Electrostatic Plasma Instabilities Excited by a High-Frequency Electric Field

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The electrostatic plasma waves excited by a uniform, alternating electric field of arbitrary intensity are studied on the basis of the Vlasov equation; their dispersion relation, which involves the determinant of either of two infinite matrices, is derived. For $\omega_0 \gg \omega_{pi}$ (ω_0 being the applied frequency and ω_{pi} the ion plasma frequency) the waves may be classified in two groups, each satisfying a simple condition; this allows writing the dispersion relation in closed form. Both groups coalesce (resonance) if (a) $\omega_0 \approx \omega_{pe}/r$ (r any integer) and (b) the wavenumber k is small. A nonoscillatory instability is found; its distinction from the DuBois–Goldman instability and its physical origin are discussed. Conditions for its excitation (in particular, upper limits to ω_0, k, and $\mathbf{k}\cdot\mathbf{v}_E$, $\mathbf{v}_E$ being the field-induced electron velocity), and simple equations for the growth rate are given off-resonance and at $\omega_0 \approx \omega_{pe}$. The dependence of both threshold and maximum growth rate on various parameters is discussed, and the results are compared with those of Silin and Nishikawa. The threshold at $\omega_0 \approx \omega_{pe}/r$, $r \neq 1$, is studied.

I. INTRODUCTION

Recently, some interest has been shown in the excitation of longitudinal plasma wave instabilities by the application of high-frequency electric fields. DuBois and Goldman[1] found that if the applied frequency ω_0 is close to the electron plasma frequency ω_{pe}, a parametric instability, coupling the field to an electron plasma wave and an ion acoustic wave, is excited for weak field intensities (field-induced electron velocity much smaller than electron thermal velocity); Lee and Su[2] showed that a relative drift between electrons and ions reduces the threshold for excitation. This instability was experimentally detected by Stern and Tzoar.[3] If the applied field is inhomogeneous, an instability due to the coupling of two electron waves appears at $\omega_0 \approx 2\omega_{pe}$.[4]

Arbitrary field intensities were considered by Silin,[5] who, on the other hand, neglected damping and thermal motion. He noticed that instabilities appear not only at $\omega_0 \approx \omega_{pe}$ but in the entire range $\omega_0 \lesssim \omega_{pe}$, with resonances occurring at $\omega_0 \approx \omega_{pe}/r$ (r is any integer). He studied the dependence of the growth rate on the field intensity and gave results in terms of series of Bessel functions; he also found a maximum growth rate. Both damping and thermal motion were taken into account by Nishikawa[6]; his analysis, however, was limited to weak fields and to the region $\omega_0 \approx \omega_{pe}$, and was based on a fluid description. He distinguished between the parametric instability and a nonoscillatory instability and discussed the dependence of both threshold and growth rate above threshold on ω_0 and k (wavenumber of excited wave) for both types of instabilities. A study based on the Vlasov equation and allowing for spatial variation of the applied field, was carried out by Jackson[7] for arbitrary intensities. He discussed both the $\omega_0 \approx 2\omega_{pe}$ and $\omega_0 \approx \omega_{pe}$ cases (for which he suggested a new instability); his results for this frequency range, however, appear to be invalid because of improper simplifications in the derivation of the dispersion relation for the waves.

Our analysis is based on the Vlasov equation and is valid for uniform fields of arbitrary intensity. In Secs. II and III we derive and discuss the general dispersion relation, which involves the determinant of either of two infinite matrices. Assuming $\omega_0 \gg \omega_{pi}$ (ω_{pi} being the ion plasma frequency) we show that its roots may be classified in two groups, each satisfying a simple condition; this allows writing the dispersion relation in closed form. In Sec. IV we find an unstable nonoscillatory root in one of the groups, and study this instability in detail; we also find that both groups coalesce (resonance) if k is much smaller than the inverse of the electron Debye length and $\omega_0 \approx \omega_{pe}/r$. The nonoscillatory instability is studied at resonance in Sec. V; the connections of some of our results with those of Silin, Nishikawa, and Jackson, and the distinctions between the DuBois–Goldman instability and the present one are considered. Finally, we summarize our results and discuss the physical mechanism for the instability in Sec. VI.

II. BASIC EQUATIONS

We consider the ideal case of a uniform, alternating electric field in an infinite, homogeneous plasma and assume that the Vlasov equation describes the time evolution of the distribution functions of ions and electrons (a weak collision term will be considered in

Sec. VI). If $\mathbf{E}\sin\omega_0 t$ is the field inside the plasma, the equilibrium distribution function of the α species, F_α (α being e, electrons, or i, ions) obeys the equation

$$\frac{\partial F_\alpha}{\partial t} + \frac{q_\alpha}{m_\alpha}\mathbf{E}\sin\omega_0 t\cdot\frac{\partial F_\alpha}{\partial \mathbf{v}} = 0. \tag{1}$$

The solution to Eq. (1) is $F_\alpha = N_\alpha F_{\alpha 0}(\mathbf{v} + \omega_0 \boldsymbol{\varepsilon}_\alpha \cdot \cos\omega_0 t)$ where $\boldsymbol{\varepsilon}_\alpha = q_\alpha \mathbf{E}/m_\alpha\omega_0^2$, $N_e q_e + N_i q_i = 0$, and $F_{\alpha 0}$ is an arbitrary function normalized to unity.

We now study the stability of the plasma around this equilibrium. Let f_α be the perturbation of the distribution function of the α species. The linearized equation for f_α is

$$\frac{\partial f_\alpha}{\partial t} + \mathbf{v}\cdot\frac{\partial f_\alpha}{\partial \mathbf{r}} + \frac{q_\alpha}{m_\alpha}\mathbf{E}\sin\omega_0 t\cdot\frac{\partial f_\alpha}{\partial \mathbf{v}} - \frac{N_\alpha q_\alpha}{m_\alpha}\frac{\partial \phi}{\partial \mathbf{r}}\cdot\frac{\partial F_{\alpha 0}}{\partial \mathbf{v}} = 0, \tag{2}$$

where the small electrostatic potential ϕ obeys Poisson's equation

$$\frac{\partial^2\phi}{\partial \mathbf{r}^2} = -4\pi \sum_\alpha q_\alpha \int f_\alpha\, d\mathbf{v}. \tag{3}$$

We introduce the transformation

$$t = t, \qquad \boldsymbol{\varrho}_\alpha = \mathbf{r} + \boldsymbol{\varepsilon}_\alpha \sin\omega_0 t,$$

$$\mathbf{u}_\alpha = \mathbf{v} + \omega_0\boldsymbol{\varepsilon}_\alpha \cos\omega_0 t$$

into Eq. (2) and define

$$\phi_k = \int \phi \exp(-i\mathbf{k}\cdot\mathbf{r})\, d\mathbf{r}, \tag{4}$$

$$f_{\alpha k} = q_\alpha \int f_\alpha \exp(-i\mathbf{k}\cdot\boldsymbol{\varrho}_\alpha)\, d\boldsymbol{\varrho}_\alpha. \tag{5}$$

There results for $f_{\alpha k}$

$$\frac{\partial f_{\alpha k}}{\partial t} + i\mathbf{k}\cdot\mathbf{u}_\alpha f_{\alpha k} - \frac{\omega_{p\alpha}^2}{k^2} i\mathbf{k}\cdot\frac{\partial F_{\alpha 0}}{\partial \mathbf{u}_\alpha}\left[\int f_{\alpha k}\, d\mathbf{u}_\alpha + \exp(-i\mathbf{k}\cdot\boldsymbol{\varepsilon}_{\alpha\beta}\sin\omega_0 t)\int f_{\beta k}\, d\mathbf{u}_\beta\right] = 0, \tag{6}$$

where ϕ_k has been eliminated between Eqs. (3) and (4); $\omega_{p\alpha}^2 = 4\pi q_\alpha^2 N_\alpha/m_\alpha$ and $\boldsymbol{\varepsilon}_{\alpha\beta} = \boldsymbol{\varepsilon}_\alpha - \boldsymbol{\varepsilon}_\beta$.

