ELECTROMAGNETIC WAVES IN A BENT PIPE OF RECTANGULAR CROSS SECTION*

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The analysis of electromagnetic wave propagation in a bent pipe of rectangular cross section, ($x=R$, $x=R+a$, $y=0$, $y=b$), is based on the Maxwell field equations, expressed in cylindrical coordinates (r, θ, y) (Fig. 1). As in the case of the straight pipe,[1] the time variation is given by the exponential $e^{j\omega t}$, where ω is the angular frequency. The angular variation is given by $e^{-\Sigma\theta}$, where Σ is the propagation constant for the bent portion. The equations may be written

$$\begin{aligned}
-\Sigma E_y - r\partial E_\theta/\partial y + rj\omega\mu H_r &= 0,\\
\partial E_r/\partial y - \partial E_y/\partial r + j\omega\mu H_\theta &= 0,\\
r\partial E_\theta/\partial r + E_\theta + \Sigma E_r + rj\omega\mu H_y &= 0,\\
-\Sigma H_y - r\partial H_\theta/\partial y - rj\omega\epsilon E_r &= 0,\\
\partial H_r/\partial y - \partial H_y/\partial r - j\omega\epsilon E_\theta &= 0,\\
r\partial H_\theta/\partial r + H_\theta + \Sigma H_r - rj\omega\epsilon E_y &= 0,\\
r\partial H_r/\partial r + H_r - \Sigma H_\theta + r\partial H_y/\partial y &= 0,\\
r\partial E_r/\partial r + E_r - \Sigma E_\theta + r\partial E_y/\partial y &= 0.
\end{aligned} \tag{1}$$

In (1) H_θ, H_r, H_y, E_θ, E_r and E_y are the components of magnetic and electric field, ϵ is the electric inductive capacity, and μ the magnetic inductive capacity. The electrical conductivity, σ, and charge density, ρ, are assumed to be zero.

The field components H_r, H_y, E_r and E_y may be expressed in terms of H_θ and E_θ by various combinations of the equations (1). These give

$$H_r(Gr^2) = -\Sigma r\partial H_\theta/\partial r - \Sigma H_\theta - r^2 j\omega\epsilon\partial E_\theta/\partial y, \tag{2a}$$

$$H_y(Gr^2) = -\Sigma r\partial H_\theta/\partial y + rj\omega\epsilon E_\theta + r^2 j\omega\epsilon\partial E_\theta/\partial r, \tag{2b}$$

$$E_r(Gr^2) = -\Sigma r\partial E_\theta/\partial r - \Sigma E_\theta + r^2 j\omega\mu\partial H_\theta/\partial y, \tag{2c}$$

$$E_y(Gr^2) = -\Sigma r\partial E_\theta/\partial y - rj\omega\mu H_\theta - r^2 j\omega\mu\partial H_\theta/\partial r, \tag{2d}$$

where

$$(Gr^2) = \Sigma^2 + r^2\omega^2\mu\epsilon.$$

Using the last two of Eqs. (1) and Eqs. (2), H_r, H_y, E_r and E_y may be eliminated and equations in H_θ and E_θ readily obtained.

$$\frac{\partial}{\partial r}\left[\frac{1}{Gr}\frac{\partial(rH_\theta)}{\partial r}\right] + H_\theta + G^{-1}\frac{\partial^2 H_\theta}{\partial y^2} + \frac{rj\omega\epsilon}{\Sigma}\frac{\partial E_\theta}{\partial y}\frac{\partial G^{-1}}{\partial r} = 0, \tag{3a}$$

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[1] This case has been discussed by Lord Rayleigh, Phil. Mag. **43**, 125–132 (1897); Brillouin, Rev. Gén. de l'Élec. **40**, 227–239 (1936); Schelkunoff, Proc. Inst. Radio Eng. **25**, 1457–1492 (1937); Chu and Barrow, Proc. Inst. Radio Eng. **26**, 1520–1555 (1938); Slater, *Microwave transmission*, McGraw-Hill, 1942, pp. 124–150.

