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ELECTROMAGNETIC SCATTERING FROM TWO ECCENTRIC METAMATERIAL CYLINDERS WITH FREQUENCY-DEPENDENT PERMITTIVITIES DIFFERING SLIGHTLY EACH OTHER

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Abstract—The interaction between a metamaterial cylindrical structure and an incident plane wave is investigated. The structure is comprised of two cylinders, one embedded into the other, whose effective characteristics vary with the operating frequency following similar laws. Such a model can be used to describe periodic structures, constituting metamaterials, with slightly different features. The well-known eigenfunction expansions are adopted, while the boundary conditions are manipulated with help of the translation theorem for cylindrical coordinates. A first-order perturbation solution is obtained leading to simple and computationally efficient formulas. The fluctuations of near-field and far-field responses with respect to the position of the internal cylinder, the permittivities and the frequency are observed and discussed.

1. INTRODUCTION

Cylindrical structures, infinite along the axial dimension, are examined very commonly in terms of electromagnetic scattering. In [1], the near-field interaction between a magnetic strip and an anisotropic cylinder is thoroughly described. Also in [2], the general case of surface currents on a metallic cylinder, coated by an inhomogeneous cladding, is investigated and analytic formulas for the radiation field are derived. Finally in [3], the closed-form solution to the scattering of a small conducting circular cylinder, located inside a dielectric slab, is presented.

Recently, there is an upsurge of interest for the materials with negative permittivities and/or permeabilities [4] and implicitly this trend has affected the studies concerning the wave scattering by cylinders. In [5], the fields in a multilayered cylinder filled with a double negative medium and a double positive medium are expressed with the help of eigenfunction expansion method. Furthermore in [6], Shooshtari and Sebak present a formulation for multiple scattering by metamaterial cylinders and produce the near-field patterns and the scattering cross sections of these electromagnetic structures.

The negative intrinsic parameters of a metamaterial are usually taken to vary with the oscillating frequency. In [7], the authors study a scatterer fabricated by a material with frequency-dependent effective permittivity and permeability that can both be negative in a certain frequency region. In [8], a model for the variation of the metamaterial's parameters with respect to the operating frequency is adopted and the resonant and antiresonant behavior of them is observed.

In this work we combine the last two topics, namely we analyze the plane wave scattering by two metamaterial cylinders, one embedded into the other, possessing frequency-dependent intrinsic parameters. A similar study but for dielectrics is contained in [9] where the translation theorem is extensively exploited. In addition, the case of a metallic internal cylinder, eccentrically coated by metamaterial, has already been investigated [10]. We choose the material characteristics of two cylinders not to be very different each other and thus we obtain a linearly approximate solution, as in [11], which is fairly simple. The proposed configuration model can be useful in real-world applications as the features of periodic structures used for the fabrication of planar metamaterials (e.g., L-C circuits) are not necessary to remain invariant across the construction plane [12].

2. PROBLEM FORMULATION

Suppose an infinite circular cylinder with radius h_1 constructed from a metamaterial with relative permittivity ϵ_1 (area 1). A second smaller cylinder with radius h_2 and relative dielectric constant ϵ_2 (area 2) is embedded into the first one at an arbitrary position. The materials of the cylinders are magnetically inert and with similar electrical properties, namely $\epsilon_1 \cong \epsilon_2$. Two cylindrical coordinate systems are defined: (ρ_1, ϕ_1, z) with its origin O_1 at the center of the first cylinder and (ρ_2, ϕ_2, z) with its origin O_2 at the center of the second one. The polar position of the point O_2 with respect to the first coordinate system is written as: $(\rho_1, \phi_1) = (\rho, \phi)$ with $\rho = \sqrt{x^2 + y^2}$ and $\phi = \arctan(x, y)$. The equivalent cartesian systems (x_1, y_1, z) and (x_2, y_2, z) can be used interchangeably. The physical configuration of the device is depicted in Fig. 1 from which it is understood that

