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Electromagnetic Steklov eigenvalues: existence and distribution in the self-adjoint case

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Abstract

In previous works, it was suggested to use Steklov eigenvalues for Maxwell equations as target signature for nondestructive testing, and it was recognized that this eigenvalue problem cannot be reformulated as a standard eigenvalue problem for a compact operator. Consequently, a modified eigenvalue problem with the desired properties was proposed. We report that apart for a countable set of particular frequencies, the spectrum of the original self-adjoint eigenvalue problem consists of three disjoint parts: The essential spectrum consisting of the origin, an infinite sequence of positive eigenvalues which accumulate only at infinity and an infinite sequence of negative eigenvalues which accumulate only at zero. The analysis is based on a suitable topological decomposition, a representation of the operator as block operator and Schur-factorizations. For each Schur-complement, the existence of an infinite sequence of eigenvalues is proven via an intermediate value technique. The modified eigenvalue problem arises as limit of one Schur-complement.

Keywords: Steklov eigenvalues, Spectral analysis, Nondestructive testing

Mathematics Subject Classification: 35J25, 35R30, 35P99

1 Introduction

An important topic in many areas of science and engineering is to determine by means of noninvasive methods if an object is subject to material defects. Many such nondestructive evaluation methods test the object with incident waves, observe the resulting scattering effect and deduce information about the medium from the resulting data. A popular approach is to use eigenvalues as target signatures to characterize the material [2–4,8–15,17]. Early methods use resonances/scattering poles as target signatures for which a fruitful theory exists [20]. However, the success of resonances/scattering poles as target signatures for electromagnetic interrogation was poor [2,13]. An alternative is to use transmission eigenvalues instead of scattering poles, and we refer to Cakoni et al. [8] for a detailed presentation of this subject. The first such methods chose the frequency as eigenvalue parameter. However, this has the drawbacks that multi-frequency data is necessary and only real eigenvalues can be measured (which is of limited use for absorbing media). More recent methods choose the eigenvalue parameter to be artificial to overcome

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these issues [2,4,10,12,16,17]. There is a lot of freedom in the construction of such methods, and a number of different versions have been proposed. Another improvement is the introduction of an additional sensitivity parameter as e.g. in [14,16,17] to tune the dependence of the eigenvalues with respect to changes in the material parameters. All of these emerging eigenvalue problems can roughly be classified in two types: transmission eigenvalue problems and Steklov eigenvalue problems. Among the important questions on these eigenvalue problems are:

- characterization of the essential spectrum (i.e. eigenvalue parameters for which the operator is not Fredholm),
- characterization of the Fredholm set (i.e. eigenvalue parameters for which the operator is Fredholm) and non-empty resolvent sets (in each connected subset),
- existence of "enough" eigenvalues (as minimal requirement to be a meaningful target signature), e.g. infinitely many,
- continuity of the eigenvalues with respect to changes in the material parameters,
- further qualitative properties of the eigenvalue distribution as e.g. eigenvalue free zones, accumulation points and estimates on the location of certain characteristic (e.g. the smallest) eigenvalues,
- reliable computational methods.

A classical approach to obtain such analytical results is to transform the eigenvalue problem first to a stencil $K - \lambda I$ with a compact operator K. While this is a very convenient technique, it is not always applicable. Indeed, this is not possible if multiple accumulation points of the spectrum exist, which was observed in [12] for electromagnetic Steklov eigenvalues. Hence, the authors of Camaño et al. [12] introduced a modification, which led to an alternative problem, which admits the desired properties. In [22], the Fredholmness and the discreteness of the spectrum of the original electromagnetic Steklov problem were reported by means of the T-coercivity technique, and in [21] the existence and stability of eigenvalues of the modified electromagnetic Steklov problem in conductive media were proven.

In this article, we consider the original electromagnetic Steklov eigenvalue problem in the self-adjoint case. We report a complete description of the spectrum (see Proposition 7): The spectrum consists of three disjoint parts: The essential spectrum consisting of the point zero, an infinite sequence of positive eigenvalues that accumulate only at infinity and an infinite sequence of negative eigenvalues that accumulate only at zero. The analysis is based on a representation of the operator as block operator. For small/big enough eigenvalue parameter, the Schur-complements with respect to different components can be built. For each Schur-complement, the existence of an infinite sequence of eigenvalues is proved via a fixed point/intermediate value technique as e.g. used in [11]. Roughly speaking, the original electromagnetic Steklov problem is a coupled system of an eigenvalue problem for a compact operator and an eigenvalue problem for the inverse of a compact operator. As a side result, we also analyze the spectrum of the modified electromagnetic Steklov eigenvalue problem, see Sect. 7. We report that the spectrum of the modified eigenvalue problem consists of an infinite sequence of eigenvalues, which accumulate only at $+\infty$.[1] Our analysis further reveals that the modified eigenvalue problem arises as asymptotic limit for $\lambda \to +\infty$ of one of the Schur-complements. This clarifies the relation between the original and the modified problem.

As the original problem contains two kinds of eigenvalue sequences, it carries more information than the modified problem and hence seems to be better suited for inverse applications. In practice, only a few eigenvalues can be identified and thus an increase of observable target signatures is of great value. For example, significant targets are the smallest negative eigenvalue and the smallest positive eigenvalue.

At last, we address the asymptotic limit for $\lambda \to 0$ of the respective Schur-complement. This eigenvalue problem is of a similar type as the modified Steklov problems considered in [14]. It has three interesting properties: It is independent of the permeability, the eigenvalues scale with the power minus two of the frequency and the eigenvalues suffice a min–max characterization.

The remainder of this article is organized as follows: In Sect. 2, we recall some results from [12] and explain the origin of the Steklov eigenvalue problem. In Sect. 3, we set our notation and formulate our assumptions on the domain and the material parameters. We also recall some classical regularity, embedding and decomposition results, which will be essential for our analysis and adapt them to our setting. In Sect. 4, we introduce the considered electromagnetic Steklov eigenvalue problem and define the associated holomorphic operator function $A_X(\cdot)$. We establish in Theorem 2 that the spectrum of $A_X(\cdot)$ is real and that $A_X(\lambda)$ is Fredholm if and only if $\lambda \neq 0$. In Sect. 5, we analyze the spectrum in a neighborhood of zero. We report in Theorem 3 that there exists $c_0 > 0$ so that $\sigma\big(A_X(\cdot)\big) \cap (0, c_0) = \emptyset$. We establish in Theorem 4 the existence of an infinite sequence of negative eigenvalues which accumulate at zero. In Sect. 6, we analyze the spectrum in a neighborhood of infinity. We report in Theorem 5 that there exists $c_\infty > 0$ so that $\sigma\big(A_X(\cdot)\big) \cap (-\infty, -c_\infty) = \emptyset$. We establish in Theorem 6 the existence of an infinite sequence of positive eigenvalues, which accumulate at $+\infty$. In Sect. 7, we collect our results in Proposition 7 and comment on the connection between the original and the modified electromagnetic Steklov eigenvalue problems.

2 Inverse scattering

In this section, we recall the discussion from Camaño et al. [12] to explain the relation between the Steklov eigenvalue problem and nondestructive inverse scattering methods. Let $\mathbb{S}^2 = \{x \in \mathbb{R}^3 \colon |x| = 1\}$. We consider a plane wave

$$u^i = \frac{i}{\omega} \operatorname{curl} \operatorname{curl} p e^{-i\omega x \cdot p} \tag{1}$$

with direction of propagation $d \in \mathbb{S}^2$, polarization vector $p \in \mathbb{R}^3 \setminus \{0\}$ and frequency $\omega \in \mathbb{R} \setminus \{0\}$. Let $\epsilon \in L^\infty(\mathbb{R}^3)$ be the relative permittivity such that $\inf_{x \in \mathbb{R}^3} \Re(\epsilon) > 0$ and $\Im(\epsilon) \geq 0$. Let $D \subset \mathbb{R}^3$ be a bounded Lipschitz domain such that $\epsilon = 1$ on $\mathbb{R}^3 \setminus D$. We consider the forward scattering problem to find u such that

[1] This result has already been reported in [12, Theorem 3.6] However, the proof thereof requires $\dim(\ker \mathbf{T})^\perp = \infty$ which the authors don't elaborate on.

$$
\begin{aligned}
&\operatorname{curl}\operatorname{curl} u-\omega^2\epsilon u=0 \quad \text{in} \quad \mathbb{R}^3,\\
&u=u^i+u^s \quad \text{in} \quad \mathbb{R}^3,\\
&\lim_{r\to\infty}(\operatorname{curl} u^s\times x-i\omega r u^s)=0
\end{aligned} \tag{2}
$$

The scattered field u^s has the following asymptotic expansion

$$
u^s(x)=\frac{e^{i\omega r}}{r}u_\infty(\hat{x},d;p)+\mathcal{O}\left(\frac{1}{r^2}\right) \tag{3}
$$

whereby we call $u_\infty(\hat{x},d;p)$ the far-field pattern of u^s for measurement direction $\hat{x}\in\mathbb{S}^2$, incident direction d and polarization p. Let $\mathcal{F}\colon L^2_t(\mathbb{S}^2)\to L^2_t(\mathbb{S}^2)$ be the far-field operator defined by

$$
(\mathcal{F}g)(\hat{x}):=\int_{\mathbb{S}^2}u_\infty(\hat{x},d,g(d))ds_d. \tag{4}
$$

We consider the following inverse problem. Let either $B=D$ or $B\subset\mathbb{R}^3$ be a bounded Lipschitz domain with D in its interior. Given the far-field pattern for all $\hat{x}$, d, p we wish to compute approximations of Steklov eigenvalues (that we shall define shortly). To this end, we introduce the ϵ-independent auxiliary scattering problem to find u_λ such that

$$
\begin{aligned}
&\operatorname{curl}\operatorname{curl} u_\lambda-\omega^2 u_\lambda=0 \quad \text{in} \quad \mathbb{R}^3\setminus\overline{B},\\
&u_\lambda=u^i+u^s_\lambda \quad \text{in} \quad \mathbb{R}^3\setminus\overline{B},\\
&\nu\times\operatorname{curl} u_\lambda+\lambda\,\nu\times u_\lambda\times\nu=0 \quad \text{on} \quad \partial B,\\
&\lim_{r\to\infty}(\operatorname{curl} u^s_\lambda\times x-i\omega r u^s_\lambda)=0
\end{aligned} \tag{5}
$$

whereby λ is a real constant. Note the sign of λ herein is reversed compared to Camaño et al. [12]. The existence and uniqueness of solutions to (2) and (5) are established in the following manner. The problems are reformulated on a ball B_R by means of the capacity operator. The injectivity is shown by testing with $u'=u$, taking the imaginary part of the sesquilinear form and applying properties of the capacity operator. If the problem is Fredholm, then the bijectivity follows. The Fredholmness of the operators related to (2) and (5) for $\lambda\leq 0$ can be shown by standard techniques. The case $\lambda>0$ can be treated as in [22]. Let $u_{\lambda,\infty}$ be the far-field pattern of u^s_λ and let

$$
(\mathcal{F}_\lambda g)(\hat{x}):=\int_{\mathbb{S}^2}u_{\lambda,\infty}(\hat{x},d,g(d))ds_d. \tag{6}
$$

Consider a Herglotz wave function

$$
v_g(x):=i\omega\int_{\mathbb{S}^2}g(d)e^{-i\omega x\cdot d}ds_d \tag{7}
$$

with Herglotz kernel g. Then, the weighted far-field pattern

$$
w_\infty(\hat{x}):=\int_{\mathbb{S}^2}u_\infty(\hat{x},d,g(d))ds_d \tag{8}
$$

is the far-field pattern of the scattered part of the solution w of

$$\begin{aligned} &\operatorname{curl}\operatorname{curl} w - \omega^2 \epsilon w = 0 \quad \text{in} \quad \mathbb{R}^3, \\ &w = v_g + w^s, \\ &\lim_{r\to\infty} (\operatorname{curl} w^s \times x - i\omega r w^s) = 0. \end{aligned} \tag{9}$$

Analogously

$$w_{\lambda,\infty}(\hat{x}) := \int_{\mathbb{S}^2} u_{\lambda,\infty}(\hat{x}, d, g(d)) ds_d \tag{10}$$

is the far-field pattern of the scattered part of the solution w_λ of

$$\begin{aligned} &\operatorname{curl}\operatorname{curl} w_\lambda - \omega^2 w_\lambda = 0 \quad \text{in} \quad \mathbb{R}^3 \setminus \overline{B}, \\ &w_\lambda = v_g + w_\lambda^s, \\ &\nu \times \operatorname{curl} w_\lambda + \lambda\, \nu \times w_\lambda \times \nu = 0 \quad \text{on} \quad \partial B, \\ &\lim_{r\to\infty} (\operatorname{curl} w_\lambda^s \times x - i\omega r w_\lambda^s) = 0 \end{aligned} \tag{11}$$

If $(\mathcal{F} - \mathcal{F}_\lambda)g = 0$, then the far-field patterns agree $w_\infty = w_{\lambda,\infty}$. Then, by Rellich's Lemma $w = w_\lambda$ in $\mathbb{R}^3 \setminus D$. It follows that w is a nontrivial solution to

$$\begin{aligned} &\operatorname{curl}\operatorname{curl} - \omega^2 \epsilon w = 0 \quad \text{in} \quad B, \\ &\nu \times \operatorname{curl} w + \lambda\, \nu \times w \times \nu = 0 \quad \text{on} \quad \partial B. \end{aligned} \tag{12}$$

The problem to find $\lambda \in \mathbb{C}$ and nontrivial w such thas (12) holds is called the electromagnetic Steklov eigenvalue problem. It was shown in [12, Sect. 4] that indeed $\mathcal{F} - \mathcal{F}_\lambda$ is injective with dense range if and only if there does not exist a Steklov eigenfunction, which has a decomposition $w = v_g + w^s$. Moreover, Theorems 4.2 and 4.4 of [12] support the claim that Steklov eigenvalues can be detected from far-field measurements. Further, it can be seen as in [10, p. 1740] that if ϵ is perturbed by $\delta\epsilon$, then for an eigenpair (λ, w) the change in the eigenvalue $\delta\lambda$ is

$$\delta\lambda \approx \frac{\langle \delta\epsilon \operatorname{curl} w, \operatorname{curl} w\rangle_{\mathbf{L}^2(B)}}{\langle \nu \times w, \nu \times w\rangle_{\mathbf{L}_t^2(\partial B)}} \tag{13}$$

up to linear terms.