$$\frac{\partial}{\partial r}\left[\frac{1}{Gr}\frac{\partial(rE_\theta)}{\partial r}\right]+E_\theta+G^{-1}\frac{\partial^2 E_\theta}{\partial y^2}-\frac{rj\omega\mu}{\Sigma}\frac{\partial H_\theta}{\partial y}\frac{\partial G^{-1}}{\partial r}=0. \tag{3b}$$

The boundary conditions for this case are

$$\begin{aligned} y&=0, \quad y=b. \qquad E_\theta=0, \quad E_r=0.\\ r&=R, \quad r=R+a. \qquad E_\theta=0, \quad E_y=0. \end{aligned} \tag{4}$$

A more useful form may be obtained from (2d):

$$\begin{aligned} y&=0, \quad y=b. \qquad E_\theta=0, \quad E_r=0.\\ r&=R, \quad r=R+a. \qquad E_\theta=0, \quad \partial(rH_\theta)/\partial r=0. \end{aligned} \tag{5}$$

By considering E_θ and H_θ as functions of y and r, these conditions establish the dependence of E_θ on $\sin k_y y$, and H_θ on $\cos k_y y$, where $k_y=n\pi/b$, n being an integer. By substituting

$$E_\theta(r,y)=E_\theta(r)\sin k_y y, \qquad H_\theta(r,y)=H_\theta(r)\cos k_y y,$$

in (3) and simplifying, equations in E_θ and H_θ as functions of r alone are obtained.

$$\frac{\partial^2 H_\theta}{\partial r^2}+A(r)\frac{\partial H_\theta}{\partial r}+B(r)H_\theta+\epsilon C(r)E_\theta=0, \tag{6a}$$

$$\frac{\partial^2 E_\theta}{\partial r^2}+A(r)\frac{\partial E_\theta}{\partial r}+B(r)E_\theta+\mu C(r)H_\theta=0, \tag{6b}$$

where the coefficients $A(r)$, $B(r)$ and $C(r)$ have the values

$$\begin{aligned} A(r)&=\frac{1}{r}+\frac{2\Sigma^2}{Gr^3}=\frac{1}{r}+\frac{2\Sigma^2}{r(\Sigma^2+r^2\omega^2\mu\epsilon)},\\ B(r)&=G-k_y^2-\frac{1}{r^2}+\frac{2\Sigma^2}{Gr^4}=\frac{\Sigma^2-1}{r^2}+\omega^2\mu\epsilon-k_y^2+\frac{2\Sigma^2}{r^2(\Sigma^2+r^2\omega^2\mu\epsilon)},\\ C(r)&=\frac{2jk_y\omega\Sigma}{Gr^2}=\frac{2jk_y\omega\Sigma}{(\Sigma^2+r^2\omega^2\mu\epsilon)}. \end{aligned} \tag{7}$$

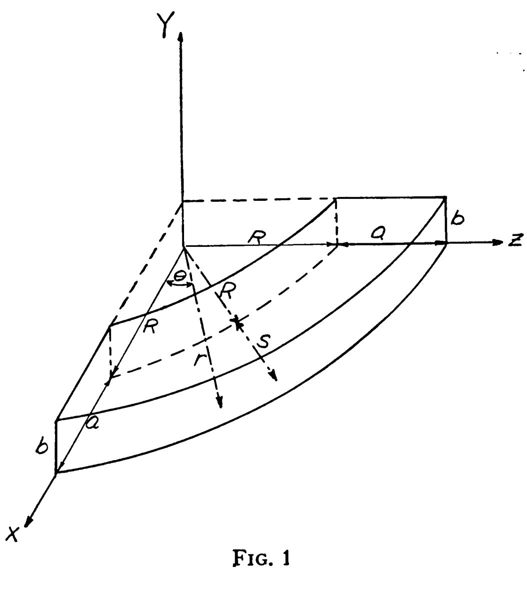

FIG. 1

Equations (6a) and (6b) show that E_θ and H_θ are not independent in the bent pipe. Furthermore, E_θ and H_θ do not vanish in this case, hence the methods of solution used for the straight pipe fail. E_θ and H_θ are not expressible in terms of Bessel functions. One possible method of solution of these equations, namely to substitute

