Electromagnetic surface and line sources under coordinate transformations

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Although the analysis of electromagnetic sources in the context of coordinate transformations and the implications for transformation optics have been discussed in the literature, a correct formulation that includes surface and line currents has not been reported. Here we derive how surface and line currents behave under coordinate transformations and validate the analysis through numerical validation of a specific example. This analysis enables transformation optics to be applied to problems that include singular source distributions, which is often the case for practical radiating systems and antennas, to make a given current distribution produce the same fields as a different current distribution by surrounding it with a material with specific electromagnetic properties.

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I. INTRODUCTION

The work of Greenleaf *et al.* [1] and Pendry *et al.* [2] introduced the way in which coordinate transformations of steady electric current and electromagnetic fields, respectively, can be physically implemented with a complex medium, offering a new paradigm for the control of electromagnetic fields around arbitrary objects. Less complete is the theory describing how electromagnetic sources, namely, currents and charges, are altered by these coordinate transformation media. Past work has examined the interaction of sources with transformation optics media in special cases. For example, Zolla *et al.* [3] showed how a line source within an electromagnetic cloaking shell radiates fields as if the source were in a different location. Greenleaf *et al.* [4], Zhang *et al.* [5], and Weder [6] analyzed the detailed behavior of sources in the interior of cloaked regions.

Luo *et al.* [7] examined the general problem of source behavior in transformation optics to show how currents and charges change under coordinate transformations. Importantly, they conceptually demonstrated how a coordinate transformation medium could be used to make one current distribution radiate like an entirely different one. Conformal antennas are one potential application of such an approach, in which currents may be constrained to a given surface but one wishes to have these currents radiate as if they were in a different location or had a different shape [7]. Kundtz *et al.* [8] demonstrated this approach through numerical simulations, confirming that complex current distributions can be made to radiate like simple ones when surrounded by a properly designed transformation optics medium. However, the analysis of [7] is incomplete in several areas and, importantly, does not give correct expressions for the of how currents constrained to surfaces or lines transform. These singular current distributions are practically important as many physical antennas and radiating systems can be usefully described in terms of surface and line currents.

In this work we add to the description of the behavior of electromagnetic sources under coordinate transformations by deriving expressions for how surface and line current distributions transform, and thus how their radiation can be intentionally manipulated through transformation optics.

The results are illustrated through two examples, namely, the mapping of a planar surface current to a nonplanar configuration and the mapping of a line current segment to a surface current on a sphere. In these examples the mechanics of transforming surface and line currents are demonstrated in two ways: treating the surface current as a vector implicitly confined to a surface, and treating the surface current as a volume current with delta functions. We show that these forms are equivalent, and which one is more useful depends on the specifics of the problem. In the former case, we present numerical simulations that demonstrate the applicability of our results to practical problems of source manipulation.

II. VOLUME CURRENTS AND CURRENT CONSERVATION

The manner in which electromagnetic field vectors transform has been analyzed both mathematically and physically in recent work [2,4,7,9–11]. In this section, we briefly summarize for completeness these results as they apply to electromagnetic sources, and we explicitly demonstrate the conservation of charge and current in transformation optics as this property plays a key role in the analysis that follows.

Current density $\mathbf{J}$ is a contravariant vector density [12], and as such it behaves in a specific way under coordinate transformations to preserve the continuity of its normal component at an interface. Mathematically, with a coordinate transformation defined by $\bar{r}'=F(\bar{r})$, the transformed current density $\mathbf{J}'$ can be written in terms of the original current density $\mathbf{J}$ as [7]

$$\mathbf{J}'(\bar{r}) = T_{\rm con}^{\rm vd}\mathbf{J}(F^{-1}(\bar{r})) \quad \text{or} \quad \mathbf{J}' = \frac{1}{\det A}A\mathbf{J}. \qquad (1)$$

In the former expression, we use the symbol $T_{\rm con}^{\rm vd}$ to generically denote the transformation operator for a contravariant (hence the subscript *con*) vector density (hence the superscript vd), and in the latter expression this operator is ex-

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pressed in terms of the Jacobian matrix, denoted here by A, of the transformation $\bar{r}' = F(\bar{r})$. Note that there are different ways to express this contravariant vector density transformation operator $T^{\text{vd}}_{\text{con}}$, including in terms of unit vectors and length scaling factors [2,10]. The $\det A$ term that appears in the denominator of Eq. (1), which also appears in the denominator of several important expressions that follow, means that transformations that are singular in critical locations (i.e., $\det A = 0$) will require special care in analyzing how sources behave (e.g., [13]). One of these examples is treated in Sec. V below, although we emphasize that the approach used may not be completely general for locally singular transformations.

