ELECTROMAGNETIC SCATTERING BY A METALLIC SPHEROID USING SHAPE PERTURBATION METHOD

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Abstract—The scattering of a plane electromagnetic wave by a perfectly conducting prolate or oblate spheroid is considered analytically by a shape perturbation method. The electromagnetic field is expressed in terms of spherical eigenvectors only, while the equation of the spheroidal boundary is given in spherical coordinates. There is no need for using any spheroidal eigenvectors in our solution. Analytical expressions are obtained for the scattered field and the scattering cross-sections, when the solution is specialized to small values of the eccentricity $h = d/(2a)$, $(h \ll 1)$, where d is the interfocal distance of the spheroid and $2a$ the length of its rotation axis. In this case exact, closed-form expressions, valid for each small h, are obtained for the expansion coefficients $g^{(2)}$ and $g^{(4)}$ in the relation $S(h) = S(0)[1 + g^{(2)}h^2 + g^{(4)}h^4 + O(h^6)]$ expressing the scattering cross-sections. Numerical results are given for various values of the parameters.

1. INTRODUCTION

Study of electromagnetic scattering by a metallic spheroid is an old problem with numerous applications. Many researchers have been involved with its solution in the past in a great number of papers, applying various methods. Among these papers are the ones included in our reference list [1–6].

In the present paper the scattering of a plane electromagnetic wave by a perfectly conducting prolate or oblate spheroid is considered. In Fig. 1 the geometry of the prolate spheroid is shown. Its interfocal distance is d, while a and b are the lengths of its major and minor semiaxes, respectively. The prolate spheroid is the only one to be considered explicitly, but corresponding formulas for the oblate one
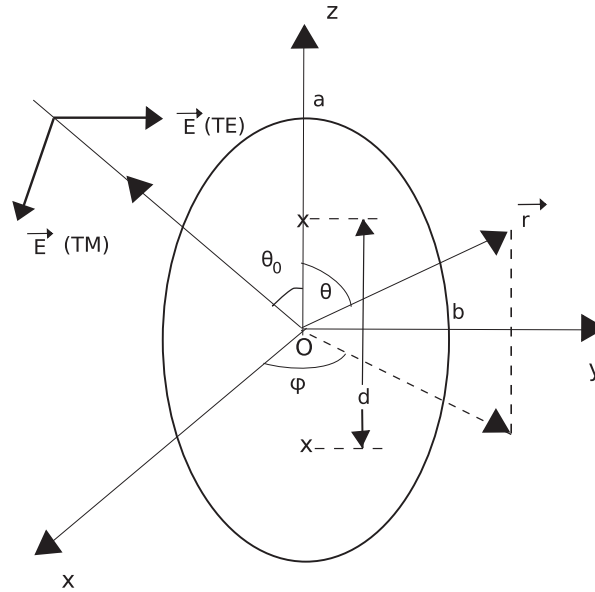
Figure 1. Geometry of the prolate spheroid.

are obtained immediately. Without loss of generality the direction of
propagation of the incident wave is assumed to be in the xz plane,
as shown in Fig. 1, making an angle θ_0 with respect to the positive z-
semiaxis. The polarized incident wave is resolved into two components:
the TE wave, for which the electric field vector is normal to the xz
plane and the TM wave, for which the electric field vector lies in the
xz plane. At an axial incidence ($\theta_0 = 0, \pi$) there is no need to consider
TE and TM waves, due to the symmetry.

The problem is solved by a shape perturbation method. The
electromagnetic field is expressed in terms of spherical eigenvectors
only, while the equation of the spheroidal boundary is given in spherical
coordinates. There is no need for using any spheroidal eigenvectors
in our solution. When the solution is specialized to small values of
the eccentricity $h = d/2a$ ($h \ll 1$) analytical expressions of the form
$S(h) = S(0)[1 + g^{(2)}h^2 + g^{(4)}h^4 + O(h^6)]$ are obtained for the various
scattering cross-sections. The expansion coefficients $g^{(2)}$ and $g^{(4)}$ are
given by exact, closed form expressions, independent of h, while $S(0)$
corresponds to a sphere with radius a ($h = 0$).

Although general exact solutions to this same problem have
already been developed [6, 7] the main advantage of such an analytical
solution lies in its general validity for each small value of h, free of

spheroidal eigenvectors. Practically this means that once $g^{(2)}$ and $g^{(4)}$ are known, $S(h)$ can be immediately evaluated by quick "back-of-the-envelope" calculations, for each small h, while all numerical techniques require repetition, from the beginning, of the complicated evaluation for each different h, independently of its size small or large. This was achieved after a great analytical effort which, made once, apparently reduces dramatically the otherwise necessary computer time for the numerical evaluation of the many spheroidal eigenvectors in the case of small h. So we can say that the present solution is much useful for a "fat" spheroid, where the use of the general solution seems to be superfluous. As far as the terms omitted in our solution are of the order h^6, or higher, it is evident that the restriction for use of small values of h is less severe than it may appear at first. Comparison with existing results found in $[1, 5–7]$ shows that even for $h = 0.866$ (maximum possible $h = 1.0$, corresponding to a rod) our method gives a very good approximation. Moreover our analytical solution can provide a benchmark, against which the more complex spheroidal function codes may be checked.

The problem is solved in Section 2, while in Section 3 numerical results are given for various values of the parameters.

2. SOLUTION OF THE PROBLEM

The prolate spheroid is examined explicitly, while corresponding formulas for the oblate one are obtained immediately.

2.1. TE Incident Wave

We start with the incident TE plane electromagnetic wave with plane of incidence the xz plane and its electric field normal to the plane of incidence (see Fig. 1). The expression for this electric field is [8]

$$\vec{E}_i = \hat{y}e^{jk(x\sin\theta_0 + z\cos\theta_0)}$$
$$= \sum_{n=1}^{\infty}\sum_{m=0}^{n}\left[c_{mn}(\theta_0)\vec{m}_{emn}^{r(1)}(r,\theta,\phi) + d_{mn}(\theta_0)\vec{n}_{omn}^{r(1)}(r,\theta,\phi)\right] \quad (1)$$

where the time factor $\exp(-j\omega t)$ is suppressed, θ_0 is the angle of incidence with respect to the positive z-semiaxis, k is the wavenumber, r, θ, ϕ are the spherical coordinates with respect to O, while

$$c_{mn}(\theta_0) = -\frac{j^n \epsilon_m (2n+1)}{n(n+1)}\frac{(n-m)!}{(n+m)!}\frac{dP_n^m(\cos\theta_0)}{d\theta_0},$$
$$d_{mn}(\theta_0) = \frac{2mj^{n-1}(2n+1)}{n(n+1)}\frac{(n-m)!}{(n+m)!}\frac{P_n^m(\cos\theta_0)}{\sin\theta_0} \quad (2)$$

and

$$\vec{m}_{\substack{e \\ o}mn}^{r(1)}(r,\theta,\phi) = \sqrt{n(n+1)}j_n(kr)\vec{C}_{mn}^{\substack{e \\ o}}(\theta,\phi) \tag{3}$$

$$\vec{n}_{\substack{e \\ o}mn}^{r(1)}(r,\theta,\phi) = n(n+1)\frac{j_n(kr)}{kr}\vec{P}_{mn}^{\substack{e \\ o}}(\theta,\phi) + \sqrt{n(n+1)}\frac{j_n^d(kr)}{kr}\vec{B}_{mn}^{\substack{e \\ o}}(\theta,\phi) \tag{4}$$

are the spherical eigenvectors of the first kind [8, 9].

The subscripts or superscripts e and o stand for even and odd functions, respectively. In $(3),(4)$, the spherical surface eigenvectors $\vec{C}$, $\vec{P}$ and $\vec{B}$ are given by the relations [9]

$$\vec{C}_{mn}^{\substack{e \\ o}} = \mp\frac{1}{\sqrt{n(n+1)}}\left[\frac{m}{\sin\theta}P_n^m{\substack{\sin \\ \cos}}m\phi\hat{\theta} \pm \frac{dP_n^m}{d\theta}{\substack{\cos \\ \sin}}m\phi\hat{\phi}\right] \tag{5}$$

$$\vec{P}_{mn}^{\substack{e \\ o}} = P_n^m{\substack{\cos \\ \sin}}m\phi\hat{r} \tag{6}$$

$$\vec{B}_{mn}^{\substack{e \\ o}} = \frac{1}{\sqrt{n(n+1)}}\left[\frac{dP_n^m}{d\theta}{\substack{\cos \\ \sin}}m\phi\hat{\theta} \mp \frac{m}{\sin\theta}P_n^m{\substack{\sin \\ \cos}}m\phi\hat{\phi}\right] \tag{7}$$

In (2)–(7) $\epsilon_0 = 1$, $\epsilon_m = 2$ $(m \geq 1)$ is the Neumann factor, P_n^m is the associated Legendre function, j_n is the spherical Bessel function of the first kind $j_n^d(x) = d[xj_n(x)/dx]$, while $\hat{r}, \hat{\theta}, \hat{\phi}$ are the unit vectors.