$$\phi_1=(\mu)^{1/2}H_\theta+(\epsilon)^{1/2}E_\theta,$$

$$\phi_2=(\mu)^{1/2}H_\theta-(\epsilon)^{1/2}E_\theta,$$

and thus to obtain separated equations in ϕ_1 and ϕ_2, is incorrect because the boundary conditions (4) are not satisfied.

Equations (6a) and (6b) have been

solved completely by a method of approximation, using the theory of the Schrödinger equation with perturbations. Only the zero order and first order terms are considered. This does not affect the generality of the solution, because in practice the radius of curvature of the pipe, R, may be chosen very large compared to the constants of the equations and to the dimension a of the pipe.

To rewrite (6a) and (6b) in the familiar Schrödinger form, let.

$$(rH_\theta)/R = \beta,$$
$$r = R + s = R(1 + s/R), \qquad 0 < s < a, \tag{8}$$
$$\gamma = \Sigma/R.$$

Thus

$$\partial^2\beta/\partial s^2 + f_1(s)\partial\beta/\partial s + g(s)\beta + h(s)\epsilon E_\theta = 0, \tag{9a}$$
$$\partial^2 E_\theta/\partial s^2 + f_2(s)\partial E_\theta/\partial s + g(s)E_\theta + h(s)\mu H_\theta = 0. \tag{9b}$$

The coefficients, to the first approximation in R^{-1}, are given by

$$f_1(s) = R^{-1}(-1 + 2\gamma^2/K^2), \qquad f_2(s) = R^{-1}(1 + 2\gamma^2/K^2),$$

$$g(s) = K^2 - k_y^2 - 2\gamma^2 s/R = k_x^2 - 2\gamma^2 s/R,$$
$$h(s) = 2jk_y\omega\gamma/RK^2,$$

where

$$K^2 = \gamma^2 + \omega^2\mu\epsilon = k_x^2 + k_y^2.$$

Continuing the approximation, E_θ and β may be written as

$$E_\theta = (E_\theta)_0 + R^{-1}(E_\theta)_1 + \cdots, \qquad \beta = \beta_0 + R^{-1}\beta_1 + \cdots, \tag{10}$$

and the perturbation of the angular coefficient for each case as

$$k_x^2 = k_s^2 + R^{-1}e_1 \quad \text{for} \quad E_\theta, \qquad k_x^2 = k_s^2 + R^{-1}h_1 \quad \text{for} \quad \beta. \tag{11}$$

By substituting (10) and (11) in (9), the zero order and first order approximations may be written separately. For E_θ these are

$$\frac{\partial^2(E_\theta)_0}{\partial s^2} + k_s^2(E_\theta)_0 = 0, \tag{12a}$$

$$\frac{\partial^2(E_\theta)_1}{\partial s^2} + \frac{\partial(E_\theta)_0}{\partial s}[1 + 2\gamma^2/K^2] + (E_\theta)_1 k_s^2$$
$$+ e_1(E_\theta)_0 - 2\gamma^2 s(E_\theta)_0 + \frac{2jk_y\gamma\omega\mu(H_\theta)_0}{RK^2} = 0. \tag{12b}$$

The zero order equation (12a) has the same form as the equation for the straight pipe, with the solution

$$(E_\theta)_0 = E_{m,n}\sin k_s s, \tag{13}$$

where m and n are integers.

