J. Phys.: Condens. Matter **11** (1999) R31–R52. Printed in the UK PII: S0953-8984(99)92078-7

REVIEW ARTICLE

Electron-drag effects in coupled electron systems

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Received 1 May 1998, in final form 11 November 1998

Abstract. The advances in the techniques used for fabrication and lithography of semiconductors have made it possible to study bi-layer systems made of two electronic layers separated by distances of several hundred ångströms. In this situation the electrons in layer 1 are distinguishable from those in layer 2, and can communicate through the direct inter-layer Coulomb interaction. In particular, if a current is applied to one of the layers, the electrons in the second will be dragged, giving rise to a transresistance ρ_D. In this article we review recent theoretical and experimental developments in the understanding of this effect. At very low temperatures it turns out that phonons dominate the transresistance. The direct Coulomb interaction and plasmon excitations are important at temperatures $T > 0.1T_F$, with T_F the Fermi temperature. If a magnetic field is applied, the transresistance is increased, in a very interesting interplay between ρ_D and Landau quantization. The non-dissipative drag is also reviewed.

1. Introduction

Electron–electron (e–e) interactions are responsible for a multitude of fascinating effects in condensed matter. They play a leading role in phenomena ranging from high-temperature superconductivity and the fractional quantum Hall effect, to Wigner crystallization, the Mott transition, and Coulomb gaps in disordered systems. The effects of this interaction on transport properties, however, are difficult to measure. A new technique has recently proven effective in measuring the scattering rates due to the Coulomb interaction directly [1].

This technique is based on an earlier proposal by Pogrebinskiĭ [4, 5]. The prediction was that for two conducting systems separated by an insulator (a semiconductor–insulator–semiconductor layer structure in particular) there will be a drag of carriers in one film due to the direct Coulomb interaction with the carriers in the other film. If layer 2 is an 'open circuit', and a current starts flowing in layer 1, there will be momentum transfer to layer 2 that will start sweeping carriers to one end of the sample, and inducing a charge imbalance across the film. The charge will continue to accumulate until the force of the resulting electric field balances the frictional force of the inter-layer scattering. In the stationary state there will be an induced, or drag voltage V_D in layer 2.

A fundamental difference emerges in the configuration of the current-drag experiment from that in which the in-plane resistance is measured. For a perfectly pure, translationally invariant system, the Coulomb interaction cannot give rise to resistance since the total current commutes with the Hamiltonian H. This means that states with a finite current are stationary states of H and will never decay, since the e–e interaction conserves not only the total momentum but also the total current. (For electrons moving in a periodic lattice, momentum and velocity are no longer proportional, and the current could in principle decay by e–e interaction.) If the layers are coupled by the Coulomb interaction, the stationary states correspond to a linear

superposition of states in which the current is shared to different extents between layers: the total current within a given layer is not conserved and can relax via inter-layer interaction.

This mechanism of current degradation was studied in the pioneering experiment of Gramila *et al* [1] for GaAs layers embedded in AlGaAs heterostructures. The separation between the layers was in the range 200–500 Å. The coupling of electrons and holes [11], and the coupling between a two-dimensional and a three-dimensional structure were also examined [2].

If we use the symbol I for the current circulating in layer 1, the drag resistance (or transresistance) is defined as

$$\rho_D = \frac{V_D}{I}.$$

The naive expectation for the temperature dependence of ρ_D is that it should vanish as T^2 at low temperatures. This results from the exclusion principle which limits the scattering to states within $k_B T$ of the Fermi surface, and there is one factor of T per layer. The first experiment seemed compatible with the e–e mechanism with some discrepancy since the observed ratio ρ_D/T^2 was not a constant at low temperatures. This experiment motivated a rather extensive theoretical effort to understand in more detail the mechanisms of cross-talking between layers between which electrons are not allowed to tunnel. The theory in turn stimulated new experiments, and in recent years we have seen Coulomb drag developing as a field in itself. In the present article we briefly review the progress made in understanding different aspects of the problem. Section 2 provides some background on the Coulomb interaction and screening for double-layer systems. In section 3 we discuss the perturbative equation for ρ_D used in most treatments. This equation can be derived in various ways. In the appendix we present the memory function derivation and the detailed low-temperature dependence assuming Coulomb scattering only.

The following sections discuss the crossover between three regimes. Phonon exchange, discussed in section 4, dominates the drag at $T < 0.1T_F$, with T_F the Fermi temperature. For $T \approx 0.1T_F$, single-particle Coulomb scattering is the dominant effect, and for $T > 0.2T_F$, plasmons (see section 6) are responsible for an enhancement of the drag current. The theoretical studies of the effects of disorder and localization are reviewed in section 5. The enhancement of ρ_D due to an applied magnetic field perpendicular to the layers, and the interplay between Landau quantization and inter-layer scattering are reviewed in section 7. Finally in section 8 we discuss the theory of non-dissipative drag and some experimental attempts to measure it.

2. Inter-layer Coulomb interaction

The leading actor in this play is the Coulomb interaction between electrons in different layers. This interaction is responsible for the scattering, and will be screened by the density fluctuations within each layer. If one neglects the interaction between layers, the effective interaction $V(q,\omega)$ within a given layer calculated in the random-phase approximation (RPA) is given by [7]

$$V(q,\omega) = \frac{V_b(q)}{1 + V_b(q)\chi(q,\omega)} \tag{1}$$

with $V_b(q) = 2\pi e^2/q$ the bare interaction and $\chi(q,\omega)$ the function characterizing the response of the charge $\delta\rho(q,\omega)$ to an external potential $\varphi(q,\omega)$: $\delta\rho(q,\omega) = -\chi(q,\omega)\varphi(q,\omega)$. For two coupled layers we also have the bare inter-layer interaction $U_b(q) = 2\pi e^2 \exp(-qd)/q$, with d the distance between layers. In addition, the response functions could be different if the

layers are not identical. The effective interaction can be written now as a 2×2 matrix [6, 8, 9]

$$\hat{V}(q,\omega) = \hat{V}_B(q)\frac{1}{1+\hat{\chi}(q,\omega)\hat{V}_B(q)} \tag{2}$$

with $[\hat{V}_B(q)]_{11} = [\hat{V}_B(q)]_{22} = V_b(q)$, $[\hat{V}_B(q)]_{12} = [\hat{V}_B(q)]_{21} = U_b(q)$, and $[\hat{\chi}(q,\omega)]_{ij} = \delta_{ij}\chi_i(q,\omega)$, with $i = 1, 2$ labelling the layer. Note that χ_1 and χ_2 could be different, so equation (2) is valid for non-identical layers. The off-diagonal element of $\hat{V}(q,\omega)$ is the screened inter-layer interaction

$$\begin{aligned}[\hat{V}(q,\omega)]_{12} &= \frac{U_b(q)}{[1+\chi_1(q,\omega)V_b(q)][1+\chi_2(q,\omega)V_b(q)] - U_b^2(q)\chi_1(q,\omega)\chi_2(q,\omega)} \\ &\equiv U^{(S)}(q,\omega).\end{aligned} \tag{3}$$

The zeros of the denominator in the above equation correspond to the collective modes of charge oscillations. It is illustrative to write the screened interaction for identical layers in the form (omitting the argument q in the interactions for simplicity)

$$U^{(S)}(q,\omega) = \frac{1}{2}\left[\frac{V_b+U_b}{1+\chi(q,\omega)(V_b+U_b)} - \frac{V_b-U_b}{1+\chi(q,\omega)(V_b-U_b)}\right] \tag{4}$$

which provides evidence for the symmetric and antisymmetric modes. The response function $\chi(q,\omega)$ has real and imaginary parts. The imaginary part corresponds to the continuum of particle–hole excitations that exist for frequencies $\omega < v_F q$. The collective modes corresponding to the poles of $U^{(S)}(q,\omega)$ will be non-decaying at $T = 0$ if they fall out of the particle–hole continuum, which guarantees that the poles are on the real axis. We can find these stable modes by expanding the real part of $\chi(q,\omega)$ for the free-electron gas for small wave-vectors and $\omega \gg v_F q$:

$$\chi(q,\omega) \simeq -\frac{n}{m}\left(\frac{q}{\omega}\right)^2 \tag{5}$$

with m the electron mass, and n the electron density [10]. In the small-q limit, $V_b + U_b \approx 4\pi e^2/q$, and $V_b - U_b \approx 2\pi e^2 d$. If we substitute this in equation (4) we find that the two poles are at frequencies $\omega_\pm(q)$ given by

$$\omega_+(q) = e\sqrt{\frac{4\pi n}{m}}\sqrt{q}$$

$$\omega_-(q) = e\sqrt{\frac{2\pi n d}{m}}q.$$

These in-phase and out-of-phase modes are usually labelled 'optical plasmon' and 'acoustic plasmon' respectively, and are stable at $T = 0$. At finite temperatures, $\chi(q,\omega)$ acquires a finite imaginary part outside of the particle–hole continuum, the poles move into the complex plane, and the plasmons acquire a finite lifetime. In section 6 we will discuss the role of plasmons in enhancing the drag resistance at temperatures of the order of the Fermi energy.

3. Frictional drag

The cross-resistance is calculated in most of the recent theoretical treatments from the following expression, valid when the inter-layer interaction is treated perturbatively [11] (see figure 1):

$$\rho_D = -\frac{\hbar^2\beta}{\pi n_1 n_2 e^2 A}\int \frac{\mathrm{d}q}{(2\pi)}\, q^3 |U^{(S)}(q,\omega)|^2 \int_{-\infty}^{\infty}\mathrm{d}\omega\, \frac{\mathrm{Im}\,\chi_1(q,\omega)\,\mathrm{Im}\,\chi_2(q,\omega)}{\sinh^2(\beta\hbar\omega/2)}. \tag{6}$$

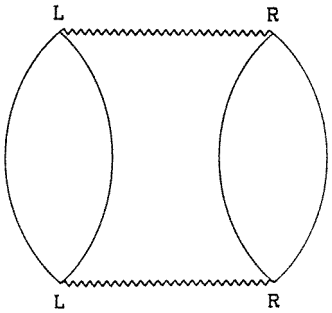

Figure 1. A schematic representation of the memory function expression for the transresistance to lowest order in the inter-layer interaction (equation (6)) (reproduced from reference [12]).