If we try to study the normal modes of Eq. (6), we get the same improper integral that appears in the case of zero applied field.[8] To specify the problem properly we consider an initial value problem and take the Laplace transform of Eq. (6); for arbitrary n, we define

$$f_{\alpha k}^n = \int_0^\infty dt\, f_{\alpha k} \exp[i(\omega + n\omega_0)t], \qquad \operatorname{Im}\omega > 0,$$

$$E^n = \int f_{ek}^n\, d\mathbf{u}_e, \qquad I^n = \int f_{ik}^n\, d\mathbf{u}_i,$$

$$A_\alpha^n = \int \frac{i f_{\alpha k}(t=0)\, d\mathbf{u}_\alpha}{\omega + n\omega_0 - \mathbf{k}\cdot\mathbf{u}_\alpha}$$

and by straightforward calculation obtain

$$D_e^n E^n = -\chi_e^n \sum_p J_{n-p} I^p + A_e^n, \tag{7a}$$

$$D_i^n I^n = -\chi_i^n \sum_p J_{p-n} E^p + A_i^n \tag{7b}$$

for $\alpha = e$ and i, respectively. In Eqs. (7a, b) above, $\mathbf{x} = \mathbf{k}\cdot\boldsymbol{\varepsilon}_{ei}$ is the argument of the Bessel functions and[9]

$$D_\alpha^n \equiv 1 + \chi_\alpha^n = 1 + \frac{\omega_{p\alpha}^2}{k^2}\int \frac{\mathbf{k}\cdot(\partial F_{\alpha 0}/\partial \mathbf{u}_\alpha)\, d\mathbf{u}_\alpha}{\omega + n\omega_0 - \mathbf{k}\cdot\mathbf{u}_\alpha}; \tag{8}$$

use has been made of the identity

$$\exp(ia\sin b) = \sum_p J_p(a)\exp(ipb).$$

We have obtained an infinite system of equations for E^n and I^n (n being an arbitrary integer, positive or negative). If the solution to this system is known, both ϕ_k and $\int f_{\alpha k}\, d\mathbf{u}_\alpha$ may be obtained by Laplace inversion. To determine their long time behavior, however, it suffices to look for the singularities of $E^0(\omega)$ and $I^0(\omega)$.[8] For E^0, for instance, we would have

$$E^0 = \frac{\Delta_0^e}{\Delta},$$

where $\Delta = \det\bar{\Delta}$, $\bar{\Delta}$ being the matrix of the coefficients of the unknowns E^n, I^n in the system (7a, b), and $\Delta_0^e = \det\bar{\Delta}_0^e$, $\bar{\Delta}_0^e$ being obtained from $\bar{\Delta}$ by replacing the coefficients of E^0 of (7a, b) by the set $\{A_e^n, A_i^n\}$. If $\partial F_{\alpha 0}/\partial \mathbf{u}_\alpha$ and $f_{\alpha k}(t = 0)$ are analytical functions of $\mathbf{u}_\alpha$, as we shall assume here, χ_α^n and A_α^n are entire functions of ω,[8] and so are Δ and Δ_0^e.[10] Thus, the singularities of $E^0(\omega)$ are just the zeros of $\Delta(\omega)$. The equation

$$\Delta(\omega) = 0 \tag{9}$$

is the dispersion relation and determines the stability of the plasma.

III. THE DISPERSION RELATION

The study of Eq. (9) is simplified by considering the homogeneous system

$$D_e^n E^n = -\chi_e^n \sum_p J_{n-p} I^p, \tag{10a}$$

$$D_i^n E^n = -\chi_i^n \sum_p J_{p-n} E^p, \tag{10b}$$

and then eliminating either I^p or E^p to obtain

$$E^n - \Gamma_e^n \sum_p \sum_m J_{n-p} J_{m-p} \Gamma_i^p E^m = 0, \quad \text{(11a)}$$

$$I^n - \Gamma_i^n \sum_p \sum_m J_{p-n} J_{p-m} \Gamma_e^p I^m = 0, \quad \text{(11b)}$$

where $\Gamma_\alpha \equiv \chi_\alpha / D_\alpha$. The matrices of coefficients of these systems are, respectively, $\bar{\Delta}_e$ and $\bar{\Delta}_i$,

$$\bar{\Delta}_e^{nn} = 1 - \Gamma_e^n \sum_p J_{n-p}^2 \Gamma_i^p \equiv 1 + d_e^{nn}, \quad \text{(12a)}$$

$$\bar{\Delta}_e^{nm} = -\Gamma_e^n \sum_p J_{n-p} J_{m-p} \Gamma_i^p \equiv d_e^{nm} \qquad (m \neq n),$$

$$\bar{\Delta}_i^{nn} = 1 - \Gamma_i^n \sum_p J_{p-n}^2 \Gamma_e^p \equiv 1 + d_i^{nn}, \quad \text{(12b)}$$

$$\bar{\Delta}_i^{nm} = -\Gamma_i^n \sum_p J_{p-n} J_{p-m} \Gamma_e^p \equiv d_i^{nm} \qquad (m \neq n);$$

the relation between $\Delta_\alpha \equiv \det \bar{\Delta}_\alpha$ and Δ is found to be

$$\Delta = \Delta_\alpha \prod_n D_e^n \prod_{n'} D_i^{n'}.$$

Thus, we have $\Delta_e = \Delta_i$ even though $\bar{\Delta}_e \neq \bar{\Delta}_i$; it follows that although according to (12a, b) Δ_α seems to have terms involving $(D_\beta^r)^{-p}$ ($\beta \neq \alpha$, r arbitrary, $p = 1, 2, 3, \cdots$), this is only true for $(D_\beta^r)^{-1}$ since the other factors do not appear in Δ_β (this not trivial result is hidden in the complicated structure of $\bar{\Delta}_\alpha$). A second point to note is that $\Delta_\alpha(\omega)$ is obviously periodic in ω, and so is $\Delta(\omega)$[7]: $\Delta(\omega + n\omega_0) = \Delta(\omega)$; from now on we may assume $|\mathrm{Re}\,\omega| \leq \omega_0/2$.

The study of Eq. (9) has been reduced to that of $\Delta_e = 0$ or $\Delta_i = 0$; the matrices $\bar{\Delta}_e$, $\bar{\Delta}_i$ have a more compact form than $\bar{\Delta}$ and make the study of the dispersion relation simpler.

We now comment on some limiting forms of Eq. (9). As $m_i \to \infty$, we have $\chi_i^n \to 0$ and $\Delta_\alpha \to 1$; Eq. (9) becomes

$$\Delta \equiv \prod_n D_e^n = 0$$

so that the electron stability is not affected by the field in this limit. The ions now only act as a uniform background of positive charge, and using a reference frame that oscillates with the electrons the net force acting upon these vanishes.