3 General setting

In this section, we set our notation and formulate assumptions on the domain and material parameters. We also recall necessary results from different literature and adapt them to our setting.

3.1 Functional analysis

Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ be generic Banach spaces. We denote the space of all bounded linear operators from X to Y as $L(X,Y)$ with operator norm $\|A\|_{L(X,Y)} := \sup_{u\in X\setminus\{0\}} \|Au\|_Y/\|u\|_X, A \in L(X,Y)$. In addition, we set $L(X) := L(X,X)$. For $A \in L(X,Y)$, we denote $A^* \in L(Y,X)$ its adjoint operator defined through $\langle u, A^*u'\rangle_X = \langle Au, u'\rangle_Y$ for all $u \in X, u' \in Y$. We call the space of compact operators as $K(X,Y) \subset L(X,Y)$ and $K(X) := K(X,X)$. We say that an operator $A \in L(X)$ is coercive, if $\inf_{u\in X\setminus\{0\}} |\langle Au, u\rangle_X|/\|u\|_X^2 > 0$. We say that $A \in L(X)$ is weakly coercive, if there exists $K \in K(X)$ so that $A+K$ is coercive. Let $\Lambda \subset \mathbb{C}$ be open and consider an operator function $A(\cdot)\colon \Lambda \to L(X)$. We call $A(\cdot)$ (weakly) coercive if $A(\lambda)$ is (weakly) coercive for all $\lambda \in \Lambda$. We denote the spectrum of $A(\cdot)$ as $\sigma\big(A(\cdot)\big) := \{\lambda \in \Lambda\colon A(\lambda) \text{ is not bijective}\}$ and the resolvent set as $\rho\big(A(\cdot)\big) := \Lambda \setminus \sigma\big(A(\cdot)\big)$. We denote $\sigma_{\mathrm{ess}}\big(A(\cdot)\big) := \{\lambda \in \Lambda\colon A(\lambda) \text{ is not Fredholm}\}$ the essential spectrum. For $A \in L(X)$, we set $\sigma(A) := \sigma\big(\cdot I - A\big)$, $\sigma_{\mathrm{ess}}(A) := \sigma_{\mathrm{ess}}\big(\cdot I - A\big)$ and $\rho(A) := \rho\big(\cdot I - A\big)$.

3.2 Lebesgue and Sobolev spaces

Let $\Omega \subset \mathbb{R}^3$ be a bounded path connected open Lipschitz domain and ν the outer unit normal vector at $\partial\Omega$. Let $C_0^\infty(\Omega)$ be the space of infinitely many times differentiable functions from Ω to $\mathbb{C}$ with compact (closure of the) support in Ω. We use standard notation for Lebesgue and Sobolev spaces $L^2(\Omega)$, $L^\infty(\Omega)$, $W^{1,\infty}(\Omega)$, $H^s(\Omega)$ defined on the domain Ω and $L^2(\partial\Omega)$, $H^s(\partial\Omega)$ defined on the boundary $\partial\Omega$. We recall the continuity of the trace operator $\operatorname{tr} \in L\big(H^s(\Omega), H^{s-1/2}(\partial\Omega)\big)$ for all $s > 1/2$. For a vector space X of scalar-valued functions, we denote its bold symbol as space of three-vector-valued functions $\mathbf{X} := X^3 = X \times X \times X$, e.g. $\mathbf{L}^2(\Omega)$, $\mathbf{H}^s(\Omega)$, $\mathbf{L}^2(\partial\Omega)$, $\mathbf{H}^s(\partial\Omega)$. For $\mathbf{L}^2(\partial\Omega)$ or a subspace, e.g. $\mathbf{H}^s(\partial\Omega)$, $s > 0$, the subscript t denotes the subspace of tangential fields. In particular, $\mathbf{L}_t^2(\partial\Omega) = \{u \in \mathbf{L}^2(\partial\Omega)\colon \nu\cdot u = 0\}$ and $\mathbf{H}_t^s(\partial\Omega) = \{u \in \mathbf{H}^s(\partial\Omega)\colon \nu\cdot u = 0\}$. Let further $H_0^1(\Omega)$ be the subspace of $H^1(\Omega)$ of all functions with vanishing Dirichlet trace, $H_*^1(\Omega)$ be the subspace of $H^1(\Omega)$ of all functions with vanishing mean, i.e. $\langle u, 1\rangle_{L^2(\Omega)} = 0$ and $H_*^1(\partial\Omega)$ be the subspace of $H^1(\partial\Omega)$ of all functions with vanishing mean $\langle u, 1\rangle_{L^2(\partial\Omega)} = 0$.

3.3 Additional function spaces

Denote $\partial_{x_i} u$ the partial derivative of a function u with respect to the variable x_i. Let

$$\begin{aligned}
&\nabla u := (\partial_{x_1} u, \partial_{x_2} u, \partial_{x_3} u)^\top,\\
&\operatorname{div}(u_1, u_2, u_3)^\top := \partial_{x_1} u_1 + \partial_{x_2} u_2 + \partial_{x_3} u_3,\\
&\operatorname{curl}(u_1, u_2, u_3)^\top := (\partial_{x_2} u_3 + -\partial_{x_3} u_2, \partial_{x_3} u_1 - \partial_{x_1} u_3, \partial_{x_1} u_2 - \partial_{x_2} u_1)^\top.
\end{aligned}$$

For $\epsilon \in \big(L^\infty(\Omega)\big)^{3x3}$ let $\operatorname{div}\epsilon\, u := \operatorname{div}(\epsilon u)$. For a bounded Lipschitz domain Ω, let ∇_∂, $\operatorname{div}_\partial$ and $\operatorname{curl}_\partial = \nu \times \nabla_\partial$ be the respective differential operators for functions defined on $\partial\Omega$. We recall that for $u \in \mathbf{L}^2(\Omega)$ with $\operatorname{curl} u \in \mathbf{L}^2(\Omega)$ the tangential trace $\operatorname{tr}_{\nu\times} u \in \mathbf{H}^{-1/2}(\operatorname{div}_\partial; \partial\Omega) := \{u \in \mathbf{H}^{-1/2}(\partial\Omega)\colon \operatorname{div}_\partial u \in H^{-1/2}(\partial\Omega)\}$, $\|u\|^2_{\mathbf{H}^{-1/2}(\operatorname{div}_\partial;\partial\Omega)} := \|u\|^2_{\mathbf{H}^{-1/2}(\partial\Omega)} + \|\operatorname{div}_\partial u\|^2_{H^{-1/2}(\partial\Omega)}$ is well defined and $\|\operatorname{tr}_{\nu\times} u\|^2_{\mathbf{H}^{-1/2}(\operatorname{div}_\partial;\partial\Omega)}$ is bounded by a constant times $\|u\|^2_{\mathbf{L}^2(\Omega)} + \|\operatorname{curl} u\|^2_{\mathbf{L}^2(\Omega)}$. Likewise for $u \in \mathbf{L}^2(\Omega)$ with $\operatorname{div} u \in L^2(\Omega)$ the normal trace $\operatorname{tr}_{\nu\cdot} u \in H^{-1/2}(\partial\Omega)$ is well defined and $\|\operatorname{tr}_{\nu\cdot} u\|^2_{H^{-1/2}(\partial\Omega)}$ is bounded by a constant times $\|u\|^2_{\mathbf{L}^2(\Omega)} + \|\operatorname{div} u\|^2_{L^2(\Omega)}$. Likewise for $u \in \mathbf{L}^2(\Omega)$ with $\operatorname{div}\epsilon\, u \in L^2(\Omega)$ the normal trace $\operatorname{tr}_{\nu\cdot\epsilon} u \in H^{-1/2}(\partial\Omega)$ is well defined and $\|\operatorname{tr}_{\nu\cdot\epsilon} u\|^2_{H^{-1/2}(\partial\Omega)}$ is bounded by a constant times $\|\epsilon u\|^2_{\mathbf{L}^2(\Omega)} + \|\operatorname{div}\epsilon\, u\|^2_{L^2(\Omega)}$. For $\mathrm{d} \in \{\operatorname{curl}, \operatorname{div}, \operatorname{div}\epsilon, \operatorname{tr}_{\nu\times}, \operatorname{tr}_{\nu\cdot}, \operatorname{tr}_{\nu\cdot\epsilon}\}$ let

$$L^2(\mathrm{d}) := \begin{cases} \mathbf{L}^2(\Omega), & \mathrm{d} = \operatorname{curl}, \\ L^2(\Omega), & \mathrm{d} = \operatorname{div}, \operatorname{div}\epsilon, \\ \mathbf{L}^2_t(\partial\Omega), & \mathrm{d} = \operatorname{tr}_{\nu\times}, \\ L^2(\partial\Omega), & \mathrm{d} = \operatorname{tr}_{\nu\cdot}, \operatorname{tr}_{\nu\cdot\epsilon} \end{cases}. \tag{14a}$$

Let

$$H(\mathrm{d};\Omega) := \{u \in \mathbf{L}^2(\Omega) \colon \mathrm{d}u \in L^2(\mathrm{d})\}, \tag{14b}$$

$$\langle u, u'\rangle_{H(\mathrm{d};\Omega)} := \langle u, u'\rangle_{\mathbf{L}^2(\Omega)} + \langle \mathrm{d}u, \mathrm{d}u'\rangle_{L^2(\mathrm{d})}, \tag{14c}$$

$$H(\mathrm{d}^0;\Omega) := \{u \in H(\mathrm{d};\Omega) \colon \mathrm{d}u = 0\}. \tag{14d}$$

Also for $\mathrm{d}_1, \mathrm{d}_2, \mathrm{d}_3, \mathrm{d}_4 \in \{\operatorname{curl}, \operatorname{div}, \operatorname{div}\epsilon, \operatorname{tr}_{\nu\times}, \operatorname{tr}_{\nu\cdot}, \operatorname{tr}_{\nu\cdot\epsilon}, \operatorname{curl}^0, \operatorname{div}^0, \operatorname{div}\epsilon^0, \operatorname{tr}^0_{\nu\times}, \operatorname{tr}^0_{\nu\cdot}, \operatorname{tr}^0_{\nu\cdot\epsilon}\}$ let

$$\begin{aligned} &H(\mathrm{d}_1, \mathrm{d}_2;\Omega) := H(\mathrm{d}_1;\Omega) \cap H(\mathrm{d}_2;\Omega), \\ &\langle u, u'\rangle_{H(\mathrm{d}_1,\mathrm{d}_2;\Omega)} := \langle u, u'\rangle_{\mathbf{L}^2(\Omega)} + \langle \mathrm{d}_1 u, \mathrm{d}_1 u'\rangle_{L^2(\mathrm{d}_1)} + \langle \mathrm{d}_2 u, \mathrm{d}_2 u'\rangle_{L^2(\mathrm{d}_2)}, \end{aligned} \tag{14e}$$

and $H(\mathrm{d}_1, \mathrm{d}_2, \mathrm{d}_3;\Omega)$, $H(\mathrm{d}_1, \mathrm{d}_2, \mathrm{d}_3, \mathrm{d}_4;\Omega)$ be defined like-wise.

3.4 Assumption on the domain and material parameters

Assumption 3.1 (*Assumption on* ϵ) Let $\epsilon \in \left(L^\infty(\Omega)\right)^{3x3}$ be a real, symmetric matrix function so that there exists $c_\epsilon > 0$ with $c_\epsilon|\xi|^2 \leq \xi^H\epsilon(x)\xi$ for all $x \in \Omega$ and all $\xi \in \mathbb{C}^3$. We further assume that there exists a Lipschitz domain $\hat\Omega \subset \Omega$ so that the closure of $\hat\Omega$ is compact in Ω and $\epsilon|_{\Omega\setminus\hat\Omega}$ equals the identity matrix $\mathrm{I}_{3\times3} \in \mathbb{C}^{3\times3}$.

Let $\check\Omega \subset \Omega$ be a Lipschitz domain so that the closure of $\check\Omega$ is compact in Ω and the closure of $\hat\Omega \subset \check\Omega$ is compact in $\check\Omega$. Let χ be infinitely many times differentiable, so that $\chi|_{\Omega\setminus\check\Omega} = 1$ and $\chi|_{\hat\Omega} = 0$.

Assumption 3.2 (*Assumption on* μ) Let $\mu^{-1} \in \left(L^\infty(\Omega)\right)^{3x3}$ be a real, symmetric matrix function so that there exists $c_\mu > 0$ with $c_\mu|\xi|^2 \leq \xi^H\mu^{-1}(x)\xi$ for all $x \in \Omega$ and all $\xi \in \mathbb{C}^3$. We further assume that $\mu|_{\Omega\setminus\hat\Omega}$ equals the identity matrix $\mathrm{I}_{3\times3} \in \mathbb{C}^{3\times3}$.