In this work we use vector and matrix notation in which general vectors are understood to be three element column vectors. Such expressions can also be written in component form. For example, under a general coordinate transformation a contravariant vector $\bar{m}$ transforms as

$$m^{i'} = \sum_i A^{i'}_i m^i, \tag{2}$$

where $A^{i'}_i = \partial u^{i'} / \partial u^i$ and the equivalent in matrix form is $\bar{m}' = A\bar{m} = T^{\text{v}}_{\text{con}}\bar{m}$. Similarly, a covariant vector $\bar{n}$ transforms as

$$n_{i'} = \sum_i A^i_{i'} n^i, \tag{3}$$

and because the matrix formed from $A^{i'}_i$ is the inverse of that formed from $A^i_{i'}$ and the sum is over the first index, the equivalent in matrix form is $\bar{n}' = (A^{-1})^T \bar{n} = T^{\text{v}}_{\text{co}}\bar{n}$. Explicit distinctions are made between covariant and contravariant vectors and each is transformed separately so that the metric of the transformation is not required when forming inner products.

Note that there are two physical meanings of the expressions in Eq. (1). In one, the components of $\mathbf{J}'$ are the contravariant components in terms of the *new* basis vectors defined by the transformation. In this case, $\mathbf{J}'$ and $\mathbf{J}$ are the same vector described in different coordinate systems. In the other, and the one relevant for transformation optics, the components of $\mathbf{J}'$ are the contravariant components of the vector in the *original* basis vectors, and thus $\mathbf{J}'$ is a new vector that shows how $\mathbf{J}$ would change in the presence of a medium derived through transformation optics. Physically, the coordinate change describes the translation of the current density from the original location to the transformed location, and the transformation operator $T^{\text{vd}}_{\text{con}}$ scales and rotates the vector so that it transforms as a contravariant vector density. For clarity in the derivations to follow we write these vector transformations in general T^{-}_{-} terms and simplify to Jacobian matrix terms in the end.

A result useful in the analysis that follows is that charge and current-density flux are conserved in transformation optics. This is stated without demonstration by Luo *et al.* [7], and because it is needed in our analysis below, it is explicitly demonstrated in the Appendix.

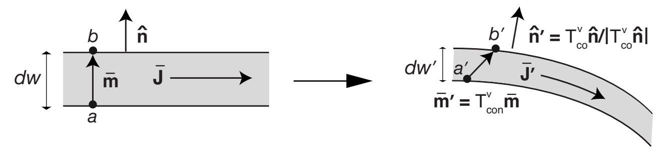

FIG. 1. An illustration of how a finite current channel behaves under coordinate transformation.

III. SURFACE AND LINE CURRENTS

As many antennas involve currents that are restricted to surfaces or lines, how surface current and line current vectors transform is both theoretically and practically relevant. Although surface currents were addressed by Luo *et al.* [7], the expressions given there are not correct in general. Closed form expressions for these operations in terms of transformation operators are derived below.

Since surface and line currents are limiting cases of volume currents, the direction of transformed surface and line currents must be also defined by Eq. (1). However, transformations of volume currents also compress or expand them into smaller or larger volumes, respectively, and the volume current magnitude must change correspondingly to conserve the total current. This idea is illustrated in Fig. 1, in which a channel of width dw containing volume current $\mathbf{J}$ is transformed to a narrower channel of width dw' containing volume current $\mathbf{J}'$. Thus to conserve the total current through this channel, $|\mathbf{J}'| > |\mathbf{J}|$, and this amplitude scaling is naturally produced by the transformation in Eq. (1).

Surface and line currents have zero extent in one or two directions, however, and thus these currents cannot be further compressed or expanded in these directions of zero extent by a coordinate transformation. Thus, to transform a surface or line current, the amplitude scaling produced by Eq. (1) corresponding to current compression or expansion in the directions normal to the surface or line current must be undone.

For a line current the implications are straightforward. Because a line current has zero extent in both directions transverse to the current flow, all of the magnitude scaling from Eq. (1) must be undone. Thus for a one-dimensional line current $\mathbf{J}_\ell$, the transformed line current $\mathbf{J}'_\ell$ is

$$\mathbf{J}'_\ell = \frac{|\mathbf{J}_\ell|}{|T^{\text{vd}}_{\text{con}}\mathbf{J}_\ell|} T^{\text{vd}}_{\text{con}}\mathbf{J}_\ell \tag{4}$$

or

$$\mathbf{J}'_\ell = \frac{|\mathbf{J}_\ell|}{\left|\dfrac{A}{\det A}\mathbf{J}_\ell\right|} \frac{A}{\det A}\mathbf{J}_\ell = \frac{|\mathbf{J}_\ell|}{|A\mathbf{J}_\ell|} A\mathbf{J}_\ell, \tag{5}$$

where again the latter expression expresses the transformation explicitly in terms of the Jacobian matrix A. The transformation simply drags and deforms an infinitely thin current-carrying wire without changing the line current magnitude. Since total current is the same as line current magnitude, this expression is consistent with current conservation.