In appendix A we reproduce the derivation of this equation given by Zheng and MacDonald who used the memory function formalism [12]. This derivation is general and contains equation (6) as the limit of lowest order in the inter-layer interaction. We will show here a more restricted derivation using the scattering formalism, which has the advantage of providing us with an interpretation of the origin of each of the terms in equation (6). The derivation is very close to the one presented in the paper by Jauho and Smith [13, 14]. Alternative treatments are the collective excitation approach [15], and the Green's function formalism starting from the Kubo formula [16–18]. In the linearized Boltzmann (or scattering) treatment, the spectral functions $\chi_i(q,\omega)$ are the susceptibilities of the non-interacting case.

We compute first the rate of change of the momentum of the electrons in layer 2 due to the scattering from electrons in layer 1:

$$\frac{\mathrm{d}\boldsymbol{P}_2}{\mathrm{d}t} = \int \frac{\mathrm{d}\boldsymbol{q}}{(2\pi)^2}\,\overline{h}\boldsymbol{q}|U(q)|^2 \int \frac{\mathrm{d}\boldsymbol{k}_1}{(2\pi)^2}\int \frac{\mathrm{d}\boldsymbol{k}_2}{(2\pi)^2}\,\left\{f_{\boldsymbol{k}_1}(1-f_{\boldsymbol{k}_1+\boldsymbol{q}})f^0_{\boldsymbol{k}_2}(1-f^0_{\boldsymbol{k}_2-\boldsymbol{q}})\right.$$
$$\left. -\ f_{\boldsymbol{k}_1+\boldsymbol{q}}(1-f_{\boldsymbol{k}_1})f^0_{\boldsymbol{k}_2-\boldsymbol{q}}(1-f^0_{\boldsymbol{k}_2})\right\}\delta(\epsilon_{\boldsymbol{k}_1}+\epsilon_{\boldsymbol{k}_2}-\epsilon_{\boldsymbol{k}_1+\boldsymbol{q}}-\epsilon_{\boldsymbol{k}_2-\boldsymbol{q}}). \tag{7}$$

In the above equation, $f_{\boldsymbol{k}}$ refers to the distribution of electron states in layer 1, and $f^0_{\boldsymbol{k}}$ to that of the electrons in layer 2. Since there is no current in layer 2, the corresponding electron distribution is the unperturbed, free-fermion equilibrium distribution. The electron distribution on layer 1, on the other hand, corresponds to a Fermi distribution displaced in k-space by an amount $m\boldsymbol{v}_1/\overline{h}$, with $\boldsymbol{v}_1$ being the drift velocity of the electrons in layer 1. The above expression can be interpreted as a sum of processes according to Fermi's golden rule: the factor $|U(q)|^2$ is the square of the matrix element of the transition that involves a momentum transfer $\overline{h}\boldsymbol{q}$, the various factors of f correspond to the probabilities of transitions from occupied to empty states, and the delta function ensures conservation of energy in the scattering process. Note that if layer 1 were in equilibrium, the magnitude in curly brackets would be zero due to the detailed balance condition

$$f^0_{\boldsymbol{k}}(1-f^0_{\boldsymbol{k}'})f^0_{\boldsymbol{k}''}(1-f^0_{\boldsymbol{k}'''}) = f^0_{\boldsymbol{k}'}(1-f^0_{\boldsymbol{k}})f^0_{\boldsymbol{k}'''}(1-f^0_{\boldsymbol{k}''}) \tag{8}$$

which is an identity if $\epsilon_{\boldsymbol{k}}+\epsilon_{\boldsymbol{k}'}+\epsilon_{\boldsymbol{k}''}+\epsilon_{\boldsymbol{k}'''}=0$. There is therefore a finite momentum transfer to layer 2 due to the asymmetry of the electron distribution in layer 1.

We now linearize in the electron distribution of layer 1:

$$f_{\boldsymbol{k}} = f^0_{\boldsymbol{k}-m\boldsymbol{v}_1/\overline{h}} = f^0_{\boldsymbol{k}} - \frac{\partial f^0_{\boldsymbol{k}}}{\partial\epsilon_{\boldsymbol{k}}}\overline{h}\boldsymbol{k}\cdot\boldsymbol{v}_1 \equiv f^0_{\boldsymbol{k}} - \frac{1}{k_BT}f^0_{\boldsymbol{k}}(1-f^0_{\boldsymbol{k}})\overline{h}\boldsymbol{k}\cdot\boldsymbol{v}_1. \tag{9}$$

Substituting the linearized expressions for f_1 in equation (7), and using the detailed balance

condition, the rate of momentum transfer has the form

$$\frac{\mathrm{d}\boldsymbol{P}_2}{\mathrm{d}t} = \frac{\boldsymbol{v}}{2}\frac{1}{k_BT}\int\frac{\mathrm{d}\boldsymbol{q}}{(2\pi)^2}\,(\hbar q)^2|V(q)|^2\int\frac{\mathrm{d}\boldsymbol{k}_1}{(2\pi)^2}\int\frac{\mathrm{d}\boldsymbol{k}_2}{(2\pi)^2}\,f^0_{\boldsymbol{k}_1}(1-f^0_{\boldsymbol{k}_1+\boldsymbol{q}}) \times f^0_{\boldsymbol{k}_2}(1-f^0_{\boldsymbol{k}_2-\boldsymbol{q}})\delta(\epsilon_{\boldsymbol{k}_1}+\epsilon_{\boldsymbol{k}_2}-\epsilon_{\boldsymbol{k}_1+\boldsymbol{q}}-\epsilon_{\boldsymbol{k}_2-\boldsymbol{q}}). \tag{10}$$

A couple of simple technical manipulations are needed to recast this expression in the form of our basic equation. First we use

$$\delta(\epsilon_{\boldsymbol{k}_1}+\epsilon_{\boldsymbol{k}_2}-\epsilon_{\boldsymbol{k}_1+\boldsymbol{q}}-\epsilon_{\boldsymbol{k}_2-\boldsymbol{q}}) = \int\hbar\,\mathrm{d}\omega\,\delta(\hbar\omega-\epsilon_{\boldsymbol{k}_1}+\epsilon_{\boldsymbol{k}_1+\boldsymbol{q}})\delta(\hbar\omega+\epsilon_{\boldsymbol{k}_2}-\epsilon_{\boldsymbol{k}_2-\boldsymbol{q}}) \tag{11}$$

and

$$f^0(\epsilon_{\boldsymbol{k}})\left[1-f^0(\epsilon_{\boldsymbol{k}}+\hbar\omega)\right] = \left[f^0(\epsilon_{\boldsymbol{k}})-f^0(\epsilon_{\boldsymbol{k}}+\hbar\omega)\right]/\left[1-\exp(-\hbar\omega/k_BT)\right]. \tag{12}$$

We then introduce the function $\chi_i^0(q,\omega)$:

$$\int\frac{\mathrm{d}\boldsymbol{k}_i}{(2\pi)^2}\,(f^0_{\boldsymbol{k}_i}-f^0_{\boldsymbol{k}_i+\boldsymbol{q}})\delta(\hbar\omega-\epsilon_{\boldsymbol{k}_1}+\epsilon_{\boldsymbol{k}_1+\boldsymbol{q}}) = \frac{1}{\pi}\mathrm{Im}\int\frac{\mathrm{d}\boldsymbol{k}_i}{(2\pi)^2}\,\frac{f^0_{\boldsymbol{k}_i}-f^0_{\boldsymbol{k}_i+\boldsymbol{q}}}{\hbar\omega-\epsilon_{\boldsymbol{k}_1}+\epsilon_{\boldsymbol{k}_1+\boldsymbol{q}}-\mathrm{i}\eta} \equiv \mathrm{Im}\,\chi_i^0(q,\omega) \tag{13}$$

and substitute in (10):

$$\frac{\mathrm{d}P_2}{\mathrm{d}t} = \frac{v_1}{2}\frac{1}{k_BT}\int\frac{\mathrm{d}\boldsymbol{q}}{(2\pi)^2}\,(\hbar q)^2|V(q)|^2\int\mathrm{d}\omega\,\frac{\mathrm{Im}\,\chi_1^0(q,\omega)\,\mathrm{Im}\,\chi_2^0(q,\omega)}{\sinh^2(\hbar\omega/k_BT)} \tag{14}$$

where we have omitted the vectorial nature of $\boldsymbol{P}_2$ and $\boldsymbol{v}_2$ since they are in the same direction. In order to compute the resistance, we need to equate the rate of momentum transfer to the total force per particle on the electrons in layer 2 due to the electric field generated, E_2:

$$\frac{\mathrm{d}P_2}{\mathrm{d}t} = +n_2eE_2 \tag{15}$$

and we get the final expression using $\rho_D = E_2/j_1$ with $j_1 = n_1ev_1$.