For zero field intensity, $x = 0$ and Δ_α becomes diagonal. We have

$$\Delta \equiv \prod_n D_e^n D_i^n (1 - \Gamma_e^n \Gamma_i^n) = \prod_n (1 + \chi_e^n + \chi_i^n) = 0, \quad \text{(13)}$$

so that the usual dispersion relation, $1 + \chi_e^0 + \chi_i^0 = 0$, is recovered. [For each root of $1 + \chi_e^0 + \chi_i^0 = 0$, Eq. (13) yields an infinite set of roots with the same imaginary part and with real parts differing by multiples of ω_0; such roots have no physical relevance and are due to the introduction of the infinite set of Laplace transforms $f_{\alpha k}^n$ from the function $f_{\alpha k}$.]

For $|x| \to \infty$, the result is not so trivial. We may take the limit inside the series in (12a, b) and find $\Delta_\alpha \to 1$. Thus,

$$\Delta \equiv \prod_n D_e^n D_i^n = 0;$$

electrons and ions oscillate independently[5] and if $F_{\alpha 0}$ is, say, Maxwellian, the plasma is stable. Actually, we shall find in the following sections that if $|x|$ exceeds a definite limit, the instability studied there is not excited. This result is discussed and clarified in the last section.

The general study of Eq. (9) remains difficult. In this paper we shall assume that $\omega_0^2 \gg \omega_{pi}^2$; the range $\omega_0/\omega_{pi} \leq O(1)$ is left for a future investigation.[11] The frequencies considered here embrace the region around ω_{pe} (of particular interest in laser-produced plasmas) because $\mu \equiv \omega_{pi}^2/\omega_{pe}^2 \ll 1$. We shall also assume that $k/k_i \leq O(1)$ ($k_\alpha^2 \equiv \omega_{p\alpha}^2/v_\alpha^2$, $v_\alpha^2 = \kappa T_\alpha/m_\alpha$). This is a rather weak restriction since most often $\beta \equiv T_i/ZT_e \leq O(1)$ (Z being the ion charge number) and therefore $k_e^2/k_i^2 \leq O(1)$. The excluded range is then $k/k_e \geq k/k_i > O(1)$; such wavenumbers may be safely disregarded *a priori*.

Under these assumptions we have $\omega_0 \gg kv_i$ and thus, for growing roots, $|\chi_i^n| = O(\omega_{pi}^2/\omega_0^2) < O(1)$ for $n \neq 0$; this may be verified by an integration by parts of the integral in (8). It follows that any root of (9) will satisfy at least one of these two conditions: (1) $|\chi_i^0(\omega)| \geq O(1)$, i.e., $|\omega/\omega_{pi}| \leq O(1)$, and (2) $|D_e^n(\omega)| \ll 1$ for some n [otherwise all d_α^{nn}, d_α^{nm} elements in (12a, b) would be small and then $\Delta_\alpha \sim 1$ so that $\Delta(\omega) \neq 0$]. Thus, all the roots may be classified in two groups. It is then possible to retain a finite number of terms in the expansion of Δ_α; we shall carry this out in the next two sections. Δ_α will be written in a form often used for infinite determinants,

$$\Delta_\alpha = 1 + \sum_n d_\alpha^{nn} + \sum_{n<m} \sum \begin{vmatrix} d_\alpha^{nn} & d_\alpha^{nm} \\ d_\alpha^{mn} & d_\alpha^{mm} \end{vmatrix} \quad \text{(14)}$$

plus terms involving three or more d_α elements. Also, in order to obtain some numerical results we shall assume $F_{\alpha 0}$ to be Maxwellian, although most of our conclusions will be independent of the particular distributions considered, with the sole restriction of $F_{\alpha 0}$ being even in $\mathbf{u}_\alpha$.

IV. NONOSCILLATORY INSTABILITY OFF-RESONANCE

In this section we shall assume that the conditions of Sec. III, $|\chi_i^0| \geq O(1)$, $|D_e^n| \ll 1$ (for, say, $n = r$), are not simultaneously satisfied. Considering the case $|\chi_i^0| \geq O(1)$, we notice that all rows in $\bar{\Delta}_e$ contain a term involving χ_i^0; thus all elements d_e^{nn}, d_e^{nm} are of $O(1)$ and there is no simple way of closing expansion (14) for Δ_e. On the other hand, only the zeroth row of $\bar{\Delta}_i$ contains χ_i^0; moreover, the nondiagonal elements in this row always appear multiplying (small) elements off this row, as follows from (14). Thus to order unity, $\Delta_i = 0$ reads

$$\Delta_i \approx 1 + d_i^{00} \equiv 1 - \Gamma_i^0 \sum_p J_p^2 \Gamma_e^p = 0; \qquad (15)$$

Eq. (15) was derived in Ref. 5 in a different way.

Before studying this equation we briefly consider the alternate possibility, $|D_e^r| \ll 1$. This is a much less interesting case and does not lead to instabilities; it helps, however, to stress the convenience of using the matrices $\bar{\Delta}_i$, $\bar{\Delta}_e$ in the study of the dispersion relation. Observe that $(D_e^r)^{-1}$ appears in all rows of $\bar{\Delta}_i$ but only in the rth row of $\bar{\Delta}_e$; this is entirely analogous to the preceding case, except that now it is convenient to use $\bar{\Delta}_e$ instead of $\bar{\Delta}_i$. We find

$$\Delta_e \approx 1 + d_e^{rr} \equiv 1 - \Gamma_e^r \sum_p J_{r-p}^2 \Gamma_i^p = 0; \qquad (16)$$

this equation only leads to a small correction in the usual stable root of the dispersion relation for $E = 0$, corresponding to electron waves.

Going back to Eq. (15), we rewrite it as

$$1 + \chi_i^0\left[\frac{J_0^2}{D_e^0} + \sum_1^\infty J_p^2\left(\frac{1}{D_e^p} + \frac{1}{D_e^{-p}}\right)\right] = 0. \qquad (17)$$

Notice now that if $F_{\alpha 0}$ is even in $\mathbf{u}_\alpha$, and $\omega = i\gamma$, γ real (nonoscillatory root), both χ_e^0 and χ_i^0 are real and χ_e^{-p} is the complex conjugate of χ_e^p ($p \neq 0$); this follows easily from (8). The left-hand side of (17) is then real; thus nonoscillatory roots are, in principle, possible. We shall study such roots in this and the following sections.