For a two-dimensional surface current $\mathbf{J}_s$ the situation is more complicated because the current magnitude scaling must be undone in only the direction normal to the surface. Let $\bar{m}$ be the vector of length dw that points across the width of a thin volume current distribution from point a to point b, as in the left panel of Fig. 1, which before transformation is parallel to the unit normal to the thin sheet $\hat{n}$. This displacement vector $\bar{m}$ is a contravariant vector [12] and thus after transformation becomes the vector between transformed points a' and b' or

$$\bar{m}' = T^{\mathrm{v}}_{\mathrm{con}}\bar{m}. \tag{6}$$

Although it still extends across the thin current sheet, $\bar{m}'$ is not necessarily normal to the sheet if the transformation is not orthogonal, as illustrated by the right panel of Fig. 1. Normal vectors are covariant vectors and thus the unit normal to this transformed thin current sheet is given by

$$\hat{n}' = \frac{T^{\mathrm{v}}_{\mathrm{co}}\hat{n}}{|T^{\mathrm{v}}_{\mathrm{co}}\hat{n}|}. \tag{7}$$

Note that $\hat{n}'$ is not simply the transformed unit vector $\hat{n}$ but has had its length rescaled so that it is still a unit vector after the transformation.

We need to determine the factor by which the width of this thin volume current has been expanded by the transformation. Let it be denoted by $s = dw'/dw$. Since $dw = |\bar{m}|$ and $dw' = \bar{m}' \cdot \hat{n}'$, we find

$$s = \frac{(T^{\mathrm{v}}_{\mathrm{con}}\bar{m}) \cdot (T^{\mathrm{v}}_{\mathrm{co}}\hat{n})}{|\bar{m}||T^{\mathrm{v}}_{\mathrm{co}}\hat{n}|} = \frac{(T^{\mathrm{v}}_{\mathrm{con}}\hat{n}) \cdot (T^{\mathrm{v}}_{\mathrm{co}}\hat{n})}{|T^{\mathrm{v}}_{\mathrm{co}}\hat{n}|}, \tag{8}$$

where in the latter expression we have used the pretransformation relation $\bar{m}/|\bar{m}| = \hat{n}$. The transformed volume current density is inherently scaled by s^{-1} due to the geometric expansion of the channel in the direction normal to the transformed thin current sheet. Consequently, this is the factor that must be removed in a transformation of a surface current, and therefore we multiply Eq. (1) by s and find that

$$\mathbf{J}'_s = \frac{(T^{\mathrm{v}}_{\mathrm{con}}\hat{n}) \cdot (T^{\mathrm{v}}_{\mathrm{co}}\hat{n})}{|T^{\mathrm{v}}_{\mathrm{co}}\hat{n}|} T^{\mathrm{vd}}_{\mathrm{con}}\mathbf{J}_s. \tag{9}$$

where $\hat{n}$ is the unit vector normal to the untransformed surface current.

This can be further simplified by noting that

$$(T^{\mathrm{v}}_{\mathrm{con}}\hat{n}) \cdot (T^{\mathrm{v}}_{\mathrm{co}}\hat{n}) = [(A^{-1})^T\hat{n}]^T(A\hat{n}) = \hat{n}^T A^{-1} A\hat{n} = 1, \tag{10}$$

which yields

$$\mathbf{J}'_s = \frac{1}{|T^{\mathrm{v}}_{\mathrm{co}}\hat{n}|} T^{\mathrm{vd}}_{\mathrm{con}}\mathbf{J}_s \tag{11}$$

or

$$\mathbf{J}'_s = \frac{1}{|(A^{-1})^T\hat{n}|}\frac{A}{(\det A)}\mathbf{J}_s \tag{12}$$

Thus, Eqs. (1), (5), and (12) describe how volume current, line current, and surface current, respectively, behave under coordinate transformations.