The scattered wave is expressed as

$$\vec{E}_s = \sum_{n=1}^{\infty}\sum_{m=0}^{n}\left[A_{mn}\vec{m}_{emn}^{r(3)}(r,\theta,\phi) + D_{mn}\vec{n}_{omn}^{r(3)}(r,\theta,\phi)\right] \tag{8}$$

where $\vec{m}_{emn}^{r(3)}$ and $\vec{n}_{omn}^{r(3)}$ are the spherical eigenvectors of the third kind with expressions similar to those in $(3),(4)$, but with the spherical Bessel functions of the first kind replaced by the spherical Hankel functions of the first kind $h_n(kr)$, with the superscript (1) omitted for simplicity.

In order to satisfy the boundary condition $\hat{n} \times (\vec{E}_i + \vec{E}_s) = 0$ at the spheroidal surface ($\hat{n}$ is the normal unit vector there), we express the equation of this surface in terms of r and θ [1, 10]

$$r = \frac{a}{\sqrt{1 - v\sin^2\theta}}, \qquad v = 1 - \frac{a^2}{b^2} \tag{9}$$

Using the expansions of (9) into a power series of $h = d/(2a)$ and keeping terms up to the order h^4, where

$$v = -h^2 - h^4 + O(h^6), \quad v^2 = h^4 + O(h^6) \tag{10}$$

we obtain

$$r = a \left[1 - \frac{h^2}{2} \sin^2 \theta - \frac{h^4}{2} \left(\sin^2 \theta - \frac{3}{4} \sin^4 \theta \right) + O(h^6) \right] \tag{11}$$

$$
\begin{aligned}
j_n(kr) \;=\; & j_n(x) - \frac{h^2}{2} x j_n'(x) \sin^2 \theta \\
& + h^4 \left\{ -\frac{x}{2} j_n'(x) \sin^2 \theta + \frac{1}{8} \left[3x j_n'(x) + x^2 j_n''(x) \right] \sin^4 \theta \right\} \\
& + O(h^6), \quad x = ka
\end{aligned} \tag{12}
$$

$$
\begin{aligned}
\frac{j_n(kr)}{kr} \;=\; & \frac{j_n(x)}{x} + \frac{h^2}{2} \left[\frac{j_n(x)}{x} - j_n'(x) \right] \sin^2 \theta \\
& + \frac{h^4}{2} \left\{ \left[\frac{j_n(x)}{x} - j_n'(x) \right] \sin^2 \theta \right. \\
& \left. - \frac{1}{4} \left[\frac{j_n(x)}{x} - j_n'(x) - x j_n''(x) \right] \sin^4 \theta \right\} + O(h^6)
\end{aligned} \tag{13}
$$

and similar expressions for $h_n(kr)$ and $h_n(kr)/kr$. The first derivative $j_n'(kr)$ is given from (12) with one more prime in each of the Bessel functions. A similar expression is obtained for $h_n'(kr)$. Finally $j_n^d(kr)/kr = j_n(kr)/kr + j_n'(kr)$ and $h_n^d(kr)/kr = h_n(kr)/kr + h_n'(kr)$ are obtained easily from the former expansions.

Substituting in $(3),(4)$ and using $\vec{m}_{emn}^{r(1)}, \vec{n}_{omn}^{r(1)}$ in (1) and $\vec{m}_{emn}^{r(3)}, \vec{n}_{omn}^{r(3)}$ in (8) we satisfy the boundary condition $\hat{n} \times (\vec{E}_i + \vec{E}_s) = \hat{n}' \times (\vec{E}_i + \vec{E}_s) = 0$ at the spheroidal surface, where [10, 11]

$$\hat{n} = \left(1 - \frac{h^4}{8} \sin^2 2\theta \right) \hat{n}', \; \hat{n}' = \hat{r} + \frac{h^2}{2} \sin 2\theta (1 + h^2 \cos^2 \theta) \hat{\theta} + O(h^6) \tag{14}$$

In the resulting equation we use the orthogonal properties of the surface eigenvectors $\vec{B}$, $\vec{C}$ and $\vec{P}$ [9] by making dot products first with $\vec{B}_{mn}^e d\Omega$ and next with $\vec{C}_{mn}^o d\Omega$, where $d\Omega = \sin \theta d\theta d\phi$, and then integrating for $\theta = 0 - \pi$ and $\phi = 0 - 2\pi$. Thus we finally obtain, respectively, the following two infinite sets of linear inhomogeneous equations for the expansion coefficients A_{mn} and D_{mn}, up to the order h^4 $(v \pm \ell \geq m' = \max\{m, 1\}, \; \ell = 0 - 4)$:

$$
\begin{aligned}
& a_{v,v-4} A_{v-4} + a_{v,v-2} A_{v-2} + a_{vv} A_v + a_{v,v+2} A_{v+2} + a_{v,v+4} A_{v+4} \\
& + d_{v,v-3} D_{v-3} + d_{v,v-1} D_{v-1} + d_{v,v+1} D_{v+1} + d_{v,v+3} D_{v+3} = N_v
\end{aligned} \tag{15}
$$

$$
\begin{aligned}
& d'_{v,v-4} D_{v-4} + d'_{v,v-2} D_{v-2} + d'_{vv} D_v + d'_{v,v+2} D_{v+2} + d'_{v,v+4} D_{v+4} \\
& + a'_{v,v-3} A_{v-3} + a'_{v,v-1} A_{v-1} + a'_{v,v+1} A_{v+1} + a'_{v,v+3} A_{v+3} = N'_v
\end{aligned} \tag{16}
$$

The subscript m is omitted from the various symbols in $(15), (16)$, for simplicity. As it is evident from (8) the subscripts of A's and D's are always non-negative and the second subscript is equal or greater than the first one, i.e., than m. In the opposite case A's and D's are equal to zero and so disappear. The same is valid also for the corresponding a's$(a'$'s$)$ and d's$(d'$'s$)$

For small values of h one can set up to the order h^4

$$a_{vv} = a_{vv}^{(0)} + a_{vv}^{(2)}h^2 + a_{vv}^{(4)}h^4 + O(h^6),$$
$$a_{v,v\pm2} = a_{v,v\pm2}^{(2)}h^2 + a_{v,v\pm2}^{(4)}h^4 + O(h^6),$$
$$a_{v,v\pm4} = a_{v,v\pm4}^{(4)}h^4 + O(h^6) \tag{17}$$

$$a'_{v,v\pm1} = a_{v,v\pm1}^{'(2)}h^2 + a_{v,v\pm1}^{'(4)}h^4 + O(h^6),$$
$$a'_{v,v\pm3} = a_{v,v\pm3}^{'(4)}h^4 + O(h^6) \tag{18}$$

$$d_{v,v\pm1} = d_{v,v\pm1}^{(2)}h^2 + d_{v,v\pm1}^{(4)}h^4 + O(h^6),$$
$$d_{v,v\pm3} = d_{v,v\pm3}^{(4)}h^4 + O(h^6) \tag{19}$$

$$d'_{vv} = d_{vv}^{'(0)} + d_{vv}^{'(2)}h^2 + d_{vv}^{'(4)}h^4 + O(h^6),$$
$$d'_{v,v\pm2} = d_{v,v\pm2}^{'(2)}h^2 + d_{v,v\pm2}^{'(4)}h^4 + O(h^6),$$
$$d'_{v,v\pm4} = d_{v,v\pm4}^{'(4)}h^4 + O(h^6) \tag{20}$$

$$N_v = N_v^{(0)} + N_v^{(2)}h^2 + N_v^{(4)}h^4 + O(h^6),$$
$$N'_v = N_v^{'(0)} + N_v^{'(2)}h^2 + N_v^{'(4)}h^4 + O(h^6) \tag{21}$$

Exact expressions for $a^{(\ell)}$'s, $d^{(\ell)}$'s, $N^{(\ell)}$'s $(\ell = 0, 2, 4)$ as well as for the corresponding primed ones are given in eqs. $(A1)$–$(A6)$ of the Appendix.