Assumption 3.3 (*Assumption on* Ω) Let $\Omega \subset \mathbb{R}^3$ be a bounded path-connected Lipschitz domain so that there exists $\delta > 0$ and the following shift theorem holds on Ω: Let $f \in L^2(\Omega)$, $g \in H^{1/2}(\partial\Omega)$ with $\langle f, 1\rangle_{L^2(\Omega)} + \langle g, 1\rangle_{L^2(\partial\Omega)} = 0$ and $w \in H^1_*(\Omega)$ be the solution to

$$-\Delta w = f \quad \text{in } \Omega, \qquad n \cdot \nabla w = g \quad \text{at } \partial\Omega. \tag{15a}$$

Then, the linear map $(f, g) \mapsto w \colon L^2(\Omega) \times H^{1/2}(\partial\Omega) \to H^{3/2+\delta}(\Omega)$ is well defined and continuous.

The above assumption holds e.g. for smooth domains and Lipschitz polyhedral [19, Corollary 23.5].

Assumption 3.4 (*Assumption on* Ω, ϵ *and* μ^{-1}) Let ϵ, μ^{-1} and Ω be so that a unique continuation principle holds, i.e. if $u \in H(\operatorname{curl};\Omega)$ solves

$$\operatorname{curl} \mu^{-1} \operatorname{curl} u - \omega^2 \epsilon u = 0 \quad \text{in } \Omega, \tag{16a}$$

$$\operatorname{tr}_{\nu\times} u = 0 \quad \text{at } \partial\Omega, \tag{16b}$$

$$\operatorname{tr}_{\nu\times} \mu^{-1} \operatorname{curl} u = 0 \quad \text{at } \partial\Omega, \tag{16c}$$

then $u = 0$.

To our knowledge, the most general todays available result on the unique continuation principle for Maxwell's equations is the one of Ball et al. [5]. It essentially requires ϵ and μ^{-1} to be piece-wise $W^{1,\infty}$.

3.5 Trace regularities and compact embeddings

We recall a classical result from Costabel [18]:

$$\operatorname{tr}_{\nu\cdot} \in L\big(H(\operatorname{curl}, \operatorname{div}, \operatorname{tr}_{\nu\times};\Omega), L^2(\partial\Omega)\big), \tag{17a}$$

$$\operatorname{tr}_{\nu\times} \in L\big(H(\operatorname{curl}, \operatorname{div}, \operatorname{tr}_{\nu\cdot};\Omega), \mathbf{L}^2_t(\partial\Omega)\big), \tag{17b}$$

and

$$\begin{aligned}&\text{The embeddings from } H(\operatorname{curl}, \operatorname{div}, \operatorname{tr}_{\nu\cdot};\Omega) \text{ and } H(\operatorname{curl}, \operatorname{div}, \operatorname{tr}_{\nu\times};\Omega)\\ &\quad\text{to } \mathbf{H}^{1/2}(\Omega) \text{ are bounded.}\end{aligned} \tag{18}$$

We adapt the trace results of Costabel to our setting in the next lemmata.

Lemma 3.5 *Let* ϵ *suffice Assumption* 3.1*. Thence* $\operatorname{tr}_{\nu\cdot} \in L\big(H(\operatorname{div}\epsilon;\Omega), H^{-1/2}(\partial\Omega)\big)$, $\operatorname{tr}_{\nu\cdot} = \operatorname{tr}_{\nu\cdot\epsilon}$, $\operatorname{tr}_{\nu\cdot\epsilon} \in L\big(H(\operatorname{div};\Omega), H^{-1/2}(\partial\Omega)\big)$*, and* $\operatorname{tr}_{\nu\cdot\epsilon} = \operatorname{tr}_{\nu\cdot}$.

Proof If $u \in H(\operatorname{div}\epsilon;\Omega)$ then $\chi u \in H(\operatorname{div};\Omega)$. Since $\chi|_{\Omega\setminus\check{\Omega}} = \epsilon|_{\Omega\setminus\check{\Omega}} = 1$, it follows $\operatorname{tr}_{\nu\times} u = \operatorname{tr}_{\nu\times} \chi u = \operatorname{tr}_{\nu\times} \epsilon\chi u$. The reverse direction follows the same way. □

Lemma 3.6 *Let* ϵ *suffice Assumption* 3.1*. Thence,*

$$\operatorname{tr}_{\nu\cdot\epsilon} \in L\big(H(\operatorname{curl}, \operatorname{div}\epsilon, \operatorname{tr}_{\nu\times};\Omega), L^2(\partial\Omega)\big), \tag{19a}$$

$$\operatorname{tr}_{\nu\times} \in L\big(H(\operatorname{curl}, \operatorname{div}\epsilon, \operatorname{tr}_{\nu\cdot\epsilon};\Omega), \mathbf{L}^2_t(\partial\Omega)\big). \tag{19b}$$

Proof Apply (17) to χu and employ Lemma 3.5. □

We deduce the next lemma from Amrouche et al. [1].

Lemma 3.7 *Let* ϵ *suffice Assumption* 3.1 *and* Ω *suffice Assumption* 3.3*. Thence,* $\operatorname{tr}_{\nu\times} \in L\big(H(\operatorname{curl}, \operatorname{div}\epsilon, \operatorname{tr}^0_{\nu\cdot\epsilon};\Omega), \mathbf{H}^\delta_t(\partial\Omega)\big)$*. In particular* $\operatorname{tr}_{\nu\times} \in L\big(H(\operatorname{curl}, \operatorname{div}\epsilon, \operatorname{tr}^0_{\nu\cdot\epsilon};\Omega), \mathbf{L}^2_t(\partial\Omega)\big)$ *is compact.*

Proof Apply the proof of [1, Proposition 3.7] to χu and employ Assumption 3.3 to obtain $\chi u \in \mathbf{H}^{1/2+\delta}(\Omega)$. Employ $\operatorname{tr} \in L\big(\mathbf{H}^{1/2+\delta}(\Omega), \mathbf{H}^\delta(\partial\Omega)\big)$ and the compact embedding $\mathbf{H}^\delta_t(\partial\Omega) \to \mathbf{L}^2_t(\partial\Omega)$. □

For ϵ satisfying Assumption 3.1, we recall from Weber [29]:

$$\text{The embeddings from } H(\operatorname{curl}, \operatorname{div}\epsilon, \operatorname{tr}^0_{\nu\cdot\epsilon}; \Omega) \quad \text{and} \quad H(\operatorname{curl}, \operatorname{div}\epsilon, \operatorname{tr}^0_{\nu\times}; \Omega) \\ \text{to } \mathbf{L}^2(\Omega) \text{ are compact.} \tag{20}$$

Lemma 3.8 *Let ϵ suffice Assumption 3.1. Thence, the embedding $H(\operatorname{curl}, \operatorname{div}\epsilon, \operatorname{tr}_{\nu\times}; \Omega) \to \mathbf{L}^2(\Omega)$ is compact.*

Proof Let $E : H(\operatorname{curl}, \operatorname{div}\epsilon, \operatorname{tr}_{\nu\cdot\epsilon}; \Omega) \to \mathbf{L}^2(\Omega) : u \mapsto u$. Let $M(\alpha)$ be the multiplication operator with symbol α. We split the identity operator in two parts $I = M(\chi) + M(1-\chi)$. Thence, $EM(\chi)$ is compact due to (18) and $EM(1-\chi)$ is compact due to (20). Hence, $E = EM(\chi) + EM(1-\chi)$ is compact too. □

3.6 Helmholtz decomposition on the boundary

We recall from Buffa, Costabel and Sheen [7, Theorem 5.5]:

$$\mathbf{L}^2_t(\partial\Omega) = \nabla_\partial H^1(\partial\Omega) \oplus^\perp \operatorname{curl}_\partial H^1(\partial\Omega). \tag{21}$$

and denote the respective orthogonal projections by

$$P_{\nabla_\partial} : \mathbf{L}^2_t(\partial\Omega) \to \nabla_\partial H^1(\partial\Omega), \quad P_{\nabla_\partial^\top} : \mathbf{L}^2_t(\partial\Omega) \to \operatorname{curl}_\partial H^1(\partial\Omega). \tag{22}$$

Recall $\operatorname{div}_\partial \operatorname{tr}_{\nu\times} \in L\big(H(\operatorname{curl}; \Omega), H^{-1/2}(\partial\Omega)\big)$. So for $u \in H(\operatorname{curl}; \Omega)$ let z be the solution to find $z \in H^1_*(\partial\Omega)$ so that

$$\langle \nabla_\partial z, \nabla_\partial z' \rangle_{\mathbf{L}^2_t(\partial\Omega)} = -\langle \operatorname{div}_\partial \operatorname{tr}_{\nu\times} u, z' \rangle_{H^{-1}(\partial\Omega)\times H^1(\partial\Omega)} \tag{23}$$

for all $z' \in H^1_*(\partial\Omega)$ and set

$$Su := \nabla_\partial z. \tag{24}$$

From the construction of S, it follows $S \in L\big(H(\operatorname{curl}; \Omega), \mathbf{L}^2_t(\partial\Omega)\big)$ and further

$$Su = P_{\nabla_\partial} \operatorname{tr}_{\nu\times} u \quad \text{for each} \quad u \in H(\operatorname{curl}, \operatorname{tr}_{\nu\times}; \Omega). \tag{25}$$

4 The electromagnetic Steklov eigenvalue problem

Let $\omega > 0$ be fixed. For $\lambda \in \mathbb{C}$, let $A(\lambda) \in L\big(H(\operatorname{curl}, \operatorname{tr}_{\nu\times}; \Omega)\big)$ be defined through

$$\langle A(\lambda)u, u' \rangle_{H(\operatorname{curl},\operatorname{tr}_{\nu\times};\Omega)} := \langle \mu^{-1} \operatorname{curl} u, \operatorname{curl} u' \rangle_{\mathbf{L}^2(\Omega)} - \omega^2 \langle \epsilon u, u' \rangle_{\mathbf{L}^2(\Omega)} \\ - \lambda \langle \operatorname{tr}_{\nu\times} u, \operatorname{tr}_{\nu\times} u' \rangle_{\mathbf{L}^2_t(\partial\Omega)} \quad \text{for all } u, u' \in H(\operatorname{curl}, \operatorname{tr}_{\nu\times}; \Omega). \tag{26}$$

The electromagnetic Steklov eigenvalue problem, which we investigate in this note, is to

$$\text{find} \quad (\lambda, u) \in \mathbb{C} \times H(\operatorname{curl}, \operatorname{tr}_{\nu\times}; \Omega) \setminus \{0\} \quad \text{so that} \quad A(\lambda)u = 0. \tag{27}$$

We note that the sign of λ herein is reversed compared to [12]. Let

$$\langle u, u' \rangle_{\tilde{X}} := \langle \mu^{-1} \operatorname{curl} u, \operatorname{curl} u' \rangle_{\mathbf{L}^2(\Omega)} + \langle \epsilon u, u' \rangle_{\mathbf{L}^2(\Omega)} + \langle \operatorname{tr}_{\nu\times} u, \operatorname{tr}_{\nu\times} u' \rangle_{\mathbf{L}^2_t(\partial\Omega)} \tag{28}$$

for all $u, u' \in H(\operatorname{curl}, \operatorname{tr}_{\nu\times}; \Omega)$. It is straightforward to see that the norms induced by $\langle \cdot, \cdot \rangle_{\tilde{X}}$ and $\langle \cdot, \cdot \rangle_{H(\operatorname{curl}, \operatorname{tr}_{\nu\times}; \Omega)}$ are equivalent. To analyze the operator $A(\lambda)$, we introduce the following subspaces of $H(\operatorname{curl}, \operatorname{tr}_{\nu\times}; \Omega)$:

$$V := H(\operatorname{curl}, \operatorname{div}\epsilon^0, \operatorname{tr}_{\nu\times}, \operatorname{tr}^0_{\nu\cdot\epsilon}; \Omega), \tag{29a}$$

$$W_1 := H(\operatorname{curl}^0, \operatorname{div}\epsilon^0, \operatorname{tr}_{\nu\times}; \Omega) \cap W_2^{\perp_{\tilde{X}}}, \tag{29b}$$

$$W_2 := H(\operatorname{curl}^0, \operatorname{tr}^0_{\nu\times}; \Omega). \tag{29c}$$

We recall [26, Theorem 4.3 and Remark 4.4]:

$$K_N(\Omega) := \{\nabla u \colon u \in H^1(\Omega), \operatorname{div}\epsilon\, u = 0 \text{ in } \Omega, \\ \operatorname{tr} u \text{ is constant on each of the connected parts of } \partial\Omega\} \tag{30}$$

and $\dim K_N(\Omega) =$ number of connected parts of $\partial\Omega - 1 < \infty$. It holds

$$W_2 = \nabla H^1_0(\Omega) \oplus^{\perp_{\tilde{X}}} K_N(\Omega). \tag{31}$$

Thus,

$$W_1 = \{\nabla u \colon u \in H^1(\Omega), \quad \operatorname{div}\epsilon\, u = 0 \text{ in } \Omega, \quad \operatorname{tr}_{\nu\cdot\epsilon} \nabla u \in L^2(\partial\Omega), \\ \langle \operatorname{tr}_{\nu\cdot\epsilon} \nabla u, 1 \rangle_{L^2(\Gamma)} = 0 \quad \text{for each } \Gamma \text{ of the connected parts of } \partial\Omega\}. \tag{32}$$

We continue with a decomposition of $H(\operatorname{curl}, \operatorname{tr}_{\nu\times}; \Omega)$, which is similar but different to Halla [22, Theorem 3.1].