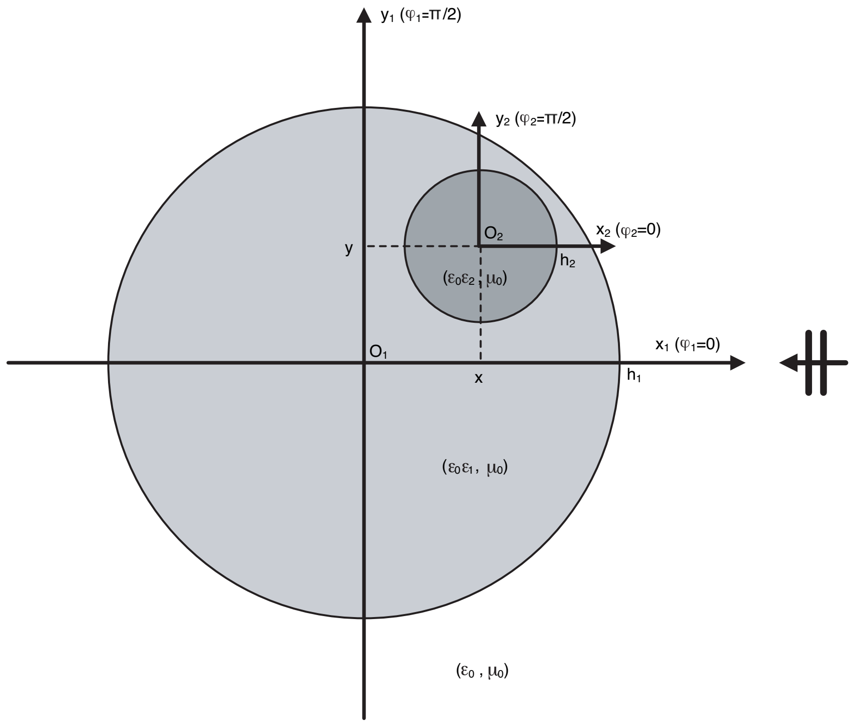

Figure 1. The physical configuration of the device. Two metamaterial cylinders, one embedded into the other, are excited by a plane wave.

$x^2 + y^2 + h_2^2 < h_1^2$ so that the second cylinder stays into the volume of the first one. The whole structure is placed into vacuum (area 0) and is excited by a unitary, z polarized plane wave traveling towards the negative x semiaxis $E_{z0,inc}(x_1, y_1) = \exp(-jk_0x_1)$. Both the shape and the excitation of the construction are invariant across the z axis. Accordingly, the only nonzero electric component is the axial one E_z and the problem is reduced to a scalar and two–dimensional one.

The wavenumbers of the metamaterials are denoted by: $k_i = k_0\sqrt{\epsilon_i}$ for $i = 1, 2$ where k_0 is the corresponding one of the free space. The permittivities ϵ_i are negative and frequency-dependent according to the following law, common in many applications [7]:

$$\epsilon_i = \epsilon_i(\omega) = \frac{\omega^2 - \omega_{Pi}^2}{\omega^2 - \omega_{Ri}^2} \tag{1}$$

It should be noted that $\omega_{Ri} < \omega < \omega_{Pi}$ for $i = 1, 2$ where ω is the operating circular frequency and thus the suppressed harmonic time dependence is taken of the form: $\exp(-j\omega t)$. The parameters ω_{Ri} and ω_{Pi} are called resonant and plasma frequencies respectively. The square roots of the permittivities possess a positive imaginary part [13] and consequently the purely imaginary wavenumbers are computed with $\Im[k_i] > 0$ for $i = 1, 2$. Our purpose is to determine the scattering field produced by the cylindrical formation on its external surface $\rho_1 = h_1$ and in the far region $\rho_1 \to +\infty$.

3. BOUNDARY CONDITIONS

We express the single electric component of the areas 1 and 2 in terms of the parameters of the second coordinate system.

$$E_{z1}(\rho_2, \phi_2) = \sum_{n=-\infty}^{+\infty} \left[A_1(n) J_n(k_1\rho_2) + B_1(n) H_n(k_1\rho_2)\right] e^{jn\phi_2} \quad (2)$$

$$E_{z2}(\rho_2, \phi_2) = \sum_{n=-\infty}^{+\infty} A_2(n) J_n(k_2\rho_2) e^{jn\phi_2} \quad (3)$$

where J_n and H_n are the Bessel and first–type Hankel functions of order n respectively. The incident and scattering quantities inside area 0 are introduced as dependent on (ρ_1, ϕ_1), possessing the following forms:

$$E_{z0,inc}(\rho_1, \phi_1) = \sum_{m=-\infty}^{+\infty} (-j)^m J_m(k_0\rho_1) e^{jm\phi_1} \quad (4)$$

$$E_{z0,scat}(\rho_1, \phi_1) = \sum_{m=-\infty}^{+\infty} \beta_0(m) H_m(k_0\rho_1) e^{jm\phi_1} \quad (5)$$

The series $A_1(n)$, $B_1(n)$, $A_2(n)$, $\beta_0(m)$ are complex determinable coefficients.