The minus sign in equation (6) means that the induced drag voltage is opposite to the resistive voltage drop in the current-carrying layer. This is so because the Coulomb-induced scattering sweeps the carriers along the dragged layer in the same direction as those in the drive layer. From the above equation we see that the drag resistance is a convolution of the density fluctuations within each layer, which at low temperatures are restricted to low frequencies by the factor $\sinh^2(\beta\hbar\omega/2)$ in the denominator. From the structure of the above equation we see that measurements of ρ_D can provide information on the inter-layer scattering mechanism as well as the in-plane fluctuations, the information on which is included in $\chi(q,\omega)$. The low-temperature prediction of Coulomb scattering alone is [1]

$$\rho_{12} = \frac{m}{ne^2}\frac{\pi\zeta(3)(k_BT)^2}{16\hbar E_F(q_{\mathrm{TF}}d)^2(k_Fd)^2} \tag{16}$$

where q_{TF} is the single-layer Thomas–Fermi screening wave-vector, E_F is the Fermi energy, d is the inter-layer spacing, and k_F is the Fermi wave-vector. Due to the finite separation of the layers, the scattering is limited to small angles [3]. Large-angle scattering events require large values of the momentum transfer q, and these processes are suppressed by the exponential dependence $\sim\mathrm{e}^{-qd}$ in the Fourier transform of the inter-layer Coulomb interaction. In two dimensions, the phase space for backscattering diverges and gives rise to logarithmic corrections to the simple T^2-dependence [19]. These corrections are missing here due to the suppression of backscattering. The screening is also enhanced at small wave-vectors, and

becomes more effective as the separation between layers increases (see equation (21)). The combination of the suppression of backscattering and the enhanced screening gives rise to the strong d^{-4}-dependence in ρ_D. It should be emphasized also that the T^2-dependence is modified when the electronic states in the layers are not free Fermi gases, and support longer-lived density fluctuations. As detailed in the following sections, this is the case, e.g., in the presence of disorder, and in the quantum Hall regime.

4. The role of phonons

The electron–electron scattering treatment predicts a transresistance ρ_D that vanishes at zero temperature as $\rho_D \sim T^2$. This dependence is roughly satisfied in the experiments, confirming the dominance of the electron–electron interaction. However, as mentioned above, the experiment also shows a noticeable deviation of ρ_D/T^2 from a constant as a function of temperature, showing a maximum for $T \sim 2$ K. For these samples the Fermi temperature is roughly 60 K. The overall temperature dependence and the position of the maximum are very similar for different values of the layer separations d, and the magnitude of ρ_D/T^2 varies very little for the three barrier thicknesses $d = 500, 225$, and 500 Å after the Coulomb scattering contribution is subtracted [3]. Also, for inter-layer separations $d = 500$ Å, the observed ρ_D is simply too big to be accounted for by the Coulomb interaction alone. This led Gramila *et al* [3] to propose an additional scattering mechanism. The obvious candidate: phonons. The phonon-mediated coupling between electrons in doped semiconductor layers separated by an insulating region of thickness $\sim$100 μm was studied by Hubner and Shockley [20]. Real phonons were found to be responsible for the inter-layer interaction. The possibility of drag due to phonons was also proposed by Gurzhi and Kopeliovich [21]. The first qualitative hint of the mechanism being phononic is the fact that the measured temperature dependence of ρ_D/T^2 resembles the acoustic-phonon-limited mobility scattering rate $\tau_{\rm ph}^{-1}$ for two-dimensional electrons in GaAs. At high temperatures, $\tau_{\rm ph}^{-1}$ is linear in T, but crosses over to a T^5- or T^7-dependence in the low-temperature Bloch–Grüneisen regime [22], where the thermal phonon wave-vectors are less than $2k_F$. For the electron density of the samples in reference [3] the crossover occurs at a few degrees Kelvin. Thus, the temperature dependences of $\tau_{\rm ph}^{-1}/T^2$ and ρ_D/T^2 are broadly similar. Furthermore, the dependence of ρ_D/T^2 on the relative electron density between the layers indicates that phonons could be playing a role. An electron in layer 1 decays through a backscattering process by emitting a phonon of wave-vector $2k_{F,1}$, and the phonon will transfer its momentum most efficiently to an electron in layer 2 if $k_{F,1} = k_{F,2}$. This implies that ρ_D/T^2 should have a maximum when the electron densities are matched in the two layers, which is experimentally observed [3]. Now, interactions of acoustic phonons with electrons are relatively weak in GaAs—to account for the value of the observed transresistance. The proposed mechanism will be the exchange of virtual phonons [16, 23], a process in which a phonon is emitted by one layer and then absorbed by the second without conserving energy from the electronic transitions. When the energy conservation constraint is relaxed, the phase space for scattering increases. Also, since the layers are separated by distances much smaller than the phonon mean free path, the phonons retain their phase coherence for the interaction between the layers. These two effects imply an enhancement in the transresistance due to virtual phonons. Tso *et al* [24] presented diagrammatic calculations including exchange of virtual phonons with a good agreement with the temperature dependence observed in the experiment (see figure 2).

The distinction between 'real' and 'virtual' phonons is not so clear cut if one extends the treatment leading to equation (6) to include a phonon-mediated Coulomb interaction. The force

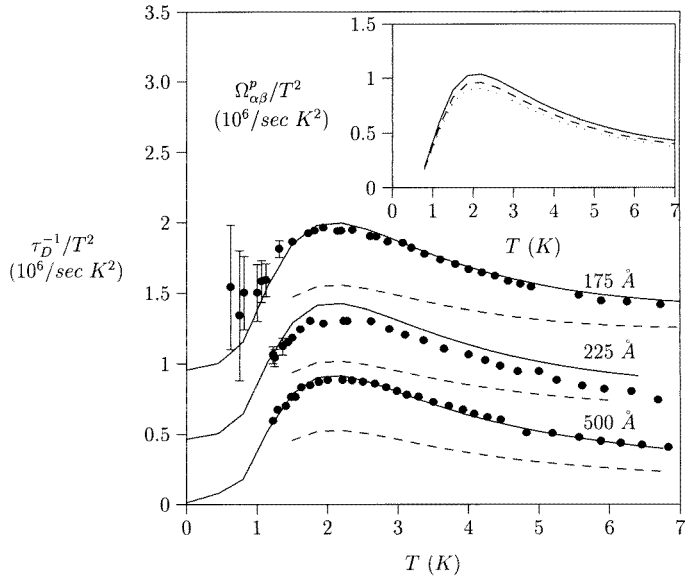

Figure 2. The scattering rate due to the Coulomb scattering and virtual phonons τ_D^{-1}/T^2 as a function of temperature for different separations. Note that $\rho_D \propto \tau_D^{-1}$. The solid circles are the experimental results of reference [1], and the solid curves are the theoretical results from reference [24]. Inset: the contribution to $\rho_D \propto \tau_D^{-1}$ due to exchange of virtual phonons as a function of temperature. (Reproduced from reference [24].)

operator on a given layer will involve the phonon operators, and the force–force correlation function will include an electron–phonon interaction $M(Q)$ and the propagator of a phonon from layer one to layer two. The result is that one can write equation (6) with $U^{(S)}(q,\omega)$ replaced by an interaction $\mathcal{D}_{12}(q,\omega)$ of the form [26]

$$\mathcal{D}_{12}(q,\omega) = \int \frac{\mathrm{d}q_z}{2\pi\hbar}\,|M(Q)|^2 \mathrm{e}^{\mathrm{i}Q_z d}\left[\frac{2\omega_Q}{\omega^2-\omega_Q^2}\right] \tag{17}$$

with $\omega_Q = v_s Q$ the frequency of the acoustic phonon (omitting the distinction between longitudinal and transverse) of wave-vector $Q = \sqrt{q^2+Q_z^2}$. The term in square brackets in the above equation is the phonon Green's function [25]. The phonon mean free path ℓ_{ph} is not included. In the above expression, one can interpret $\hbar\omega$ as the energy transferred between layers and $\hbar\omega_Q$ as the energy of the intermediate phonon. When the denominator in the square brackets of equation (17) vanishes, energy is conserved in the intermediate state. As pointed out by Bønsager *et al* [26], this expression contains both the real and virtual phonons: real phonons correspond to Im $\mathcal{D}_{12}$, whereas virtual phonons correspond to Re $\mathcal{D}_{12}$. If we insert in equation (17) the long-wavelength limit [27] $|M(Q)|^2 = \hbar Q D^2/2\rho v_s$, we obtain [26]

$$\mathcal{D}_{12}(q,\omega) \approx \frac{\omega^2}{q\sqrt{1-\omega^2/(v_s q)^2}}\exp\!\left(-qd\sqrt{1-\omega^2/(v_s q)^2}\right) \tag{18}$$

indicating that the effective interaction diverges as $q \to \omega/v_s$. Even though $\mathcal{D}_{12}(q,\omega)$ involves a small prefactor, if we substitute in equation (6) the expression obtained for the coupling, the divergence $|\mathcal{D}_{12}(q,\omega)|^2 \sim |\omega - v_s q|^{-1}$ gives rise to a divergent ρ_D. Although this is a spurious divergence that is removed by inclusion of a finite mean free path of the phonons or dynamical screening of the interaction, the large contribution explains partly why a 'weak'

phonon-mediated interaction can compete with the Coulomb interaction as a mechanism for drag. For a phonon mean free path $\ell_{\rm ph}$ below a critical value which for GaAs can be of the order of 0.2 mm, the predicted distance dependence of the drag is $\ln(\ell_{\rm ph}/d)$. The experimental distance dependence is yet to be clarified, although the evidence is for a weak dependence of the phonon contribution [28].

