Electromagnetic scattering by magnetic spheres

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A number of unusual electromagnetic scattering effects for magnetic spheres are described. When $\epsilon = \mu$, the backscatter gain is zero; the scattered radiation is polarized in the same sense as the incident radiation. In the small-particle (or long-wavelength) limit, conditions are described for zero forward scatter, for complete polarization of scattered radiation in other directions, and for asymmetry of forward scatter to backscatter. The special case in the small-particle limit of $m = 1$, i.e., $\mu = 1/\epsilon$, provides interesting special instances of complete polarization and forward-scatter-to-backscatter asymmetry.

1. INTRODUCTION

Some unusual properties have recently been noted[1] of the Fresnel equations governing reflection of plane electromagnetic waves at a plane interface whenever the real refractive indices of the two media are equal, $m_1 = m_2$, but the magnetic permeabilities are unequal, i.e., since $m = \sqrt{\mu\epsilon}$, $\mu_1/\mu_2 = \epsilon_2/\epsilon_1$. In such a case the reflected ray is independent of the incident polarization and of the angle of incidence. This condition, although unattainable in the visible spectrum, is possible for infrared and millimeter waves. Although μ and ϵ are usually complex, nearly lossless materials do exist, so that there is the possibility of compounding materials approximating the conditions discussed below.

This has stimulated us to explore for unusual electromagnetic scattering effects by spheres composed of magnetic materials, and we have encountered a number of such effects, not only for unit refractive index of the particle relative to that of the medium, $m = 1$ but for some other conditions as well. These are described as follows. In Section 2 a sphere with equal values of the relative dielectric constant and relative magnetic permeability, $\epsilon = \mu$, is shown to exhibit zero backscatter and no depolarization. Sections 3 and 4 deal with the small-particle (or long-wavelength) limit. Conditions for zero scatter, no depolarization, and asymmetric forward-scatter-to-backscatter ratio by small magnetic spheres are treated in Section 3. Finally, in Section 4, the condition treated earlier,[1] $m = 1$, is discussed.

2. SCATTERING BY SPHERES OF ARBITRARY SIZE WITH $\epsilon = \mu$

When a plane wave of unit irradiance and wavelength in the medium λ is incident upon an isotropic homogeneous sphere of radius a and refractive index m, the scattered radiant intensity is composed of two polarized components[2]

$$I_1 = (\lambda^2/4\pi^2r^2)|S_1|^2 \sin^2 \phi, \tag{1}$$

$$I_2 = (\lambda^2/4\pi^2r^2)|S_2|^2 \cos^2 \phi, \tag{2}$$

where r is the distance to the observer, ϕ is the angle between the electric vector of the incident wave and the scattering plane, and I_1 and I_2 are the polarized components with electric vectors perpendicular and parallel to the scattering plane. The amplitude functions are

$$S_1 = \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} [a_n\pi_n(\cos\theta) + b_n\tau_n(\cos\theta)], \tag{3}$$

$$S_2 = \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} [a_n\tau_n(\cos\theta) + b_n\pi_n(\cos\theta)], \tag{4}$$

where the angular functions

$$\pi_n(\cos\theta) = P_n{}^1(\cos\theta)/\sin\theta \tag{5}$$

and

$$\tau_n(\cos\theta) = \mathrm{d}/\mathrm{d}\theta[P_n{}^1(\cos\theta)] \tag{6}$$

are defined in terms of the associated Legendre functions of the first degree. The angle θ is between the forward and scattering directions.

The physical parameters that describe the particle, namely, m, a, and λ, are embedded within the scattering coefficients a_n and b_n. The magnetic permeability of the particle relative to the medium is usually suppressed since it is unity at optical wavelengths. However, we now retain it in the following form, which differs somewhat from that of Stratton,[3]

$$a_n = \frac{\mu\psi_n(\alpha)\psi_n'(\beta) - m\psi_n(\beta)\psi_n'(\alpha)}{\mu\zeta_n(\alpha)\psi_n'(\beta) - m\psi_n(\beta)\zeta_n'(\alpha)}, \tag{7}$$

$$b_n = \frac{m\psi_n(\alpha)\psi_n'(\beta) - \mu\psi_n(\beta)\psi_n'(\alpha)}{m\zeta_n(\alpha)\psi_n'(\beta) - \mu\psi_n(\beta)\zeta_n'(\alpha)}, \tag{8}$$

where the radial functions $\psi_n(x)$ and $\zeta_n(x)$ are the Ricatti Bessel and Hankel functions of arguments $\alpha = 2\pi a/\lambda$ and $\beta = m\alpha$. It will be recalled that a_n and b_n can be considered the electric and magnetic moments of multipole sources located at the origin.

Whenever $\epsilon = \mu$, it follows that $a_n = b_n$, so the backscatter gain given by[2]

$$G(180°) = (4/\alpha^2)\left|\sum_{n=1}^{\infty} (n + 1/2)(-1)^n(b_n - a_n)\right|^2 \tag{9}$$

equals zero. This is a general result, valid for spheres of any size; indeed, the same would be true for any axially symmetric body illuminated along the axis of symmetry.