Equation (12) is consistent with the electromagnetic boundary condition associated with surface currents, namely, $\mathbf{J}_s = \hat{n} \times \Delta\mathbf{H}$. The vectors $\hat{n}$ and $\mathbf{H}$ are both covariant vectors and their cross product is a contravariant vector density [12]. Thus after a coordinate transformation this boundary condition transforms to

$$T^{\mathrm{vd}}_{\mathrm{con}}\mathbf{J}_s = T^{\mathrm{v}}_{\mathrm{co}}\hat{n} \times \Delta(T^{\mathrm{v}}_{\mathrm{co}}\mathbf{H}). \tag{13}$$

While true, the above needs to have the term $T^{\mathrm{v}}_{\mathrm{co}}\hat{n}$ renormalized to unit length to be useful as a boundary condition, and thus

$$\frac{1}{|T^{\mathrm{v}}_{\mathrm{co}}\hat{n}|}T^{\mathrm{vd}}_{\mathrm{con}}\mathbf{J}_s = \mathbf{J}'_s = \frac{1}{|T^{\mathrm{v}}_{\mathrm{co}}\hat{n}|}T^{\mathrm{v}}_{\mathrm{co}}\hat{n} \times \Delta(T^{\mathrm{v}}_{\mathrm{co}}\mathbf{H}) = \hat{n}' \times \Delta\mathbf{H}', \tag{14}$$

which gives the same expression for the transformed surface current as Eq. (12) and preserves this boundary condition under coordinate transformations.

Equation (5) for a line current is also equivalent to Eq. (12) when the transformation-induced current compression is undone in two directions as it must be for a transformed line current. Let $\hat{n}_1$ and $\hat{n}_2$ be orthogonal unit vectors that are each orthogonal to a line current $\mathbf{J}_\ell = |\mathbf{J}_\ell|\hat{u}_3$, where $\hat{u}_3 = \hat{n}_1 \times \hat{n}_2$. The line current scaling factor in Eq. (5) can thus be rewritten as

$$\frac{|\mathbf{J}_\ell|}{\left|\frac{A}{\det A}\mathbf{J}_\ell\right|} = \frac{|\mathbf{J}_\ell|}{\left|\frac{A}{\det A}\right||\mathbf{J}_\ell|\hat{u}_3|} = \frac{1}{\left|\frac{A}{\det A}\hat{u}_3\right|}$$
$$= \frac{1}{|(A^{-1})^T\hat{n}_1 \times (A^{-1})^T\hat{n}_2|} = \frac{1}{|(A^{-1})^T\hat{n}_1||(A^{-1})^T\hat{n}_2|}. \tag{15}$$

Therefore the line current scaling factor in Eq. (5) is thus identical to the product of two surface current scaling factors in orthogonal directions, as expected.

We wish to emphasize that applying the source transformation expressions in Eqs. (1), (5), and (12) to transformations in which $\det(A)=0$ in certain locations will likely require special care. It has been shown that such locally singular transformations can result in unusual material properties or wave behavior in these regions [13,14], and placing singular or nonsingular sources in these regions seems likely to lead to further anomalous behavior. The precise form of this behavior appears to depend on the details of the transformation [13,14], and thus we do not attempt to treat this issue here in a general way.

IV. SURFACE CURRENT UNDER A NONORTHOGONAL TRANSFORMATION

We demonstrate and validate through simulation the above analysis on a surface current confined to $y=0$, on the surface of a perfect magnetic conductor (PMC) half-space, which is transformed to the triangular bump illustrated in Fig. 2. The overall transformation is limited to a two-dimensional box with sides $-1 < x < 1$ and $0 < y < 1$ and is described by

$$x' = x,$$
$$y' = [1 - a(1 - |x|)]y + a(1 - |x|),$$
$$z' = z. \tag{16}$$

For this transformation the Jacobian matrix and its inverse transpose are

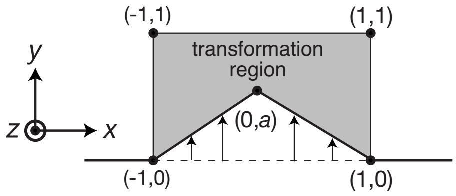

FIG. 2. An illustration of the coordinate transformation described by Eq. (16).

$$A=\begin{bmatrix}1 & 0 & 0\\ c & b & 0\\ 0 & 0 & 1\end{bmatrix}\quad (A^{-1})^T=\begin{bmatrix}1 & -c/b & 0\\ 0 & 1/b & 0\\ 0 & 0 & 1\end{bmatrix}, \tag{17}$$

with $b=1-a(1-|x'|)$ and $c=a\,\mathrm{sgn}(x)y'-1/b$ for convenience. Note that $\det(A)=b$.