5. The effect of disorder and localization

The effect of disorder was first studied by Zheng and MacDonald [12], who included the fact that the density response function at small frequencies and small wave-vectors is given by

$$\chi_i(q,\omega) = \frac{{\rm d}n}{{\rm d}\mu}\frac{Dq^2}{Dq^2 - {\rm i}\omega} \tag{19}$$

with D being the diffusion constant [29] given by $D = l^2/2\tau$, l being the mean free path, $\tau = l/v_F$ the scattering time, and ${\rm d}n/{\rm d}\mu$ the density of states. This formula for the density response function is valid for $q < 1/l$ and $\omega < 1/\tau$. Following the derivation of $\rho_D \sim T^2$ presented in appendix A, Zheng and MacDonald obtain that the low-temperature dependence is modified to $\rho_D \sim T^2 \log T$. This temperature dependence can be seen to result from the low-frequency and low-wave-vector behaviour of ${\rm Im}\,\chi_i(q,\omega) \sim \omega/q^2$ for $q > (\omega/D)^{1/2}$. (Note that in the ballistic regime ${\rm Im}\,\chi_i(q,\omega) \sim \omega/q$.) The contributions to the integral in q for $q < (\omega/D)^{1/2}$ can therefore be neglected. We can obtain the low-temperature behaviour of ρ_D as follows [12, 30]:

$$\rho_D \sim \frac{e^2}{(k_{\rm TF}d)^2}\beta\int_0^{1/\tau}{\rm d}\omega\,\frac{\omega^2}{{\rm e}^{\beta\hbar\omega} + {\rm e}^{-\beta\hbar\omega} - 2}\int_{(\omega/D)^{1/2}}^{1/l}\frac{{\rm d}q}{q} \approx \frac{e^2}{(k_{\rm TF}d)^2}T^2\log T. \tag{20}$$

In the above expression we have considered the low-wave-vector contribution from the Coulomb potential as a constant, $U^{(S)}(q,\omega) \sim e^2/[k_{\rm TF}(k_{\rm TF}d)^2]$. Note the change in distance dependence of ρ_D from d^{-4} to d^{-2}. This is a consequence of having treated the Coulomb scattering as a screened interaction. The logarithmic term originates in the spectra of particle–hole excitations for a ballistic and for a diffusive system being different, and from the dimensionality of the system. The spectrum of excitations and the dimensions (two in this case) conspire to give the logarithmic term in the temperature dependence. At low temperatures one can ignore the contribution of ${\rm Im}\,\chi$ to the screened potential given by equation (3), and replace the inter-layer interaction by

$$U^{(S)}(q,\omega) = \frac{\pi e^2 q}{k_{\rm TF}^2\sinh qd} \tag{21}$$

with $k_{\rm TF} \equiv 2\pi e^2\,{\rm d}n/{\rm d}\mu$ the single-layer Thomas–Fermi screening wave-vector.

The case of strong disorder, where the localization length ξ of the states within each layer is of the order of the distance between layers, was considered by Shimshoni [31]. For the case of Anderson insulators, the density response function within each layer is taken from the self-consistent theory of localization of Vollhardt and Wölfle [32]:

$$\chi(q,\omega) = \frac{{\rm d}n}{{\rm d}\mu}\frac{Dq^2}{Dq^2 - [{\rm i}\omega + \tau(\omega^2 - \omega_0^2)]} \tag{22}$$

with ω_0 a restoring frequency that incorporates the effects of localization. Within the scheme of Vollhardt and Wölfle, ω_0 is related to the localization length ξ through $\xi = v_F/(\sqrt{2}\omega_0)$. The parameter ω_0 in the above expression is responsible for the quadratic vanishing of the conductivity $\sigma(\omega)$ at low frequencies:

$${\rm Re}\,\sigma(\omega) = e^2\lim_{q\to 0}(\omega/q^2)\,{\rm Im}\,\chi(q,\omega) \sim [D/(\tau^2\omega_0^4)]\omega^2. \tag{23}$$

Note that the logarithmic divergence at low q is now cut off by $1/\xi$, and the T^2-dependence is retained. In the strongly localized regime, when the localization length is of the order of the inter-layer spacing, the distance dependence of ρ_D is modified. Screening is not effective in this regime, so we can evaluate ρ_D using the unscreened interaction $U_q = 2\pi e^2 \mathrm{e}^{-qd}/q$. In this regime, the low-q dependence Im $\chi(q,\omega)$ will be given by

$$\mathrm{Im}\,\chi(q,\omega) \approx \frac{\mathrm{d}n}{\mathrm{d}\mu}\frac{\omega q^2}{D[1/\xi^2+q^2]^2} \approx \frac{\mathrm{d}n}{\mathrm{d}\mu}\frac{1}{D}\omega q^2\xi^4. \tag{24}$$

Replacing the above expression in equation (6), we obtain [31]

$$\rho_D \approx \frac{5}{32\pi}\frac{h}{e^2}\left(\frac{k_B T}{\hbar D n}\right)^2 (k_{\mathrm{TF}}d)^2\left(\frac{\xi}{d}\right)^8. \tag{25}$$

If the localization length $\xi \gg d$, one has to include the effects of screening for wave-vectors $q > 1/(\xi^2 k_{\mathrm{TF}})$, and the distance dependence changes to $\ln(k_{\mathrm{TF}}\xi)/(k_{\mathrm{TF}}d)^2$. When $\xi \to \infty$ we recover the diffusive result. Another case treated in reference [31] is that of the so-called Efros–Shklovskii insulators, where Coulomb interactions are important and increase the conductivity at finite frequency in such a way that $\sigma(\omega \to 0) \sim \omega$ as opposed to ω^2. This has the consequence that the integration in ω is infrared divergent and is cut off, at finite T, by incoherent phonon processes. As a consequence the drag resistance diverges at zero temperature in the following way:

$$\rho_D \propto f(d,\xi)\left(\frac{T}{T_0}\right)^3 \exp\left[\left(\frac{T_0}{T}\right)^{1/2}\right]$$

with $f(d,\xi) = 1/(\xi d)^2$ for $\xi \ll d$ and $f(d,\xi) = 1/\xi^4$ for $\xi \gg d$. Also $k_B T_0 = e^2/\epsilon\xi$, with ϵ the dielectric constant.

6. Plasmon enhancement

The discussion of plasmon modes of section 2 focuses on the small-q dispersion at zero temperature. Since the drag resistance is given by an integral involving Im χ in the integrand, at low temperature there is no contribution from the plasmons, since they appear in a region of the plane (q,ω) where Im $\chi = 0$. The function Im χ 'counts' the number of particle–hole excitations of momentum q and frequency ω: it is non-zero in regions of the (q,ω) plane where particle–hole excitations are allowed. The plasmons are collective modes that are outside of the particle–hole continuum. For a more detailed discussion of this point the reader is referred to Pines and Nozières [33].

Flensberg and Hu [34,35] made the very interesting observation that at higher temperatures ($T \sim T_F$), Im $\chi \neq 0$ in the region of the plasmon pole, and therefore there will be a large contribution at intermediate values of q and one should observe an enhancement of the resistance [36]. Qualitatively one can understand this effect as a Schottky-like peak that develops from a thermally populated dissipation channel that is not available at zero temperature. In the plasmon-pole approximation [35] one approximates the dielectric constant $\epsilon(q,\omega)$ (which is the denominator in equation (3)) by

$$\epsilon(q,\omega) = 2V_b(q)\mathrm{e}^{-qd}|\beta_\pm(q)[\omega-\omega_\pm(q)] + \mathrm{i}\,\mathrm{Im}\,\chi(q,\omega_\pm)| \tag{26}$$

with $\beta_\pm = [\mathrm{d}\,\mathrm{Re}\,\chi(q,\omega)/\mathrm{d}\omega]|_{\omega=\omega_\pm}$. For small values of the imaginary part of χ one can approximate the Lorentzian by δ-functions and obtain

$$|U^{(S)}(q,\omega)|^2 \approx \frac{\pi}{4\,\mathrm{Im}\,\chi(q,\omega_\pm)|\beta_\pm(q)|}\delta(\omega-\omega_\pm(q)) \tag{27}$$

which we can insert in equation (6) to obtain the plasmon contributions to the drag rate:

$$\rho_D = \frac{\hbar^2\beta}{\pi n_1 n_2 e^2 A}\int_0^{q_{c,\pm}} \frac{\mathrm{d}q}{(2\pi)}\, q^3 \frac{\mathrm{Im}\,\chi[q,\omega_\pm(q)]}{4|\beta_\pm(q)|\sinh^2[\beta\hbar\omega_\pm(q)/2]}. \tag{28}$$

The parameter $q_{\pm,c}$ defines the values of q for which the plasmon ceases to exist. The contribution to the drag from the above equation involves an integral of Im χ for frequencies and wave-vectors corresponding to the plasmon modes. From the above discussion, the contribution of the plasmons to ρ_D is zero at $T = 0$. At small temperatures, Im χ at $\omega_\pm(q)$ is small because the carriers do not have sufficient energy to be excited far above the Fermi surface. However, at intermediate temperatures (of the order of the Fermi temperature) there are enough thermally excited particles to give a large Im χ at the plasmon poles and the drag will be dominated by this contribution. The numerical calculations indicate a maximum in ρ_D at a temperature of the order of $0.5T_F$. For electron densities of the order of $n = 1.5 \times 10^{11}$ cm^{-2} for GaAs quantum wells, $T_F \sim 50$ K. For $T > 0.5T_F$ the plasmon modes are no longer well defined since they can decay by emitting single-particle excitations. This implies that the enhancement diminishes at high temperatures.

