t)$ on the induction theory will clearly be obtained by taking the extreme distribution e. This theoretical maximum for $i_1(t)$ is shown as column e' in Table III and as curve e' in fig. 2. It will be seen that the observed values of $i_1(t)$ are very close to this theoretical maximum, but do not exceed it, except in one case ($t = 24$), and it seems quite possible that the exceptionally high value of the observed $i_1(t)$ in this case is due to some irregular fluctuation, which the analysis of the observations did not completely eliminate. This may be regarded as good confirmation of the induction hypothesis; at the same time it shows that the observed results for P_3^2 and D_{st} will not, *of themselves*, indicate an upper limit to the value of κ for depths greater than about $0{\cdot}1a_0$. It should also be noted that results differing inappreciably from the above will be obtained if, instead of taking κ to be infinite below $r = 0{\cdot}903a_0$, as in e, we take it to have any value greater than about 10^{-11} e.m.u. Hence the observations so far considered are compatible with the conductivity of the earth below a depth of about $0{\cdot}1a_0$ having any value greater than about 10^{-11} e.m.u.

The least value which $\kappa_0\delta$ can have in the $(\kappa_0\delta,\ m,\ k,\ q)$-model is about 2×10^{-6} e.m.u. $\times$ cm., because the observed results for P_3^2 then allow m to be as great as 37, and we find that this value of m is just large enough to give a suitable internal field for D_{st}. Thus, with this value of $\kappa_0\delta$, which corresponds to an ocean of uniform depth of about $\frac{1}{4}$ mile, we find that the amplitude ratio (mod E'/I') and the phase difference ($\arg E'/I'$) just inside the shell, corresponding to the observed values $(2{\cdot}2, -18^\circ)$ for P_3^2 outside the shell, are respectively $2{\cdot}4$ and -15°. Various distributions of κ in the conducting core which give rise to these values of mod E'/I' and arg E'/I' are shown in Table IIB. Of these, only the distribution for which $m = 37$, $k = 4 \times 10^{-14}$ e.m.u., $q = 1$ (represented diagrammatically by curve d of fig. 1), gives a satisfactory time function $i_1(t)$. This is shown as column d' in Table III, and as curve d' in fig. 2. It will be seen from this figure that the curves d' and e' differ on the whole by less than 1γ, and, with one exception ($t = 24$), the observed values of $i_1(t)$ lie on or between these curves. It thus appears that any distribution between those represented by d and e of fig. 1 will lead, on the induction theory, to the correct internal fields for both P_3^2 and D_{st}. In the case of e, the highly conducting inner core is partially shielded from the periodic variations by the oceanic shell; in the case of d, this shielding is due only in part to the oceanic shell, and in part to the outer layers of moderate conductivity of the core itself.

The value 2×10^{-6} e.m.u. $\times$ cm. for $\kappa_0 \delta$ in the distribution d does not, however, necessarily represent a lower limit to the possible influence of the oceans, because there is still the possibility that the observations are compatible with a distribution in which κ is, throughout the upper layers, greater than the value which has to be taken when it is supposed expressible in the form $k\rho^{-m}$. In connexion with this, we find that the distribution in which $\kappa = 2 \cdot 3 \times 10^{-13}$ e.m.u. for $r > 0 \cdot 89 a_0$, and $\kappa = \infty$ (or any value greater than 10^{-11} e.m.u.) for $r < 0 \cdot 89 a_0$, gives the correct amplitude ratio and phase difference for P_3^2, but leads to a time factor $i_1(t)$ for D_{st} which lies between the curves c' and d' of fig. 2, and is just slightly too low to give a satisfactory agreement with the observed $i_1(t)$. Now the only way of improving this result for $i_1(t)$ would be by increasing (slightly) the radius of the highly conducting core, but this would upset the result for P_3^2, unless at the same time κ were increased at or near the surface, and decreased a little lower down. This shows that the observed results are not entirely compatible with any distribution in which κ does not somewhere decrease (either suddenly or gradually) with increasing depth, before the depth (about $0 \cdot 1 a_0$) is reached where it rapidly increases.