A's and D's are obtained from the solution of the sets (15) and (16) by Cramer's rule, following steps similar to the ones in $[12, 13]$. The determinant Δ of a's, a''s, d's and d''s is $[10, 12]$

$$\Delta = P(a_{ss})P(d'_{ss})\left\{1 - \sum_{w=m'}^{\infty}\left\{\frac{a'_{w,w+1}d_{w+1,w}}{d'_{ww}a_{w+1,w+1}} + \frac{a'_{w+1,w}d_{w,w+1}}{a_{ww}d'_{w+1,w+1}}\right.\right.$$
$$\left.\left. + \frac{a_{w,w+2}a_{w+2,w}}{a_{ww}a_{w+2,w+2}} + \frac{d'_{w,w+2}d'_{w+2,w}}{d'_{ww}d'_{w+2,w+2}}\right]\right\} \tag{22}$$

where

$$
\begin{aligned}
P(a_{ss}) &= a_{m',m'}a_{m'+1,m'+1}a_{m'+2,m'+2}\cdots,\\
P(d'_{ss}) &= d'_{m',m'}d'_{m'+1,m'+1}d'_{m'+2,m'+2}\cdots
\end{aligned}
\tag{23}
$$

The determinant Δ_{Ap} originating from Δ after the substitution of the column of the coefficients a_{vp} and a'_{vp} of A_p by the column of N_v's and N'_v's is [13]

$$
\Delta_{Ap} = P(a_{ss})P(d'_{ss})\frac{N_p}{a_{pp}}
$$

$$
\cdot\Bigg\{ 1 - \sum_{\substack{w=m'\\w\neq p-1}}^{\infty}\frac{a'_{w,w+1}d_{w+1,w}}{d'_{ww}a_{w+1,w+1}} - \sum_{\substack{w=m'\\w\neq p}}^{\infty}\frac{a'_{w+1,w}d_{w,w+1}}{a_{ww}d'_{w+1,w+1}}
$$

$$
- \sum_{\substack{w=m'\\w\neq p-2,p}}^{\infty}\frac{a_{w,w+2}a_{w+2,w}}{a_{ww}a_{w+2,w+2}} - \sum_{w=m'}^{\infty}\frac{d'_{w,w+2}d'_{w+2,w}}{d'_{ww}d'_{w+2,w+2}}
$$

$$
- \sum_{t=p\pm2,p\pm4}\frac{a_{pt}N_t}{a_{tt}N_p} - \sum_{t=p\pm1,p\pm3}\frac{d_{pt}N'_t}{d'_{tt}N_p}
$$

$$
+ \frac{a_{p\pm2,p\pm4}a_{p,p\pm2}N_{p\pm4}}{a_{p\pm4,p\pm4}a_{p\pm2,p\pm2}N_p} + \frac{a'_{p\pm1,p\pm2}d_{p,p\pm1}N_{p\pm2}}{a_{p\pm2,p\pm2}d'_{p\pm1,p\pm1}N_p}
$$

$$
+ \frac{a_{p,p\pm2}d_{p\pm2,p\pm3}N'_{p\pm3}}{a_{p\pm2,p\pm2}d'_{p\pm3,p\pm3}N_p} + \frac{d'_{p\pm1,p\pm3}d_{p,p\pm1}N'_{p\pm3}}{d'_{p\pm3,p\pm3}d'_{p\pm1,p\pm1}N_p}
$$

$$
+ \frac{a_{p,p\pm2}d_{p\pm2,p\pm1}N'_{p\pm1}}{a_{p\pm2,p\pm2}d'_{p\pm1,p\pm1}N_p} + \frac{d'_{p\pm1,p\mp1}d_{p,p\pm1}N'_{p\mp1}}{d'_{p-1,p-1}d'_{p+1,p+1}N_p}
\tag{24}
$$

where the pairs of signs mean summations of two terms in each case, one with the upper signs and the other with the lower signs.

The determinant Δ_{Dp} originating from Δ after the substitution of the column of the coefficients d_{vp} and d'_{vp} of D_p, by the column of N_v's and N'_v's is obtained from (24) by simply making the substitutions $a \leftrightarrow d'$, $d \leftrightarrow a'$ and $N \leftrightarrow N'$.

Using the expansions (17)–(21) into (22)–(24) A_p is obtained from an expression of the form

$$
A_p = \frac{\Delta_{Ap}}{\Delta} = A_p^{(0)} + A_p^{(2)}h^2 + A_p^{(4)}h^4 + O(h^6)
\tag{25}
$$

where

$$
A_p^{(0)} = Z_p^{(0)}
\tag{26}
$$

$$A_p^{(2)} = Z_p^{(2)} - Z_p^{(0)} \sum_{\substack{s=p\pm2 \\ t=p\pm1}} \left(Y_s^a + Y_t^d\right) \tag{27}$$

$$A_p^{(4)} = Z_p^{(4)} - Z_p^{(2)} \sum_{\substack{s=p\pm2 \\ t=p\pm1}} \left(Y_s^a + Y_t^d\right)$$

$$-Z_p^{(0)}\Bigg\{ \sum_{\substack{s=p\pm2 \\ t=p\pm1}} \left(X_s^a + X_t^d + W_s^a + W_t^d\right)$$

$$+ \sum_{\substack{s=p\pm4 \\ t=p\pm3}} \left(V_s^a + V_t^d\right) - \frac{a_{p\pm2,p\pm4}^{(2)} a_{p,p\pm2}^{(2)} N_{p\pm4}^{(0)}}{a_{p\pm4,p\pm4}^{(0)} a_{p\pm2,p\pm2}^{(0)} N_p^{(0)}}$$

$$- \frac{a_{p\pm1,p\pm2}^{\prime(2)} d_{p,p\pm1}^{(2)} N_{p\pm2}^{(0)}}{a_{p\pm2,p\pm2}^{(0)} d_{p\pm1,p\pm1}^{\prime(0)} N_p^{(0)}} - \frac{a_{p,p\pm2}^{(2)} d_{p\pm2,p\pm3}^{(2)} N_{p\pm3}^{\prime(0)}}{a_{p\pm2,p\pm2}^{(0)} d_{p\pm3,p\pm3}^{\prime(0)} N_p^{(0)}}$$

$$- \frac{d_{p\pm1,p\pm3}^{\prime(2)} d_{p,p\pm1}^{(2)} N_{p\pm3}^{\prime(0)}}{d_{p\pm1,p\pm1}^{\prime(0)} d_{p\pm3,p\pm3}^{\prime(0)} N_p^{(0)}} - \frac{a_{p,p\pm2}^{(2)} d_{p\pm2,p\pm1}^{(2)} N_{p\pm1}^{\prime(0)}}{a_{p\pm2,p\pm2}^{(0)} d_{p\pm1,p\pm1}^{\prime(0)} N_p^{(0)}}$$

$$- \frac{d_{p\pm1,p\mp1}^{\prime(2)} d_{p,p\pm1}^{(2)} N_{p\mp1}^{\prime(0)}}{d_{p-1,p-1}^{\prime(0)} d_{p+1,p+1}^{\prime(0)} N_p^{(0)}} \Bigg\} \tag{28}$$

with $Z_p = N_p/a_{pp} = Z_p^{(0)} + Z_p^{(2)}h^2 + Z_p^{(4)}h^4 + O(h^6)$. The various symbols appearing in (26)–(28) are defined in eqs (A18)–(A22) of the Appendix.

Results analogous to (25)–(28) are obtained for the evaluation of $D_p = \Delta_{Dp}/\Delta = D_p^{(0)} + D_p^{(2)}h^2 + D_p^{(4)}h^4 + O(h^6)$ by simply making the substitutions $a \leftrightarrow d'$, $d \leftrightarrow a'$ and $N \leftrightarrow N'$ in these equations, as well as in eqs. (A18)–(A22) of the Appendix.