By imposing continuity of the electric field and its derivative with respect to ρ_2 (due to the magnetic inertia of the bodies) at $\rho_2 = h_2$, the unknown $A_2(n)$ is eliminated and a proportional relation between the other two coefficients $A_1(n)$, $B_1(n)$ is derived:

$$B_1(n) = A_1(n) \frac{k_1 J'_n(k_1h_2) J_n(k_2h_2) - k_2 J'_n(k_2h_2) J_n(k_1h_2)}{k_2 J'_n(k_2h_2) H_n(k_1h_2) - k_1 H'_n(k_1h_2) J_n(k_2h_2)} \quad (6)$$

To manipulate the boundary conditions at $\rho_1 = h_1$, an alternative expansion for the electric field into area 1 shall be adopted:

$$E_{z1}(\rho_1, \phi_1) = \sum_{m=-\infty}^{+\infty} \left[\alpha_1(m) J_m(k_1\rho_1) + \beta_1(m) H_m(k_1\rho_1)\right] e^{jm\phi_1} \quad (7)$$

The standard textbook of Stratton [14, 372–374] provides a detailed derivation of the addition theorem for circularly cylindrical waves. Through integral representations of Bessel functions and a plane wave expansion, important relations expressing cylindrical eigenfunctions of a coordinate system in terms of the corresponding ones of another system with translated axis are produced. In our case the formulas are specialized to give:

$$J/H_n(k_1\rho_2)e^{jn\phi_2} = \sum_{m=-\infty}^{+\infty} J_{m-n}(k_1\rho)e^{-j(m-n)\phi} J/H_m(k_1\rho_1)e^{jm\phi_1} \quad (8)$$

The latter equation is valid only if $\rho_1 > \rho|\cos(\phi_1 - \phi)|$. As we are interested for the enforcement of the conditions on the external surface, this demand is complied.

If one substitutes (8) in (2), one receives the explicit forms for the constants $\alpha_1(m)$, $\beta_1(m)$:

$$\alpha/\beta_1(m) = \sum_{n=-\infty}^{+\infty} A/B_1(n) J_{m-n}(k_1\rho)e^{-j(m-n)\phi} \quad (9)$$

In case the aforementioned translation formulas are written for the first cylindrical coordinate system, the opposite relations are found via a symmetric procedure:

$$A/B_1(n) = \sum_{m=-\infty}^{+\infty} \alpha/\beta_1(m) J_{m-n}(k_1\rho)e^{j(m-n)\phi} \quad (10)$$

The demands for continuity of the electric field and its normal derivative across the bound $\rho_1 = h_1$ and the elimination of the parameters $\beta_0(m)$, yield to the following expression:

$$\alpha_1(m) = P(m, k_0, k_1) + Q(m, k_0, k_1)\beta_1(m) \quad (11)$$

where:

$$P(m, k_0, k_1) = \frac{2}{j\pi h_1} \frac{(-j)^m}{k_1 J'_m(k_1h_1)H_m(k_0h_1) - k_0 H'_m(k_0h_1)J_m(k_1h_1)} \quad (12)$$

$$Q(m, k_0, k_1) = \frac{k_0 H'_m(k_0h_1)H_m(k_1h_1) - k_1 H'_m(k_1h_1)H_m(k_0h_1)}{k_1 J'_m(k_1h_1)H_m(k_0h_1) - k_0 H'_m(k_0h_1)J_m(k_1h_1)} \quad (13)$$

The coefficients of the scattering field into vacuum area, $\beta_0(m)$, are connected with $\alpha_1(m)$ and $\beta_1(m)$ through the relation:

$$\beta_0(m) = \frac{\alpha_1(m)J_m(k_1h_1) + \beta_1(m)H_m(k_1h_1) - (-j)^m J_m(k_0h_1)}{H_m(k_0h_1)} \quad (14)$$

With successive substitutions of (9), (6) and (10) in (11), an infinite set of equations for $\alpha_1(m)$ is obtained:

$$\alpha_1(m) = P(m,k_0,k_1) + Q(m,k_0,k_1)e^{-jm\phi} \cdot \sum_{n=-\infty}^{+\infty} R(n,k_1,k_2) J_{m-n}(k_1\rho) \sum_{u=-\infty}^{+\infty} \alpha_1(u) J_{u-n}(k_1\rho) e^{ju\phi} \tag{15}$$

with integer m and:

$$R(n,k_1,k_2) = \frac{k_1 J_n'(k_1h_2) J_n(k_2h_2) - k_2 J_n'(k_2h_2) J_n(k_1h_2)}{k_2 J_n'(k_2h_2) H_n(k_1h_2) - k_1 H_n'(k_1h_2) J_n(k_2h_2)} \tag{16}$$