The plasmon enhancement theory was put to test in the experiments by Hill *et al* [37], with very good qualitative agreement. The temperature required to excite a plasmon appears to be lower than the value predicted by the theory, and the magnitude of the drag is larger than

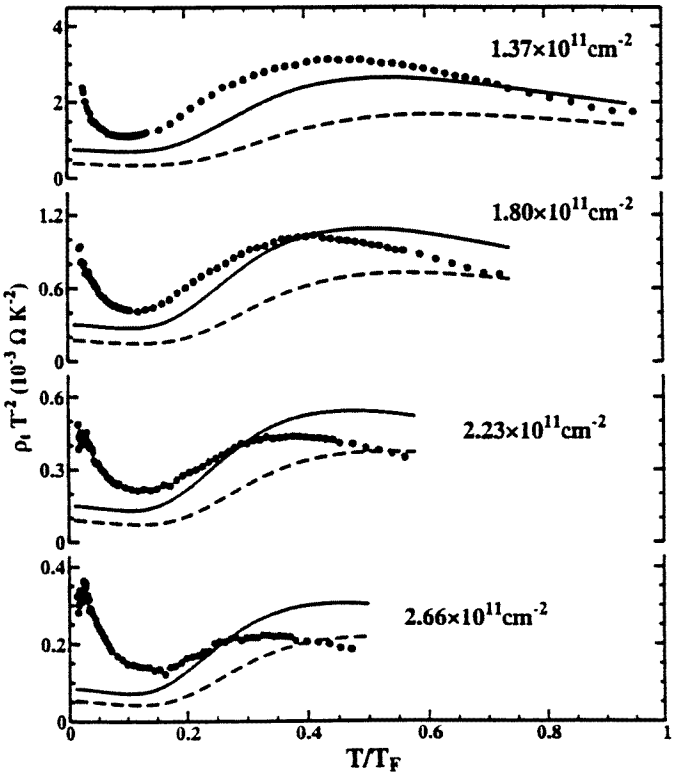

Figure 3. The scaled transresistivity $\rho_t T^{-2}$ ($\rho_t \equiv \rho_D$) versus the reduced temperature for different densities (the densities in the two layers are the same). The dashed (solid) curves show the RPA (Hubbard) calculations of Flensberg and Hu [35], and the circles show the experimental results of reference [37]. (Reproduced from reference [37].)

the prediction over most of the temperature range. This seems to imply that one should go beyond the random-phase approximation. The Hubbard approximation [35] provides a better fit to the data, but there is still room for improvement (see figure 3).

Improvements over RPA were considered by Świerkowski *et al* [38] in connection with drag experiments on electron–hole systems by Sivan *et al* [11]. The transresistance was calculated using equation (6), modifying the effective inter-layer interaction using the local field approach of Singwi, Tosi, Land and Sjölander (STLS) [39–41]. This approach includes correlations neglected in RPA, and the net effect is an increase of the effective interaction. These corrections use zero-temperature local field corrections. Tso *et al* [42] studied a generalized RPA including exchange processes to infinite order in the Hartree–Fock potential, and emphasized the fact that RPA is good only for very high densities. Thus, the RPA treatment of the Coulomb scattering overestimates the screening and renders the coupling weaker. The agreement with the experiments of reference [11] is good at low temperatures. Güven and Tanatar [43] treated the plasmons and the phonons on an equal footing using the Hubbard approximation, and found an enhancement due to the coupled plasmon–phonon modes. We still need better approximations beyond RPA in the higher-temperature regime to achieve a better quantitative understanding of the plasmon enhancement effects.

7. Drag in the presence of magnetic fields

7.1. Hall drag

In the presence of a magnetic field applied perpendicular to the planes, there could in principle exist a trans-Hall resistance—that is, a voltage in the y-direction of layer 2 when a current is applied in the x-direction of layer 1. This drag Hall resistance is zero if computed in the lowest order in the inter-layer interaction. The reason for this is that in the lowest order the electron distribution in layer 2 is the equilibrium one, and also that the momentum is being transferred from layer 1 in the direction of the current. Since there is no net current in layer 2, there is no Lorentz force and therefore no net Hall voltage. To see how this emerges formally, we can repeat the steps that lead to equation (6). In that case we obtain

$$\rho_D^{xy} = \frac{\hbar^2\beta}{\pi n_1 n_2 e^2 A}\int \frac{\mathrm{d}\boldsymbol{q}}{(2\pi)^2}\, q_x q_y |U^{(S)}(q,\omega)|^2 \int_{-\infty}^{\infty}\mathrm{d}\omega\, \frac{\operatorname{Im}\chi_1(\boldsymbol{q},\omega)\operatorname{Im}\chi_2(\boldsymbol{q},\omega)}{\sinh^2(\beta\hbar\omega/2)} \tag{29}$$

where the functions $\chi_1(\boldsymbol{q},\omega)$ have to be evaluated in the presence of a magnetic field $\boldsymbol{B}(\boldsymbol{x}) = \boldsymbol{\nabla}\times\boldsymbol{A}(\boldsymbol{x}_i)$. The reason for the equation retaining its form is the fact that the force operator is still given by equation (A12) even in the presence of a magnetic field since $[\boldsymbol{A}(\boldsymbol{x}_i),\rho_{\boldsymbol{q}}] = 0$. In other words, the inter-layer force does not depend on the applied field. Due to rotational invariance $\chi_1(\boldsymbol{q},\omega) = \chi_1(|\boldsymbol{q}|,\omega)$, and the integral above vanishes by parity. The result $\rho_D^{xy} = 0$ was also shown by Kamenev and Oreg [30] using a diagrammatic approach. However, as emphasized by Hu [44] this is not a general result. In particular, Hu showed, using an approach based on the Boltzmann equation, that if one includes an *energy-dependent* lifetime $\tau(\epsilon)$ one obtains a finite ρ_D^{xy}. No experiment has reported so far a finite Hall drag, at least in the situation were one can guarantee that there is no tunnelling between layers. Patel *et al* [45] measured frictional drag in the presence of a field for modulation-doped GaAs/AlGaAs double quantum wells separated by a barrier of 100 Å for which tunnelling is significant. They found oscillatory behaviour as a function of magnetic field in both the longitudinal and transverse drag resistivities. However, since for this inter-layer separation tunnelling is significant [46], our analysis does not apply to this case.

7.2. Magneto-Coulomb drag

The physics of electrons in two dimensions under quantum conditions of temperature and magnetic field has been the subject of intensive research in recent years, and the reader is referred to some of the very good reviews on the progress in the understanding of the integer and fractional quantum Hall effects [47, 48]. In this section we review the research on current drag in the presence of magnetic fields.

In the presence of a magnetic field, the polarization function $\chi(q,\omega)$ assumes different forms depending on the number of filled Landau levels and on the strength of the disorder. For example, for non-interacting electrons, if the density and magnetic fields are such that the outermost occupied Landau level is completely filled, the lowest energy for a particle–hole excitation is $\hbar\omega_c$, with $\omega_c = eB/mc$ the cyclotron frequency. Therefore Im χ will consist of a series of delta functions separated by $\omega_c = eB/mc$. If the outermost occupied Landau level is partially filled there will be excitations of zero frequency that correspond to transitions between orbits in the same Landau level. The presence of disorder smears these delta functions. One therefore expects that the transresistance will be sensitive to Landau-level quantization for fields large enough that the cyclotron frequency is much larger than the disorder-induced lifetime of the orbits. This problem was studied by Bønsager *et al* [49–51] and Wu *et al* [52] and Qin [53], who treated the individual layers as non-interacting electrons in the presence of disorder consisting of short-range impurities. In the presence of disorder the one-particle density of states $g(\epsilon)$ is broadened, and Bønsager *et al* chose to substitute the comb of delta functions centred at $\epsilon_n = (n + \frac{1}{2})\hbar\omega_c$ for a Gaussian density of states as derived by Gerhardts [54]:

$$g(\epsilon) = \frac{\sqrt{2/\pi}}{2\pi\ell_B^2\Gamma_0}\sum_m \exp\left[-2\left(\frac{\epsilon-\epsilon_m}{\Gamma_0}\right)^2\right] \tag{30}$$

with $\ell_B = \sqrt{\hbar/eB}$ the magnetic length, and $\Gamma_0 = (2/\pi)\hbar\omega_c(\hbar/\tau)$, τ being the transport lifetime in the absence of magnetic field. The equations for the transresistance were solved numerically, and revealed a very interesting 'twin-peak' structure as a function of filling fraction: as the filling factor is changed from an odd value (where the highest Landau level is half-filled in the spin-unpolarized situation) towards an even value, the transresistance goes through a maximum before it is suppressed. The authors explain this non-monotonic effect as being the result of the competition between the phase space for scattering and the strength of the effective interaction. When the Landau levels are completely filled, the chemical potential is in a gap, there is no dissipation, and the transresistance vanishes. This corresponds to the plateau regions of the quantum Hall effect. Also, the screening is strongly depressed in the region where the density of states is small. In the transition region there are excitations of zero energy, and since the density of states increases as the centre of the band is approached (half-filled Landau levels) one gets an increase in $\chi(q,\omega)$ contributing to an increase in ρ_D. On the other hand, an increase in $\chi(q,\omega)$ means an enhancement of screening, or equivalently a decrease in the effective interaction. The prediction is therefore that ρ_D should roughly fulfil the relation

$$\rho_D = g_1 g_2 |W_{12}|^2 \tag{31}$$

with g_i the density of states at the Fermi energy for layer i, and W_{12} the effective inter-layer interaction. The striking twin-peak structure was observed in the experiments performed by Rubel *et al* [55] with good quantitative agreement found with the theory. The twin-peak structure is observed up to a filling fraction $\nu = 15$. There is a marked increase in the value of the transresistance from $\rho_D = 8$ mΩ at zero field and $T = 3.1$ K to values of the order of

1–2 Ω, as expected from the theory (see figure 4). The calculation does not include the effect of localization, and one should keep in mind that the g_i of equation (31) refers to a density of *extended* states. The enhancement of ρ_D in the critical region was studied by Shimshoni and Sondhi [56]. They show that anomalously slow relaxation of density fluctuations at criticality yields a power law $\rho_D \sim T^{2-\eta}$, where η is the anomalous diffusion exponent.