To illustrate the analysis, we consider two different vector directions for the original $y=0$ surface current, $\mathbf{J}_{s1}=J_0\hat{z}$ and $\mathbf{J}_{s2}=J_0\hat{x}$. These currents will radiate orthogonally polarized uniform plane waves when on a PMC surface. In both cases the unit normal to the original surface current is $\hat{n}=\hat{y}$. Applying Eq. (12), the resulting general transformed surface current vector is

$$\mathbf{J}_s'=\frac{A}{b}\mathbf{J}_s\frac{1}{|(A^{-1})^T\hat{y}|}=\frac{A}{\sqrt{c^2+1}}\mathbf{J}_s, \tag{18}$$

since $|(A^{-1})^T\hat{y}|=b^{-1}\sqrt{c^2+1}$. We now revert to the original basis and coordinates by dropping the primes and thus let the above expression represent a new surface current in the original coordinates.

The resulting transformed current densities $\mathbf{J}_{s1}'$ and $\mathbf{J}_{s2}'$ are confined in both cases for $-1<x<1$ to the $y=a(1-|x|)$ surface (the transformed $y=0$ plane) and are given by

$$\mathbf{J}_{s1}'=\frac{J_0}{\sqrt{c^2+1}}\hat{z},$$

$$\mathbf{J}_{s2}'=\frac{J_0}{\sqrt{c^2+1}}(\hat{x}+c\hat{y}). \tag{19}$$

On the $y=a(1-|x|)$ plane, $c=-a\,\mathrm{sgn}(x)$ and we thus have on the $y=a(1-|x|)$ surface for $-1<x<1$

$$\mathbf{J}_{s1}'=\frac{J_0}{\sqrt{a^2+1}}\hat{z},$$

$$\mathbf{J}_{s2}'=\frac{J_0}{\sqrt{a^2+1}}[\hat{x}-a\,\mathrm{sgn}(x)\hat{y}]. \tag{20}$$

The surface currents are thus dragged to a new location and direction by the transformation. For case 1, the total transformed current flowing in the z direction between $x=-1$ and $x=1$ is $2J_0$ and equals the untransformed current. For case 2, the total transformed current per unit z width is J_0 and also equals the untransformed current per unit width. Thus, in both cases, the total current is conserved, and the $|(A^{-1})^T\hat{y}|$ scaling factor from Eq. (12) plays a critical role in correctly scaling the transformed current.

Case 1 is straightforward to simulate numerically using the COMSOL Multiphysics solver and we do so now to demonstrate and validate the analysis. The relative permittivity and permeability of the transformation electromagnetic medium inside the region defined by $-1<x<1$ and $0<y<1$ is given by [15]

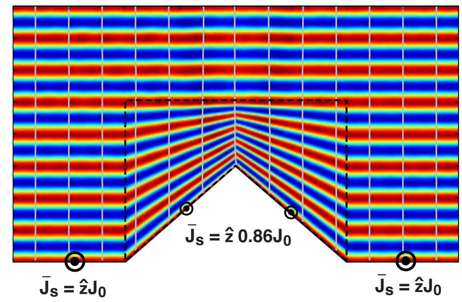

FIG. 3. (Color online) The numerically simulated electric field produced by a nonuniform surface current on the lower boundary of the domain. In the presence of the transformation electromagnetic medium contained in the region bounded by the dashed lines, the radiated fields are identical to those produced by a uniform and flat surface current in free space. The computed power flow direction indicated by the gray lines remains purely in the y direction, even in the anisotropic transformation medium.

$$\epsilon_r'=\mu_r'=\frac{AA^T}{\det A}=\frac{1}{b}\begin{bmatrix}1 & c & 0\\ c & c^2+b^2 & 0\\ 0 & 0 & 1\end{bmatrix}. \tag{21}$$

We choose the specific value $a=0.6$, which results in the transformed current density $\mathbf{J}_{s1}'=0.86J_0\hat{z}$ along the triangular bump from $-1<x<1$. The source frequency is 1.5 GHz. Figure 3 shows a time snapshot of the resulting electric field (E_z) distribution produced by this nonuniform surface current distribution radiating on a PMC region. In the free space region outside the transformed area, the electric field is exactly the uniform plane wave that would be produced by a uniform surface current along the $y=0$ plane radiating in free space. Small nonuniformities are present that we attribute to the discrete finite element approach used in the simulation. This confirms that the combination of the transformation medium from Eq. (21) with the deformed and appropriately scaled surface current from Eq. (12) yields the expected result. If the correct scaling factor of 0.86 from Eq. (12) is not applied to the transformed surface current, the resulting simulated fields (not shown) are clearly not uniform in amplitude and are not equal to the fields produced by the untransformed uniform surface current in free space.