Using now the asymptotic expansions for the Hankel functions in (8) the scattered far electric field is obtained, given by the expression

$$\vec{E}_s = \frac{e^{jkr}}{kr}\vec{f}(\theta,\phi) = \frac{e^{jkr}}{kr}[f_\theta(\theta,\phi)\hat{\theta} + f_\phi(\theta,\phi)\hat{\phi}] \tag{29}$$

where $\vec{f}(\theta,\phi)$ is the scattering amplitude and

$$f_\theta(\theta,\phi) = \sum_{n=1}^{\infty} \sum_{m=0}^{n} (-j)^{n+1} \left[-A_{mn}\frac{m}{\sin\theta}P_n^m + jD_{mn}\frac{dP_n^m}{d\theta} \right] \sin m\phi \tag{30}$$

$$f_\phi(\theta,\phi) = \sum_{n=1}^{\infty} \sum_{m=0}^{n} (-j)^{n+1} \left[-A_{mn}\frac{dP_n^m}{d\theta} + jD_{mn}\frac{m}{\sin\theta}P_n^m \right] \cos m\phi \tag{31}$$

The backscattering (σ_b) and the forward (σ_f) scattering cross-sections are defined as follows:

$$\begin{aligned}
\frac{\sigma_b}{\lambda^2} &= \frac{1}{\pi}\left[|f_\theta(\pi-\theta_0,\pi)|^2 + |f_\phi(\pi-\theta_0,\pi)|^2\right] \\
\frac{\sigma_f}{\lambda^2} &= \frac{1}{\pi}\left[|f_\theta(\theta_0,0)|^2 + |f_\phi(\theta_0,0)|^2\right]
\end{aligned} \tag{32}$$

Eqs. (30) and (31) can be set in the form

$$f_{\substack{\theta \\ \phi}}(\theta,\phi) = f_{\substack{\theta \\ \phi}}^{(0)}(\theta,\phi) + f_{\substack{\theta \\ \phi}}^{(2)}(\theta,\phi)h^2 + f_{\substack{\theta \\ \phi}}^{(4)}(\theta,\phi)h^4 + O(h^6) \tag{33}$$

where $f_\theta^{(\ell)}$ and $f_\phi^{(\ell)}$ are given by (30) and (31), respectively, by replacing A_{mn} with $A_{mn}^{(\ell)}$ and D_{mn} with $D_{mn}^{(\ell)}$, $\ell = 0, 2, 4$. Using (33), eqs. for σ_b and σ_f in (32) can be set in the form

$$\begin{aligned}
\frac{\sigma}{\lambda^2} &= \frac{1}{\pi}\left[f_\theta f_\theta^* + f_\phi f_\phi^*\right] \\
&= \frac{1}{\pi}\left\{\left|f_\theta^{(0)}\right|^2 + \left|f_\phi^{(0)}\right|^2 + 2\mathrm{Re}\left[f_\theta^{(0)*}f_\theta^{(2)} + f_\phi^{(0)*}f_\phi^{(2)}\right]h^2 \right. \\
&\quad \left. + \left[\left|f_\theta^{(2)}\right|^2 + \left|f_\phi^{(2)}\right|^2 + 2\mathrm{Re}\left(f_\theta^{(0)*}f_\theta^{(4)} + f_\phi^{(0)*}f_\phi^{(4)}\right)\right]h^4\right\} + O(h^6)
\end{aligned} \tag{34}$$

where Re represents the real part and the asterisk denotes the complex conjugate.

The total scattering cross-section is defined as

$$\begin{aligned}
\frac{Q_t}{\lambda^2} &= \frac{1}{4\pi^2}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}|\vec{f}(\theta,\phi)|^2\sin\theta d\theta d\phi \\
&= \frac{1}{4\pi^2}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\left[f_\theta f_\theta^* + f_\phi f_\phi^*\right]\sin\theta d\theta d\phi
\end{aligned} \tag{35}$$

Substituting from (30), (31) into (35) and using the orthogonal properties of the Legendre and the trigonometric functions we finally obtain

$$\begin{aligned}
\frac{Q_t}{\lambda^2} &= \frac{1}{\pi}\sum_{n=1}^{\infty}\sum_{m=0}^{n}\frac{n(n+1)}{\epsilon_m(2n+1)}\frac{(n+m)!}{(n-m)!}\left(|A_{mn}|^2 + |D_{mn}|^2\right) \\
&= \left(\frac{Q_t^{(0)}}{\lambda^2} + \frac{Q_t^{(2)}}{\lambda^2}h^2 + \frac{Q_t^{(4)}}{\lambda^2}h^4\right) + O(h^6)
\end{aligned} \tag{36}$$

where $|A_{mn}|^2 + |D_{mn}|^2 = A_{mn}A_{mn}^* + D_{mn}D_{mn}^*$, so the last expansion is found by steps similar to the ones in (34).

Eq. (36) can be set in the form

$$Q_t = Q_t^{(0)} \left[1 + g_Q^{(2)}h^2 + g_Q^{(4)}h^4 + O(h^6)\right], \ g_Q^{(\ell)} = Q_t^{(\ell)}/Q_t^{(0)}, \ \ell = 2, 4 \tag{37}$$

The same is valid also for σ/λ^2 in (34) and consequently for σ_b and σ_f in (32) which can be written as

$$\sigma = \sigma^{(0)} \left[1 + g_\sigma^{(2)}h^2 + g_\sigma^{(4)}h^4 + O(h^6)\right] \tag{38}$$

Evidently $Q_t^{(0)}$ and $\sigma^{(0)}$ correspond to a sphere with radius $a(h = 0)$.

A check for the correctness of the results is by the fulfillment of the forward scattering theorem [14] which in the present case has the form

$$Q_t/\lambda^2 = \text{Im} \left[f_\phi(\theta_0, 0)\right]/\pi \tag{39}$$

where Im represents the imaginary part. Its validity was verified numerically for various values of the parameters.

For the oblate spheroid simply h^2 is replaced with $-h^2$ in each case.

2.2. TM Incident Wave

In this case the plane of incidence is again the xz plane, while the electric field lying on this plane has the expression [15]

$$\vec{E}_i = j \sum_{n=1}^{\infty} \sum_{m=0}^{n} \left[d_{mn}(\theta_0)\vec{m}_{omn}^{r(1)}(r, \theta, \phi) + c_{mn}(\theta_0)\vec{n}_{emn}^{r(1)}(r, \theta, \phi)\right] \tag{40}$$

The scattered wave is expressed as

$$\vec{E}_s = \sum_{n=1}^{\infty} \sum_{m=0}^{n} \left[A_{mn}\vec{m}_{omn}^{r(3)}(r, \theta, \phi) + D_{mn}\vec{n}_{emn}^{r(3)}(r, \theta, \phi)\right] \tag{41}$$

where A_{mn} and D_{mn} are different from the corresponding coefficients in (8).

Satisfying the boundary condition $\hat{n} \times (\vec{E}_i + \vec{E}_s) = 0$ at the spheroidal surface and following the same procedure as for the TE wave we obtain again the sets of eqs. (15) and (16), with the only differences that d's and d''s simply change their signs, while N's and N''s are obtained from the corresponding ones for the TE wave by simply

replacing c_{mn} and d_{mn} by jd_{mn} and $-jc_{mn}$, respectively. Eqs. (17)–(38) are again valid with the former changes, while eq. (39) is replaced by

$$Q_t/\lambda^2 = \mathrm{Im}[f_\theta(\theta_0, 0)]/\pi \qquad (42)$$

For the oblate spheroid h^2 is replaced by $-h^2$ in each case.

3. NUMERICAL RESULTS AND DISCUSSION

In Table 1 the values of $g^{(2)}$ and $g^{(4)}$ appearing in eqs. (37), (38) are given, for various values of θ_0 and for $a/\lambda = 0.7$. The results are symmetric about $\theta_0 = 90°$, as it is imposed by the geometry of the scatterer.

In Figs. 2–4 the scattering cross-sections are plotted versus θ_0, for $h = 0.2, 0.4$, for a TE and a TM incident wave and for a prolate or an oblate spheroid if $a/\lambda = 0.7$, where $\lambda = 2\pi/k$ is the wavelength of the electromagnetic field. In each figure the corresponding scattering cross-section for $h = 0$ (sphere with radius a) is also plotted. The results are symmetric about $\theta_0 = 90°$.