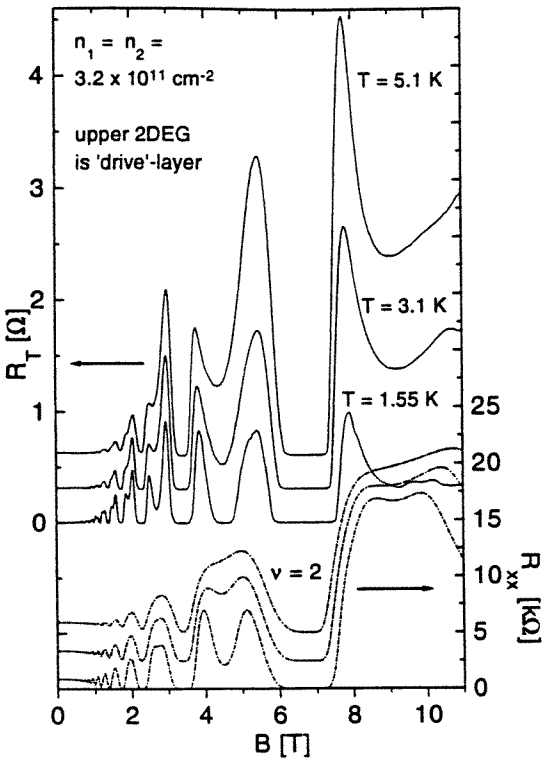

Figure 4. The transresistance $R_T(\rho_D)$ as a function of magnetic field B for a coupled electron gas with a separation barrier of 30 nm, shown for different temperatures (plotted with offsets for clarity). The electron density is $n = 3.2 \times 10^{11}$ cm^{-2} in both layers. The longitudinal resistance is also shown. (Reproduced from reference [55].)

The temperature dependence of ρ_D also shows different behaviours with and without an applied magnetic field. In the discussion of section 5 we found a low-temperature dependence $\rho_D \sim T^2 \ln T$ in the diffusive regime. The range of temperatures for which this dependence applies corresponds to $k_B T \lesssim \hbar/\tau$, with τ a scattering time. This defines a diffusive temperature that for high-mobility samples is of the order of 50 mK. Moreover, Zheng and MacDonald [12] estimate—after including the different distance dependences in the ballistic regime and the diffusive regime—a crossover temperature of 10^{-100} K! At low temperatures for $B \neq 0$ one expects the same $T^2 \ln T$ behaviour since the motion is also diffusive, but the temperature scale where the effect sets in can be higher and experimentally accessible. The reason for this is that in the presence of a field one can think of diffusion as hopping between adjacent orbits of radius $\ell_B \ll v_F \tau$. This implies that the diffusive form of the polarizability of equation (19) is now valid for $q \lesssim 1/\ell_B$. The numerical solutions given by Bønsager *et al* [49] indicate that, for a choice of parameters corresponding to identical layers with a density $n = 3 \times 10^{15}$ m^{-2} and an inter-layer distance $d = 800$ Å, the diffusive behaviour sets

in at $T = 0.4$ K.

Another very interesting aspect of the temperature dependence at finite magnetic fields predicted by Bønsager *et al* [49] and observed in the experiments carried out by Rubel *et al* [55] is the dynamic screening leading to a maximum in ρ_D/T^2, the maximum being at T_F. The theoretical reason for this is the same as that leading to plasmon enhancement discussed in section 6.

Wu *et al* [52] also studied the interplay between Landau quantization and transresistivity by solving equation (6) numerically in the presence of a magnetic field. They find pronounced oscillations as a function of field but not the 'twin-peak' structure. Also, experiments carried out by Hill *et al* [57] show oscillations in ρ_D without the 'twin-peak' structure. The oscillations are pronounced at low temperatures $T \lesssim 4$ K. For high temperatures the Landau quantization effects are washed out and ρ_D has a dependence

$$\rho_D \propto T B^2$$

which has not been addressed theoretically yet.

7.3. The case of $\nu = 1/2$

The case of Landau levels of filling fraction $\nu = 1/2$ (half-filled) or other even denominators is special [47]. The quantized Hall effect does not occur in this case, and the low-energy physics is that of a Fermi liquid, behaving in many ways as electrons in zero magnetic field [58]. The theoretical approach that has proved most useful for understanding this state is the fermion Chern–Simons theory [59], which is based in turn on the composite-fermion theory developed by Jain [60].

For the fractional Hall effect, Orgad and Levit [61] and Duan [62] studied the Coulomb drag for edge excitations using a Chern–Simons theory. The case of inter-layer friction was considered for $\nu = 1/2$ by Sakhi [63], Ussishkin and Stern [9], and Kim and Millis [64]. The dominant low-temperature behaviour for ρ_D is found to be $\sim T^{4/3}$. This temperature dependence results from the slow diffusion of the density modes at filling fraction $1/2$. In the composite-fermion picture, at $\nu = 1/2$ the density response at small frequencies and small wave-vectors is of the form [58]

$$\chi(q,\omega) \approx \frac{q^3}{q^3/\chi_0 - 8\pi \mathrm{i}\hbar\omega k_F} \tag{32}$$

with $\chi_0 = \mathrm{d}n/\mathrm{d}\mu$ the electronic compressibility. The form of this is similar to that for the diffusive regime at $B = 0$ of equation (19) with an effective diffusion constant that vanishes linearly with q. This means that the long-wavelength density fluctuations relax very slowly, leading to an increase in the transresistance. To obtain the low-temperature dependence we proceed in the same way as for the diffusive case, except that now, for small q, $\mathrm{Im}\,\chi \sim 1/q^3$, with the divergence now being cut off at $q \approx 2(\chi_0\hbar k_F)^{1/3}\omega^{1/3} \equiv k_F(\omega/\omega_0)^{1/3}$. The low-temperature behaviour is given by

$$\rho_D \sim \frac{e^2}{(k_{\mathrm{TF}}d)^2}\beta\int_0^\infty \mathrm{d}\omega\,\frac{\omega^2}{\mathrm{e}^{\beta\hbar\omega}+\mathrm{e}^{-\beta\hbar\omega}-2}\int_{k_F(\omega/\omega_0)^{1/3}}\mathrm{d}q\,\frac{1}{q^3} \approx \frac{e^2}{(k_{\mathrm{TF}}d)^2}T^{4/3}. \tag{33}$$

The case of $\nu = 1/2$ poses some fundamental questions that are unresolved at the time of writing this review. In the recent experiments carried out by Lilly *et al* [28] for modulation-doped GaAs/Al_xGa_{1-x}As double quantum wells separated by 200 Å, the drag resistance ρ_D has a qualitatively similar behaviour to the longitudinal resistance of a single isolated layer ρ_{xx} when the magnetic field is varied: where ρ_{xx} is at its maximum due to the quantum Hall effect (for example at $\nu = 1$ and $2/3$), so is ρ_D. As pointed out in reference [28], this is not

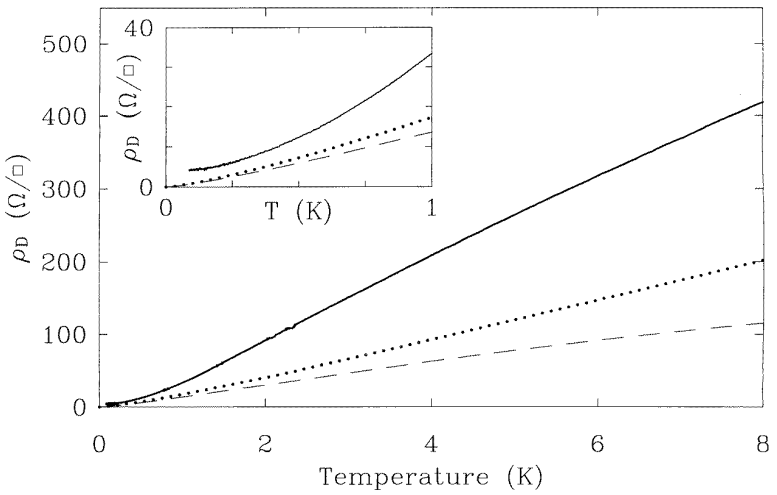

Figure 5. The measured temperature dependence of ρ_D at $\nu = 1/2$ (solid curve). The broken curves are calculations from references [9, 63] of ρ_D assuming two different values of the composite-fermion mass (dotted, $m^* = 12m_b$; dashed, $m^* = 4m_b$; where m_b is the GaAs band mass). (Reproduced from reference [28].)

surprising since both resistances are controlled by the density of states available for scattering. There is, however, a notable difference between their respective temperature dependences. Whereas ρ_{xx} increases by only 6% as the temperature is lowered from $T = 4$ K to 0.2 K, the drag resistance decreases by a factor of 40 over this temperature range. The numerical value of the drag resistance at $B = 11$ T (the field corresponding to $\nu = 1/2$) and at 4 K is about 2000 times larger than ρ_D at zero field. In addition, the temperature dependence of ρ_D differs qualitatively at $\nu = 1/2$ from the corresponding resistance at zero field: the experiment does not show evidence of a 'phonon peak' in ρ_D/T^2, suggesting that phonons are relatively unimportant as a scattering mechanism in the $\nu = 1/2$ case.

The most intriguing part of this story is the evidence of a finite value of $\rho_D \approx 5\ \Omega$ when extrapolated to zero temperature (see figure 5). This is clearly in conflict with the above discussion leading to the temperature dependence of equation (33) which predicts a vanishing resistance at zero temperature. Since the scattering giving rise to ρ_D is inelastic, and the common view is that all inelastic processes cease to be effective at zero temperature, this experiment encourages us to reanalyse the mechanisms of inter-layer dissipation at zero temperature. This interesting experiment places current drag in the same arena as some recent efforts to understand the issues of dephasing [65, 66] and resistance at zero temperature [67] due to 'inelastic' mechanisms. Some very recent papers address the $\nu = 1/2$ case from different angles: Ussishkin and Stern [68] attribute the anomaly to pairing fluctuations whereas Yang [69] suggests that the low-temperature behaviour of the drag resistance is due to the inter-layer distance being close to the critical value at which the two layers form a collective incompressible state.