Assuming $F_{\alpha 0}$ to be Maxwellian, we have

$$\chi_\alpha^0 = \frac{g_\alpha k_\alpha^2}{k^2}, \qquad g_\alpha = g\left(\frac{\gamma}{kv_\alpha}\right),$$

where

$$g(y) \equiv 1 - \left(\frac{\pi}{2}\right)^{1/2} y \exp\left(\frac{y^2}{2}\right)\left[1 - \operatorname{erf}\left(\frac{y}{2^{1/2}}\right)\right];$$

this function is monotonically decreasing in the entire $(-\infty, \infty)$ range (for positive argument g decays smoothly from 1 at zero to zero at $+\infty$). We also have[12]

$$\chi_e^{\pm p} = (R_p \pm iI_p)k_e^2/k^2,$$

$$R_p \equiv R[(p\omega_0 + i\gamma)/2^{1/2}kv_e],$$

$$I_p \equiv I[(p\omega_0 + i\gamma)/2^{1/2}kv_e],$$

where

$$R(y) \equiv 1 - 2y \exp(-y^2) \int_0^y \exp(z^2)\, dz,$$

$$I(y) \equiv \pi^{1/2} y \exp(-y^2).$$

Using these expressions and defining $s \equiv k^2/k_e^2$, Eq. (17) becomes

$$1 + \frac{g_i}{\beta}\left(\frac{J_0^2}{s + g_e} + 2\sum_1^\infty J_p^2 \frac{s + R_p}{(s + R_p)^2 + I_p^2}\right) = 0. \qquad (18)$$

The dependence of the bracket on γ (through g_e, R_p, and I_p) may be neglected. Note that if $\gamma \ll kv_e$, we may write $g_e \approx 1$; now, for $\beta \leq O(1)$, we have $\gamma \ll kv_e$ whenever $\gamma/kv_i \leq O(1)$, while if $\gamma \gg kv_i$ we have $g_i \approx k^2v_i^2/\gamma^2 = \mu\beta k^2 v_e^2/\gamma^2$ and, therefore,

$$\frac{g_i}{\beta}\frac{J_0^2}{s + g_e} \approx \frac{\mu J_0^2}{s + g_e}\frac{k^2 v_e^2}{\gamma^2},$$

which is small compared with unity, and thus negligible in (18), unless $\gamma \ll kv_e$. In a similar manner we may write, in (18),

$$R_p \approx R\left(\frac{p\omega_0}{2^{1/2}kv_e}\right) \equiv \bar{R}_p,$$

$$I_p \approx I\left(\frac{p\omega_0}{2^{1/2}kv_e}\right) \equiv \bar{I}_p;$$

this follows from the conditions $\omega_0 \gg \omega_p$, and $|\chi_i^0| \geq O(1)$ [i.e., $\gamma/\omega_{pi} \leq O(1)$].

We have then[13]

$$\gamma = kv_i g^{-1}(\beta/A), \qquad (19)$$

where g^{-1} is the inverse function of g and

$$A = -\frac{J_0^2}{1 + s} - 2\sum_1^\infty J_p^2 \frac{s + \bar{R}_p}{(s + \bar{R}_p)^2 + \bar{I}_p^2}. \qquad (20)$$

With a table of g, Eqs. (19) and (20) give γ/kv_i explicitly as a function of k/k_e, $(v_E/v_e)\cos\psi$, ω_0/ω_{pe} and T_i/ZT_e, where $\mathbf{v}_E = \omega_0 \boldsymbol{\varepsilon}_{ei}$, and ψ is the angle between $\mathbf{k}$ and $\mathbf{E}$.

Some interesting conclusions may immediately be obtained from the above result. Since $1 \geq g_i > 0$ for $0 \leq \gamma < \infty$, it follows that this instability will be excited only if $1 \geq \beta/A > 0$ or

$$\beta - A \equiv \beta + \frac{J_0^2}{1 + s} + 2\sum_1^\infty J_p^2 \frac{s + \bar{R}_p}{(s + \bar{R}_p)^2 + \bar{I}_p^2} \leq 0. \qquad (21)$$

It follows from (21) that $s + \bar{R}_p < 0$ for at least one value of p; since the minimum of $R(y)$ is -0.285 (at $y \approx 1.50$) we must have $s < 0.285$ or

$$k < 0.53k_e. \tag{22}$$

A similar and more interesting result may be derived for ω_0/ω_{pe}. Multiplying $s + \bar{R}_p < 0$ by $p^2\omega_0^2/k^2v_e^2$ we get

$$\frac{p^2\omega_0^2}{\omega_{pe}^2} + 2\left(\frac{p^2\omega_0^2}{2k^2v_e^2}\right)R\left(\frac{p\omega_0}{2^{1/2}kv_e}\right) < 0.$$

Since the minimum of $2y^2R(y)$ is -1.64 (at $y \approx 2.01$) we must have $\omega_0^2/\omega_{pe}^2 < 1.64$ or [14]

$$\omega_0 < 1.28\omega_{pe}. \tag{23}$$

Both (22) and (23) are, of course, necessary but not sufficient conditions for the excitation of the instability.

Condition (21) may be rewritten

$$\beta s + 1 - \frac{J_0^2}{1+s} - 2\sum_1^\infty J_p^2 \frac{s\bar{R}_p + \bar{R}_p^2 + \bar{I}_p^2}{(s+\bar{R}_p)^2 + \bar{I}_p^2} \leq 0. \tag{24}$$

Take $|x| \ll 1$ and retain terms $O(x^2)$; according to (24) we must have $\bar{I}_1 \ll 1$, $|s + \bar{R}_1| \ll 1$ and this leads to $\omega_0 \gg kv_e$ and $s \ll 1$. Since $\bar{I}_1$ is then exponentially small and $\bar{R}_1 \approx -s\omega_{pe}^2(\omega_0^2 - 3k^2v_e^2)^{-1}$, we have

$$s\beta + s + \frac{x^2}{2} + \frac{x^2}{2}\frac{\omega_{pe}^2}{\omega_0^2 - \omega_{ke}^2} \leq 0$$

or

$$\frac{v_E^2}{v_e^2} \geq 2(1+\beta)\frac{(\omega_{ke}^2 - \omega_0^2)}{\omega_{pe}^2}, \tag{25}$$

where $\omega_{ke}^2 \equiv \omega_{pe}^2 + 3k^2v_e^2$ and we took $\mathbf{k}$ along $\mathbf{E}$. The equal sign in (25) gives the threshold in the field intensity for the onset of the instability; (25) also gives the auxiliary condition

$$\omega_0^2 < \omega_{ke}^2 \equiv \omega_{pe}^2(1 + 3s),$$

which is in agreement with (23) since $s \ll 1$.

In the opposite limit of large $|x|$, with other parameters fixed, (24) cannot be satisfied; an explicit formula for the maximum $|x|$ possible is difficult to obtain. In the special case of large ω_0/kv_e, however, definite results may be presented. We then have $s \ll 1$, $\bar{R}_p \approx -s\omega_{pe}^2(p^2\omega_0^2 - 3k^2v_e^2)^{-1}$; $\bar{I}_p$ is exponentially small. Using the identity[5]

$$\sum_p J_p^2\sigma^2(\sigma^2 - p^2)^{-1} \equiv \pi\sigma(\sin\pi\sigma)^{-1}J_\sigma J_{-\sigma}, \tag{26}$$

condition (24) becomes

$$s\beta + 1 - \frac{J_0^2}{1+s} + \frac{\omega_{pe}^2}{\omega_{ke}^2}\left(J_0^2 - \frac{\pi\sigma}{\sin\pi\sigma}J_\sigma J_{-\sigma}\right) \leq 0, \tag{27}$$

where here $\sigma = \omega_{ke}/\omega_0$. For $|x| \ll 1$, (25) is recovered. For $\sigma \gg 1$ and $x/\sigma \equiv \operatorname{sech}\alpha < 1$, we use known asymptotic expansions for the Bessel functions[15] to get

$$\frac{\pi\sigma}{\sin\pi\sigma}J_\sigma J_{-\sigma} \approx \frac{1}{\tanh\alpha} \equiv \frac{1}{(1 - x^2/\sigma^2)^{1/2}};$$

condition (27) gives then $x^2/\sigma^2 \geq 2s(3 - 2J_0^2 + \beta)$ or

$$\frac{v_E^2}{v_e^2} \geq \frac{2(3 - 2J_0^2 + \beta)\omega_{ke}^2}{\omega_{pe}^2}. \tag{28}$$

The limit of (28) for $|x| \to 0$ agrees with the limit of (25) for $\omega_0/\omega_{pe} \to 0$ ($\sigma \to \infty$). The parenthesis in (28) oscillates between $1 + \beta$ and $3 + \beta$.