Similar equations may be written for β, giving

$$\beta_0 = \lim \frac{r}{R}(H_\theta)_0 = H_{m,n}\cos k_s s = (H_\theta)_0. \tag{14}$$

Eq. (12b) may be rewritten as

$$\frac{\partial^2(E_\theta)_1}{\partial s^2} + k_s^2(E_\theta)_1 = -e_1(E_\theta)_0 + 2\gamma^2 s(E_\theta)_0 - (1 + 2\gamma^2/K^2)\frac{\partial(E_\theta)_0}{\partial s} - \frac{2jk_y\omega\mu\gamma(H_\theta)_0}{K^2}. \tag{15}$$

This is the general form of the Schrödinger equation with perturbations, where the usual perturbation factor λ is equal to $1/R$.

By using the orthogonality condition for the Schrödinger theory, the value of e_1 may be readily determined:

$$\int_0^a (-e_1 + 2\gamma^2 s)E_{m,n}^2 \sin^2 k_s s ds - \int_0^a (1 + 2\gamma^2/K^2)k_s E_{m,n}^2 \cos k_s s \sin k_s s ds - \int_0^a 2jk_y\omega\mu\gamma K^{-2}H_{m,n}E_{m,n} \cos k_s s \sin k_s s ds = 0.$$

Therefore

$$e_1 = \gamma^2 a.$$

By using this value and (13), the first approximation (15) may be solved for $(E_\theta)_1$. The solution, satisfying the boundary conditions, is given by

$$\begin{aligned}(E_\theta)_1 = {} & E_{m,n} \cos k_s s[(\gamma^2 s)(a - s)(2k_s)^{-1}] \\ & + E_{m,n} \sin k_s s\left[(s)(2k_s^2)^{-1}\left\{\gamma^2 - \left[1 + \frac{2\gamma^2}{K^2}\right]k_s^2\right\}\right] \\ & - H_{m,n} \sin k_s s[(jk_y s\omega\mu\gamma)(K^2 k_s)^{-1}].\end{aligned} \tag{16a}$$

In like manner, from the β approximation equations,

$$h_1 = \gamma^2 a.$$

Since $e_1 = h_1$, there is no change in the angle variable during the perturbations. The solution of the β equation, satisfying the boundary conditions, and corresponding to (16a) is

$$\begin{aligned}\beta_1 = {} & E_{m,n} \cos k_s s[(jk_y s\omega\epsilon\gamma)(k_s K^2)^{-1}] - E_{m,n} \sin k_s s[(jk_y\omega\epsilon\gamma)(k_s^2 K^2)^{-1}] \\ & + H_{m,n} \cos k_s s[(s/2)(1 - 2\gamma^2/K^2 + \gamma^2/k_s^2)] \\ & + H_{m,n} \sin k_s s[(2k_s)^{-1}(-\gamma^2 as - 1 + \gamma^2 s^2 + 2\gamma^2/K^2 - \gamma^2/k_s^2)].\end{aligned} \tag{16b}$$

The complete solutions of (6a) and (6b), including both the zero order and first order approximations, may be written as

$$\begin{aligned}E_\theta = {} & \{E_{m,n} \sin k_s s[1 + c_1 s] + E_{m,n} \cos k_s s[s(a - s)c_2] \\ & - H_{m,n} \sin k_s s[\mu c_3 s]\}\{\sin k_y y e^{j\omega t - \Sigma\theta}\},\end{aligned} \tag{17a}$$

$$\begin{aligned}H_\theta = {} & \{H_{m,n} \cos k_s s[1 + c_1 s] - H_{m,n} \sin k_s s[s(a - s)c_2 - c_4] \\ & - E_{m,n} \sin k_s s[\epsilon c_3/k_s] + E_{m,n} \cos k_s s[\epsilon c_3 s]\}\{\cos k_y y e^{j\omega t - \Sigma\theta}\},\end{aligned} \tag{17b}$$

where

$$c_1 = \frac{\gamma^2}{2Rk_s^2} - \frac{1}{2R} - \frac{\gamma^2}{RK^2}, \qquad c_2 = \frac{\gamma^2}{2Rk_s},$$

$$c_3 = \frac{jk_y\omega\gamma}{Rk_sK^2}, \qquad c_4 = \frac{-\gamma^2}{2Rk_s^3} - \frac{1}{2Rk_s} + \frac{\gamma^2}{RK^2k_s}\cdot$$