We note that the resulting electric and magnetic field directions are also those expected from theory (e.g., [11]). Both $\mathbf{E}$ and $\mathbf{H}$ are covariant vectors and thus transform as $\mathbf{E}'=(A^{-1})^T\mathbf{E}$. For the untransformed problem, the radiated plane-wave fields are $\mathbf{E}(y)=-\hat{z}E_0\exp(-jky)$ and $\mathbf{H}(y)=-\hat{x}H_0\exp(-jky)$. As seen in Fig. 3, the electric field inside the transformation medium is spatially compressed in the y direction as expected from the original coordinate transformation. The direction of $\mathbf{E}'$ inside this region is given by $(A^{-1})^TE_0\hat{z}=E_0\hat{z}$ and is thus unaltered except for the spatial compression. Similarly, the direction of $\mathbf{H}'$ inside the transformation medium is given by $(A^{-1})^TH_0\hat{x}=H_0\hat{x}$ and is thus also unaltered except for the spatial compression. This results in the $\mathbf{E}'\times\mathbf{H}'$ power flow direction in the transformation medium being solely in the y direction, while the wave vector or phase normal clearly has an x component as well. The anisotropy of the transformation medium is responsible for this effect.

Note that if one defines the original surface current in terms of delta functions, for instance,

$$\mathbf{J}(\bar{r}) = J_0\delta(y)\hat{x}, \tag{22}$$

then this is an expression for volume current density, and Eq. (1), not Eq. (12) must be used to compute the transformed current. Applying Eq. (1) to case 2 above with the coordinate transformation in Eq. (16) yields, for the transformed current density,

$$\mathbf{J}'(\bar{r}) = \frac{J_0}{b}\delta\left(\frac{y-a(1-|x|)}{b}\right)(\hat{x}+c\hat{y}). \tag{23}$$

This shifts the surface current from $y=0$ to $y=a(1-|x|)$, as expected. Accounting for the scaling properties of the delta function [16] and noting again that $c=-a\,\mathrm{sgn}(x)$ along $y=a(1-|x|)$ gives

$$\mathbf{J}'(\bar{r}) = J_0\delta(y-a(1-|x|))[\hat{x}-a\,\mathrm{sgn}(x)\hat{y}], \tag{24}$$

which becomes

$$\mathbf{J}'(\bar{r}) = \frac{J_0}{\sqrt{a^2+1}}\delta\left(\frac{y-a(1-|x|)}{\sqrt{a^2+1}}\right)[\hat{x}-a\,\mathrm{sgn}(x)\hat{y}] \tag{25}$$

after the delta function is scaled to unit amplitude. This expression for the transformed surface current is identical to that for case 2 in Eq. (20). The scaling properties of the delta function thus naturally yield the same scaling factor to the surface current magnitude that is given explicitly in Eq. (12).

V. LINE TO SURFACE CURRENT TRANSFORMATION

Some source transformations, for example one that transforms a line to a surface current, are perhaps more easily handled through the delta function approach and applying Eq. (1). Luo *et al.* [7] considered a three-dimensional transformation of a finite line current to the surface of a sphere via

$$x = a(r')\cos\phi'\sin\theta', \quad y = a(r')\sin\phi'\sin\theta', \quad z = b(r')\cos\theta', \tag{26}$$

$$a(r') = \frac{R_2}{R_2-R_1}(r'-R_1) = k_1(r'-R_1),$$

$$b(r') = \frac{R_2-d/2}{R_2-R_1}(r'-R_1) + \frac{d}{2} = k_2(r'-R_1) + \frac{d}{2}, \tag{27}$$

in which the constants k_1 and k_2 are defined implicitly. Note that Luo *et al.* [7] did not specify the mapping of the polar or azimuthal angles in their presentation of this transformation. For this transformation the inverse Jacobian matrix is given by

$$A^{-1} = \begin{bmatrix} \frac{\partial x}{\partial r'} & \frac{\partial x}{\partial \phi'} & \frac{\partial x}{\partial \theta'} \\ \frac{\partial y}{\partial r'} & \frac{\partial y}{\partial \phi'} & \frac{\partial y}{\partial \theta'} \\ \frac{\partial z}{\partial r'} & \frac{\partial z}{\partial \phi'} & \frac{\partial z}{\partial \theta'} \end{bmatrix} = \begin{bmatrix} k_1\cos\phi'\sin\theta' & -a\sin\phi'\sin\theta' & a\cos\phi'\cos\theta' \\ k_1\sin\phi'\sin\theta' & a\cos\phi'\sin\theta' & a\sin\phi'\cos\theta' \\ k_2\cos\theta' & 0 & -b\sin\theta' \end{bmatrix}. \tag{28}$$