Theorem 1 *Let ϵ suffice Assumption 3.1 and μ suffice Assumption 3.2. Thence,*

$$H(\operatorname{curl}, \operatorname{tr}_{\nu\times}; \Omega) = (V \oplus W_1) \oplus^{\perp_{\tilde{X}}} W_2 \tag{33}$$

in the following sense. There exist projections $P_V, P_{W_1}, P_{W_2} \in L\big(H(\operatorname{curl}, \operatorname{tr}_{\nu\times}; \Omega)\big)$ with $\operatorname{ran} P_V = V$, $\operatorname{ran} P_{W_1} = W_1$, $\operatorname{ran} P_{W_2} = W_2$, $W_1, W_2 \subset \ker P_V$, $V, W_2 \subset \ker P_{W_1}$, $V, W_1 \subset \ker P_{W_2}$ *and $u = P_V u + P_{W_1} u + P_{W_2} u$ for each $u \in H(\operatorname{curl}, \operatorname{tr}_{\nu\times}; \Omega)$. Thus, the norm induced by*

$$\langle u, u' \rangle_X := \langle P_V u, P_V u' \rangle_{\tilde{X}} + \langle P_{W_1} u, P_{W_1} u' \rangle_{\tilde{X}} + \langle P_{W_2} u, P_{W_2} u' \rangle_{\tilde{X}}, \tag{34}$$

$u, u' \in H(\operatorname{curl}, \operatorname{tr}_{\nu\times}; \Omega)$, is equivalent to $\|\cdot\|_{H(\operatorname{curl}, \operatorname{tr}_{\nu\times}; \Omega)}$.

Proof 1. Step: Let P_{W_2} be the $\tilde{X}$-orthogonal projection onto W_2. Hence, $P_{W_2} \in L\big(H(\operatorname{curl}, \operatorname{tr}_{\nu\times}; \Omega)\big)$ is a projection with range W_2 and kernel

$$W_2^{\perp_{H(\operatorname{curl}, \operatorname{tr}_{\nu\times}; \Omega)}} \supset V, W_1.$$

2a. Step: Let $u \in H(\mathrm{curl}, \mathrm{tr}_{\nu\times}; \Omega)$. Note that due to $\mathrm{div}\epsilon(u - P_{W_2}u) = 0$ and Lemma 3.6 it hold $\mathrm{tr}_{\nu\cdot\epsilon}(u - P_{W_2}u) \in L^2(\partial\Omega)$ and $\langle \mathrm{tr}_{\nu\cdot\epsilon}(u - P_{W_2}u), 1\rangle_{L^2(\Gamma)} = 0$ for each Γ of the connected parts of $\partial\Omega$. Let $w_* \in H^1_*(\Omega)$ be the unique solution to

$$-\,\mathrm{div}\epsilon\,\nabla w_* = 0 \quad \text{in } \Omega, \qquad \nu\cdot\epsilon\nabla w_* = \mathrm{tr}_{\nu\cdot\epsilon}(u - P_{W_2}u) \quad \text{at } \partial\Omega.$$

Let $P_{W_1}u := \nabla w_*$. By construction of P_{W_1} and due to Lemma 3.6, it holds $P_{W_1} \in L\big(H(\mathrm{curl}, \mathrm{tr}_{\nu\times}; \Omega)\big)$ and $\mathrm{ran}\, P_{W_1} \subset W_1$. Let $u \in W_1$. Then, $P_{W_2}u = 0$ and hence $P_{W_1}u = u$. Thus, P_{W_1} is a projection and $\mathrm{ran}\, P_{W_1} = W_1$.

2b. Step: If $u \in W_2$ then $u - P_{W_2}u = 0$, further $\mathrm{tr}_{\nu\cdot\epsilon}(u - P_{W_2}u) = 0$ and thus $P_{W_1}u = 0$. Thus, $W_2 \subset \ker P_{W_1}$. If $u \in V$ then $P_{W_2}u = 0$, further $\mathrm{tr}_{\nu\cdot\epsilon}(u - P_{W_2}u) = \mathrm{tr}_{\nu\cdot\epsilon}\, u = 0$ and thus $P_{W_1}u = 0$. Hence, $V \subset \ker P_{W_1}$.

3. Step: Let $u \in H(\mathrm{curl}, \mathrm{tr}_{\nu\times}; \Omega)$ and $P_V u := u - P_{W_1}u - P_{W_2}u$. It follows $P_V \in L\big(H(\mathrm{curl}, \mathrm{tr}_{\nu\times}; \Omega)\big)$, $P_V u \in V$ and $P_V P_V u = P_V u$. If $u \in V$, then $P_V u = u$, and hence, $\mathrm{ran}\, P_V = V$. It follows further $W_1, W_2 \subset \ker P_V$.

4. Step: By means of the triangle inequality and a Young inequality, it holds.

$$\|u\|^2_{\tilde X} = \|P_V u + P_{W_1}u + P_{W_2}u\|^2_{\tilde X} \le 3\big(\|P_V u\|^2_{\tilde X} + \|P_{W_1}u\|^2_{\tilde X} + \|P_{W_2}u\|^2_{\tilde X}\big) = 3\|u\|^2_X.$$

On the other hand, due to the boundedness of the projections

$$\begin{aligned}\|u\|^2_X &= \|P_V u\|^2_{\tilde X} + \|P_{W_1}u\|^2_{\tilde X} + \|P_{W_2}u\|^2_{\tilde X}\\ &\le \big(\|P_V\|^2_{L(\tilde X)} + \|P_{W_1}\|^2_{L(\tilde X)} + \|P_{W_2}\|^2_{L(\tilde X)}\big)\|u\|^2_{\tilde X}.\end{aligned}$$

Thus, $\|\cdot\|_X$ is equivalent to $\|\cdot\|_{\tilde X}$. Since $\|\cdot\|_{\tilde X}$ is equivalent to $\|\cdot\|_{H(\mathrm{curl},\mathrm{tr}_{\nu\times};\Omega)}$, $\|\cdot\|_X$ is also equivalent to $\|\cdot\|_{H(\mathrm{curl},\mathrm{tr}_{\nu\times};\Omega)}$. ◻

Let us look at $A(\lambda)$ in light of this substructure of $H(\mathrm{curl}, \mathrm{tr}_{\nu\times}; \Omega)$. To this end, we consider the space

$$X := H(\mathrm{curl}, \mathrm{tr}_{\nu\times}; \Omega), \qquad \langle\cdot,\cdot\rangle_X \quad \text{as defined in (34).} \tag{35}$$

It follows that P_V, P_{W_1} and P_{W_1} are even orthogonal projections in X. Let further $A_X(\cdot), A_c, A_\epsilon, A_{\mathrm{tr}} \in L(X)$ be defined through

$$\langle A_X(\lambda)u, u'\rangle_X := \langle A(\lambda)u, u'\rangle_{H(\mathrm{curl},\mathrm{tr}_{\nu\times};\Omega)} \quad \text{for all } u, u' \in X, \lambda \in \mathbb{C} \tag{36a}$$

$$\langle A_c u, u'\rangle_X := \langle \mu^{-1}\,\mathrm{curl}\, u, \mathrm{curl}\, u'\rangle_{\mathbf{L}^2(\Omega)} \quad \text{for all } u, u' \in X, \tag{36b}$$

$$\langle A_\epsilon u, u'\rangle_X := \langle \epsilon u, u'\rangle_{\mathbf{L}^2(\Omega)} \quad \text{for all } u, u' \in X, \tag{36c}$$

$$\langle A_{\mathrm{tr}} u, u'\rangle_X := \langle \mathrm{tr}_{\nu\times}\, u, \mathrm{tr}_{\nu\times}\, u'\rangle_{\mathbf{L}^2_t(\partial\Omega)} \quad \text{for all } u, u' \in X. \tag{36d}$$

From the definitions of V, W_1 and W_2, we deduce that

$$\begin{aligned}A_X(\lambda) &= (P_V + P_{W_1} + P_{W_2})(A_c - \omega^2 A_\epsilon - \lambda A_{\mathrm{tr}})(P_V + P_{W_1} + P_{W_2})\\ &= P_V A_c P_V - \omega^2\big(P_V A_\epsilon P_V + P_{W_1} A_\epsilon P_{W_1} + P_{W_2} A_\epsilon P_{W_2}\big)\\ &\quad - \lambda(P_V + P_{W_1}) A_{\mathrm{tr}} (P_V + P_{W_1})\end{aligned}$$

$$
\begin{aligned}
&= P_V A_c P_V - \omega^2 (P_V A_\epsilon P_V + P_{W_1} A_\epsilon P_{W_1} + P_{W_2} A_\epsilon P_{W_2}) \\
&\quad - \lambda \big(P_V A_{\mathrm{tr}} P_V + P_{W_1} A_{\mathrm{tr}} P_{W_1} + P_V A_{\mathrm{tr}} P_{W_1} + P_{W_1} A_{\mathrm{tr}} P_V \big).
\end{aligned} \tag{37}
$$

If we identify $X \sim V \times W_1 \times W_2$ and $X \ni u \sim (v, w_1, w_2) \in V \times W_1 \times W_2$, we can identify $A_X(\lambda)$ with the block operator

$$
\begin{pmatrix}
P_V (A_c - \omega^2 A_\epsilon - \lambda A_{\mathrm{tr}})|_V & -\lambda P_V A_{\mathrm{tr}}|_{W_1} & \\
-\lambda P_{W_1} A_{\mathrm{tr}}|_V & -P_{W_1} (\omega^2 A_\epsilon + \lambda A_{\mathrm{tr}})|_{W_1} & \\
 & & -\omega^2 P_{W_2} A_\epsilon|_{W_2}
\end{pmatrix}. \tag{38}
$$

Theorem 2 *Let ϵ suffice Assumption 3.1, μ suffice Assumption 3.2 and Ω suffice Assumption 3.3. Thence, $A_X(\lambda)$ is Fredholm if and only if $\lambda \in \mathbb{C} \setminus \{0\}$. If in addition Assumption 3.4 holds true, then $\sigma\big(A(\cdot)\big) \subset \mathbb{R}$ and $\sigma\big(A(\cdot)\big) \setminus \{0\}$ consists of an at most countable set of eigenvalues with finite algebraic multiplicity which have no accumulation point in $\mathbb{R} \setminus \{0\}$.*

Proof The first statement follows from Theorem 3.2 and Corollary 3.4 of [22]. The second statement can be seen as in the proof of Corollary 3.3 of [22]. □

From (37) or (38), we recognize that any eigenfunction $u \in X$ satisfies $P_{W_2} u = w_2 = 0$. Hence, to study the eigenvalues of $A_X(\cdot)$ it suffices to study

$$
\begin{aligned}
&(P_V + P_{W_1}) A_X(\lambda)|_{V \oplus W_1} \\
&\quad \sim \begin{pmatrix}
P_V (A_c - \omega^2 A_\epsilon - \lambda A_{\mathrm{tr}})|_V & -\lambda P_V A_{\mathrm{tr}}|_{W_1} \\
-\lambda P_{W_1} A_{\mathrm{tr}}|_V & -P_{W_1} (\omega^2 A_\epsilon + \lambda A_{\mathrm{tr}})|_{W_1}
\end{pmatrix}.
\end{aligned} \tag{39}
$$

5 Spectrum in the neighborhood of zero

First, we establish in Theorem 3 the absence of eigenvalues of $A_X(\cdot)$ in $(0, c)$ for sufficiently small $c > 0$. Later on in Theorem 4, we establish the existence of an infinite sequence of negative eigenvalues of $A_X(\cdot)$ which accumulate at zero.

5.1 Spectrum right of zero

We will require in this section the following additional assumption.

Assumption 5.1 (ω^2 *is no Neumann eigenvalue*) $P_V A_c|_V - \omega^2 P_V A_\epsilon|_V \in L(V)$ is bijective.