4. LINEARIZED SOLUTION

It is already mentioned that the dielectric permittivities of the materials in areas 1 and 2 are chosen close each other. This property is exploited in order to educe a simpler solution, by following the novel approach of Uzunoglu and Fikioris [11]. The only quantity involving the wavenumber k_2 in the basic concurrent expression (15) is the function $R(n,k_1,k_2)$ which vanishes for $k_2 = k_1$. That means that this parameter indicates the effect of the internal cylinder on the field developed in case all the volume of the first cylinder was filled by material of permittivity ϵ_1. Due to the closeness of the dielectric constants, it is permissible to make a linear approximation of the k_2-dependent quantity $R(n,k_1,k_2)$ in the neighborhood of the point $k_2 = k_1$:

$$R(n,k_1,k_2) \cong (k_2 - k_1) R'(n,k_1,k_1) \tag{17}$$

where the prime $\star'$ denotes the derivative of R with respect to k_2 and thus:

$$R'(n,k_1,k_1) = \frac{j\pi h_2}{2}\Big[k_1h_2\Big(J_n^2(k_1h_2) + J_{n+1}^2(k_1h_2)\Big) - 2nJ_n(k_1h_2)J_{n+1}(k_1h_2)\Big] \tag{18}$$

By substituting the approximate formula (17) in the concurrent expression (15) and by ignoring the second order terms multiplied with $(k_2-k_1)^2$, one obtains the following simplified form for $\alpha_1(m)$ with integer m:

$$\alpha_1(m) = P(m,k_0,k_1) + (k_2-k_1)Q(m,k_0,k_1)e^{-jm\phi} \cdot \sum_{n=-\infty}^{+\infty} R'(n,k_1,k_1) J_{m-n}(k_1\rho) \sum_{u=-\infty}^{+\infty} P(u,k_0,k_1) J_{u-n}(k_1\rho) e^{ju\phi} \tag{19}$$

The general term of the double series (19) denoted as: $\sum_{n=-\infty}^{+\infty}\sum_{u=-\infty}^{+\infty} S(m,n,u)$ should be asymptotically evaluated for $n,u \to \pm\infty$. With the help of well–known approximate expansions of Bessel functions for large orders [15], the undermentioned formulas are derived:

$$|S(m,n,u)| = O\left[|n|^{m+u-4}\left(\frac{e^2k_1^2h_2\rho}{4n^2}\right)^{2|n|}\right], \quad n \to \pm\infty \tag{20}$$

$$|S(m,n,u)| = O\left[|u|^{n-1/2}\left(\frac{ek_0\rho}{2|u|}\right)^{|u|}\right], \quad u \to \pm\infty \tag{21}$$

It is apparent that the double infinite sum is rapidly converging and therefore the parameters $\alpha_1(m)$ for integer m can be easily calculated. The same happens for the complex coefficients $\beta_0(m)$ through the combination of (11), (14). In this way the scattering field on the external surface of the cylindrical structure $\rho_1 = h_1$ is computable. As far as the scattering cross section $\sigma(\phi_1)$ of the device is concerned, the following expression is utilized:

$$\sigma(\phi_1) = \frac{4}{k_0}\sum_{m=-\infty}^{+\infty}\beta_0(m)(-j)^m e^{jm\phi_1} \tag{22}$$

5. NUMERICAL EXAMPLES

We are interested in studying the variation of the electric field on the surface of the device at the direction $\phi = \pi$, $E_{z0}(h_1, \pi)$ expressing the output response of the system to the plane wave excitation input. Also, the backscattering cross section of the investigated structure $\sigma(0)$ is another interesting quantity. Before proceeding to the discussion of the numerical results, we should first clarify the intervals determining the magnitudes of the problem parameters. The plasma and resonant frequency of the first material remain constant throughout the numerical procedures: $\omega_{P1} = 45\pi\,\text{Grad/sec}$, $\omega_{R1} = 35\pi\,\text{Grad/sec}$, close to values dictated by [8]. The corresponding frequencies of the second material are chosen not very different from the first one's $0.9\omega_{P1} < \omega_{P2} < 1.1\omega_{P1}$, $0.9\omega_{R1} < \omega_{R2} < 1.1\omega_{R1}$ so that the linear approximation of (17) is permissible. A typical value for the external radius of the device could be $h_1 = 10\,\text{cm}$ similar to the dimension of a related structure depicted in Fig. 15 of [16]. The radius of the second cylinder h_2 is taken smaller but comparable with

h_1. Needless to say that the truncation parameters in (19), (22) has been selected so as the corresponding series are convergent.