8. Non-dissipative drag

The possibility of a drag effect at zero temperature was considered by Rojo and Mahan [70], who considered two coupled mesoscopic [71] rings that can individually sustain persistent

currents. The mechanism giving rise to drag in a non-dissipative system is also based on the inter-ring or inter-layer Coulomb interaction, the difference from the dissipative case being the coupling between real or virtual interactions. One geometry in which this effect comes to life is that of two collinear rings of perimeter L, with a Bohm–Aharonov Φ_1 flux threading *only one* of the rings (which we will call ring one). This is of course a difficult geometry to attain experimentally, but has the advantage of making the analysis more transparent. Two coplanar rings also show the same effect [70]. If the rings are uncoupled in the sense that the Coulomb interaction is zero between electrons in different rings, and the electrons are non-interacting within the rings, then a persistent current $J_0 = -c\,\mathrm{d}E/\mathrm{d}\Phi_1 = ev_F/L$ will circulate in ring one [72]. If the Coulomb interaction between rings is turned on, the Coulomb interaction induces coherent charge fluctuations between the rings, and the net effect is that ring two acquires a finite persistent current. The magnitude of the persistent drag current J_D can be computed by treating the modification of the ground-state energy in second-order perturbation theory $\Delta E_0^{(2)}$, and evaluating

$$J_D = -e\left.\frac{\mathrm{d}\,\Delta E_0^{(2)}}{\mathrm{d}\Phi_2}\right|_{\Phi_2=0} \tag{34}$$

with Φ_2 an auxiliary flux threading ring two that we remove after computing the above derivative. In other words, the persistent drag current is equivalent to an induced diamagnetic current that is finite even when the flux is zero in that system. (Note that a diamagnetic current is in general given by the differential change in energy with respect to a change in magnetic field.) The correction to the energy resembles the van der Waals interaction, and its relevance to systems that can individually break time-reversal symmetry was studied in references [73, 74]. The second-order correction is given by

$$\Delta E_0^{(2)} = -\sum_q |U_q|^2 \int_0^\infty \mathrm{d}\omega \int_0^\infty \mathrm{d}\omega' \,\frac{S_1(q,\omega)S_2(-q,\omega')}{\omega+\omega'} \tag{35}$$

with $S_i(q,\omega)$ the dynamical structure factor of ring i (see equation (A14)). For a mesoscopic ring with an applied Bohm–Aharonov flux, one has to retain the discreteness of the spectrum:

$$S_i(q,\omega) = S_i^{(0)}\left(q, \omega - \frac{\hbar^2}{2m}q\frac{\phi_i}{2\pi L}\right) \tag{36}$$

with $S_i^{(0)}(q,\omega)$ the structure factor at zero flux, $\phi_i = \Phi_i/\phi_0$, and $\phi_0 = hc/e$ the flux quantum. Note that, for a mesoscopic system, due to the presence of the flux, $S_i^{(0)}(q,\omega) \neq S_i^{(0)}(-q,\omega)$. For the inter-ring interaction we take the unscreened Coulomb interaction (screening is not effective in one dimension), so $U_q = K_0(qd)$, with d the distance between rings and $K_0(x)$ the modified zeroth-order Bessel function. In the limit where the inter-particle distance is much smaller than the distance between rings ($k_F d \gg 1$), we obtain

$$J_D = J_0 \frac{1}{(k_F a_0)^2}\frac{1}{(k_F d)^2} \tag{37}$$

with a_0 being the Bohr radius. For dimensions corresponding to those of the experiments measuring persistent currents [75], $J_D \simeq 10^{-4}J_0$. The drag current is itself mesoscopic, and therefore vanishes in the limit of infinite length. For a ring with a single channel carrying the current, an extension of the argument presented by Vignale [76] for a bound on the value of the persistent currents gives $J_D < J_0$ in general. The non-dissipative drag for two concentric rings was studied by Shahbazyan and Ulloa [77] using a Luttinger-liquid formulation for mesoscopic systems [78]. They found that the inter-ring interaction modifies the period of the Aharonov–Bohm oscillations. Related work on the effect of the interactions on the flux dependence was reported by Canali *et al* [79].

One basic difference between the dissipative drag in semiconductor systems and the non-dissipative drag in mesoscopic systems appears if one opens ring two so that no current can circulate, and computes the induced charge modulation, which will play the role of a drag voltage. The induced 'voltage' is zero in this case. This can be seen by starting with a set-up that in the absence of flux in system one is 'parity even'. By this we mean that the charge distribution in wire two is symmetric around the centre. Let us call the time-reversal and parity operators $\mathcal{T}$ and $\mathcal{P}$, respectively. We want to compute the induced dipole moment in ring two, $x_2 = \langle\Psi_0|\hat{x}_2|\Psi_0\rangle$. The coordinate operator satisfies $\mathcal{P}T\hat{x}_2(\mathcal{P}T)^{-1} = -\hat{x}_2$ while the wave-function is invariant under $\mathcal{P}T$, which implies that $x = 0$. The charge distribution in ring one remains uniform, and there is no induced voltage.

Natural candidates for use in the study of non-dissipative drag, other than mesoscopic systems, are the superconductors, which can sustain macroscopic persistent currents. The extension to this case was carried out by Duan and Yip [80]. For wires the corrections to the zero-point energy of the charge fluctuations (plasmon modes) were computed for when two superconducting wires individually carry a supercurrent. In analogy with the discussion of section 2, the dispersion of the coupled modes is given by the determinantal equation

$$\begin{vmatrix} (\omega - qv_1)^2 - s^2q^2 & -X \\ -X & (\omega - qv_1)^2 - s^2q^2 \end{vmatrix} = 0 \tag{38}$$

with s the plasmon velocity of a single wire; $X = 4\pi n_0 e^2 q^2 K_0(qd)/m$ comes from the inter-wire interaction, and $v_{1,2}$ are the superconducting velocities of the two wires. For the case of wires, the two coupled modes of frequencies $\omega_\pm(q)$ are linear in q. The zero-point energy

$$E_0 = \sum_q \frac{1}{2}\left[\hbar\omega_+(q) + \hbar\omega_-(q)\right]$$

depends now on the relative superfluid velocities. The superfluid velocity is $\hbar(\boldsymbol{\nabla}\Psi - 2e\boldsymbol{A}/c)/2m$ with Ψ the order parameter. The supercurrent in wire two, I_2, is computed in an analogous way to equation (34) by taking the derivative of the free energy with respect to the vector potential in wire 2, and

$$I_2 = \frac{e}{m}(\rho_{22}v_2 + \rho_{21}v_1)$$

with $\rho_{12} = \hbar n_0^2 e^4/16\pi m^2 s^5 d^2$ representing the drag term. If one starts with a situation in which there is no current in either wire and slowly increases the current in wire one, the prediction is that a current will start to flow in wire two of magnitude $e\rho_{21}v_1/m$. Superconductivity is essential to this effect, in order that the wire can be trapped in a metastable state. Extensions of the above arguments to superfluid Bose systems were presented by Shevchenko and Terent'ev [81] and Tanatar and Das [82]. The case of the transresistance of an excitonic condensate with electrons in one layer and holes in the other layer was studied by Vignale and MacDonald [83], who found a discontinuous jump in ρ_D at the condensation temperature.

Two groups attempted to measure the non-dissipative drag. Giordano and Monnier [84] measured the drag between a superconductor (Al) and a normal metal (Sb), and found a non-reciprocal drag effect that was finite only in the transition region—that is, at temperatures close to the critical temperature of the Al layer. Similar results were reported by Huang *et al* [85] with the superconducting system being AlO_x and Au used as the normal metal. An interpretation of the non-reciprocity effect in terms of inductive coupling of the spontaneously generated vortices in the superconductor and the normal metal was proposed in reference [86]. It seems clear that these experiments do not provide evidence of supercurrent drag.

9. Summary

We have reviewed the recent theoretical and experimental effort in the growing field of current drag. For electronic systems separated sufficiently that there is no tunnelling between them, we found that through studies of the transresistance ρ_D one can extract information not only about the direct Coulomb interaction between electrons in different systems, but also about the collective modes of the coupled systems, and about phonons (virtual and real) that can propagate through the barrier separating the systems in question. The magnetic field B has a non-trivial effect on the drag resistance. The value of ρ_D is larger at finite fields than its value at zero field, and shows a large enhancement in the transition region between quantum Hall plateaus. This effect is understood qualitatively in terms of an interplay between changes in the phase space available for scattering and variations in the effective interaction with B. Some theoretical predictions, like those in connection with the non-dissipative drag, and the ones for strongly disordered systems, are still awaiting experimental verification. The recent experiment on the $\nu = 1/2$ case poses a fundamental question connected with the general theory of transport: is it possible to have an intrinsic resistance at zero temperature due to the electron–electron interaction? The magnitude of the challenge seems to equal that of the progress made so far in the field.

Acknowledgments

We acknowledge interactions with W Dietsche, J P Eisenstein, A J Leggett, A H MacDonald, G D Mahan, C Proetto, E Shimshoni, S Ulloa, P Vasilopoulos and G Vignale.

Appendix

In this section we follow the derivation of Zheng and MacDonald [12] of the inter-planar resistance using the memory function formalism. The formalism of Mori [87] uses the projector technique to write the response function in terms of a memory function. A simple case is a classical particle subject to stochastic forces, for which one is interested in the velocity–velocity correlation function $\phi(t) = \langle v(t)v(0)\rangle$. The Laplace transform is

$$\phi(s) = \frac{\langle v^2(0)\rangle}{s + K(s)} \tag{A1}$$

with $K(s)$ the Laplace transform of the memory function

$$K(t) = \frac{1}{mk_BT}\langle F(t)F(0)\rangle. \tag{A2}$$

Here $F(t)$ is the force acting on the particle, and $m\langle v^2(0)\rangle = k_BT$. The friction coefficient is then given by a force–force correlation function.