Now taking $\sigma \to \infty$, $x/\sigma \equiv \sec\alpha > 1$, we get

$$\frac{\pi\sigma J_\sigma J_{-\sigma}}{\sin\pi\sigma} \approx \frac{2\cos[\sigma(\tan\alpha - \alpha) - \frac{1}{4}\pi]\cos[\sigma(\tan\alpha - \alpha + \pi) - \frac{1}{4}\pi]}{(x^2/\sigma^2 - 1)^{1/2}\sin\pi\sigma};$$

condition (27) then gives

$$k_\parallel v_E \leq \omega_{pe}\{1 + 4\cos^2[\sigma(\tan\alpha - \alpha) - \tfrac{1}{4}\pi] \cdot \cos^2[\sigma(\tan\alpha - \alpha + \pi) - \tfrac{1}{4}\pi]\sin^{-2}\pi\sigma\}^{1/2} \tag{29}$$

and

$$[1 + \sin 2\sigma(\tan\alpha - \alpha)]\cot\pi\sigma + \cos 2\sigma(\tan\alpha - \alpha) > 0,$$

where $k_\parallel$ is the component of $\mathbf{k}$ along $\mathbf{E}$. Condition (29) has the desired form, $|x| \leq |x|_{\max}$.

Notice that the above results show that for both $x/\sigma < 1$ and $x/\sigma > 1$, the condition for instability is easily satisfied when x/σ approaches unity. Indeed one may show that for $\sigma \to \infty$, the maximum growth rate occurs at $x/\sigma \approx 1$ or

$$k_\parallel v_E \approx \omega_{pe}. \tag{30}$$

Finally, we draw attention to the following points. First, conditions (25), (29), and (30) have a rough similarity to results of the well known theory of the two-stream instability; this question will be discussed in Sec. VI. Second, (29) and (25) indicate that $|x|_{\max}$ has a marked increase when σ approaches any integer ($r\omega_0 \approx \omega_{pe}$) and that the threshold decreases as ω_0 approaches ω_{pe} [changes in the threshold for $r\omega_0 \approx \omega_{pe}$ ($r \neq 1$) were lost in (25) because only terms $O(x^2)$ were retained]; we notice, however, that $r\omega_0 \approx \omega_{pe}$ implies $|D_e^r| \ll 1$ for $s \ll 1$, while it has been assumed that $|D_e^n| \geq O(1)$ for all n. Nonetheless, it is apparent that when we have both $|\chi_i^0| \geq O(1)$ and $|D_e^r| \ll 1$ the two groups of roots coalesce and some resonance develops, which makes a more detailed analysis of the region $r\omega_0 \approx \omega_{pe}$ imperative; this will be carried out in the next section.

V. NONOSCILLATORY INSTABILITY AT RESONANCE

We shall assume here that the conditions $|\chi_i^0| \geq O(1)$ and $|D_e^n| \ll 1$ (for, say, $n = r$) are simultaneously satisfied; as indicated in the preceding section, the resulting resonance leads to important effects. Since $\gamma \ll \omega_0$, it follows from $|D_e^r| \ll 1$ that $r\omega_0 \gg kv_e$, $k^2 \ll k_e^2$, and $r\omega_0 \approx \omega_{ke}$. In this section, therefore, we consider $\omega_0 \approx \omega_{pe}/r$ and wavelengths large compared with the electron Debye length.

The analysis of Sec. IV fails then in two respects. First, some simplifications performed on Eq. (15) and relating to D_e^r are no longer valid. Second, Eq. (15) itself ceases to be valid; some additional terms must be included in the expansion of Δ_i.

The closing of that expansion was based on d_i^{nm} ($n \neq 0$) being small ($|\chi_i^n/D_e^n| \ll 1$). For sufficiently small D_e^r, that condition will no longer be satisfied and a finite expression for Δ_i for arbitrary x will not be practicable.[16] To avoid this difficulty we impose a lower limit to D_e^r by assuming $|\chi_i^n| \ll |D_e^r| \ll 1$. Thus, $d_i^{nm} (n \neq 0)$ is still small; the restrictions arising from this assumption are indicated below. The assumption itself will be relaxed later, in the study of the range of field intensities just above threshold, for which $|x| \ll 1$.

Δ_i, however, will not be given by $\Delta_i \approx 1 + d_i^{00}$ as in (15), because d_i^{0m} ($m \neq 0$) now contains terms with the large factors $(D_e^{\pm r})^{-1}$, whose product with single elements of other rows will be of order of $\chi_i^n(D_e^r D_e^{-r})^{-1}$. This is not necessarily small compared with unity or even with d_i^{00} $\{d_i^{00} \equiv -\Gamma_i^0[\sum_{p\neq\pm r} J_p^2\Gamma_e^p + J_r^2(\Gamma_e^r + \Gamma_e^{-r})]$, but, in general, we have $|\Gamma_e^r + \Gamma_e^{-r}| \ll |\Gamma_e^{\pm r}|$; we may even have $|\Gamma_e^r + \Gamma_e^{-r}| = O(1)$ so that $\sum_{\neq p\pm r} J_p^2\Gamma_e^p$ must be retained in $d_i^{00}]\}$. It follows that retaining dominant terms only, the dispersion relation will read

$$\Delta_i \approx 1 - \Gamma_i^0 \sum_p J_p^2\Gamma_e^p + \Gamma_i^0 J_r^2 \Gamma_e^r \Gamma_e^{-r} \cdot \sum_{m\neq 0} \Gamma_i^m (J_{m+r} + J_{m-r})^2 = 0. \quad (31)$$

Introducing $\bar{R}_p$, $\bar{I}_p$, s, ω_{ke}, and g_α from the preceding section we have

$$\Gamma_e^0 \approx (1+s)^{-1}, \qquad \chi_i^0 = \frac{g_i}{s\beta},$$

$$\Gamma_e^p \approx \Gamma_e^{-p} \approx \omega_{pe}^2(\omega_{ke}^2 - p^2\omega_0^2)^{-1}, \quad (p \neq 0, \pm r); \quad (32)$$

we also have $\Gamma_i^m \approx \chi_i^m \approx -\omega_{pi}^2/m^2\omega_0^2$ ($m \neq 0$). Using (32) we obtain

$$\sum_{p\neq\pm r} J_p^2\Gamma_e^p \approx \frac{J_0^2}{1+s} + \frac{\omega_{pe}^2}{\omega_{ke}^2} \sum_{p\neq\pm r,0} \frac{J_p^2\sigma^2}{\sigma^2 - p^2}$$

$$\approx J_0^2(1-s) - \frac{\omega_{pe}^2}{\omega_{ke}^2} J_0^2 - \sum_{p\neq\pm r} \frac{J_p^2 r^2}{r^2 - p^2} + O\left(\frac{\omega_{ke}}{\omega_0} - r\right),$$

where $\sigma = \omega_{ke}/\omega_0$. Since s, $|\omega_{ke}/\omega_0 - r| \ll 1$, we have to lowest order,

$$\sum_{p\neq\pm r} J_p^2\Gamma_e^p \approx \sum_{p\neq\pm r} \frac{J_p^2 r^2}{r^2 - p^2} - s,$$

where the last term has been included to make this approximation uniformly valid for $|x| \leq O(1)$.