Due to Assumption 5.1, we know that $P_V (A_c - \omega^2 A_\epsilon)|_V$ is invertible. Thus, by a Neumann series argument $P_V (A_c - \omega^2 A_\epsilon - \lambda A_{\mathrm{tr}})|_V \in L(V)$ is invertible too for all

$$
|\lambda| < \frac{1}{\| (P_V (A_c - \omega^2 A_\epsilon)|_V)^{-1} P_V A_{\mathrm{tr}}|_V \|_{L(V)}} \tag{40}
$$

and thence it holds

$$
\begin{aligned}
&\| (P_V (A_c - \omega^2 A_\epsilon - \lambda A_{\mathrm{tr}})|_V)^{-1} \|_{L(V)} \\
&\quad \le \frac{1}{1 - \lambda \| (P_V (A_c - \omega^2 A_\epsilon)|_V)^{-1} P_V A_{\mathrm{tr}}|_V \|_{L(V)}}.
\end{aligned} \tag{41}
$$

For λ satisfying (40), we build the Schur-complement of $(P_V + P_{W_1})A_X(\lambda)|_{V\oplus W_1}$ with respect to $P_V u = v$:

$$A_{W_1}(\lambda) := -\omega^2 P_{W_1} A_\epsilon|_{W_1} - \lambda(P_{W_1} A_{\mathrm{tr}}|_{W_1} + H_{W_1}(\lambda)) \in L(W_1), \tag{42a}$$

$$H_{W_1}(\lambda) := \lambda P_{W_1} A_{\mathrm{tr}} (P_V(A_c - \omega^2 A_\epsilon - \lambda A_{\mathrm{tr}})|_V)^{-1} P_V A_{\mathrm{tr}}|_{W_1} \in L(W_1). \tag{42b}$$

It is straightforward to see, that for λ satisfying (40), λ is an eigenvalue to $A_X(\cdot)$ if and only if λ is an eigenvalue to $A_{W_1}(\cdot)$. Hence, to study the eigenvalues of $A_X(\cdot)$ in a neighborhood of zero, it completely suffices to study the eigenvalues of $A_{W_1}(\cdot)$ in a neighborhood of zero. For

$$|\lambda| < \frac{1}{2\|(P_V(A_c - \omega^2 A_\epsilon)|_V)^{-1} P_V A_{\mathrm{tr}}|_V\|_{L(V)}} \tag{43}$$

we deduce

$$\|H_{W_1}(\lambda)\|_{L(W_1)} \le \lambda 2 \|P_V\|_{L(X)} \|P_{W_1}\|_{L(X)} \|A_{\mathrm{tr}}\|^2_{L(X)}. \tag{44}$$

Let $B_{\mathrm{tr}} \in L\big(X, \mathbf{L}^2_t(\partial\Omega)\big) \colon u \mapsto \mathrm{tr}_{\nu\times} u$ so that

$$A_{\mathrm{tr}} = B^*_{\mathrm{tr}} B_{\mathrm{tr}}. \tag{45}$$

Lemma 5.2 *Let Assumptions 3.1 hold true. Thence, $P_{W_1} A_{\mathrm{tr}}|_{W_1}$ is strictly positive definite, i.e.*

$$\inf_{w_1 \in W_1 \setminus \{0\}} \frac{\langle (P_{W_1} A_{\mathrm{tr}}|_{W_1}) w_1, w_1 \rangle_X}{\|w_1\|^2_X} > 0. \tag{46}$$

Proof A_{tr} is self-adjoint and positive semi-definite due to (45) and hence so is $P_{W_1} A_{\mathrm{tr}}|_{W_1}$. $P_{W_1} A_{\mathrm{tr}}|_{W_1}$ is weakly coercive due to Lemma 3.8 and $\operatorname{curl} w_1 = 0$ for each $w_1 \in W_1$. $P_{W_1} A_{\mathrm{tr}}|_{W_1}$ is injective since $w_1 \in W_1 \cap \ker(P_{W_1} A_{\mathrm{tr}}|_{W_1})$ implies $w_1 \in W_2$ and hence $w_1 = 0$. Since $P_{W_1} A_{\mathrm{tr}}|_{W_1}$ is self-adjoint, positive semi-definite and bijective, it is already strictly positive definite. □

Lemma 5.3 *Let Assumptions 3.1–3.3 and 5.1 hold true. Thence, there exists $c_0 > 0$ so that $P_{W_1} A_{\mathrm{tr}}|_{W_1} + H_{W_1}(\lambda)$ is strictly positive definite, i.e.*

$$\inf_{w_1 \in W_1 \setminus \{0\}} \frac{\langle (P_{W_1} A_{\mathrm{tr}}|_{W_1} + H_{W_1}(\lambda)) w_1, w_1 \rangle_X}{\|w_1\|^2_X} > 0, \tag{47}$$

for each $\lambda \in (-c_0, c_0)$.

Proof It is straightforward to see that $H_{W_1}(\lambda)$ self-adjoint for $\lambda \in \mathbb{R}$ satisfying (40). The inverse triangle inequality, Lemma 5.2 and (43), (44) yield the claim. □

Theorem 3 *Let Assumptions 3.1–3.4 and 5.1 hold true and c_0 be as in Lemma 5.3. Thence, $\sigma\big(A_X(\cdot)\big) \cap (0, c_0) = \emptyset$.*

Proof For $\lambda \in (0, c_0)$, we can build the Schur-complement $A_{W_1}(\lambda)$ of $A_X(\lambda)$ with respect to $P_V u = v$ and $A_X(\lambda)$ is bijective if and only if $A_{W_1}(\lambda)$ is so. It follows from the definition (42a) of $A_{W_1}(\lambda)$ and Lemma 5.3 that $A_{W_1}(\lambda)$ is strictly positive definite for $\lambda \in (0, c_0)$ and hence bijective. □

5.2 Spectrum left of zero

To study the eigenvalues of $A_{W_1}(\cdot)$ in $(-c_0, 0)$, we introduce

$$A_{W_1}(\tau, \lambda) := -\omega^2 P_{W_1} A_\epsilon|_{W_1} - \tau(P_{W_1} A_{\mathrm{tr}}|_{W_1} + H_{W_1}(\lambda)). \tag{48}$$

We notice that $\lambda \in (-c_0, 0)$ is an eigenvalue of $A_{W_1}(\cdot)$, if and only if τ is an eigenvalue of $A_{W_1}(\cdot, \lambda)$ and $\tau = \lambda$. We prove the existence of infinite eigenvalues of $A_{W_1}(\cdot)$ in $(-c_0, 0)$ by the fixed point technique outlined in [11].

Lemma 5.4 *Let Assumptions 3.1–3.4 and 5.1 hold true and c_0 be as in Lemma 5.3. Let $\lambda \in (-c_0, c_0)$. The spectrum of $A_{W_1}(\cdot, \lambda)$ consists of $\sigma_{\mathrm{ess}}\big(A_{W_1}(\cdot, \lambda)\big) = \{0\}$ and an infinite sequence of negative eigenvalues $(\tau_n(\lambda))_{n\in\mathbb{N}}$ which accumulate at zero.*

Proof Due to Lemma 5.3 $(P_{W_1} A_{\mathrm{tr}}|_{W_1} + H_{W_1}(\lambda))^{-1/2}$ is well defined and self-adjoint. It holds $\dim W_1 = \infty$ due to (32). The spectra of $A_{W_1}(\cdot, \lambda)$ and

$$\begin{aligned}&(P_{W_1} A_{\mathrm{tr}}|_{W_1} + H_{W_1}(\lambda))^{-1/2} A_{W_1}(\cdot, \lambda)(P_{W_1} A_{\mathrm{tr}}|_{W_1} + H_{W_1}(\lambda))^{-1/2}\\ &\quad = -\omega^2 (P_{W_1} A_{\mathrm{tr}}|_{W_1} + H_{W_1}(\lambda))^{-1/2} P_{W_1} A_\epsilon|_{W_1} (P_{W_1} A_{\mathrm{tr}}|_{W_1} + H_{W_1}(\lambda))^{-1/2} - \cdot I_{W_1}\end{aligned}$$

coincide. The latter is the pencil of a standard eigenvalue problem for a compact self-adjoint non-positive injective operator on an infinite-dimensional Hilbert space and respective properties follow. □

Lemma 5.5 *Let Assumptions 3.1–3.4 and 5.1 hold true and c_0 be as in Lemma 5.3. Let the sequence of negative eigenvalues $(\tau_n(\lambda))_{n\in\mathbb{N}}$ to the operator function $A_{W_1}(\cdot, \lambda)$ be ordered non-decreasingly with multiplicity taken into account. The function $(-c_0, c_0) \to \mathbb{R} : \lambda \mapsto \tau_n(\lambda)$ is continuous for each $n \in \mathbb{N}$.*

Proof Follows from the ordering of $(\tau_n(\lambda))_{n\in\mathbb{N}}$ and [23, Sect. 3] or [27, Proposition 5.4]. □

Theorem 4 *Let Assumptions 3.1–3.4 and 5.1 hold true. Thence, there exists an infinite sequence $(\lambda_n)_{n\in\mathbb{N}}$ of negative eigenvalues to $A_X(\cdot)$ which accumulate at zero.*

Proof Let $(\tau_n(\lambda))_{n\in\mathbb{N}}$ be as in Lemma 5.5. Let $\lambda \in (-c_0, 0)$. Let $n_1 \in \mathbb{N}$ be so that $\lambda < \tau_{n_1}(\lambda)$. Consider the function $f_1(t) := \tau_{n_1}(t) - t$. It holds: f_1 is continuous on $(-c_0, c_0)$ due to Lemma 5.5, $f_1(\lambda) > 0$ and $f_1(0) = \tau_{n_1}(0) < 0$. It follows from the intermediate value theorem that there exists $\lambda_1 \in (\lambda, 0)$ with $f_1(\lambda_1) = 0$, i.e. λ_1 is an eigenvalue to $A_{W_1}(\cdot)$. Let now $\lambda \in (\lambda_1, 0)$ and $n_2 \in \mathbb{N}$ be so that $\lambda < \tau_{n_2}(\lambda)$. We can repeat the former procedure to construct a second eigenvalue $\lambda_2 \in (\lambda_1, 0)$ to $A_{W_1}(\cdot)$. Since $\lambda_2 \in (\lambda_1, 0)$, λ_2 is distinct from λ_1. We can repeat the former procedure inductively to construct a sequence $(\lambda_n \in (-c_0, 0))_{n\in\mathbb{N}}$ of pairwise distinct eigenvalues to $A_{W_1}(\cdot)$. As already discussed, the spectra of $A_{W_1}(\cdot)$ and $A_X(\cdot)$ coincide on the ball (40). Since $[-c_0, 0]$ is compact and the sequence $(\lambda_n \in (-c_0, 0))_{n\in\mathbb{N}}$ has an infinite index set, $(\lambda_n)_{n\in\mathbb{N}}$ admits a cluster point in

$[-c_0, 0]$. Due to Theorem 2 $\sigma\big(A_X(\cdot)\big)$ admits no cluster points in $\mathbb{C} \setminus \{0\}$. Thus, $(\lambda_n)_{n\in\mathbb{N}}$ accumulate at zero. The claim is proven. □

6 Spectrum in the neighborhood of infinity

First, we establish in Theorem 5 the absence of eigenvalues of $A_X(\cdot)$ in the interval $(-\infty, -c)$ for sufficiently large $c > 0$. Later on in Theorem 6, we establish the existence of an infinite sequence of positive eigenvalues of $A_X(\cdot)$ which accumulate at $+\infty$.

6.1 The spectrum near negative infinity

We require the following additional assumption for Theorem 5.

Assumption 6.1 (ω^2 *is no Dirichlet eigenvalue*) There exists no nontrivial solution $u \in H(\operatorname{curl}, \operatorname{tr}^0_{\nu\times}; \Omega)$ to $\operatorname{curl}\mu^{-1}\operatorname{curl} u - \omega^2\epsilon u = 0$ in Ω.

Lemma 6.2 (Nitsche penalty technique) *Let Assumptions 3.1–3.3 hold true. Let $f \in \mathbf{L}^2(\Omega)$ and $u \in H(\operatorname{curl}, \operatorname{tr}^0_{\nu\times}; \Omega)$ be the solution to $\operatorname{curl}\mu^{-1}\operatorname{curl} u + \epsilon u = f$ in Ω. For $\lambda > 0$ let $u_\lambda \in H(\operatorname{curl}, \operatorname{tr}_{\nu\times}; \Omega)$ be the solution to*

$$\langle \mu^{-1}\operatorname{curl} u_\lambda, \operatorname{curl} u'\rangle_{\mathbf{L}^2(\Omega)} + \langle \epsilon u_\lambda, u'\rangle_{\mathbf{L}^2(\Omega)} + \lambda\langle \operatorname{tr}_{\nu\times} u_\lambda, \operatorname{tr}_{\nu\times} u'\rangle_{\mathbf{L}^2_t(\partial\Omega)} = \langle f, u'\rangle_{\mathbf{L}^2(\Omega)}$$

for all $u' \in H(\operatorname{curl}, \operatorname{tr}_{\nu\times}; \Omega)$. Then, there exist $C, \lambda_0 > 0$ so that $\|u - u_\lambda\|_{H(\operatorname{curl},\operatorname{tr}_{\nu\times};\Omega)} \le C/\lambda$ for all $\lambda > \lambda_0$.