By using (2) and the approximations (8) and (10), the components H_r, H_y, E_r and E_y are seen to become

$$\begin{aligned}
-K^2H_r = \{&H_{m,n}\sin k_ss[-\gamma k_s(1 + c_1s - s/R + 2\gamma^2s/RK^2) \\
&+ c_2(2s - a)\gamma - j\omega\mu\epsilon c_3k_ys] \\
&+ H_{m,n}\cos k_ss[\gamma\{c_1 + R^{-1} - c_2s(a - s)k_s\} + \gamma k_sc_4] \\
&+ E_{m,n}\sin k_ss[j\omega\epsilon k_y(1 + c_1s + 2\gamma^2s/RK^2) - k_sc_3\gamma\epsilon s] \\
&+ E_{m,n}\cos k_ss[j\omega\epsilon k_ys(a - s)c_2]\}\cos k_yye^{j\omega t-\Sigma\theta},
\end{aligned} \tag{18a}$$

$$\begin{aligned}
K^2H_y = \{&E_{m,n}\sin k_ss[j\omega\epsilon c_1 - k_ss(a - s)c_2 + R^{-1}\} - (c_3k_y\gamma\epsilon)(k_s)^{-1}] \\
&+ E_{m,n}\cos k_ss[j\omega\epsilon k_s(1 + c_1s + 2\gamma^2s/RK^2) - j\omega\epsilon(2s - a)c_2 + k_yc_3\gamma\epsilon s] \\
&- H_{m,n}\sin k_ss[j\omega\mu\epsilon c_3 + \gamma k_y\{s(a - s)c_2 - c_4\}] \\
&+ H_{m,n}\cos k_ss[-j\omega\mu\epsilon k_sc_3s + \gamma k_y(1 + c_1s - s/R + 2\gamma^2s/RK^2)]\} \\
&\cdot\sin k_yye^{j\omega t-\Sigma\theta},
\end{aligned} \tag{18b}$$

$$\begin{aligned}
-K^2E_r = \{&E_{m,n}\sin k_ss[\gamma\{c_1 - c_2s(a - s)k_s + R^{-1}\} - (k_yc_3j\omega\mu\epsilon)(k_s)^{-1}] \\
&+ E_{m,n}\cos k_ss[\gamma k_s(1 + c_1s - s/R + 2\gamma^2s/RK^2) + j\omega\mu\epsilon k_yc_3s - c_2\gamma(2s - a)] \\
&- H_{m,n}\sin k_ss[c_3\gamma\mu + j\omega\mu k_y\{s(a - s)c_2 - c_4\}] \\
&+ H_{m,n}\cos k_ss[j\omega\mu k_y(1 + c_1s + 2\gamma^2s/RK^2) - k_sc_3\gamma\mu s]\} \\
&\cdot\sin k_yye^{j\omega t-\Sigma\theta}
\end{aligned} \tag{18c}$$

$$\begin{aligned}
-K^2E_y = \{&E_{m,n}\sin k_ss[\gamma k_y(1 + c_1s - s/R + 2\gamma^2s/RK^2) - c_3k_sj\omega\mu\epsilon s] \\
&+ E_{m,n}\cos k_ss[\gamma k_ys(a - s)c_2] \\
&+ H_{m,n}\sin k_ss[-j\omega\mu k_s(1 + c_1s + 2\gamma^2s/RK^2) + j\omega\mu c_2(2s - a) - k_yc_3s\gamma\mu] \\
&+ H_{m,n}\cos k_ss[j\omega\mu\{c_1 - k_ss(a - s)c_2 + R^{-1}\} + j\omega\mu k_sc_4]\} \\
&\cdot\cos k_yye^{j\omega t-\Sigma\theta}.
\end{aligned} \tag{18d}$$

The solutions for the field components (18) satisfy the Maxwell field equations (1) within the approximation conditions imposed on the solution of the problem.