To apply Eq. (1) we will need $A/\det(A)$. For our particular problem, the source and transformation possess complete azimuthal symmetry and thus we can derive $A/\det(A)$ by making the convenient assumption that $\phi'=0$, which results in

$$\frac{A}{\det A} = \begin{bmatrix} -ab\sin^2\theta' & 0 & -a^2\sin\theta'\cos\theta' \\ 0 & -k_1 b\sin^2\theta' - k_2 a\cos^2\theta' & 0 \\ -k_2 a\sin\theta'\cos\theta' & 0 & k_1 a\sin^2\theta' \end{bmatrix}. \tag{29}$$

As in Fig. 4, we begin with an infinitely thin linear volume current density in the original domain of

$$\mathbf{J}(\bar{r}) = \hat{z}\frac{1}{2\pi\rho}I_0\delta(\rho)\Pi\left(\frac{z}{d}\right), \tag{30}$$

where $\rho=\sqrt{x^2+y^2}$ and the unit pulse or rect function $\Pi(x)=1$ for $-\frac{1}{2}<x<\frac{1}{2}$ and $=0$ otherwise [16]. This creates a line current of total current I_0 that extends to $\pm d/2$ in the z direction.

Noting that the transformation defines $\rho=a(r')\sin\theta'$ and $z=b(r')\cos\theta'$, and following Eq. (1), the original volume current can be written to

$$\mathbf{J}'(\bar{r}') = \frac{A}{\det A}\mathbf{J}(F^{-1}(\bar{r})) = I_0\frac{\delta(a\sin\theta')}{2\pi a\sin\theta'}\Pi\left(\frac{b\cos\theta'}{d}\right) \times \begin{pmatrix} -a^2\sin\theta'\cos\theta' \\ 0 \\ k_1 a\sin^2\theta' \end{pmatrix}, \tag{31}$$

and when we revert to the original coordinates and basis by

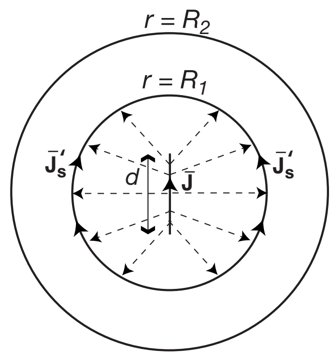

FIG. 4. An illustration of the transformation in Eq. (26), which maps a finite length line current to a spherical surface.

dropping the primes, the original current density transforms to the new current density

$$\mathbf{J}'(\bar{r}) = I_0 \frac{\delta(a \sin\theta)}{2\pi a \sin\theta} \Pi\left(\frac{b \cos\theta}{d}\right) \begin{pmatrix} -a^2 \sin\theta \cos\theta \\ 0 \\ k_1 a \sin^2\theta \end{pmatrix}. \tag{32}$$

This can be simplified through several steps. First, we use the scaling properties of the delta function to find

$$\delta(a \sin\theta) = \delta(k_1(r-R_1)\sin\theta) = \frac{\delta(r-R_1)}{k_1 \sin\theta}. \tag{33}$$

When all of the terms are combined, the first component of the transformed current has a radial dependence proportional to $(r-R_1)\delta(r-R_1)=0$, and thus this term vanishes (except at $\theta=0$ because of a remaining $\sin\theta$ term in the denominator, but this is not significant). After fully expanding the remaining third component and simplifying, we find that the transformed current distribution is

$$\mathbf{J}'(\bar{r}) = \frac{I_0}{2\pi} \delta(r-R_1) \Pi\left(\frac{b(r)\cos\theta}{d}\right) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{34}$$

The basis vectors for the above expression are defined by the transformation from Cartesian to spherical coordinates using $A_{cs}/\det(A_{cs})$ according to Eq. (1) and where A_{cs} is the Jacobian matrix for the Cartesian to spherical transformation. One could thus either multiply Eq. (34) by $A_{cs}^{-1}/\det(A_{cs}^{-1})$ to convert back to a Cartesian basis or use the third column of the matrix $A_{cs}^{-1}/\det(A_{cs}^{-1})$ as the needed basis vector in Eq. (34). Both result in

$$\mathbf{J}'(\bar{r}) = \frac{I_0}{2\pi} \delta(r-R_1) \Pi\left(\frac{b(r)\cos\theta}{d}\right) \frac{-\cos\phi\cos\theta\hat{x} - \sin\phi\cos\theta\hat{y} + \sin\theta\hat{z}}{r \sin\theta} = -\hat{\theta} \frac{I_0}{2\pi r \sin\theta} \delta(r-R_1) \Pi\left(\frac{b(r)\cos\theta}{d}\right), \tag{35}$$

and when we substitute in $r=R_1$ to reflect the delta function distribution, the pulse distribution in theta becomes $\Pi(\frac{\cos\theta}{2})$ and thus simply extends across the entire domain of $\theta=0$ to π. The transformed current distribution then can be written simply as

$$\mathbf{J}'(\bar{r}) = -\hat{\theta} \frac{I_0}{2\pi R_1 \sin\theta} \delta(r-R_1). \tag{36}$$

Thus, the original line current of magnitude I_0 is now a surface current spread uniformly over the surface of the $r=R_1$ sphere with the same total current through any transformed contour.