Proof We are not aware of a direct appropriate reference for this lemma. Although we believe that the technique applied in this proof is common knowledge. We introduce mixed equations for u (and u_λ) as e.g. in [28] as follows. Let $\hat{f} \in X$ be so that $\langle \hat{f}, u'\rangle_X = \langle f, u'\rangle_{\mathbf{L}^2(\Omega)}$ for all $u' \in X$. Due to $u \in H(\operatorname{curl}, \operatorname{tr}^0_{\nu\times}; \Omega)$ and Assumption 3.2, it follows $\phi := \nu \times \operatorname{tr}_{\nu\times} \mu^{-1}\operatorname{curl} u \in \mathbf{L}^2_t(\partial\Omega)$. It holds $\phi_\lambda := \nu \times \operatorname{tr}_{\nu\times} \mu^{-1}\operatorname{curl} u_\lambda = \lambda \operatorname{tr}_{\nu\times} u_\lambda \in \mathbf{L}^2_t(\partial\Omega)$ too. Integration by parts yields that $(u, \phi), (u_\lambda, \phi_\lambda) \in X \times \mathbf{L}^2_t(\partial\Omega)$ solve

$$\begin{pmatrix} A_c + A_\epsilon & B^*_{\mathrm{tr}} \\ B_{\mathrm{tr}} & 0 \end{pmatrix}\begin{pmatrix} u \\ \phi \end{pmatrix} = \begin{pmatrix} \hat{f} \\ 0 \end{pmatrix} \tag{49}$$

and

$$\begin{pmatrix} A_c + A_\epsilon & B^*_{\mathrm{tr}} \\ B_{\mathrm{tr}} & -\lambda^{-1} I_{\mathbf{L}^2_t(\partial\Omega)} \end{pmatrix}\begin{pmatrix} u_\lambda \\ \phi_\lambda \end{pmatrix} = \begin{pmatrix} \hat{f} \\ 0 \end{pmatrix} \tag{50}$$

respective. Both (49) and (50) are stable saddle point problems [6, Theorem 4.3.1]. Since (50) is a perturbation of (49) by magnitude λ^{-1}, the claim follows. □

Theorem 5 *Let Assumptions 3.1–3.4 and 6.1 hold true. Thence, there exists $c > 0$ so that $A_X(\lambda)$ is bijective for all $\lambda \in (-\infty, -c)$.*

Proof Assume the contrary. Thus, there exists a sequence $(\lambda_n < 0)_{n\in\mathbb{N}}$ with $\lim_{n\in\mathbb{N}} \lambda_n = -\infty$, so that $A_X(\lambda_n)$ is not bijective. Due Theorem 2 $(\lambda_n)_{n\in\mathbb{N}}$ are eigenvalues of $A_X(\cdot)$. Hence, let $(u_n \in X)_{n\in\mathbb{N}}$ be a corresponding sequence of normalized eigenfunctions: $A_X(\lambda_n)u_n = 0$ and $\|u_n\|_X = 1$ for each $n \in \mathbb{N}$. It follows

$$u_n = (\omega^2 + 1)(A_c + A_\epsilon + |\lambda_n| A_{\mathrm{tr}})^{-1} A_\epsilon u_n. \tag{51}$$

As already discussed at the end of Sect. 4, it holds $u_n \in V \oplus W_1$ for each $n \in \mathbb{N}$. Denote $E \in L\big(X, \mathbf{L}^2(\Omega)\big)$ the embedding operator and $M_\epsilon \in L\big(\mathbf{L}^2(\Omega)\big)$ the multiplication operator with symbol ϵ. Thus, $A_\epsilon = E^* M_\epsilon E$. Due to Lemma 3.8, there exist $f \in \mathbf{L}^2(\Omega)$ and a subsequence $(n(m))_{m\in\mathbb{N}}$ so that $\lim_{m\in\mathbb{N}} E u_{n(m)} = f$. Let $u \in H(\mathrm{curl}, \mathrm{tr}^0_{\nu\times}; \Omega)$ be the solution to $\mathrm{curl}\, \mu^{-1} \mathrm{curl}\, u + \epsilon u = \epsilon f$ in Ω. It follows from Lemma 6.2 and (51) that $\lim_{m\in\mathbb{N}} u_{n(m)} = (\omega^2 + 1) u$ in X. Since $\mathrm{curl}\, \mu^{-1} \mathrm{curl}\, u_{n(m)} - \omega^2 \epsilon u_{n(m)} = 0$ in Ω for each $m \in \mathbb{N}$, it follows that $\mathrm{curl}\, \mu^{-1} \mathrm{curl}\, u - \omega^2 \epsilon u = 0$ in Ω as well. Due to Assumption 6.1 it holds $u = 0$, which is a contradiction to $\|u_{n(m)}\|_X = 1$ for each $m \in \mathbb{N}$. The claim is proven. □

6.2 The spectrum near positive infinity

$P_{W_1} A_{\mathrm{tr}}|_{W_1} \in L(W_1)$ is strictly positive definite due to Lemma 5.2. Hence, there exists $c_\infty > 0$ so that

$$P_{W_1}(\omega^2 A_\epsilon + \lambda A_{\mathrm{tr}})|_{W_1} = \lambda P_{W_1}(\omega^2 \lambda^{-1} A_\epsilon + A_{\mathrm{tr}})|_{W_1} \tag{52}$$

is coercive and thus bijective for each $\lambda \in \mathbb{C}$ with $|\lambda| > c_\infty$. (Since A_ϵ is positive semi definite, it follows even that $P_{W_1}(\omega^2 A_\epsilon + \lambda A_{\mathrm{tr}})|_{W_1}$ is coercive for each $\lambda \in \mathbb{C} \setminus \mathbb{R}_0^-$. However, we will not use this fact.) Hence for $|\lambda| > c_\infty$ we build and study the Schur-complement of $(P_V + P_{W_1}) A_X(\lambda)|_{V \oplus W_1}$ with respect to $P_{W_1} u = w_1$:

$$A_V(\lambda) := P_V(A_c - \omega^2 A_\epsilon)|_V - \lambda K_V(\lambda) \in L(V), \tag{53a}$$

$$K_V(\lambda) := P_V(A_{\mathrm{tr}} - A_{\mathrm{tr}} S_V(\lambda) P_{W_1} A_{\mathrm{tr}})|_V \in L(V), \tag{53b}$$

$$S_V(\lambda) := \Big(P_{W_1}(\omega^2 \lambda^{-1} A_\epsilon + A_{\mathrm{tr}})|_{W_1}\Big)^{-1} \in L(W_1). \tag{53c}$$

It is straightforward to see that for λ satisfying $|\lambda| > c_\infty$, λ is an eigenvalue to $A_X(\cdot)$ if and only if λ is an eigenvalue to $A_V(\cdot)$. Hence to study the eigenvalues of $A_X(\cdot)$ in a neighborhood of infinity, it completely suffices to study the eigenvalues of $A_V(\cdot)$ in a neighborhood of infinity. It will be more convenient to work with λ^{-1} instead of λ. Hence let

$$\tilde{A}_V(\tilde{\lambda}) := \tilde{\lambda} A_V(\tilde{\lambda}^{-1}) = \tilde{\lambda} P_V(A_c - \omega^2 A_\epsilon)|_V - \tilde{K}_V(\tilde{\lambda}) \in L(V), \tag{54a}$$

$$\tilde{K}_V(\tilde{\lambda}) := K_V(\tilde{\lambda}^{-1}) = P_V(A_{\mathrm{tr}} - A_{\mathrm{tr}} \tilde{S}_V(\tilde{\lambda}) P_{W_1} A_{\mathrm{tr}})|_V \in L(V), \tag{54b}$$

$$\tilde{S}_V(\tilde{\lambda}) := S_V(\tilde{\lambda}^{-1}) = \Big(P_{W_1}(\omega^2 \tilde{\lambda} A_\epsilon + A_{\mathrm{tr}})|_{W_1}\Big)^{-1} \in L(W_1), \tag{54c}$$

for $\tilde{\lambda} \in \mathbb{C}$ with $|\tilde{\lambda}| < c_\infty^{-1}$. Again, it is straightforward to see that $\tilde{\lambda}$ with $0 < |\tilde{\lambda}| < c_\infty^{-1}$ is an eigenvalue to $\tilde{A}_V(\cdot)$ if and only if $\tilde{\lambda}^{-1}$ with $|\tilde{\lambda}^{-1}| > c_\infty$ is an eigenvalue to $A_V(\cdot)$. Thus, we study the eigenvalues of $\tilde{A}_V(\cdot)$ in the ball

$$B_{c_\infty^{-1}} := \{z \in \mathbb{C} \colon |z| < c_\infty^{-1}\}. \tag{55}$$

To this end, we introduce

$$\tilde{A}_V(\tilde{\tau}, \tilde{\lambda}) := \tilde{\tau} P_V(A_c - \omega^2 A_\epsilon)|_V - \tilde{K}_V(\tilde{\lambda}). \tag{56}$$

We note that $\tilde{\lambda} \in B_{c_\infty^{-1}}$ is an eigenvalue of $\tilde{A}_V(\cdot)$, if and only if $\tilde{\tau}$ is an eigenvalue of $\tilde{A}_V(\cdot, \tilde{\lambda})$ and $\tilde{\tau} = \tilde{\lambda} \in B_{c_\infty^{-1}}$.

We would like to proceed as in Sect. 5. Operator $\tilde{K}_V(\tilde{\lambda})$ is compact due Lemma 3.7. However, different to Sect. 5, $P_V(A_c - \omega^2 A_\epsilon)|_V$ is (for arbitrary $\omega > 0$) not definite! Moreover, $\tilde{K}_V(\tilde{\lambda})$ is not injective! Indeed, $\{\operatorname{curl} f : f \in (C_0^\infty(\hat{\Omega} \setminus \check{\Omega}))^3\} \subset \ker \tilde{K}_V(\tilde{\lambda})$. Therefore, we introduce the abstract Lemma 6.3. Subsequently, we prove that the conditions of Lemma 6.3 are satisfied and the lemma can be employed for our particular application. We derive the results aimed at in Lemma 6.12 and consequently continue the analysis in the same manner as in Sect. 5.

Lemma 6.3 *Let Y be a separable Hilbert space. Let $G \in L(Y)$ be compact, self-adjoint and $I + G$ be bijective. Let $K \in L(Y)$ be compact, self-adjoint, positive semi-definite and so that $\ker K = \ker(K^{1/2}(I+G)K^{1/2})$ and $\dim(\ker K)^\perp = \infty$. Let $P_{(\ker K)^\perp}$ be the orthogonal projection onto $(\ker K)^\perp$ and $P_{(\ker K)^\perp}(I+G)|_{(\ker K)^\perp}$ be bijective.*

Then, the spectra of $(I+G)K$ and $K^{1/2}(I+G)K^{1/2}$ coincide and consist of the essential spectrum $\{0\}$ and an infinite sequence $(\tau_n \in \mathbb{R})_{n\in\mathbb{N}}$ of nonzero eigenvalues. Apart from a finite set, all $(\tau_n)_{n\in\mathbb{N}}$ are positive and it holds $\lim_{n\in\mathbb{N}} \tau_n = 0$.

Proof *1. Step:* If $(\tau, y) \in \mathbb{C}\setminus\{0\} \times Y\setminus\{0\}$ solves

$$\Big(\tau I - (I+G)K\Big)y = 0,$$

then $K^{1/2}y \neq 0$ and

$$0 = K^{1/2}\Big(\tau I - (I+G)K\Big)y = \Big(\tau I - K^{1/2}(I+G)K^{1/2}\Big)K^{1/2}y.$$

Vice versa, if $(\tau, y') \in \mathbb{C}\setminus\{0\} \times Y\setminus\{0\}$ solves

$$\Big(\tau I - K^{1/2}(I+G)K^{1/2}\Big)y',$$

then $(I+G)K^{1/2}y' \neq 0$ and

$$0 = \Big(I+G)K^{1/2}(\tau I - K^{1/2}(I+G)K^{1/2}\Big)y' = \Big(\tau I - (I+G)K\Big)(I+G)K^{1/2}y'.$$

By assumption, $(I+G)Ky = 0$ if and only if $K^{1/2}(I+G)K^{1/2}y = 0$. Thus, the spectra of $(I+G)K$ and $K^{1/2}(I+G)K^{1/2}$ coincide.

2. Step: Since $K^{1/2}(I+G)K^{1/2}$ is compact and self-adjoint and Y is separable with $\dim Y \geq \dim(\ker K)^\perp = \infty$ the Spectral Theorem for compact, self-adjoint operators yields: The spectrum of $K^{1/2}(I+G)K^{1/2}$ consists of the essential spectrum $\{0\}$ and an infinite sequence of eigenvalues $(\tau_n \in \mathbb{R})_{n\in\mathbb{N}}$ (with multiplicity taken into account), $\lim_{n\in\mathbb{N}} \tau_n = 0$ and there exists an orthonormal basis $(y_n)_{n\in\mathbb{N}}$ of corresponding eigenelements. Due to $\dim(\ker K)^\perp = \infty$, there exists an infinite index set $\mathbb{M} \subset \mathbb{N}$ so that $\tau_m \neq 0$ for each $m \in \mathbb{M}$.

3. Step: It remains to prove that all $(\tau_m)_{m\in\mathbb{M}}$ apart from a finite set are positive. To this end, we apply a technique which is inspired by [25, Sect. 3]. Let

$$\tilde{Y} := \overline{\operatorname{span}\{y_m : m \in \mathbb{M}\}}^{\mathrm{cl}} = (\ker K^{1/2}(I+G)K^{1/2})^{\perp} = (\ker K)^{\perp}$$

and denote $P_{\tilde{Y}}$ the orthogonal projection onto $\tilde{Y}$. We note that for each $y \in Y$, $y^0 \in \ker K$ it holds

$$\langle K^{1/2}y, y^0\rangle_Y = \langle y, K^{1/2}y^0\rangle_Y = 0.$$

Thus, $\operatorname{ran} K^{1/2} \subset (\ker K)^{\perp} = \tilde{Y}$ and so $(\tau I + K)^{1/2}\tilde{Y} \subset \tilde{Y}$. Let $G = G_+ - G_-$ so that G_+ and G_- are compact, self-adjoint and positive semi-definite, i.e. a decomposition of G in the positive and the negative part. For $\tau > 0$, we compute

$$\begin{aligned}
\big(\tau I + K^{1/2}(I+G)K^{1/2}\big)|_{\tilde{Y}} &= \big(\tau I + K + K^{1/2}GK^{1/2}\big)|_{\tilde{Y}} \\
&= (\tau I + K)^{1/2}\Big(I - (P_{\tilde{Y}}(\tau I + K)|_{\tilde{Y}})^{-1/2} \\
&\quad K^{1/2}(G_+^{1/2}G_+^{1/2} - G_-^{1/2}G_-^{1/2})K^{1/2}(P_{\tilde{Y}}(\tau I + K)|_{\tilde{Y}})^{-1/2}\Big) \\
&\quad \times (\tau I + K)^{1/2}|_{\tilde{Y}}.
\end{aligned}$$