It follows from the above that, if the oceans were assumed to have no appreciable influence, it would be necessary to suppose that there is a region at or near the earth's surface where κ is greater than $2 \cdot 3 \times 10^{-13}$ e.m.u., and that, between this region and the deeper and more highly conducting region, there is one where κ is distinctly less than $2 \cdot 3 \times 10^{-13}$ e.m.u. This, however, seems unlikely, especially as the measured conductivity of surface rocks seldom exceeds 10^{-15} e.m.u. It seems more probable that the oceans have an effect of the same order as that represented by the thin shells in d or e above. An independent estimate of this effect, based on an investigation of the induced currents in a suitable *non-uniform* thin shell, would obviously be of value in this connexion.

11. The depth of penetration of the induced currents

The inferences, which can be drawn from the above results as to the distribution of conductivity within the earth, are of course restricted by the fact that the induced currents will not penetrate (appreciably) beyond a certain depth. In the case of the extreme theoretical distribution *e* the currents do not, in fact, penetrate below the surface of the conducting core. In the other cases considered, the general character of the induced current distribution has been determined qualitatively in § 6.

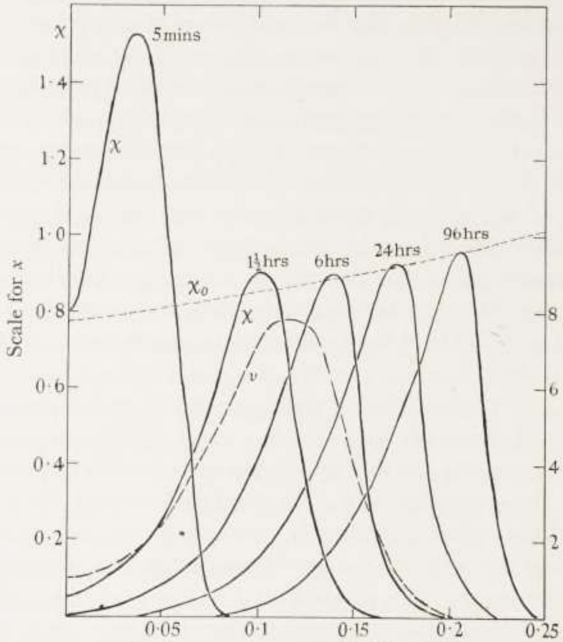

FIG. 3. Distribution of induced current density when κ, within the earth, is given by curve *d*, fig. 1. Curve v shows the amplitude of the induced current as compared with the surface current for the periodic variation P_3^2. The curves χ show the distribution of current at various times, following a sudden change in the aperiodic field P_1 (cf. equations (11, 1) and (6, 44)–(6, 48)).

For periodic fields, such as P_3^2, the amplitude of the current density is proportional to mod v, where v is given as a function of ρ by (6, 17); this was evaluated for the case when κ has the distribution *d* (in which $m = 37$, $k = 4 \times 10^{-14}$ e.m.u., $q = 1$), the harmonic being P_3^2. The result is shown as the curve v in fig. 3. It will be seen that the maximum current density is at a depth of about $0{\cdot}12a_0$ (or 800 km.), and the main part of the induced current flows between the depths $0{\cdot}05a_0$ and $0{\cdot}15a_0$, the current being negligible at about $0{\cdot}20a_0$. This indicates that the information as to the conductivity of the earth, which is likely to be afforded by the diurnal variations, relates to depths distinctly less than one-fifth of the earth's radius.