Since $|D_e^{\pm r}| \ll 1$, the approximation (32) fails for $\Gamma_e^{\pm r}$, as indicated at the beginning of this section; first, the imaginary part of $\chi_e^{\pm r}$ can no longer be neglected, and second, γ must be retained in the argument of $\chi_e^{\pm r}$. The large phase-velocity asymptotic expansion of $\chi_e^{\pm r}$ is still valid, however, because $r\omega_0 \gg kv_e$, so that we have

$$\chi_e^r \approx \frac{-\omega_{pe}^2}{(r\omega_0 + i\gamma + i\nu_c)(r\omega_0 + i\gamma) - 3k^2v_e^2} + i\left(\frac{\pi}{2}\right)^{1/2} \frac{r\omega_0 k_e^2}{kv_e k^2} \exp\left(\frac{-r^2\omega_0^2}{2k^2v_e^2}\right); \quad (33)$$

we have neglected γ in the last term above. Notice the introduction of a collisional damping, ν_c, in (33); collisions have been neglected until now but, at resonance, damping effects must be considered (principally in the determination of the threshold intensity) and for sufficiently small s, ν_c may dominate the Landau damping, ν_L. The first term in (33) may also be obtained from a macroscopic formulation, in which ion–electron friction is easily taken into account; the dependence of χ_e^r on ν_c given above was obtained in this way.

Since the imaginary part of $\chi_e^{\pm r}$ is small compared with the real part, the effects of ν_c and ν_L are additive and we can define a total damping rate, $\nu \equiv \nu_c + \nu_L$; ν_L may be approximately written

$$\nu_L \approx \left(\frac{\pi}{2}\right)^{1/2} \omega_{pe} s^{-3/2} \exp\left(\frac{-r^2\omega_0^2}{2s\omega_{pe}^2}\right). \quad (34)$$

We then have

$$\Gamma_e^r + \Gamma_e^{-r} \approx 2\frac{\omega_{pe}^2(\lambda + \gamma^2 + \gamma\nu)}{\lambda^2 + r^2\omega_0^2(2\gamma + \nu)^2},$$

$$\Gamma_e^r\Gamma_e^{-r} \approx \frac{\omega_{pe}^4}{\lambda^2 + r^2\omega_0^2(2\gamma + \nu)^2},$$

where $\lambda \equiv \omega_{ke}^2 - \omega_0^2$, is a measure of the sharpness of the resonance; it is now apparent that the failure of (32) for $\Gamma_e^{\pm r}$ occurs when $|\bar{\lambda}|$ is comparable to, or smaller than, $\bar{\gamma}$ and $\bar{\nu}$ where $\bar{\lambda} \equiv \lambda/\omega_{pe}^2$, $\bar{\gamma} \equiv \gamma/\omega_{pe}$, and $\bar{\nu} \equiv \nu/\omega_{pe}$. One may also verify that the condition $|\chi_i^n/D_e^{\pm r}| \ll 1$, used to close the expansion of Δ_i, is violated only when all three quantities, $\bar{\lambda}$, $\bar{\gamma}$, and $\bar{\nu}$ are comparable to, or smaller than, $\mu \equiv \omega_{pi}^2/\omega_{pe}^2$.

With the above results one finds that after multiplying by D_i^0, Eq. (31) becomes

$$1 + \frac{g_i}{s\beta} B_r = 0, \tag{35}$$

where

$$B_r = 1 + s - G_r - 2J_r^2 \frac{\bar{\lambda} + \bar{\gamma}^2 + \bar{\gamma}\bar{\nu} + \mu r^2 H_r/2}{\bar{\lambda}^2 + (2\bar{\gamma} + \bar{\nu})^2}; \tag{36}$$

the second term in B_r must be retained only if $|x| \ll 1$, and, in the last term, $r\omega_0/\omega_{pe}$ has been approximated by unity. The functions $H_r(x)$ and $G_r(x)$ are

$$H_r \equiv \sum_{m \neq 0} (J_{m+r} + J_{m-r})^2 m^{-2}, \tag{37}$$

$$G_r \equiv \sum_{p \neq \pm r} \frac{J_p^2 r^2}{r^2 - p^2};$$

in the Appendix we show that

$$G_r = -\frac{J_r^2}{2} + r!\, rJ_r \left(\frac{2}{x}\right)^r \sum_0^{r-1} \frac{J_p (x/2)^p}{p!\,(r-p)}. \tag{38}$$

For $r = 1$, H_r can also be given in finite terms; using the relations $J_{m+1} + J_{m-1} = 2mJ_m/x$, and $\sum_p J_p^2 = 1$ we find $H_1 = 4(1 - J_0^2)x^{-2}$.

We shall now consider Eq. (35) in some detail in the important case $r = 1$. Introducing G_1 from (38) into (36) we find

$$B_1 = 1 - \frac{2J_0J_1}{x} + s - 2J_r^2 \frac{\bar{\lambda} - \frac{1}{4}\bar{\nu}^2 + 2\mu(1 - J_0^2)x^{-2}}{\bar{\lambda}^2 + (2\bar{\gamma} + \bar{\nu})^2};$$

we have neglected $\bar{\lambda}^2$ in the numerator of the last term, because $\bar{\lambda}^2 \ll |\bar{\lambda}|$. As in Sec. IV, a condition for the excitation of the instability follows from $0 < g_i \leq 1$,

$$s\beta + B_1 \leq 0; \tag{39}$$

since $1 - 2J_0J_1/x \geq 0$, we obtain the important result

$$\bar{\lambda} + 2\mu(1 - J_0^2)x^{-2} - \tfrac{1}{4}\bar{\nu}^2 > 0$$

or

$$\omega_0^2 < \omega_{ke}^2 + 2\omega_{pi}^2(1 - J_0^2)x^{-2} - \tfrac{1}{4}\nu^2. \tag{40}$$

Although more definite than (23), this is still a necessary but not sufficient condition for instability.

To find the threshold for excitation we take $|x| \ll 1$ in (39) and obtain

$$s(\beta + 1) + \frac{x^2}{2} - \frac{x^2}{2} \frac{\bar{\lambda} + \mu}{\bar{\lambda}^2 + \bar{\nu}^2} \leq 0$$

or

$$\frac{v_E^2}{v_e^2} \geq 2(\beta + 1) \frac{\bar{\lambda}^2 + \bar{\nu}^2}{\bar{\lambda} + \mu - \bar{\nu}^2}. \tag{41}$$