By means of the Spectral Theorem for compact, self-adjoint operators, we deduce that $(P_{\tilde{Y}}(\tau I + K)|_{\tilde{Y}})^{-1/2}K^{1/2}$ converges point-wise to $P_{\tilde{Y}}$ for $\tau \to 0+$. Since $G_\pm^{1/2}$ is compact, it follows that $(P_{\tilde{Y}}(\tau I + K)|_{\tilde{Y}})^{-1/2}K^{1/2}G_\pm^{1/2}$ converges to $P_{\tilde{Y}}G_\pm^{1/2}$ in $L(Y)$ for $\tau \to 0+$. Hence,

$$\Big((P_{\tilde{Y}}(\tau I + K)|_{\tilde{Y}})^{-1/2}K^{1/2}G_\pm^{1/2}\Big)^* = G_\pm^{1/2}K^{1/2}(P_{\tilde{Y}}(\tau I + K)|_{\tilde{Y}})^{-1/2}P_{\tilde{Y}}$$

converges to $(P_{\tilde{Y}}G_\pm^{1/2})^* = G_\pm^{1/2}P_{\tilde{Y}}$ in $L(Y)$. Thus

$$P_{\tilde{Y}}\Big(I - (P_{\tilde{Y}}(\tau I + K)|_{\tilde{Y}})^{-1/2}K^{1/2}GK^{1/2}(P_{\tilde{Y}}(\tau I + K)|_{\tilde{Y}})^{-1/2}\Big)|_{\tilde{Y}} \tag{57}$$

converges in norm to $P_{\tilde{Y}}(I - G)|_{\tilde{Y}}$. Hence, there exists $c > 0$ so that (57) is bijective for all $\tau \in (0, c)$. Since for each $\tau \in (0, c)$, $(\tau I + K^{1/2}(I+G)K^{1/2})|_{\tilde{Y}} \in L(\tilde{Y})$ is a composition of three bijective operators in $L(\tilde{Y})$, it is bijective. Due to $\lim_{m \in \mathbb{M}} \tau_m = 0$ there can only exist a finite number of $m \in \mathbb{M}$ with $\tau_m < 0$. □

Lemma 6.4 *Let Assumptions 3.1–3.3 hold true. Thence $\tilde{K}_V(\tilde{\lambda})$ is compact, self-adjoint and positive semi-definite for each $\tilde{\lambda} \in [0, c_\infty^{-1})$. It holds further $\ker \tilde{K}_V(\tilde{\lambda}) = \ker B_{\mathrm{tr}}$ for each $\tilde{\lambda} \in (0, c_\infty^{-1})$.*

Proof Let $\tilde{\lambda} \in [0, c_\infty^{-1})$. $\tilde{K}_V(\tilde{\lambda})$ is compact due Lemma 3.7. It follows from the definition of $\tilde{K}_V(\tilde{\lambda})$ that $\tilde{K}_V(\tilde{\lambda})$ is self-adjoint. Let $v \in V$ and $w_1 := \tilde{S}_V(\tilde{\lambda})P_{W_1}B_{\mathrm{tr}}^*B_{\mathrm{tr}}v$. We compute

$$\begin{aligned}
\langle B_{\mathrm{tr}}w_1, B_{\mathrm{tr}}w_1\rangle_{\mathbf{L}_t^2(\partial\Omega)} &\le \langle B_{\mathrm{tr}}w_1, B_{\mathrm{tr}}w_1\rangle_{\mathbf{L}_t^2(\partial\Omega)} + \omega^2\tilde{\lambda}\langle \epsilon w_1, w_1\rangle_{\mathbf{L}^2(\Omega)} \\
&= \langle (A_{\mathrm{tr}} + \omega^2\tilde{\lambda}A_\epsilon)w_1, w_1\rangle_X \\
&= \langle (A_{\mathrm{tr}} + \omega^2\tilde{\lambda}A_\epsilon)\tilde{S}_V(\tilde{\lambda})P_{W_1}B_{\mathrm{tr}}^*B_{\mathrm{tr}}v, w_1\rangle_X \\
&= \langle B_{\mathrm{tr}}v, B_{\mathrm{tr}}w_1\rangle_{\mathbf{L}_t^2(\partial\Omega)} \le \|B_{\mathrm{tr}}v\|_{\mathbf{L}_t^2(\partial\Omega)}\|B_{\mathrm{tr}}w_1\|_{\mathbf{L}_t^2(\partial\Omega)}
\end{aligned}$$

and hence $\|B_{\mathrm{tr}}\tilde{S}_V(\tilde{\lambda})P_{W_1}B_{\mathrm{tr}}^*B_{\mathrm{tr}}v\|_{\mathbf{L}_t^2(\partial\Omega)} = \|B_{\mathrm{tr}}w_1\|_{\mathbf{L}_t^2(\partial\Omega)} \le \|B_{\mathrm{tr}}v\|_{\mathbf{L}_t^2(\partial\Omega)}$. Thus

$$\begin{aligned}\langle B_{\mathrm{tr}}\tilde{S}_V(\tilde{\lambda})P_{W_1}B_{\mathrm{tr}}^*B_{\mathrm{tr}}v, B_{\mathrm{tr}}v\rangle_{\mathbf{L}_t^2(\partial\Omega)} &\le \|B_{\mathrm{tr}}\tilde{S}_V(\tilde{\lambda})P_{W_1}B_{\mathrm{tr}}^*B_{\mathrm{tr}}v\|_{\mathbf{L}_t^2(\partial\Omega)}\|B_{\mathrm{tr}}v\|_{\mathbf{L}_t^2(\partial\Omega)}\\ &\le \|B_{\mathrm{tr}}v\|_{\mathbf{L}_t^2(\partial\Omega)}\|B_{\mathrm{tr}}v\|_{\mathbf{L}_t^2(\partial\Omega)}.\end{aligned}$$

Hence

$$\begin{aligned}\langle\tilde{K}_V(\tilde{\lambda})v, v\rangle_X &= \langle(A_{\mathrm{tr}} - A_{\mathrm{tr}}\tilde{S}_V(\tilde{\lambda})P_{W_1}A_{\mathrm{tr}}v, v\rangle_X\\ &= \langle B_{\mathrm{tr}}v, B_{\mathrm{tr}}v\rangle_{\mathbf{L}_t^2(\partial\Omega)} - \langle B_{\mathrm{tr}}\tilde{S}_V(\tilde{\lambda})P_{W_1}B_{\mathrm{tr}}^*B_{\mathrm{tr}}v, B_{\mathrm{tr}}v\rangle_{\mathbf{L}_t^2(\partial\Omega)} \ge 0. \end{aligned}\tag{58}$$

Let $\tilde{\lambda} \in (0, c_\infty^{-1})$. Let $B_{\mathrm{tr}}v \neq 0$. If $P_{W_1}B_{\mathrm{tr}}^*B_{\mathrm{tr}}v = 0$ it follows $\tilde{S}_V(\tilde{\lambda})P_{W_1}B_{\mathrm{tr}}^*B_{\mathrm{tr}}v = 0$ and (58) is strict. So let $P_{W_1}B_{\mathrm{tr}}^*B_{\mathrm{tr}}v \neq 0$. It follows $w_1 \neq 0$ and hence $\langle\epsilon w_1, w_1\rangle_{\mathbf{L}^2(\Omega)} > 0$. Since $w_1 \in W_1$, it also holds $\|B_{\mathrm{tr}}w_1\|_{\mathbf{L}_t^2(\partial\Omega)} \neq 0$. So in this case $\|B_{\mathrm{tr}}\tilde{S}_V(\tilde{\lambda})P_{W_1}B_{\mathrm{tr}}^*B_{\mathrm{tr}}v\|_{\mathbf{L}_t^2(\partial\Omega)} < \|B_{\mathrm{tr}}v\|_{\mathbf{L}_t^2(\partial\Omega)}$ and (58) is strict too. Thus $\tilde{K}_V(\tilde{\lambda})v \neq 0$. On the other hand: If $B_{\mathrm{tr}}v = 0$, then also $\tilde{K}_V(\tilde{\lambda})v = 0$ due to the definition of $\tilde{K}_V(\tilde{\lambda})$. Thus $\ker\tilde{K}_V(\tilde{\lambda}) = \ker B_{\mathrm{tr}}$ for each $\tilde{\lambda} \in (0, c_\infty^{-1})$. □

Lemma 6.5 *Let Assumptions 3.1–3.3 hold true. Thence $\tilde{K}_V(0) = B_{\mathrm{tr}}^*P_{\nabla_\partial}B_{\mathrm{tr}}$.*

Proof Let $P \in L\big(\mathbf{L}_t^2(\partial\Omega)\big)$ be the $\mathbf{L}_t^2(\partial\Omega)$-orthogonal projection onto the closure of $\operatorname{ran} B_{\mathrm{tr}}|_{W_1}$. It follows from the definition of $\tilde{K}_V(0)$ that $\tilde{K}_V(0) = B_{\mathrm{tr}}^*(I-P)B_{\mathrm{tr}}$. The claim is proven, if we show that $\operatorname{ran} B_{\mathrm{tr}}|_{W_1} = \operatorname{curl}_\partial H^1(\partial\Omega)$. It follows from the definition of W_1 that $\operatorname{ran} B_{\mathrm{tr}}|_{W_1} \subset \operatorname{curl}_\partial H^1(\partial\Omega)$. Let $\phi \in \operatorname{curl}_\partial H^1(\partial\Omega)$ and $\psi \in H^1(\partial\Omega)$ so that $\phi = \operatorname{curl}_\partial\psi = v \times \nabla_\partial\psi$. Let $\tilde{w} \in H^1(\Omega)$ solve $\operatorname{div}\epsilon\nabla\tilde{w} = 0$ in Ω and $\operatorname{tr}\tilde{w} = \psi$ at $\partial\Omega$. With (31), it follows $\nabla\tilde{w} - P_{W_2}\nabla\tilde{w} =: w \in W_1$ and $B_{\mathrm{tr}}w = \phi$. Thus, $\operatorname{ran} B_{\mathrm{tr}}|_{W_1} = \operatorname{curl}_\partial H^1(\partial\Omega)$ and

$$\tilde{K}_V(0) = B_{\mathrm{tr}}^*(I-P)B_{\mathrm{tr}} = B_{\mathrm{tr}}^*(I - P_{\nabla_\partial^\perp})B_{\mathrm{tr}} = B_{\mathrm{tr}}^*P_{\nabla_\partial}B_{\mathrm{tr}}.$$

□

Lemma 6.6 *Let Assumptions 3.1, 3.2, 3.3 hold true. Thence,*

$$\dim(\ker B_{\mathrm{tr}}|_V)^{\perp_V} = \dim(\ker P_{\nabla_\partial}B_{\mathrm{tr}}|_V)^{\perp_V} = \infty.$$

Proof Let $(f_n)_{n\in\mathbb{N}}$ be an orthonormal basis of $\nabla_\partial H^1(\partial\Omega) \subset \mathbf{L}_t^2(\partial\Omega)$. Let $u_n \in H(\operatorname{curl};\Omega)$ be so that $\operatorname{tr}_{v\times} u_n = f_n$. Hence $u_n \in X$. It follows

$$P_{\nabla_\partial}B_{\mathrm{tr}}(P_V u_n + \ker P_{\nabla_\partial}B_{\mathrm{tr}}|_V) = f_n.$$

Thus if $\sum_{n=1}^N c_n(P_V u_n + \ker P_{\nabla_\partial}B_{\mathrm{tr}}|_V)$ would be a nontrivial linear combination of zero in $V/(\ker P_{\nabla_\partial}B_{\mathrm{tr}}|_V)$, then $\sum_{n=1}^N c_n f_n$ would be a nontrivial linear combination of zero in $\nabla_\partial H^1(\partial\Omega)$. Hence, $\dim V/(\ker P_{\nabla_\partial}B_{\mathrm{tr}}|_V) = +\infty$. Since $\ker B_{\mathrm{tr}} \subset \ker P_{\nabla_\partial}B_{\mathrm{tr}}$, it follows $\dim V/(\ker B_{\mathrm{tr}}|_V) \ge \dim V/(\ker P_{\nabla_\partial}B_{\mathrm{tr}}|_V)$ and thus the dimension of $\dim V/(\ker B_{\mathrm{tr}}|_V$ is infinite too. The claim follows from $\dim V/Z = \dim Z^{\perp_V}$ for any closed subspace $Z \subset V$. □

We require the following additional assumption for Lemma 6.8.

Assumption 6.7 (ω^2 *is no "Dirichlet" eigenvalue*) Let

$$Z_1 := \{z \in V : B_{\mathrm{tr}} z = 0\} = H(\mathrm{curl}, \mathrm{div}\epsilon^0, \mathrm{tr}^0_{\nu\times}, \mathrm{tr}^0_{\nu\cdot\epsilon}; \Omega)$$

and denote P_{Z_1} the X-orthogonal projection onto Z_1. The operator

$$P_{Z_1} A_c|_{Z_1} - \omega^2 P_{Z_1} A_\epsilon|_{Z_1} \in L(Z_1)$$

is bijective.

Lemma 6.8 *Let Assumptions 3.1–3.3, 5.1 and 6.7 hold true. Let $\tilde\lambda \in (0, c_\infty^{-1})$. Thence*

$$\ker \Big(\tilde K_V(\tilde\lambda)^{1/2} \big(P_V(A_c - \omega^2 A_\epsilon)|_V\big)^{-1} \tilde K_V(\tilde\lambda)^{1/2}\Big) = \ker \tilde K_V(\tilde\lambda).$$

Proof Let $v \in \ker \Big(\tilde K_V(\tilde\lambda)^{1/2} \big(P_V(A_c - \omega^2 A_\epsilon)|_V\big)^{-1} \tilde K_V(\tilde\lambda)^{1/2}\Big)$ and

$$z := \big(P_V(A_c - \omega^2 A_\epsilon)|_V\big)^{-1} \tilde K_V(\tilde\lambda)^{1/2}\Big) v.$$

It follows $B_{\mathrm{tr}} z = 0$ due to $\ker K_V(\tilde\lambda)^{1/2} = \ker K_V(\tilde\lambda)$ and Lemma 6.4. Due to the definitions of z and Z_1, $z \in Z_1$ solves

$$(P_{Z_1} A_c - \omega^2 P_{Z_1} A_\epsilon) z = 0.$$

It follows from Assumption 6.7 that $z = 0$. Thus, $v \in \ker K_V(\tilde\lambda)^{1/2} = \ker K_V(\tilde\lambda)$. □

We require the following additional assumption for Lemma 6.10.