The distribution of induced currents, when the field is aperiodic, is less simple in character than the above. It has been shown in § 6 that the pulse of induced current due to a sudden change in the inducing field will, in special circumstances, *increase* in intensity as it penetrates inwards. This will be so when $n = 1$ (as in the case of the storm-time variations) and when the conductivity increases downwards at a rate greater than that corresponding to $\kappa = k\rho^{-2}$. In the particular case when the distribution of κ is of the form d, the equation (6, 33) shows that the induced current, resulting from an instantaneous change of 1γ in e_1 at time $t = 0$, is everywhere perpendicular to the radius vector and eastwards, and is of magnitude

$$C = \frac{105 \times 10^{-5}}{16\pi a_0} \chi(t, \rho) \sin\theta = 3{\cdot}3 \times 10^{-14} \chi(t, \rho) \sin\theta, \tag{11, 1}$$

where $\chi(t, \rho)$ is given in various forms in (6, 36)–(6, 45). This function was calculated as a function of ρ for a series of values of t, and is shown graphically in fig. 3. This figure shows that the sheet current distribution, which is initially produced on the surface, rapidly penetrates inwards, reaching a depth of some 200 km. within 5 min. The maximum current density at first decreases as it penetrates inwards, but after a depth of about 900 km. is reached (in about 6 hr.) it begins to increase again, being, in fact, proportional to $1/\rho$. Also the speed with which the point of maximum current density moves inwards now decreases rapidly, being proportional to ρ^{36}. The total induced current is, of course, continually diminishing, so that the maximum gets sharper and sharper as it penetrates farther inwards.

The exact calculation of the current distribution corresponding to the actual e_1 of the storm-time variations would be tedious, but its general nature and order of magnitude can be inferred from the above. The first rapid decrease in e_1 (corresponding to the initial *increase* in the horizontal force) produces a pulse of westward current, which is transmitted inwards in the way just described. This is followed, within about 4 hr., by a much greater increase in e_1, which produces a correspondingly greater pulse of eastward current, following the westward one inwards and gradually overtaking it. The subsequent slow recovery of the external field then produces a less intensive but more extended pulse of westward current, which again gradually overtakes the previous one, and eventually cancels it out. Fig. 3 shows that the first pulse of current will have reached a depth of about $0{\cdot}2a_0$ after 48 hr. Hence the storm-time changes, during the first 48 hr., will be affected only by the conductivity at depths less than $0{\cdot}2a_0$, and will not yield any information about the conductivity at depths greater than this.

To estimate the order of magnitude of the induced currents, we note from fig. 3 that the maximum value of $\chi(t, \rho)$, following an instantaneous change in e, has decreased in 5 min. from infinity to about 1·5, while in $1\frac{1}{2}$ hr. it has decreased to about 0·9, and does not vary much from this value for the next 96 hr. We infer from (11, 1) that the maximum current density, induced by the regular part of the storm-time changes, is about 3×10^{-14} e.m.u. per 1γ change in e_1 for the more rapid changes and rather less for the others. Since the maximum variation in e_1 for the average storms now under consideration is about 28γ, the corresponding maximum current density is about 10^{-12} e.m.u. This is, as would be expected, somewhat larger than the value 3×10^{-13} e.m.u., found by Chapman and Price when considering the uniform core model, which gave too small a value for $i_1(t)$. (The value given in their paper on p. 456 should, of course, read 3×10^{-13} e.m.u., *not* 3×10^{13} e.m.u.)

12. The distribution of electrical conductivity within the earth

The foregoing results show that the internal fields of both the daily variations and the storm-time variations can be satisfactorily explained as due to electric currents induced in the earth by the corresponding primary external fields, provided a suitable non-uniform distribution is assumed for κ. Apart from general considerations, the precise nature of the internal field in the case of the storm-time variations leaves no reasonable doubt that this induction hypothesis is the correct one, and consequently those assumed distributions of κ, which have been found suitable for *both* S and D_{st}, indicate the nature of the actual distribution of κ within the earth.