From Figs. 2–4 it is seen that, for the values of the parameters used, the deviation of the spheroid from the corresponding sphere
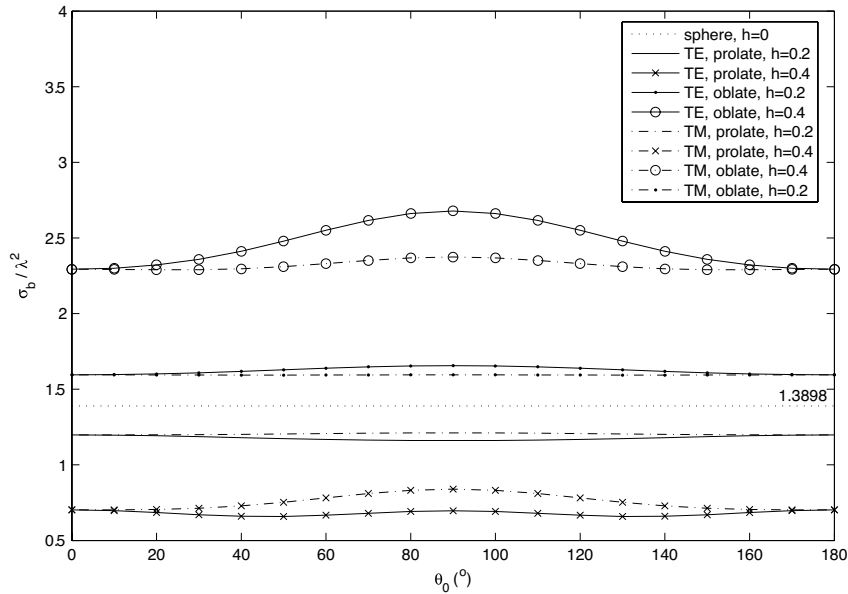

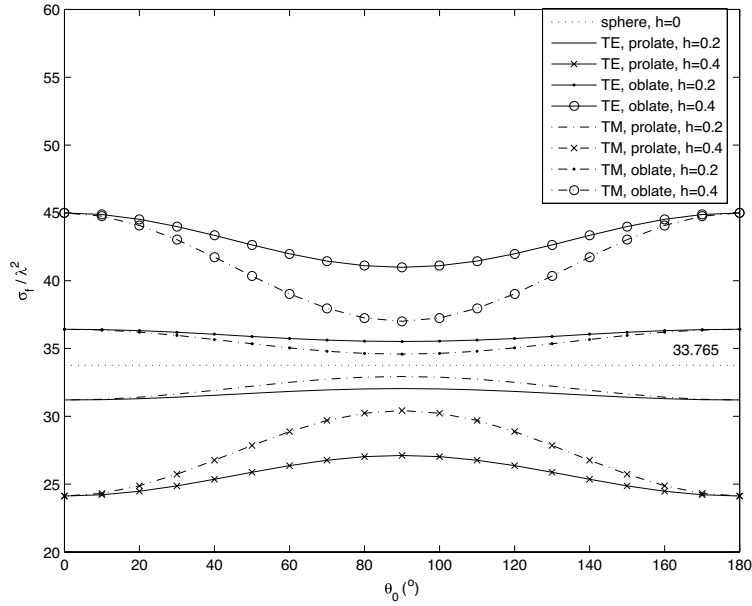**Figure 2.** Backscattering cross-section for $a/\lambda = 0.7$.

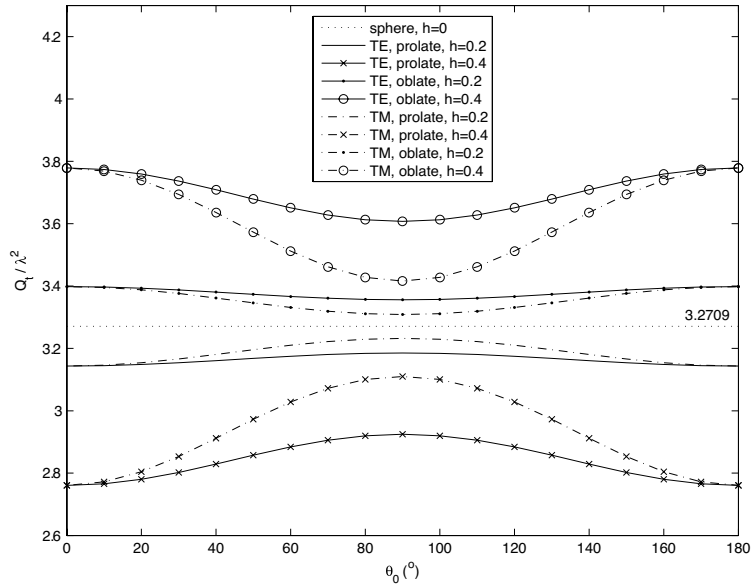**Figure 3.** Forward scattering cross-section for $a/\lambda = 0.7$.

**Figure 4.** Total scattering cross-section for $a/\lambda = 0.7$.

Table 1. Values of g's for $a/\lambda = 0.7$.

$\theta_0\,(^o)$	$g^{(2)}_{\sigma_b}$	$g^{(2)}_{\sigma_f}$	$g^{(2)}_{Q_t}$	$g^{(4)}_{\sigma_b}$	$g^{(4)}_{\sigma_f}$	$g^{(4)}_{Q_t}$
TE wave
0	-3.5771	-1.9316	-0.9731	3.0362	0.9243	-0.0129
10	-3.6036	-1.9121	-0.9635	3.0613	0.9057	-0.0131
20	-3.6800	-1.8558	-0.9356	3.1806	0.8523	-0.0139
30	-3.7969	-1.7695	-0.8929	3.4988	0.7712	-0.0165
40	-3.9403	-1.6637	-0.8406	4.1128	0.6730	-0.0216
50	-4.0929	-1.5510	-0.7848	5.0371	0.5699	-0.0296
60	-4.2363	-1.4452	-0.7325	6.1601	0.4745	-0.0395
70	-4.3532	-1.3589	-0.6898	7.2579	0.3976	-0.0492
80	-4.4295	-1.3026	-0.6620	8.0629	0.3479	-0.0563
90	-4.4560	-1.2830	-0.6523	8.3588	0.3308	-0.0590
TM wave
0	-3.5771	-1.9316	-0.9731	3.0362	0.9243	-0.0129
10	-3.5733	-1.8918	-0.9526	3.0249	0.8994	-0.0052
20	-3.5623	-1.7770	-0.8935	3.0285	0.8260	0.0134
30	-3.5454	-1.6011	-0.8030	3.1375	0.7089	0.0318
40	-3.5246	-1.3854	-0.6919	3.4426	0.5575	0.0377
50	-3.5025	-1.1558	-0.5737	3.9745	0.3871	0.0238
60	-3.4818	-0.9401	-0.4627	4.6690	0.2181	-0.0081
70	-3.4648	-0.7642	-0.3721	5.3749	0.0741	-0.0478
80	-3.4538	-0.6495	-0.3130	5.9033	-0.0230	-0.0803
90	-3.4500	-0.6096	-0.2925	6.0993	-0.0573	-0.0927

decreases/increases the scattering cross-sections, with respect to those of the sphere, in the case of a prolate/oblate spheroid.

In Figs. 5 the backscattering cross-section is plotted versus $ka\,(= 2\pi a/\lambda)$ for $\theta_0 = 0°$ in (a) (in this case there is no need to consider TE and TM waves) and for $\theta_0 = \pi/3, \pi/2$ in (b), while in Figs. 6–8 the scattering cross-sections are plotted versus ka for various values of θ_0 and h.