This gives the threshold together with the condition

$$\bar{\lambda} + \mu - \bar{\nu}^2 > 0, \tag{42}$$

which, even though only valid for $|x| \ll 1$, is more restrictive than (40). The threshold goes to infinity at $\bar{\lambda} + \mu - \bar{\nu}^2 = 0$ and has a minimum at

$$\bar{\lambda} = -\mu + (\mu^2 + \bar{\nu}^2)^{1/2}. \tag{43}$$

Our condition $|\chi_i^n/D_e^{\pm r}| \ll 1$, $n \neq 0$, requires either $\bar{\lambda}$ or $\bar{\nu}$ much larger than μ (since $\bar{\gamma} = 0$ at threshold). For $\bar{\nu} \gg \mu$ we have $\bar{\lambda} \approx \bar{\nu}$ from (43) and the minimum threshold is found to be

$$\frac{v_E^2}{v_e^2} = 2(1 + \beta)\bar{\nu}; \tag{44}$$

on the other hand, for $\bar{\nu}/\mu \leq O(1)$ the minimum occurs at a value of $\bar{\lambda}$ for which our condition is violated and (31) is invalid. In the small $|x|$ range considered, however, the expansion of Δ_i can be closed without using said condition; assuming $\bar{\nu}/\mu = O(1)$ and retaining only terms $O(x^2)$ in Δ_i at threshold we obtain

$$\frac{v_E^2}{v_e^2} = 2(1 + \beta) \frac{(\bar{\lambda} + \mu)^2 + \bar{\nu}^2}{\bar{\lambda} + \mu - \bar{\nu}^2}. \tag{45}$$

The minimum of v_E^2/v_e^2 is attained at $\bar{\lambda} = \bar{\nu} - \mu$ and its value is again given by (44). In fact, (45) is valid under the much weaker restriction $\bar{\nu}/\mu^2 \geq O(1)$ except in the frequency range $|\bar{\lambda} + \mu| \ll \bar{\nu}$, which does not embrace the minimum threshold; it follows that this minimum is given correctly by (44) and is independent of the mass ratio, μ, for almost all realistic values of $\bar{\nu}/\mu$. In Ref. 6, Eq. (44) was obtained from an expression for the threshold given by

$$\frac{v_E^2}{v_e^2} = 2(1 + \beta) \frac{\bar{\lambda}^2 + \bar{\nu}^2}{\bar{\lambda}}; \tag{46}$$

our analysis shows that although (46) is often invalid, Eq. (44) is practically always true.

Let us now study (39) in the opposite limit, $|x| \gg 1$. We obtain approximately

$$1 - \frac{4}{\pi |x|} \cos^2\left(x - \frac{3\pi}{4}\right) \frac{\bar{\lambda} + 2\mu x^{-2}}{\bar{\lambda}^2 + \bar{\nu}^2} \leq 0, \tag{47}$$

which requires

$$\bar{\lambda} + 2\mu x^{-2} > 0. \tag{48}$$

With the equal sign, (47) is an equation for $|x|$ that has many roots, separating alternate stable and unstable ranges of the x variable. The largest root, $|x|_L$, may be obtained by taking $1 - \sin 2x \approx 2$ (small relative variations in x substantially affect the value of $\sin 2x$, because $|x| \gg 1$), so that

$$\frac{\pi}{4}(\bar{\lambda}^2 + \bar{\nu}^2)\,|x|_L^3 - \bar{\lambda}\,|x|_L^2 - 2\mu = 0.$$

$|x|_L$ has a maximum at $\bar{\lambda} = \bar{\nu}$, which is given by $|x|_L = 2/\pi\bar{\nu}$ or

$$k_\parallel v_E = \frac{2}{\pi\bar{\nu}}\,\omega_{pe}. \qquad (49)$$

For larger values of $k_\parallel v_E$ the plasma is stable.

We now consider the question of the maximum growth rate. We make the ansatz that this maximum is such that $\bar{\gamma} \gg (s\beta\mu)^{1/2}$, i.e., $\gamma \gg kv_i$. Then $g_i \approx k^2v_i^2/\gamma^2 = s\beta\mu/\bar{\gamma}^2$, and multiplying Eq. (35) for $r = 1$ by $[\bar{\lambda}^2 + (2\bar{\gamma} + \bar{\nu})^2]\bar{\gamma}^2$ we find

$$[\bar{\lambda}^2 + (2\bar{\gamma} + \bar{\nu})^2][\bar{\gamma}^2 + \mu(1 - 2J_0J_1/x)]$$
$$- 2\mu J_1^2[\bar{\lambda} - \bar{\nu}^2/4 + 2\mu(1 - J_0^2)x^{-2}] = 0. \qquad (50)$$

Deriving (50) with respect to $\bar{\lambda}$ and setting $d\bar{\gamma}/d\bar{\lambda} = 0$, we find

$$\bar{\lambda}\bar{\gamma}^2 = \mu J_1^2; \qquad (51)$$

therefore, $\bar{\gamma}^2 \gg \mu \gg s\beta\mu$ and our ansatz is verified. It also follows that $\bar{\lambda} \gg \mu$; this, together with $\bar{\gamma}^2 \gg \mu$, simplifies (50), which becomes

$$[\bar{\lambda}^2 + (2\bar{\gamma} + \bar{\nu})^2]\bar{\gamma}^2 = 2\mu J_1^2\bar{\lambda}. \qquad (52)$$

From (51) and (52) for $\bar{\gamma}$ and $\bar{\lambda}$ we get

$$\bar{\gamma}^3 + \frac{\bar{\nu}\bar{\gamma}^2}{2} - \frac{\mu J_1^2}{2} = 0, \qquad (53)$$

$$\bar{\lambda} = \bar{\nu} + 2\bar{\gamma}.$$

Equation (53) gives, for the maximum growth rate,

$$\bar{\gamma} = \frac{\bar{\nu}}{2}\,\phi\!\left(\frac{4J_1^2\mu}{\bar{\nu}^3}\right), \qquad (54)$$

where $\phi(\tau)$ is given by the positive root of

$$\phi^3 + \phi^2 - \tau = 0. \qquad (55)$$

For small $\mu/\bar{\nu}^3$,

$$\bar{\gamma} \approx |J_1|\left(\frac{\mu}{\bar{\nu}}\right)^{1/2}, \qquad (56)$$

and for large $\mu/\bar{\nu}^3$

$$\bar{\gamma} \approx \left(\frac{\mu J_1^2}{2}\right)^{1/3}. \qquad (57)$$

The last is the result given in Ref. 5. We notice that the maximum growth rate is very weakly dependent on $\bar{\nu}$ up to relatively large values of the damping: Equation (57) is the limit for vanishing $\bar{\nu}$, while for $\bar{\nu}$ as large as $(2J_1^2\mu)^{1/3}$ we get $\bar{\gamma} \approx 0.8(\mu J_1^2/2)^{1/3}$.

Let us briefly consider the threshold at a $r \neq 1$ resonance. From Eqs. (35)–(38) we have

$$s(1 + \beta) - \frac{x^2}{2(r^2 - 1)} - \frac{2x^{2r}}{2^{2r}(r!)^2}\,\frac{\bar{\lambda} + \mu}{\bar{\lambda}^2 + \bar{\nu}^2} \leq 0,$$

for $|x| \ll 1$; the threshold is given by the equal sign,

$$1 + \beta - \frac{r^2}{r^2 - 1}\,\frac{v_E^2}{2v_e^2} - \frac{r^{2r}s^{r-1}}{2^{2r-1}(r!)^2}\,\frac{v_E^{2r}}{v_e^{2r}}\,\frac{\bar{\lambda} + \mu}{\bar{\lambda}^2 + \bar{\nu}^2} = 0. \qquad (58)$$