Assumption 6.9 (ω^2 *is no "hybrid" eigenvalue*) Let

$$Z_2 := \{z \in V : P_{\nabla_\partial} B_{\mathrm{tr}} z = 0\}$$

and denote P_{Z_2} the X-orthogonal projection onto Z_2. The operator

$$P_{Z_2} A_c|_{Z_2} - \omega^2 P_{Z_2} A_\epsilon|_{Z_2} \in L(Z_2)$$

is bijective.

Lemma 6.10 *Let Assumptions 3.1–3.3, 5.1 and 6.9 hold true. Thence,*

$$\ker \Big(\tilde K_V(0)^{1/2} \big(P_V(A_c - \omega^2 A_\epsilon)|_V\big)^{-1} \tilde K_V(0)^{1/2}\Big) = \ker \tilde K_V(0).$$

Proof Let $v \in \ker \Big(\tilde K_V(0)^{1/2} \big(P_V(A_c - \omega^2 A_\epsilon)|_V\big)^{-1} \tilde K_V(0)^{1/2}\Big)$ and

$$z := \big(P_V(A_c - \omega^2 A_\epsilon)|_V\big)^{-1} \tilde K_V(0)^{1/2}\Big) v.$$

It follows $P_{\nabla_\partial} B_{\mathrm{tr}} z = 0$ due to $\ker K_V(0)^{1/2} = \ker K_V(0)$ and Lemma 6.5. Due to the definitions of z and Z_2, $z \in Z_2$ solves

$$(P_{Z_2} A_c - \omega^2 P_{Z_2} A_\epsilon) z = 0.$$

It follows from Assumption 6.9 that $z = 0$. Thus, $v \in \ker K_V(0)^{1/2} = \ker K_V(0)$. □

We require the following additional assumption for Lemma 6.12.

Assumption 6.11 (ω^2 *is no "projected" eigenvalue*) The operators

$$P_{Z_1}\big(P_V(A_c-\omega^2 A_\epsilon)|_V\big)^{-1}|_{Z_1}\in L(Z_1)\ \text{and}\ P_{Z_2}\big(P_V(A_c-\omega^2 A_\epsilon)|_V\big)^{-1}|_{Z_2}\in L(Z_2)$$

are bijective.

We note that

$$P_V A_c|_V = P_V(I - A_\epsilon - A_{\mathrm{tr}})|_V \tag{59}$$

and consequently

$$\begin{aligned} P_V(A_c-\omega^2 A_\epsilon)|_V\big)^{-1} &= I|_V - P_V(A_c-\omega^2 A_\epsilon)|_V\big)^{-1} P_V((\omega^2+1)A_\epsilon + A_{\mathrm{tr}})|_V \\ &=: I_V + G. \end{aligned} \tag{60}$$

Lemma 6.12 *Let Assumptions 3.1–3.4 and 5.1, 6.7, 6.9, 6.11 hold true. Let $\tilde\lambda \in [0, c_\infty^{-1})$. The spectrum of $\tilde A_V(\cdot,\tilde\lambda)$ consists of $\sigma_{\mathrm{ess}}\big(\tilde A_V(\cdot,\tilde\lambda)\big) = \{0\}$ and an infinite sequence of nonzero eigenvalues $(\tilde\tau_n(\tilde\lambda))_{n\in\mathbb{N}}$ with $\lim_{n\in\mathbb{N}} \tilde\tau_n(\tilde\lambda) = 0$. Apart from a finite number, all nonzero eigenvalues $(\tilde\tau_n(\tilde\lambda))_{n\in\mathbb{N}}$ are positive.*

Proof We note that $\tilde A_V(\cdot,\tilde\lambda)$ and $P_V(A_c-\omega^2 A_\epsilon)|_V\big)^{-1}\tilde A_V(\cdot,\tilde\lambda)$ have the very same spectral properties. We aim to apply Lemma 6.3 to

$$P_V(A_c-\omega^2 A_\epsilon)|_V\big)^{-1}\tilde A_V(\cdot,\tilde\lambda) = \tilde\tau I - (I+G)\tilde K_V(\tilde\lambda)$$

with G defined as in (60). G is compact due to Lemmas 3.8 and 3.7. Since $P_V(A_c - \omega^2 A_\epsilon)|_V\big)^{-1}$ and the identity are self-adjoint, the self-adjointness of G follows from (60). $I+G$ is bijective due to its definition and Assumption 5.1. $\tilde K_V(\tilde\lambda)$ is compact, self-adjoint and positive semi-definite due to Lemma 6.4. It holds $\ker\Big(\tilde K_V(\tilde\lambda)^{1/2}(I+G)\tilde K_V(\tilde\lambda)^{1/2}\Big) = \ker \tilde K_V(\tilde\lambda)$ due to Lemmas 6.8 and 6.10. It holds $\dim(\ker \tilde K_V(\tilde\lambda))^\perp = \infty$ due to Lemma 6.6. $P_{(\ker \tilde K_V(\tilde\lambda))^\perp}(I+G)|_{(\ker \tilde K_V(\tilde\lambda))^\perp}$ is bijective due to Assumption 6.11. Hence, the conditions of Lemma 6.3 are satisfied and the claim follows. □

Lemma 6.13 *Let Assumptions 3.1–3.3 and 5.1, 6.7, 6.9, 6.11 hold true. For $\tilde\lambda \in [0, c_\infty^{-1})$, let $(\tilde\tau_n^+(\tilde\lambda))_{n\in\mathbb{N}}$ be a non-increasing ordering with multiplicity taken into account of the positive eigenvalues of $\tilde A_V(\cdot,\tilde\lambda)$. Thence, for each $n\in\mathbb{N}$ the function $\tilde\tau_n^+ \colon [0, c_\infty^{-1}) \to \mathbb{R}^+$ is continuous.*

Proof We note that for each $n \in \mathbb{N}$ it holds $\inf_{\tilde\lambda\in[0,c_\infty^{-1})} \tilde\tau_n^+(\tilde\lambda) > 0$: Indeed, the existence of $\tilde\lambda_0 \in [0, c_\infty^{-1})$, $n\in\mathbb{N}$ so that $\lim_{\tilde\lambda\to\tilde\lambda_0+} \tilde\tau_n^+(\tilde\lambda) = 0$ would imply that for $\tilde\lambda = \tilde\lambda_0$ there would exist only a finite number of positive eigenvalues, which is a contradiction to Lemma 6.12. The continuity of $\tilde\tau_n^+$ follows with Sanchez Hubert and Sánchez-Palencia [27, Proposition 5.4]. We note that a delicate part of Sanchez Hubert and Sánchez-Palencia [27, Proposition 5.4] is the existence of eigenvalues. However, the existence of eigenvalues is already established by Lemma 6.12. We only require the continuity result of Sanchez Hubert and Sánchez-Palencia [27, Proposition 5.4]. □

Theorem 6 *Let Assumptions 3.1–3.4 and 5.1, 6.7, 6.9, 6.11 hold true. Thence, there exists an infinite sequence $(\lambda_n)_{n\in\mathbb{N}}$ of positive eigenvalues to $A_X(\cdot)$ which accumulate at $+\infty$.*

Proof Proceed as in the proof of Theorem 4. □

7 Conclusion

We conclude with a summary of Theorems 2–6 and some remarks on assumptions and the relation to the modified electromagnetic Steklov eigenvalue considered in [12,22].

7.1 Main result

We formulate the individual results of the previous sections in the following proposition.

Proposition 7 *Let Assumptions 3.1–3.4 and 5.1, 6.1, 6.7, 6.9, 6.11 be satisfied. Then, it holds*

$$\sigma\big(A_X(\cdot)\big) = \sigma_{\mathrm{ess}}\big(A_X(\cdot)\big) \dot\cup \bigcup_{n\in\mathbb{N}} \{\lambda_n^{-0}\} \dot\cup \bigcup_{n\in\mathbb{N}} \{\lambda_n^{+\infty}\} \tag{61}$$

and $\sigma_{\mathrm{ess}}\big(A_X(\cdot)\big) = \{0\}$. The sequence $(\lambda_n^{-0})_{n\in\mathbb{N}}$ consists of pairwise distinct negative eigenvalues with finite algebraic multiplicity so that $\lim_{n\in\mathbb{N}} \lambda_n^{-0} = 0$. The sequence $(\lambda_n^{+\infty})_{n\in\mathbb{N}}$ consists of pairwise distinct positive eigenvalues with finite algebraic multiplicity so that $\lim_{n\in\mathbb{N}} \lambda_n^{+\infty} = +\infty$.

Proof Follows from Theorems 2–6. □

7.2 Remarks to the assumptions

The condition in Assumptions 3.1 and 3.2 that μ and ϵ equal the identity matrix in a neighborhood of the boundary is used to obtain extra regularity of traces. If this extra regularity can be derived by other means, then the mentioned assumption becomes obsolete.

Each of the Assumptions 5.1, 6.1, 6.7, 6.9, 6.11 can be formulated in the following manner: Y is a Hilbert space, $A \in L(Y)$ is weakly coercive, $K(\cdot)\colon \Lambda \subset \mathbb{C} \to K(Y)$ is holomorphic and it is imposed that $A - K(\omega^2)$ is bijective. Consequently, for fixed domain Ω and fixed material parameters μ^{-1}, ϵ there exists only a countable set of frequencies ω for which the Assumptions 5.1, 6.1, 6.7, 6.9, 6.11 are not satisfied (see e.g. [24, Proposition A.8.4]).

7.3 Modified electromagnetic Steklov eigenvalues

The modified electromagnetic Steklov eigenvalue problem considered in [22] is to find $(\lambda, u) \in \mathbb{C} \times H(\mathrm{curl}; \Omega) \setminus \{0\}$ so that

$$\langle \mu^{-1} \operatorname{curl} u, \operatorname{curl} u' \rangle_{\mathbf{L}^2(\Omega)} - \omega^2 \langle \epsilon u, u' \rangle_{\mathbf{L}^2(\Omega)} - \lambda \langle Su, Su' \rangle_{\mathbf{L}^2_t(\partial\Omega)} = 0 \tag{62}$$

for all $u' \in H(\mathrm{curl}; \Omega)$ (with S defined as in (24)). It can easily be seen that the eigenvalue problem decouples with respect to the decomposition $H(\mathrm{curl}; \Omega) = H(\mathrm{curl}, \operatorname{div}\epsilon^0, \mathrm{tr}^0_{\nu\cdot\epsilon}; \Omega) \oplus \nabla H^1(\Omega)$. Thus, the eigenvalue problem can be reformulated to find $(\lambda, u) \in \mathbb{C} \times H(\mathrm{curl}, \operatorname{div}\epsilon^0, \mathrm{tr}^0_{\nu\cdot\epsilon}; \Omega) \setminus \{0\}$ so that

$$
\begin{aligned}
0 &= \langle \mu^{-1} \operatorname{curl} u, \operatorname{curl} u' \rangle_{\mathbf{L}^2(\Omega)} - \omega^2 \langle \epsilon u, u' \rangle_{\mathbf{L}^2(\Omega)} - \lambda \langle Su, Su' \rangle_{\mathbf{L}^2_t(\partial\Omega)} \\
&= \langle \mu^{-1} \operatorname{curl} u, \operatorname{curl} u' \rangle_{\mathbf{L}^2(\Omega)} - \omega^2 \langle \epsilon u, u' \rangle_{\mathbf{L}^2(\Omega)} - \lambda \langle P_{\nabla_\partial} \operatorname{tr}_{\nu\times} u, \operatorname{tr}_{\nu\times} u' \rangle_{\mathbf{L}^2_t(\partial\Omega)} \\
&= \langle \lambda \tilde{A}_V(\lambda^{-1}, 0) u, u' \rangle_X
\end{aligned}
\tag{63}
$$

for all $u' \in H(\operatorname{curl}, \operatorname{div}\epsilon^0, \operatorname{tr}^0_{\nu\cdot\epsilon}; \Omega)$. Thence if the respective assumptions are satisfied, Lemma 6.12 yields that the spectrum consists of an infinite sequence of eigenvalues $(\lambda_n)_{n\in\mathbb{N}}$ which accumulate only at $+\infty$. A similar existence result has been reported in [12, Theorem 3.6]. Though it seems to us that the proof of [12, Theorem 3.6] requires $\dim(\ker \mathbf{T})^\perp = \infty$ which the authors don't elaborate on. The former observation admits to interpret the modified electromagnetic Steklov eigenvalue problem as asymptotic limit of the original electromagnetic Steklov eigenvalue problem for large eigenvalue parameter λ. We have seen that (at least in the self-adjoint case) the original electromagnetic Steklov eigenvalue problem yields two kinds of spectra. Contrary the modified electromagnetic Steklov eigenvalue problem yields only one kind of spectrum. This suggests that for inverse scattering applications the original version is more advantageous than the modified version, because it contains more information, though the approximation of the modified eigenvalue problem is better understood than for the original version [22].

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