Since $S_1 = S_2$ whenever $\epsilon = \mu$, it follows from Eqs. (1) and (2) for linearly polarized incident radiation that the scattered radiation will be similarly polarized for all scattering angles, i.e.,

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$$I_2/I_1 = \cot^2 \phi. \quad (10)$$

Indeed, the state of polarization of the scattered and incident radiation will be the same even when the incident radiation is elliptically polarized since $S_1 = S_2$, so the scattering process does not impose an additional phase shift between the two components S_1 and S_2.

3. SMALL-PARTICLE LIMIT FOR $\mu \neq 1$

The small-particle limit is obtained by inserting expansions of the radial functions in powers of α and β into the scattering coefficients a_n and b_n and then expanding these in powers of α. For small values of α, the leading terms dominate, viz.,

$$a_1 = \frac{2i}{3}\alpha^3\left(\frac{\epsilon - 1}{\epsilon + 2}\right), \quad (11)$$

$$b_1 = \frac{2i}{3}\alpha^3\left(\frac{\mu - 1}{\mu + 2}\right). \quad (12)$$

These correspond to electric and magnetic dipole sources. [We call attention to an error in Stratton's (Ref. 3, p. 571) expression for the magnetic dipole that, in our notation, he gives as $-(i/3)\alpha^3(\mu - 1/\mu + 2)$.]

This result is already unusual since, for nonmagnetic materials, the leading term in the expansion of b_1 is the α^5 term, so the magnetic dipole does not make a significant contribution in the small-particle limit. The radiation that is scattered by a small nonmagnetic sphere resembles that which is due only to an electric dipole and is described by

$$I_1 = (\lambda^2/4\pi^2r^2)\alpha^6\left(\frac{\epsilon - 1}{\epsilon + 2}\right)^2 \sin^2\phi, \quad (13)$$

$$I_2 = (\lambda^2/4\pi^2r^2)\alpha^6\left(\frac{\epsilon - 1}{\epsilon + 2}\right)^2 \cos^2\theta\cos^2\phi. \quad (14)$$

However, for magnetic materials there is a contribution from both the electric and magnetic dipoles, so

$$I_1 = (\lambda^2/4\pi^2r^2)\alpha^6\left[\left(\frac{\epsilon - 1}{\epsilon + 2}\right) + \left(\frac{\mu - 1}{\mu + 2}\right)\cos\theta\right]^2 \sin^2\phi, \quad (15)$$

$$I_2 = (\lambda^2/4\pi^2r^2)\alpha^6\left[\left(\frac{\epsilon - 1}{\epsilon + 2}\right)\cos\theta + \left(\frac{\mu - 1}{\mu + 2}\right)\right]^2 \cos^2\phi. \quad (16)$$

This results in a number of interesting effects in addition to the general ones discussed in Section 2 for $\epsilon = \mu$. For example, when $\epsilon = (4 - \mu)/(2\mu + 1)$, it follows that $a_1 = -b_1$, so the forward scatter is zero, i.e., $I_1(0°) = I_2(0°) = 0$. Also for this condition, just as for $\epsilon = \mu$, the polarization of the scattered radiation is the same as that of the incident radiation (including elliptically polarized incident radiation), i.e., as before

$$I_2/I_1 = \cot^2 \phi. \quad (10)$$

For small nonmagnetic spheres there is the well-known condition of complete polarization at $\theta = 90°$ since I_2 is zero at this angle provided that ϵ is not too large. In the case of small magnetic spheres it follows from Eqs. (15) and (16) that there are two conditions for complete polarization of the scattered radiation. $I_1 = 0$ for

$$\cos\theta_1 = -(a_1/b_1) = -\left(\frac{\epsilon - 1}{\epsilon + 2}\right)\bigg/\left(\frac{\mu - 1}{\mu + 2}\right) \quad (17)$$

and $I_2 = 0$ for

$$\cos\theta_2 = -(b_1/a_1) = -\left(\frac{\mu - 1}{\mu + 2}\right)\bigg/\left(\frac{\epsilon - 1}{\epsilon + 2}\right). \quad (18)$$

Obviously it is not possible for I_1 and I_2 each to be equal to zero for the same set ϵ, μ since this would require that $\cos\theta_1$ and $\cos\theta_2$ be reciprocals.