A further check for the correctness of our method has been made by comparison of our results with existing ones found in [1, 5– 7]. So in Figs. 8a and 8b we reproduce, with very good agreement, the normalized backscattering cross-sections $\sigma_b/\pi a^2$ and $\sigma_b a^2/\pi b^4$, respectively, versus ka for $\theta_0 = 0°$, shown in Figs. 6 and 7, respectively of [1]. The values of h vary from 0.31 ($b/a = 0.95$) up to 0.66 ($b/a = 1.2$). It should be noticed here that for $b/a < 1$, corresponding to a prolate spheroid, $h = \sqrt{1 - (b/a)^2}$, while for $b/a > 1$, corresponding to
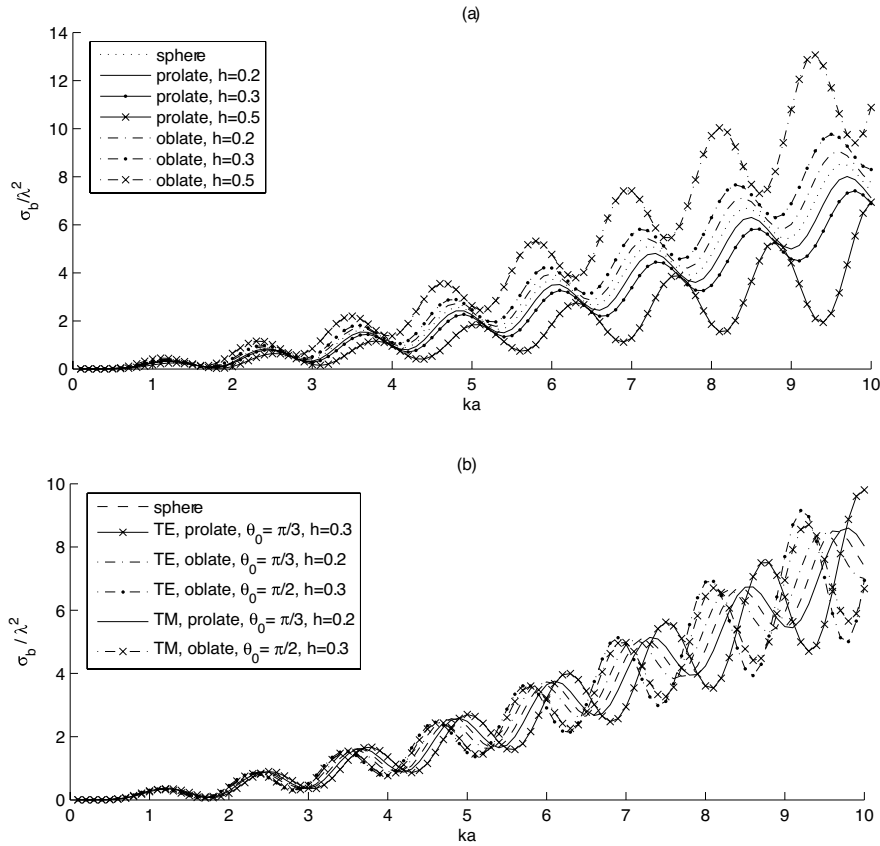
Figure 5. Backscattering cross-section (a) for $\theta_0 = 0$ (axial incidence). (b) for $\theta_0 = \pi/3, \pi/2$.

an oblate spheroid, $h = \sqrt{(b/a)^2 - 1}$. Moreover, we have reproduced exactly the backscattering cross-sections for TE and TM waves incident on a prolate spheroid, given in Figs. 4.7a, b of [7], where $a/b = 1.01$ ($h = 0.14$) and $ka = 1, 2$, respectively. In Fig. 9 we give the backscattering cross-section for a TE wave incident on a prolate spheroid with $a/b = 2$ ($h = 0.866$) and $ka = 1$. Finally in Figs. 10 we give the bistatic cross-sections $\pi\sigma(\theta, 0)/\lambda^2$ for a TE wave incident on a prolate spheroid with $a/b = 2$, $ka = 1$ and $\theta_0 = 0°, 20°, 40°$ and $80°$. Comparison of Fig. 9 with Fig. 4.5b of [7] (see also Fig. 2 of [6]) and of Figs. 10a, b, c and d, with Figs. 4.2b (see also Fig. 3a of [5]), 4.3a, 4.3b and 4.4b of [7] shows that our method gives very good results even for such a great value of h (maximum possible $h = 1.0$, corresponding to
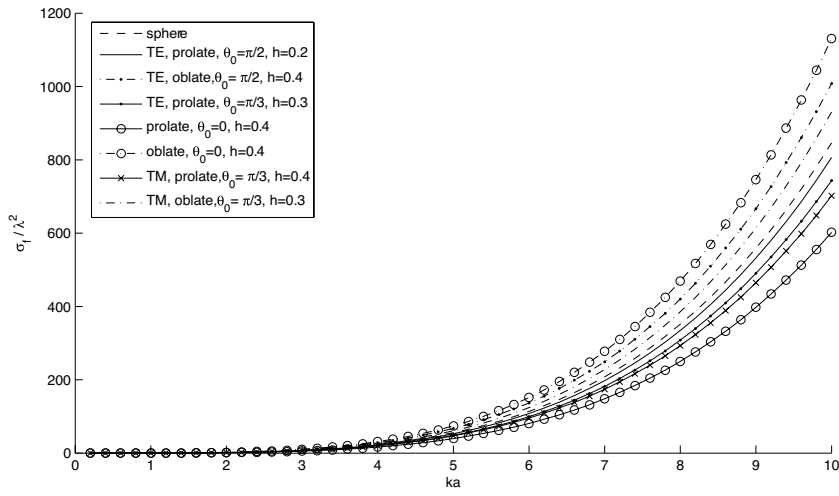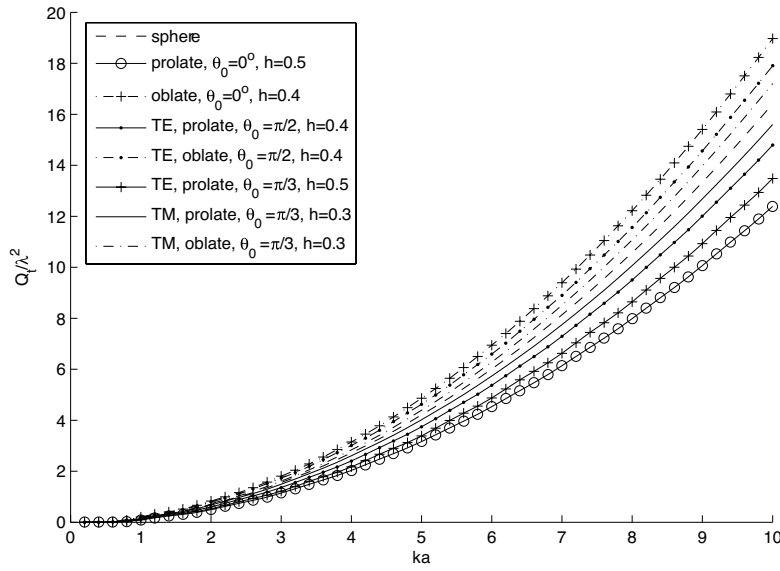
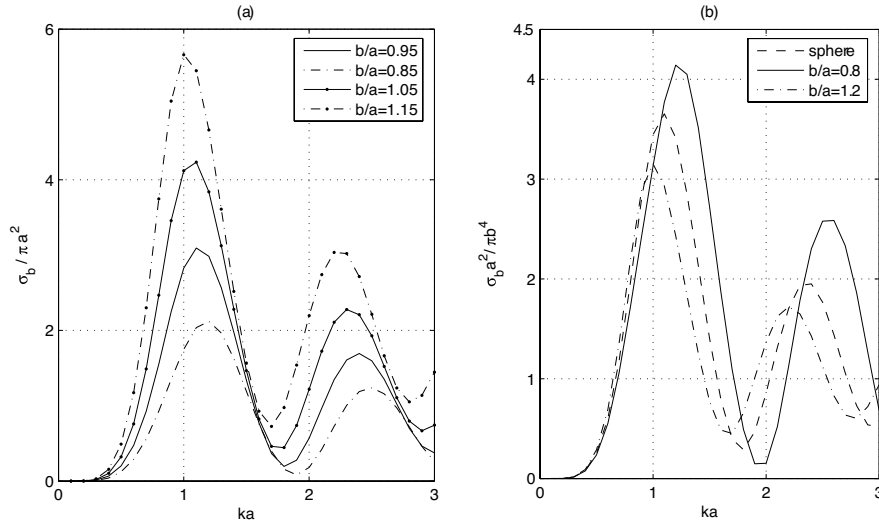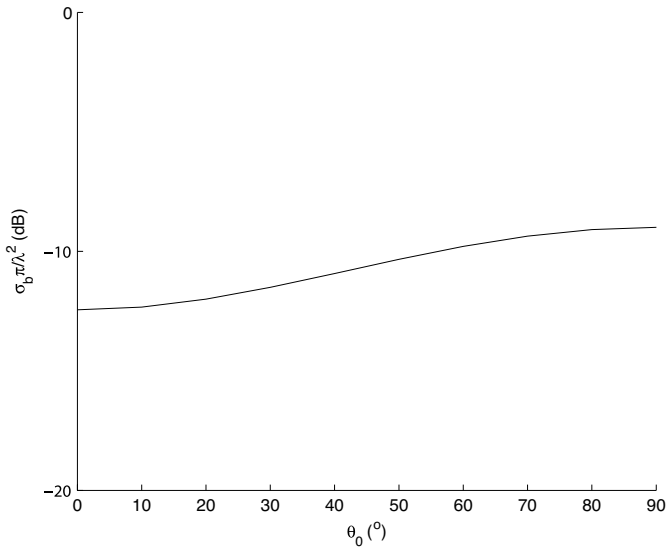**Figure 6.** Forward scattering cross-section.
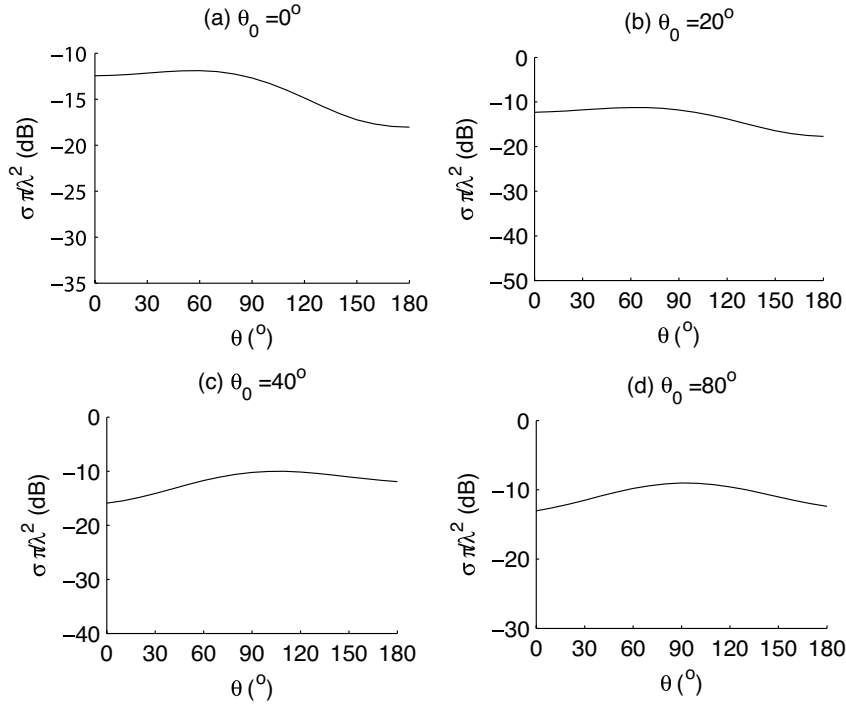