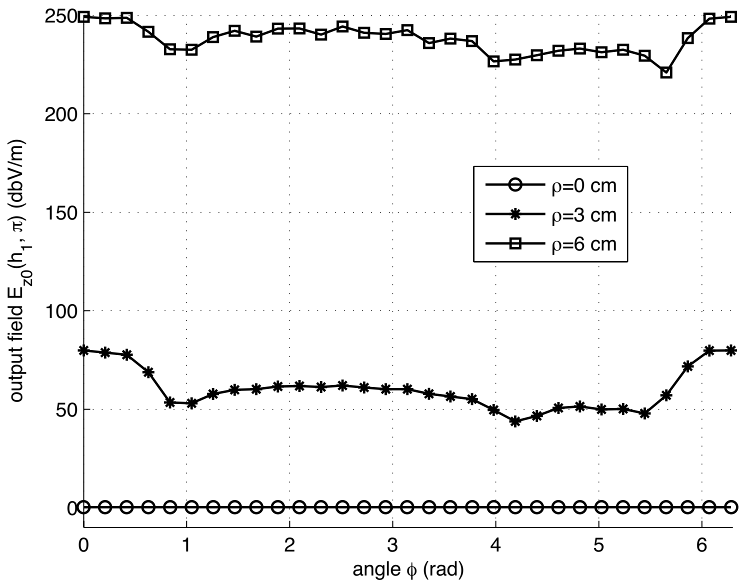

Figure 2. The output field on the surface of the structure as function of the azimuthal position of the inhomogeneity for various radial positions. Plot parameters: $\omega = 38\pi$ Grad/sec, $\omega_{R1} = 35\pi$ Grad/sec, $\omega_{P1} = 45\pi$ Grad/sec, $\omega_{R2} = 0.95\omega_{R1}$, $\omega_{P2} = 1.05\omega_{P1}$, $h_1 = 10$ cm, $h_2 = 2$ cm.

In Fig. 2 we show the output field $E_{z0}(h_1, \pi)$ as function of the azimuthal position ϕ of the smaller cylinder for various radial positions ρ of it. One can observe more rapid oscillations with increasing ρ which is natural as the same azimuthal change in ϕ modifies to a greater extent the shape of the device. In addition, a larger ρ leads to a much more significant output signal measured on the negative x semi axis which means that the position of the second cylinder can have a reinforcing effect. It is also interesting that substantial values are noticed for $\phi \cong 0$, which underlines the amplifying influence for the system response played by the cylindrical inhomogeneity.

In Fig. 3 the backscattering cross section $\sigma(0)$ is represented in a contour plot with respect to the characteristic frequencies of the second cylinder. It is remarkable that the far–field signature of the scatterer is negatively related to both ω_{R2}, ω_{P2}, namely the recorded quantity gets reinforced when the resonant and plasma frequencies of