The most obvious and important deduction as to the conductivity of the earth, which can be drawn from the above results, is that it begins to rise very rapidly with increasing depth at a distance of about $0{\cdot}1a_0$ (600 or 700 km.) from the surface, because all the assumed distributions (cf. fig. 1), which give results in satisfactory agreement with the observations, have this feature in common. This is in general agreement with the conclusion reached by Chapman and Price that there is a rapid rise in κ with increasing depth somewhere below 250 km., but it now appears that the depth at which this rapid increase occurs is some 600 or 700 km. from the surface. The conductivity below this depth must be at least as high as 10^{-11} e.m.u., and may continue to rise to much higher values. If it does continue to rise, e.g. as in the distribution *d*, then the induced currents will not penetrate to any appreciable extent beyond about $0{\cdot}2a_0$. Hence the information as to the conductivity afforded by these variations will not extend to depths beyond $0{\cdot}2a_0$ unless more elaborate calculations using, e.g., the other harmonics in the daily variations show that κ does not rise much above 10^{-11} e.m.u. between $0{\cdot}1a_0$ and $0{\cdot}2a_0$.

The estimate of the conductivity at depths less than $0{\cdot}1a_0$ is affected by the possible influence, on the magnetic variations considered, of electric currents induced in the oceans. If this influence is negligible, the mean conductivity of the earth, down to a depth of about $0{\cdot}1a_0$, must be of the order 2×10^{-13} e.m.u., and the conductivity in some region near the surface must be somewhat *higher* than this (§ 10). The observations are, however, also compatible with the oceans having an effect equivalent to a uniform ocean covering the whole earth to a depth not greater than 1 km. If their actual effect is of this order, the conductivity of the earth for some distance below the surface must be so small that it has practically no effect on the particular variations considered, i.e. κ in this region must be not greater than about 10^{-15} e.m.u. This would imply that κ down to depths of 200 or 300 km., and *possibly* as far as 600 km., would be of the same order as the measured conductivity of rocks at the earth's surface. This distribution of κ seems more probable than the alternative one described above.

The above information as to the distribution of κ has been derived mainly from the harmonic P_3^2 in the daily variations, and from the storm-time variations. It might be rendered more precise by considering other (sufficiently reliable) harmonics in the diurnal variations. An indication of this is afforded by the phase differences of the harmonics P_2^1, P_3^2, P_4^3, P_5^4, the observed values of which are 13°, 18°, 21°, 25°. Now the values for these phase differences calculated by Chapman from his original uniform core model ($q = 0{\cdot}96$, $\kappa = 3{\cdot}6 \times 10^{-13}$ e.m.u.) were 19°, 19°, 20°, 21°, and did not show a sufficiently rapid increase with the increasing order of the harmonics. On the other hand, the values we find for these phase differences, when the extreme distribution *e* is used, are 8°, 18°, 25°, 30°, the increase being now considerably too rapid. This shows that the distribution *e* is too extreme; it has too much conductivity in the surface shell and therefore also too much conductivity in the core. Hence the effective mean conductivity of the oceanic shell is probably less than the value 5×10^{-6} e.m.u. × cm., assigned to it in *e*. Also a definite upper limit to the value of κ at about 700 km. could probably be determined by further calculations for these four harmonics. It is, however, doubtful whether it would be possible to distinguish between a gradual increase in κ with increasing depth as, for example, in *d*, or a very abrupt increase, as in *e*, but to some finite value instead of infinity. In any case, the conclusions based on the present series of observations should not be pushed too far. It is desirable to check their reliability before more extensive calculations are made, and a fresh analysis of a more recent set of magnetic data is now in progress at the Imperial College. It may also be noted that if the non-cyclic variation could be analysed into external and internal parts in the same way as the D_{st} variation—of which it is evidently a later phase—the relations between these parts would be a further check on the results, and would probably extend our knowledge of κ to slightly greater depths.