For the quantum mechanical case, the calculation of an inter-layer resistance is then performed by identifying the relevant memory function. The starting point is Kubo's formula for the conductivity:

$$\sigma_{ij}(\omega) = \frac{\beta}{A}\int_0^\infty \mathrm{d}t\ \mathrm{e}^{\mathrm{i}\omega t}\langle J_j|J_i(t)\rangle \tag{A3}$$

where $\beta = 1/k_BT$, A is the area of each of the two-dimensional systems considered, i and j are the layer indices, and J is the zero-wave-vector component of the total-current operator. The inner product appearing in (A3) is

$$\langle \hat{J}_j|\hat{J}_i(t)\rangle = \frac{1}{\beta}\int_0^\beta \mathrm{d}\lambda\ \mathrm{Tr}\left[\rho_0 \exp(\lambda H)\hat{J}_j \exp(-\lambda H)\hat{J}_i(t)\right] \tag{A4}$$

with $\hat{J}_i(t) = \mathrm{e}^{(\mathrm{i}/\hbar)Ht}\hat{J}_i\mathrm{e}^{-(\mathrm{i}/\hbar)Ht} = \mathrm{e}^{\mathrm{i}\mathcal{L}t}\hat{J}_i$, and $\mathcal{L}$ is the Liouville superoperator defined through its action on an operator $\hat{O}$ as $\mathcal{L}\hat{O} = (1/\hbar)[H, \hat{O}]$. The next step is to use Mori's projector method to write the inverse matrix $\sigma^{-1}_{i,j}(\omega)$ in terms of a memory function, and hence obtain the transresistance. Zheng and MacDonald define a superoperator $\mathcal{P}$ that projects onto the current:

$$\mathcal{P} = \frac{|J_1\rangle\langle J_1|}{\langle J_1|J_1\rangle} + \frac{|J_2\rangle\langle J_2|}{\langle J_2|J_2\rangle} \equiv 1 - \mathcal{Q} \tag{A5}$$

with $\langle J_i|J_j\rangle = \delta_{ij}(A/\beta)\chi_i$, and $\chi_i = n_i e^2/m$.

With this definition we have (for $\mathrm{Im}\,\omega > 0$)

$$\sigma_{ij}(\omega) = \frac{\beta}{A}\int_0^\infty \mathrm{d}t\ \mathrm{e}^{\mathrm{i}\omega t}\langle J_i|\mathrm{e}^{-\mathrm{i}\mathcal{L}t}|J_j\rangle = \frac{\beta}{A}\langle J_i|\frac{\mathrm{i}}{\omega-\mathcal{L}}|J_j\rangle \tag{A6}$$

$$= \frac{\beta}{A}\langle J_i|\frac{\mathrm{i}}{\omega-\mathcal{Q}L-\mathcal{P}L}|J_j\rangle \tag{A7}$$

$$= \frac{\beta}{A}\langle J_i|\frac{\mathrm{i}}{\omega-\mathcal{Q}L}|J_j\rangle + \frac{\beta}{A}\langle J_i|\frac{\mathrm{i}}{\omega-\mathcal{L}}\mathcal{P}L\frac{1}{\omega-\mathcal{Q}L}|J_j\rangle \tag{A8}$$

$$= \delta_{ij}\frac{\mathrm{i}\chi_i}{\omega} + \sum_{k=1}^{2}\frac{\sigma_{ik}(\omega)}{\omega}\frac{\beta}{A}\frac{1}{\chi_k}\langle J_k|\mathcal{L}\frac{\mathrm{i}}{\omega-\mathcal{Q}L}\mathcal{L}|J_j\rangle \tag{A9}$$

where we have used $\mathcal{P}\dot{J}_i = \mathrm{i}\mathcal{P}\mathcal{L}J_i = 0$, which results, in the present case, from the fact that $[J_i, J_j] = 0$, meaning that the expressions derived are also valid in the presence of a magnetic field that breaks time-reversal invariance. We are interested in the real part of σ^{-1}, which gives the resistance. From the above equations, we have

$$\rho_{12}(\omega) = \frac{\beta}{A}\frac{1}{\chi_1\chi_2}\,\mathrm{Re}\,\langle\dot{J}_1|\frac{\mathrm{i}}{\omega-\mathcal{Q}L}|\dot{J}_2\rangle \tag{A10}$$

$$= \frac{\beta}{A}\frac{1}{n_1 n_2 e^2}\int_0^\infty \mathrm{d}t\ \mathrm{e}^{\mathrm{i}\omega t}\langle F_1|\mathrm{e}^{-\mathrm{i}\mathcal{Q}Lt}|F_2\rangle \tag{A11}$$

where the force operator is $F_i = -m/e\dot{J}_i$. The contribution to the force due to the inter-layer interaction potential U_q is

$$\boldsymbol{F}_1 = -\boldsymbol{F}_2 = \frac{1}{A}\frac{\mathrm{i}}{\hbar}\sum \boldsymbol{q}U_q\rho^{(1)}_{\boldsymbol{q}}\rho^{(2)}_{-\boldsymbol{q}}. \tag{A12}$$

The first approximation is to replace $\mathrm{e}^{-\mathrm{i}\mathcal{Q}L}$ by $\mathrm{e}^{-\mathrm{i}\mathcal{L}}$. The leading order in U_q corresponds to the correlation function in (A11) evaluated in the uncoupled case. Using a representation in terms of exact eigenstates, one finds (for the static limit)

$$\rho_{12} = \frac{\hbar^2\beta}{\pi n_1 n_2 e^2 A}\int\frac{\mathrm{d}\boldsymbol{q}}{(2\pi)^2}\,q^2U_q\int_{-\infty}^{\infty}\mathrm{d}\omega\,\frac{\mathrm{Im}\,\chi_1(q,\omega)\,\mathrm{Im}\,\chi_2(q,\omega)}{\sinh^2(\beta\hbar\omega/2)} \tag{A13}$$

with $\chi_i(q,\omega)$ the density–density response function [7] of layer i:

$$\mathrm{Im}\,\chi_i(q,\omega) = \frac{\pi}{A}(1-\mathrm{e}^{\hbar\omega\beta})\frac{1}{Z_i}\sum_{m,n}\mathrm{e}^{-\beta E_m}|\langle n|\rho_i(q)|m\rangle|^2\delta[\omega-(E_m-E_n)/\hbar] \tag{A14}$$

$$\equiv (1-\mathrm{e}^{\hbar\omega\beta})S(q,\omega) \tag{A15}$$

and Z_i the partition function of the isolated layer.

Here we show that ρ_{12} vanishes as T^2. For simplicity we treat the layers as identical. The approximation used in the literature [3, 88] is to replace $\chi_i(q,\omega)$ by its non-interacting, zero-temperature value [10]:

$$\mathrm{Im}\,\chi(q,\omega) = \frac{m}{\pi q\hbar^2}\left[\Theta(k_F-|x_-|)\sqrt{k_F^2-x_-^2} - \Theta(k_F-|x_+|)\sqrt{k_F^2-x_+^2}\right] \tag{A16}$$

with $x_\pm = m\omega/\hbar q \pm q/2$. At low temperatures, the factor $1/\sinh^2(\beta\hbar\omega/2)$ in (6) ensures that the values of ω are small. Then we can separate the density response functions as

$$\mathrm{Im}\,\chi(q,\omega) = \frac{2m^2\Theta(2k_F - q)}{\pi\hbar^3 q\sqrt{(2k_F)^2 - q^2}}\omega \equiv F(q)\omega. \tag{A17}$$

The integral over ω is

$$\beta\int_{-\infty}^{\infty} \mathrm{d}\omega\, \frac{[\mathrm{Im}\,\chi(q,\omega)]^2}{\sinh^2(\beta\hbar\omega/2)} = (k_B T)^2 \frac{4\pi^2}{3\hbar^3} F(q)^2. \tag{A18}$$

The remaining integral is

$$I = \int \mathrm{d}q\; q^3 U_b^2(q) F(q)^2. \tag{A19}$$

Here, one can replace the bare inter-layer interaction with

$$U_b(q) \to U_q^{(SC)} = \frac{2\pi e^2}{q}\frac{\mathrm{e}^{-qd}}{\epsilon_\infty \epsilon(q)} \tag{A20}$$

where $\epsilon(q)$ is the effective dielectric function for the two parallel conducting planes, and d is the distance between planes. In the random-phase approximation,

$$\epsilon(q) = 1 - 2U_b(q)\chi(q,0) + U_b^2(q)\chi^2(q,0)(1 - \mathrm{e}^{-2qd}). \tag{A21}$$

In the long-wavelength limit, $U_b(q)\chi(q,0) = -q_{\mathrm{TF}}/q$, with $q_{\mathrm{TF}} = 2/a_0^*$ the screening wave-vector in two dimensions. The last term in $\epsilon(q)$ dominates whenever $q_{\mathrm{TF}} \gg 1$. In this limit, the integral I can be approximated as

$$I \simeq \frac{4e^4m^4}{\epsilon_\infty^2\hbar^8 k_F^2 q_{\mathrm{TF}}^4}\int_0^\infty q^3\,\mathrm{d}q\;\frac{\mathrm{e}^{-2qd}}{(1-\mathrm{e}^{-2qd})^2} \tag{A22}$$

$$= \frac{3m^4\zeta(3)}{2\epsilon_\infty^2\hbar^8 k_F^2 q_{\mathrm{TF}}^4 d^4}. \tag{A23}$$

Collecting the integrals, we obtain

$$\rho_{12} = \frac{m}{ne^2}\frac{\pi\zeta(3)(k_B T)^2}{16\hbar E_F (q_{\mathrm{TF}}d)^2 (k_F d)^2} \tag{A24}$$

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