Still another way in which the scattering by small magnetic spheres differs from nonmagnetic spheres is that, whereas in the latter case the radiant intensity is symmetric about 90°, in the magnetic case there may be either preferential backscatter or preferential forward scatter. For example, from Eqs. (15) and (16) the ratio of forward scatter to backscatter is given by

$$I_1(0°)/I_1(180°) = \left[\left(\frac{\epsilon - 1}{\epsilon + 2}\right) + \left(\frac{\mu - 1}{\mu + 2}\right)\right]^2 \bigg/ \left[\left(\frac{\epsilon - 1}{\epsilon + 2}\right) - \left(\frac{\mu - 1}{\mu + 2}\right)\right]^2. \quad (19)$$

These effects are somewhat reminiscent of what occurs for small nonmagnetic spheres with large values of ϵ. For example, in the limit that $\epsilon = \infty$, the magnetic dipole makes a contribution to the scattering[2] in the small-particle limit so that the scattered radiant intensity is no longer symmetrical about 90°. In this case the forward-scatter-to-backscatter ratio $I_1(0°)/I_1(180°)$ is 1/9, and the scattered radiation is completely polarized at $\theta = 60°$ when $I_2 = 0$. The range of applicability of the dielectric dipole limit for large but finite values of ϵ was explored in detail earlier.[4]

4. SMALL-PARTICLE LIMIT FOR $m = 1$

We now consider the special case that $m = \sqrt{\mu\epsilon} = 1$; i.e., $\epsilon = 1/\mu$. There are two interesting facets, namely, the conditions for complete polarization and the forward-scatter-to-backscatter symmetry. It will be convenient, at the outset, to express the scattered radiant intensities in terms of only one of the two parameters (ϵ, μ), and we have selected μ:

$$I_1 = (\lambda^2/4\pi^2r^2)(\mu - 1)^2\left[\frac{\cos\theta}{\mu + 2} - \frac{1}{2\mu + 1}\right]^2 \sin^2\phi, \quad (20)$$

$$I_2 = (\lambda^2/4\pi^2r^2)(\mu - 1)^2\left[\frac{1}{\mu + 2} - \frac{\cos\theta}{2\mu + 1}\right]^2 \cos^2\phi. \quad (21)$$

The conditions for complete polarization now become, for $I_1 = 0$,

$$\cos\theta_1 = (\mu + 2)/(2\mu + 1), \quad (22)$$

and, for $I_2 = 0$,

$$\cos\theta_2 = (2\mu + 1)/(\mu + 2). \quad (23)$$

These conditions of course are special cases of Eqs. (17) and (18) under the constraint that $\epsilon = 1/\mu$. The ranges of scattering angles for which there is complete polarization are shown in Table 1 for all values of μ. Of course, real materials would be limited to a small part of the total range.

Table 1. Ranges of Magnetic Permeability μ and Scattering Angle $\theta_{1,2}$ for Which I_1 or $I_2 = 0$[a]

Permeability	Scattering Angle
$1 \leq \mu \leq \infty$	$0° \leq \theta_1 \leq 60°$
$0 \leq \mu \leq 1$	$60° \geq \theta_2 \geq 0°$
$-0.5 \leq \mu \leq 0$	$90° \geq \theta_2 \geq 60°$
$-1 \leq \mu \leq -0.5$	$0° \leq \theta_2 \leq 90°$
$-\infty \leq \mu \leq -1$	$60° \geq \theta_1 \geq 0°$

[a] $(m = 1; \mu = 1/\epsilon)$.

Another interesting aspect of the condition $m = 1$ is that this leads, for all positive values of μ, to extremely strong preferential backscattering. It follows from either Eq. (22) or Eq. (23) that

$$I_1(0°)/I_2(180°) = 1/9(\mu - 1/\mu + 1)^2. \tag{24}$$

For large values of μ (with correspondingly small values of ϵ) the forward-to-back asymmetry is 1/9, which is precisely the result obtained for nonmagnetic spheres with $\epsilon \to \infty$. However, as μ becomes smaller the asymmetry increases without limit as μ approaches unity. Of course, the scattering becomes small in that limit $(m = 1, \mu \to 1, \epsilon \to 1)$. Yet, for values as appreciable as $\mu = 1.2$, $I_1(0°)/I_2(180°)$ is only 9.2×10^{-3}.

This preferential backscatter may lead to multiple-scattering effects that are different from those encountered for nonmagnetic media for which the scattering either is symmetric about $\theta = 90°$ or else is more intense in the forward direction. For the simplest approach the ratio of scattering into the forward hemisphere to that into the back hemisphere is a parameter of interest. This is given as follows:

$$R_{fb} = \frac{\int_0^{\pi/2} (I_1 + I_2)\sin\theta d\theta}{\int_{\pi/2}^{\pi} (I_1 + I_2)\sin\theta d\theta}, \tag{25}$$

which, on insertion of Eq. (20) and Eq. (21) and then carrying out of the indicated operations, leads to

$$R_{fb} = \frac{4\mu^2 + \mu + 4}{16\mu^2 + 31\mu + 16}. \tag{26}$$

As an example, the ratio for $\mu = 1.2$ is 0.14. Unlike $I_1(0°)/I_1(180°)$, the integrated ratio R_{fb} varies over a much narrower range, from 1/7 for $\mu = 1$ to 1/4 for $\mu = \infty$.

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