Figure 7. Total scattering cross-section.

Figure 8. Normalized backscattering cross-section for $\theta_0 = 0$ (axial incidence) (a) $\sigma_b/\pi a^2$ (b) $\sigma_b a^2/\pi b^4$.

Figure 9. Backscattering cross section for a TE wave incident on a prolate spheroid with $a/b = 2$ ($h = 0.866$) and $ka = 1$.

Figure 10. Bistatic cross sections for a TE wave incident on a prolate spheroid with $\phi = 0$, $a/b = 2$ ($h = 0.866$) and $ka = 1$. (a) $\theta_0 = 0°$, (b) $\theta_0 = 20°$, (c) $\theta_0 = 40°$, (d) $\theta_0 = 80°$.

a rod), in the case of a TE wave. The same is not valid for a TM wave and for $h = 0.866$, used in Figs. 9 and 10. The values of $h = 0.2 - 0.5$ used in our Figs. 2–7 are much lower than $h = 0.866$ used in Figs. 9, 10, thus keeping the errors low enough in each case.

APPENDIX A.

The expressions for the various symbols appearing in eqs. (17)–(21) are the following:

$$a_{vv}^{(0)} = h_v(x)I_1(v, v), \;\; a_{vs}^{(2)} = H_s^{(2)}(x)[m^2 I(v, s) + I_2(v, s)], \;\; s = v, v \pm 2,$$
$$a_{vs}^{(4)} = a_{vs}^{(2)} + H_s^{(4)}(x)\left[m^2 I_3(v, s) + I_4(v, s)\right], \;\; s = v, v \pm 2, \; v \pm 4$$

$$\text{(A1)}$$

$$d_{vs}^{(2)} = mI_5(v, s)\left\{2\left[H_s^{'(2)}(x) + \Lambda_s^{(2)}(x)\right] + s(s+1)h_s(x)/x\right\}, \;\; s = v \pm 1,$$

$$d_{vs}^{(4)} = d_{vs}^{(2)} + mI_6(v,s)\left\{4\left[H_s^{'(4)}(x) + \Lambda_s^{(4)}(x)\right] + s(s+1)\left[\Lambda_s^{(2)}(x) - h_s(x)/x\right]\right\}$$
$$s = v \pm 1, v \pm 3 \quad \text{(A2)}$$

$$N_v^{(0)} = -c_{mv}j_v(x)I_1(v,v),$$
$$N_v^{(2)} = -\sum_{s=v,v\pm2} c_{ms}J_s^{(2)}(x)\left[m^2 I(v,s) + I_2(v,s)\right]$$
$$-m\sum_{s=v\pm1} d_{ms}I_5(v,s)\left\{2\left[J_s^{'(2)}(x) + K_s^{(2)}(x)\right] + s(s+1)j_s(x)/x\right\},$$
$$N_v^{(4)} = N_v^{(2)} - \sum_{s=v,v\pm2,v\pm4} c_{ms}J_s^{(4)}(x)\left[m^2 I_3(v,s) + I_4(v,s)\right]$$
$$-m\sum_{s=v\pm1,v\pm3} d_{ms}I_6(v,s)\left\{4\left[J_s^{'(4)}(x) + K_s^{(4)}(x)\right]\right.$$
$$\left. + s(s+1)\left[K_s^{(2)}(x) - j_s(x)/x\right]\right\} \quad \text{(A3)}$$

$$a_{vs}^{'(2)} = 2mI_5(v,s)H_s^{(2)}(x), \quad s = v \pm 1,$$
$$a_{vs}^{'(4)} = a_{vs}^{'(2)} + 4mI_6(v,s)H_s^{(4)}(x), \quad s = v \pm 1, v \pm 3 \quad \text{(A4)}$$

$$d_{vv}^{'(0)} = \left[h_v'(x) + h_v(x)/x\right]I_1(v,v),$$
$$d_{vs}^{'(2)} = \left[m^2 I(v,s) + I_2(v,s)\right]\left[H_s^{'(2)}(x) + \Lambda_s^{(2)}(x)\right]$$
$$+ s(s+1)I_7(v,s)h_s(x)/x, \quad s = v, v \pm 2,$$
$$d_{vs}^{'(4)} = d_{vs}^{'(2)} + \left[m^2 I_3(v,s) + I_4(v,s)\right]\left[H_s^{'(4)}(x) + \Lambda_s^{(4)}(x)\right]$$
$$+ s(s+1)\left[\Lambda_s^{(2)}(x) - h_s(x)/x\right]I_8(v,s), \quad s = v, v \pm 2, v \pm 4 \quad \text{(A5)}$$

$$N_v^{'(0)} = -d_{mv}\left[j_v'(x) + j_v(x)/x\right]I_1(v,v),$$
$$N_v^{'(2)} = -\sum_{s=v\pm1} 2mc_{ms}J_s^{(2)}(x)I_5(v,s)$$
$$-\sum_{s=v,v\pm2} d_{ms}\left\{\left[J_s^{'(2)}(x) + K_s^{(2)}(x)\right]\left[m^2 I(v,s) + I_2(v,s)\right]\right.$$
$$\left. + s(s+1)I_7(v,s)j_s(x)/x\right\},$$
$$N_v^{'(4)} = N_v^{'(2)} - \sum_{s=v\pm1,v\pm3} 4mc_{ms}J_s^{(4)}(x)I_6(v,s)$$
$$-\sum_{s=v,v\pm2,v\pm4} d_{ms}\left\{\left[J_s^{'(4)}(x) + K_s^{(4)}(x)\right]\left[m^2 I_3(v,s) + I_4(v,s)\right]\right.$$
$$\left. + s(s+1)\left[K_s^{(2)}(x) - j_s(x)/x\right]I_8(v,s)\right\} \quad \text{(A6)}$$

In (A1)–(A6) we have made the substitutions

$$
J_s^{(2)}(x) = -xj_s'(x)/2, \; J_s^{(4)}(x) = \left[3xj_s'(x) + x^2 j_s''(x)\right]/8,
$$
$$
J_s'^{(2)}(x) = -xj_s''(x)/2, \; J_s'^{(4)}(x) = \left[3xj_s''(x) + x^2 j_s'''(x)\right]/8
$$
(A7)

$$
K_s^{(2)}(x) = [j_s(x)/x - j_s'(x)]/2,
$$
$$
K_s^{(4)}(x) = \left[-j_s(x)/x + j_s'(x) + x^2 j_s''(x)\right]/8
$$
(A8)

while $H_s^{(2)}(x), H_s^{(4)}(x), H_s'^{(2)}(x), H_s'^{(4)}(x)$ and $\Lambda_s^{(2)}(x), \Lambda_s^{(4)}(x)$ are given by (A7) and (A8) respectively, if $j_s(x), j_s'(x), j_s''(x)$ and $j_s'''(x)$ are replaced by $h_s(x), h_s'(x), h_s''(x)$ and $h_s'''(x)$, respectively.