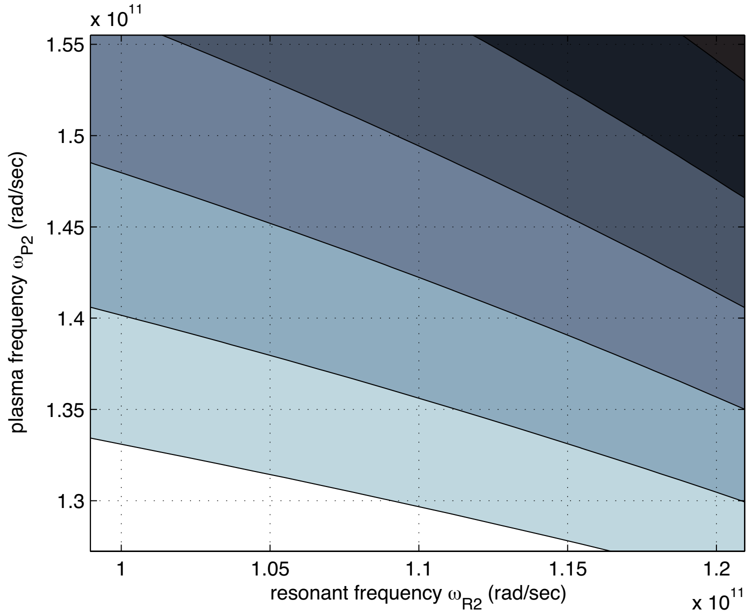

Figure 3. Contour plot of the backscattering cross section with respect to the characteristic frequencies of the second cylinder. Plot parameters: $\omega = 40\pi\,\mathrm{Grad/sec}$, $\omega_{R1} = 30\pi\,\mathrm{Grad/sec}$, $\omega_{P1} = 50\pi\,\mathrm{Grad/sec}$, $h_1 = 10\,\mathrm{cm}$, $h_2 = 2\,\mathrm{cm}$, $\rho = 3\,\mathrm{cm}$, $\phi = \pi$.

the second material are smaller than the corresponding ones of the first material. Furthermore, the slope of the levels becomes steeper when the characteristic frequencies possess larger values, as a result the changes in plasma and resonant frequency become less and more effective respectively.

In Fig. 4 we present the variation of $\sigma(0)$ with respect to the operating frequency ω for several radii of the second cylinder h_2. With increasing frequency, the radiated field of the structure in the far region diminishes as the corresponding permittivities of (1) do so. With smaller values of ω (close to resonant frequencies), the differences between the curves are larger, while all of them have similar behavior for $\omega \to \omega_{P1}, \omega_{P2}$. When the operating frequency is kept fixed, the response is proportionate to the size of the second cylinder, a fact that assists the operation of the device.

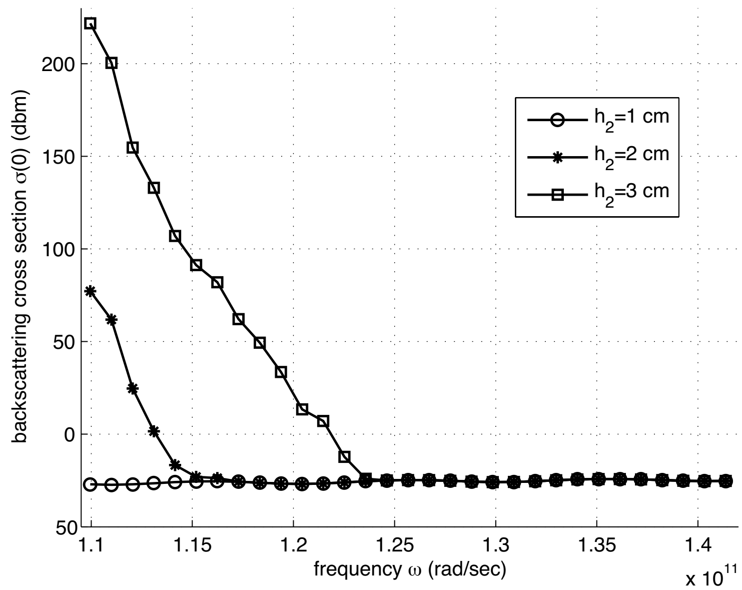

Figure 4. The backscattering cross section as function of the operating frequencies for various inhomogeneity radii. Plot parameters: $\omega_{R1} = 30\pi\,\mathrm{Grad/sec}$, $\omega_{P1} = 50\pi\,\mathrm{Grad/sec}$, $\omega_{R2} = 0.95\omega_{R1}$, $\omega_{P2} = 1.05\omega_{P1}$, $h_1 = 10\,\mathrm{cm}$, $\rho = 3\,\mathrm{cm}$, $\phi = \pi$.

6. CONCLUSION

A metamaterial cylinder which is coated eccentrically by another cylinder with similar characteristic frequencies, scatters an incident plane wave. Such a model can have practical applications in planar metamaterial structures, a region of which possesses slightly different features. The solution based on the translation theorem is simplified via a perturbation procedure. The output surficial field and the backscattering cross section are represented as functions of the problem's parameters and several conclusions are drawn.

Similar techniques could be applied to treat the corresponding waveguiding problem of a fiber with an axial cylindrical inhomogeneity which would affect the propagation along the dielectric waveguide. In addition, the internal cylinder could have multiple layers without changing the method of manipulating the boundary conditions on the outer surface. Finally, the linear approximation technique could be applied for an external cylinder inside an infinite coat with similar characteristics to study its effect on the fields of the internal vacuum cavity.

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