For the special cases of $H_{m,n}$ and $E_{m,n}$ when one of the integers m or n is zero, the components may be obtained from (18). For $m=0$ and n not equal to zero:

$$\begin{aligned}
E_\theta &= H_{0,n}[j\omega\mu\gamma k_ys(a - s)/RK_m^2]\sin k_yye^{j\omega t-\Sigma\theta}, \\
H_\theta &= H_{0,n}[1 - s/R - \gamma^2as^2/2R + \gamma^2s^3/3R]\cos k_yye^{j\omega t-\Sigma\theta}, \\
K_m^2H_r &= H_{0,n}[\gamma s(a - s)R^{-1}(\omega^2\mu\epsilon k_y^2/K_m^2 + \gamma^2)]\cos k_yye^{j\omega t-\Sigma\theta}, \\
K_m^2H_y &= H_{0,n}[k_y\gamma R^{-1}(-\omega^2\mu\epsilon a/K_m^2 + R - \gamma^2as^2/2 + \gamma^2s^3/3)]\sin k_yye^{j\omega t-\Sigma\theta}, \\
-K_m^2E_r &= H_{0,n}[j\omega\mu k_yR^{-1}(\gamma^2a/K_m^2 + R - s - \gamma^2as^2/2 + \gamma^2s^3/3)]\sin k_yye^{j\omega t-\Sigma\theta}, \\
E_y &= 0,
\end{aligned}$$

where

$$K_m^2 = (K^2)_{m=0} = k_y^2 + \gamma^2 a/R.$$

For m not equal to zero, $n=0$:

$$E_\theta = 0,$$

$$H_\theta = \{H_{m,0}\cos k_s s[1 + c_1 s] - H_{m,0}\sin k_s s[(a-s)c_2 s - c_4]\}\cos k_y y e^{j\omega t - \Sigma\theta},$$

$$-K_n^2 H_r = \{H_{m,0}\sin k_s s[c_2\gamma(2s-a) - \gamma k_s(1 + c_1 s - s/R + 2\gamma^2 s/RK_n^2)]$$
$$+ H_{m,0}\cos k_s s[\gamma\{c_1 + R^{-1} - c_2 s(a-s)k_s\} + k_s c_4\gamma]\}\cos k_y y e^{j\omega t - \Sigma\theta},$$

$$H_y = 0, \qquad E_r = 0,$$

$$-K_n^2 E_y = \{H_{m,0}\sin k_s s[-j\omega\mu k_s(1 + c_1 s + 2\gamma^2 s/RK_n^2) + j\omega\mu c_2(2s-a)]$$
$$+ H_{m,0}\cos k_s s[j\omega\mu\{c_1 - k_s s(a-s)c_2 + R^{-1}\} + j\omega\mu c_4 k_s]\cos k_y y e^{j\omega t - \Sigma\theta},$$

where the c_1, c_2, c_3, c_4 are calculated for n vanishing, and

$$K_n^2 = (K^2)_{n=0} = k_s^2 + \gamma^2 a/R.$$

It should be noted that both the $E_{0,n}$ and $E_{m,0}$ are missing.

A consideration of the continued propagation of E and H waves from a straight pipe into a bent pipe yields some interesting results. A pure $E_{m,n}$ or $H_{m,n}$ wave in the straight pipe will be reflected, partially, at the junction with the bent pipe. After reflection the amplitudes are proportional to a/R, and intensities to a^2/R^2, hence, for the first approximation, the reflected portion may be neglected. Thus a pure $E_{m,n}$ or a pure $H_{m,n}$ wave in the straight pipe may be traced into the bent pipe, where it will become a mixed E and H wave.

For a mixed E and H wave in the straight portion, the intensities are proportional to a/R and must be considered. A mixed E and H wave in the straight pipe, because of the reflected portion at the junction, sets up an undetermined condition within the pipe, not predictable from the results of this paper.

If the propagation constant is measured along the center line, $a/2$, of the bent pipe, there is no change in its value from that of the straight pipe.