The integrals $I(v,s)$ and $I_i(v,s)$, $i = 1 - 8$, appearing in (A1)–(A6) are evaluated by using the recurrence relations and the orthogonal properties of Legendre functions [9, 16]. Their expressions are:

$$
I(v,s) = \int_{-1}^{1} P_v^m P_s^m d\eta =
\begin{cases}
\dfrac{2}{2v+1}\dfrac{(v+m)!}{(v-m)!}, & s = v \\[2mm]
0, & s \neq v
\end{cases}
$$
(A9)

$$
\begin{aligned}
I_1(v,s) &= \int_{-1}^{1} \left[(1-\eta^2)\frac{dP_v^m}{d\eta}\frac{dP_s^m}{d\eta} + \frac{m^2}{1-\eta^2}P_v^m P_s^m\right] d\eta \\
&= v(v+1)I(v,s)
\end{aligned}
$$
(A10)

$$
\begin{aligned}
I_2(v,s) &= \int_{-1}^{1} (1-\eta^2)^2 \frac{dP_v^m}{d\eta}\frac{dP_s^m}{d\eta} d\eta \\
&= \begin{cases}
\left[\dfrac{(v+1)^2(v^2-m^2)}{2v-1} + \dfrac{v^2[(v+1)^2-m^2]}{2v+3}\right]\dfrac{I(v,v)}{2v+1}, & s = v \\[3mm]
-\dfrac{(v+1)(v-2)(v-m)(v-m-1)}{(2v-3)(2v-1)}I(v,v), & s = v-2 \\[3mm]
-\dfrac{v(v+3)(v+m+1)(v+m+2)}{(2v+3)(2v+5)}I(v,v), & s = v+2
\end{cases}
\end{aligned}
$$
(A11)

$$I_3(v,s) = \int_{-1}^{1}(1-\eta^2)P_v^m P_s^m d\eta$$

$$= \begin{cases} 2\dfrac{v^2+m^2+v-1}{(2v-1)(2v+3)}I(v,v), & s = v \\[3mm] -\dfrac{(v-m-1)(v-m)}{(2v-1)(2v-3)}I(v,v), & s = v-2 \\[3mm] -\dfrac{(v+m+1)(v+m+2)}{(2v+3)(2v+5)}I(v,v), & s = v+2 \end{cases} \quad \text{(A12)}$$

$$I_4(v,s) = \int_{-1}^{1}(1-\eta^2)^3\frac{dP_v^m}{d\eta}\frac{dP_s^m}{d\eta}d\eta$$

$$= \frac{1}{(2v+1)(2s+1)}[(v+1)(v+m)(s+1)(s+m)I_3(v-1,s-1)$$
$$+v(v-m+1)s(s-m+1)I_3(v+1,s+1)$$
$$-(v+1)(v+m)s(s-m+1)I_3(v-1,s+1)$$
$$-v(v-m+1)(s+1)(s+m)I_3(v+1,s-1)] \quad \text{(A13)}$$

$$I_5(v,s) = \int_{-1}^{1}\eta P_v^m P_s^m d\eta = \begin{cases} \dfrac{v-m}{2v-1}I(v,v), & s = v-1 \\[3mm] \dfrac{v+m+1}{2v+3}I(v,v), & s = v+1 \end{cases} \quad \text{(A14)}$$

$$I_6(v,s) = \int_{-1}^{1}\eta(1-\eta^2)P_v^m P_s^m d\eta$$

$$= \frac{1}{2v+1}[(v+m)I_3(v-1,s)+(v-m+1)I_3(v+1,s)] \quad \text{(A15)}$$

$$I_7(v,s) = \int_{-1}^{1}\eta(1-\eta^2)\frac{dP_v^m}{d\eta}P_s^m d\eta$$

$$= \frac{1}{2v+1}[(v+1)(v+m)I_5(v-1,s)-v(v-m+1)I_5(v+1,s)]$$
$$\text{(A16)}$$

$$I_8(v,s) = \int_{-1}^{1}\eta(1-\eta^2)^2\frac{dP_v^m}{d\eta}P_s^m d\eta$$

$$= \frac{1}{(2v+1)(2s+1)}[(v+1)(v+m)(s+m)I_3(v-1,s-1)$$
$$-v(v-m+1)(s-m+1)I_3(v+1,s+1)$$
$$+(v+1)(v+m)(s-m+1)I_3(v-1,s+1)$$
$$-v(v-m+1)(s+m)I_3(v+1,s-1)] \quad \text{(A17)}$$

Finally, the symbols appearing in eqs. (26)–(28) have the expressions

$$Z_p^{(0)} = N_p^{(0)}/a_{pp}^{(0)}, \ Z_p^{(2)} = \left[a_{pp}^{(0)} N_p^{(2)} - a_{pp}^{(2)} N_p^{(0)} \right] \Big/ \left[a_{pp}^{(0)} \right]^2,$$

$$Z_p^{(4)} = \left\{ \left[a_{pp}^{(0)} \right]^2 N_p^{(4)} - a_{pp}^{(0)} a_{pp}^{(4)} N_p^{(0)} - a_{pp}^{(0)} a_{pp}^{(2)} N_p^{(2)} + \left[a_{pp}^{(2)} \right]^2 N_p^{(0)} \right\} \Big/ \left[a_{pp}^{(0)} \right]^3 \tag{A18}$$

$$Y_s^a = \frac{a_{ps}^{(2)} N_s^{(0)}}{a_{ss}^{(0)} N_p^{(0)}}, \ s = p \pm 2, \quad Y_t^d = \frac{d_{pt}^{(2)} N_t^{'(0)}}{d_{tt}^{'(0)} N_p^{(0)}}, \ t = p \pm 1 \tag{A19}$$

$$X_s^a = \left\{ a_{ss}^{(0)} N_p^{(0)} \left[a_{ps}^{(2)} N_s^{(2)} + a_{ps}^{(4)} N_s^{(0)} \right] \right.$$
$$\left. - a_{ps}^{(2)} N_s^{(0)} \left[a_{ss}^{(0)} N_p^{(2)} + a_{ss}^{(2)} N_p^{(0)} \right] \right\} \Big/ \left[a_{ss}^{(0)} N_p^{(0)} \right]^2, \ s = p \pm 2,$$

$$X_t^d = \left\{ d_{tt}^{'(0)} N_p^{(0)} \left[d_{pt}^{(2)} N_t^{'(2)} + d_{pt}^{(4)} N_t^{'(0)} \right] \right.$$
$$\left. - d_{pt}^{(2)} N_t^{'(0)} \left[d_{tt}^{'(0)} N_p^{(2)} + d_{tt}^{'(2)} N_p^{(0)} \right] \right\} \Big/ \left[d_{tt}^{'(0)} N_p^{(0)} \right]^2, \ t = p \pm 1 \tag{A20}$$

$$W_s^a = -\frac{a_{ps}^{(2)} a_{sp}^{(2)}}{a_{pp}^{(0)} a_{ss}^{(0)}}, \ s = p \pm 2, \quad W_t^d = -\frac{a_{tp}^{'(2)} d_{pt}^{(2)}}{a_{pp}^{(0)} d_{tt}^{'(0)}}, \ t = p \pm 1 \tag{A21}$$

$$V_s^a = \frac{a_{ps}^{(4)} N_s^{(0)}}{a_{ss}^{(0)} N_p^{(0)}}, \ s = p \pm 4, \quad V_t^d = \frac{d_{pt}^{(4)} N_t^{'(0)}}{d_{tt}^{'(0)} N_p^{(0)}}, \ t = p \pm 3 \tag{A22}$$

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