VI. CONCLUSIONS

Building on the ideas first described by Luo *et al.* [7] and numerically simulated by Kundtz *et al.* [8], we have derived how surface and line currents behave under coordinate transformations as applied in transformation electromagnetics. These cases require special handling because they cannot be stretched by transformations in their directions of zero extent. This effect appears as additional scaling factors that must be added to the expression for the transformation behavior of volume current. The relevance and correctness of these scaling factors is illustrated through a specific example supported by numerical simulations.

We have also shown that surface and line current transformations can be obtained by treating these singular distributions as volume current distributions containing explicit delta functions. This approach, demonstrated in two specific cases, can be easier to apply in situations where the transformation changes the character of the current distribution, such as one that maps a line current to a surface current. Collectively, the analysis presented here contributes to the theoretical picture of how a current distribution can be made to produce the same fields as a different current distribution by surrounding it with a material with electromagnetic properties determined by transformation optics.

APPENDIX: CHARGE AND CURRENT CONSERVATION IN TRANSFORMATION ELECTROMAGNETICS

Here we demonstrate charge and current conservation under transformation electromagnetics. Given a charge-density distribution $\rho(\bar{r})$ and a volume defined in terms of a function $v(\bar{r})$ such that $v(\bar{r})=1$ inside the volume and $v(\bar{r})=0$ outside, then the charge in this volume is given by

$$Q=\int \rho(\bar{r})v(\bar{r})dV, \tag{A1}$$

where the integral is over all space. After applying the coordinate transformation $\bar{r}'=F(\bar{r})$ through the transformation optics approach, the new charge-density distribution is given by Luo *et al.* [7],

$$\rho'(\bar{r})=\frac{1}{\det A}\rho(F^{-1}(\bar{r})). \tag{A2}$$

This form results from ρ being a scalar density [12], and represents the physical displacement of the charge from one location to another and the magnitude scaling that result from the transformation. The new charge in the transformed volume is thus given by

$$Q'=\int \frac{1}{\det A}\rho(F^{-1}(\bar{r}))v(F^{-1}(\bar{r}))dV, \tag{A3}$$

where again the integral is over all space and the original volume has been transformed to a new volume defined by $v(F^{-1}(\bar{r}))=1$. Now change coordinates to $\bar{r}'=F(\bar{r})$ and, in doing so, dV becomes $dV(\det A)$ and thus

$$Q'=\int \rho(F^{-1}(\bar{r}))v(F^{-1}(\bar{r}))dV=\int \rho(\bar{r}')v(\bar{r}')dV=Q, \tag{A4}$$

and thus total charge in a volume is conserved through the translation and scaling of the volume and the charge density.

If charge is conserved and the Maxwell equations are still satisfied after the transformation, then total current must also be conserved. But this can also be shown directly by a similar approach in which we define

$$I=\int s(\bar{r})\mathbf{J}(\bar{r})\cdot d\bar{s}, \tag{A5}$$

where $s(\bar{r})=1$ defines the integration surface. After transformation, the total current through the new $s(F^{-1}(\bar{r}))=1$ surface is

$$I'=\int s(F^{-1}(\bar{r}))\frac{1}{\det A}A\mathbf{J}(F^{-1}(\bar{r}))\cdot d\bar{s}, \tag{A6}$$

where Eq. (1) has been used to give the new current density after the transformation. Again change coordinates to $\bar{r}'=F(\bar{r})$, and because the infinitesimal surface element is a covariant vector capacity [12], $d\bar{s}$ becomes $(\det A)(A^{-1})^T d\bar{s}$. Thus

$$I'=\int s(\bar{r}')A\mathbf{J}(\bar{r}')\cdot[(A^{-1})^T d\bar{s}]=\int s(\bar{r}')\mathbf{J}(\bar{r}')\cdot d\bar{s}=I, \tag{A7}$$

where we have used $A\mathbf{J}\cdot[(A^{-1})^T d\bar{s}]=J^T A^T (A^{-1})^T ds=\mathbf{J}\cdot d\bar{s}$. Thus, total current through a surface is conserved through the translation and scaling of the surface and the vector current density.

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