Summary

The results of previous investigations by Chapman and Price of the induced fields and current distributions, associated with the magnetic daily and storm-time variations, suggest that more precise information as to the distribution of electrical conductivity (κ) within the earth might be obtained by considering electromagnetic induction in a non-uniform sphere. The general theory for any non-uniform conductor is here considered, and the formal solution for any conductor with spherical symmetry is obtained. Detailed formulae for the induced field and current distribution, in the special case when $\kappa = k\rho^{-m}$, where k and m are constants, are obtained and applied to the terrestrial magnetic variations. The results obtained support the view, expressed by Chapman and Price, that there is a considerable increase of κ with increasing depth, beyond 250 km. It seems, however, that the really important increase in κ takes place at about 700 km. depth, beyond which κ is at least as great as 10^{-11} e.m.u., while above this depth the mean conductivity may be of the same order as for rocks on the earth's surface (10^{-16} or 10^{-15} e.m.u.). This suggests that there is some change in the composition of the earth (e.g. to a more metallic content) at a depth of about 700 km.; seismological evidence appears to indicate that such a transition occurs at a much greater depth. The results

also show that there is an effective distribution of κ at or near the surface of the earth, and it seems most probable that this represents the influence of the relatively highly conducting oceans. The induced currents do not penetrate appreciably beyond a depth of about one-fifth of the earth's radius, so that the knowledge of κ afforded by the daily variations and the storm-time variations will be restricted to an outer shell of this thickness.

(*Note added in proof* 20 *September* 1938.) Following publication of the abstract of this paper, Dr K. E. Bullen has kindly drawn our attention to further seismological evidence which indicates a change in the composition of the earth at a depth of the same order as that obtained by us. In particular he has shown (Bullen 1936, 1937) that unless there is a sharp increase in density at a depth of the order of several hundred km. the deduced moment of inertia of the central core involves a most improbable distribution of matter. Also the work of Byerly (1926), Jeffreys and Bullen (1933, 1935) and Lehmann (1934) on seismic waves indicates a change in the elastic properties at a depth which, on the assumption of a *sudden* change, Jeffreys (1937) estimates at 474 ± 20 km.

References

Berlage 1937 *Beitr. Geophys.* **50**, 7–17.
Bromwich 1916 *Proc. Lond. Math. Soc.* Series 2, **15**, 401–48.
Bullen 1936 *Mon. Not. R. Astr. Soc. Geophys. Supp.* **3**, 395.
— 1937 *Trans. Roy. Soc. N.Z.* **67**, 122.
Byerly 1926 *Bull. Seism. Soc. Amer.* **16**, 209.
Carson 1926 "Electric Circuit Theory and the Operational Calculus." New York: McGraw-Hill Book Co.
Chapman 1918 *Proc. Roy. Soc.* A, **95**, 61.
— 1919 *Philos. Trans.* A, **218**, 1.
Chapman and Price 1930 *Philos. Trans.* A, **229**, 427–60.
Chapman and Whitehead 1922 *Trans. Camb. Phil. Soc.* **22**, 463–82.
Goldstein 1932 *Proc. Lond. Math. Soc.* Series 2, **34**, 103–25.
Gutenberg and Richter 1938 *Mon. Not. R. Astr. Soc. Geophys. Supp.* **5**, 363.
Heaviside "Electromagnetic Theory", **2** (1899), **3** (1912), reprinted 1922. London: Benn.
Jeffreys 1929 "The Earth", 2nd ed. Camb. Univ. Press.
— 1931 "Operational Methods in Mathematical Physics", 2nd ed. Camb. Univ. Press.
— 1937 *Mon. Not. R. Astr. Soc. Geophys. Supp.* **4**, 57.
Jeffreys and Bullen 1933 *Nature, Lond.*, **131**, 97.
— — 1935 *Bureau Centr. Seism. Trav. Sci.* **11**.
Lamb 1883 *Philos. Trans.* **174**, 526.
Lehmann 1934 *Geod. Inst. Copenhagen*, **5**.
Price 1930 *Proc. Lond. Math. Soc.* Series 2, **31**, 217–24.
— 1931 *Proc. Lond. Math. Soc.* Series 2, **33**, 233–45.
Schuster 1889 *Philos. Trans.* A, **180**, 467.
Sutton 1934 *J. Lond. Math. Soc.* **9**, 131–7.
Whittaker and Watson 1920 "Modern Analysis", 3rd ed. Camb. Univ. Press.