To find the minimum threshold, we first maximize $(\bar{\lambda} + \mu)(\bar{\lambda}^2 + \bar{\nu}^2)^{-1}$ with respect to $\bar{\lambda}$; that maximum is $(2\bar{\nu})^{-1}$ at $\bar{\lambda} = \bar{\nu}(\mu \ll \bar{\nu})$. To go further let us consider the case $r = 2$; at $\bar{\lambda} = \bar{\nu}$ we have,

$$\frac{v_E^2}{v_e^2} = \left[\left(\frac{4\bar{\nu}}{3s}\right)^2 + \frac{4\bar{\nu}}{s}(1 + \beta)\right]^{1/2} - \frac{4\bar{\nu}}{3s}.$$

This equation gives v_E^2/v_e^2 as a function of s; the minimum of v_E^2/v_e^2 occurs at the minimum of $\bar{\nu}(s)/s$. Using (34) we find

$$2s\bar{\nu}_c = (1 - 5s)\bar{\nu}_L \qquad (\bar{\nu}_c = \nu_c/\omega_{pe},\ \bar{\nu}_L = \nu_L/\omega_{pe})$$

as the equation determining s. For $10^{-2} > \bar{\nu}_c > 10^{-4}$,[17] we obtain $s^{-1} \approx 2(\ln\bar{\nu}_c^{-1} + 8)$ and finally get

$$\frac{v_E^2}{v_e^2} \approx 3(\ln\bar{\nu}_c^{-1} + 8)^{1/2}(1 + \beta)^{1/2}\bar{\nu}_c^{1/2}$$
$$\cdot[1 - (\ln\bar{\nu}_c^{-1} + 8)^{1/2}\bar{\nu}_c^{1/2}(1 + \beta)^{-1/2}]; \qquad (59)$$

the second term in the bracket may be neglected for $\bar{\nu}_c < 10^{-3}$. From (25) and (59), and considering that now $\omega_0 \approx \omega_{pe}/2$, we obtain

$$\frac{E_m^2(\omega_{pe})}{E_m^2(\omega_{pe}/2)} \approx \frac{8}{3}(1 + \beta)^{1/2}\bar{\nu}_c^{1/2}(\ln\bar{\nu}_c^{-1} + 8)^{-1/2} \qquad (60)$$

as the ratio of the minimum field energies for excitation at ω_{pe} and $\omega_{pe}/2$. This ratio is close to $\bar{\nu}_c^{1/2}$; the minimum threshold appears to increase substantially from ω_{pe} to $\frac{1}{2}\omega_{pe}$. Indeed, the resonance weakens so fast with increasing r, that for $r > 3$ its effect is unimportant and the threshold is approximately given by the off-resonance formula, Eq. (25); for $r = 3$, the resonant effect is weak for large $\bar{\nu}_c(\sim 10^{-2})$, while for small $\bar{\nu}_c(\sim 10^{-4})$ it produces a minimum threshold which is 0.25 (0.38) times smaller than that given by (25), for $\beta = 1$ (0).

Finally, assume $\bar{\gamma} \gg \bar{\nu}$ in Eq. (50); writing $\gamma = \omega/i$ we have

$$\frac{\omega^2}{\omega_{pe}^2} = A \pm (A^2 + B)^{1/2}, \qquad (61)$$

$$A = \frac{\bar{\lambda}^2}{8} + \frac{\mu}{2}\left(1 - \frac{2J_0J_1}{x}\right),$$

$$B = \frac{\mu J_1^2}{2}\left(\bar{\lambda} + 2\mu\,\frac{1 - J_0^2}{x^2}\right). \quad (62)$$

Since A is positive ($1 > 2J_0J_1/x$), Eq. (61) has an unstable root if $B > 0$, i.e., $\omega_0^2 < \omega_{ke}^2 + 2\omega_{pi}^2(1 - J_0^2)x^{-2}$; this is our nonoscillatory instability. Another instability appears when $B < -A^2$, but now Re $\omega \neq 0$ [it may be shown that in the (x, $\bar{\lambda}$) range giving the largest $\bar{\gamma}$, in which damping and thermal motion may be neglected, (61) is valid even if Re $\omega \neq 0$]. Notice that (a) the frequency ranges for the two instabilities do not overlap, and (b) $B > 0$ yields an upper but no lower bound for ω_0, while $B < -A^2$ yields both upper and lower bounds (we comment on this in Sec. VI); one can show that, approximately, we have

$$\omega_{ke}^2 + \frac{\omega_{pi}^2}{2}\,\frac{x - 4J_1(xJ_0 - J_1)}{xJ_1^2} < \omega_0^2 < \omega_{ke}^2 + (32J_1^2)^{1/3}\omega_{pe}^{1/3}\omega_{pi}^{2/3} \quad (63)$$

for the second instability, if $x = O\,(1)$.

The above result for strong fields agrees with Nishikawa's discovery, for weak fields, of one oscillatory and one nonoscillatory instability at resonance. Equations similar to (61) were derived by Silin and Jackson, but no such clear distinction of two instabilities was made. Writing ω_{pe}^2 for ω_{ke}^2 ($T_e = 0$), Eq. (3.30) of Ref. 5 (for $n = 1$) is identical to (61) above, although Silin did not sum the Bessel series H_1 and G_1. Equation (28) of Ref. 7 differs from (61): both J_0J_1/x and $(1 - J_0^2)x^{-2}$ in (62) are replaced by $(J_0^2 + J_1^2)/2$. The error can be traced to Jackson's simplification of the system (10a, b) [Eq. (20) in his paper]. He neglected χ_α^m, $|m| > 1$, after stating that it is quite small; this is incorrect. For his case, $\omega_0 \approx \omega_{pe}$, in particular, $\chi_\alpha^m \approx \chi_\alpha^{\pm 1}m^{-2}$; thus χ_e^m is not small, while χ_i^m is of the order of $\chi_i^{\pm 1}$, which he correctly retained. He further assumed $\omega^2 \approx \omega_L^2$ and $(\omega - \omega_0)^2 \approx \omega_H^2[\omega_L^2$ and ω_H^2 given in his Eqs. (23) and (31)] in some terms of Eq. (27) of his paper, while in general, $(|\omega| - \omega_L)\omega_L^{-1} \geq O\,(1)$ as in his Eq. (29).

VI. CONCLUSION

In this paper, we derive two infinite matrices and, equating to zero the determinant of either one, obtain the dispersion relation for the electrostatic waves of a plasma in the presence of a uniform, alternating electric field; those matrices are convenient for analysis because of their compactness. Assuming $\omega_0 \gg \omega_{pi}$, we show that the waves may be classified in two groups, each satisfying a simple condition; this allows us to close the expansions in the dispersion relation. We find a nonoscillatory instability and give an explicit expression for the growth rate (which, for cold plasmas, agrees with Silin's off-resonance result) and for the threshold for excitation; we also find upper bounds for ω_0, k, and $\mathbf{k}\cdot\mathbf{v}_E$ ($\mathbf{v}_E$ being the field-induced electron velocity). When $\omega_0 \approx \omega_{pe}/r$ (r is any integer) and $k \ll \omega_{pe}/v_e$, a resonance develops (the two groups of roots of the dispersion relation coalesce) that decreases the threshold and increases the upper bound of $\mathbf{k}\cdot\mathbf{v}_E$. For $r = 1$, we give precise results for both quantities; although the threshold depends on the ion mass, its minimum does not (it agrees with the result of Nishikawa, who neglected the high-frequency motion of the ions). We obtain the maximum growth rate and show that under most conditions the result is independent of the damping, and agrees with that given by Silin. We derive definite formulas for the upper bound of ω_0. We find the threshold for $r = 2$; we also find that resonances with $r > 3$ have negligible effects, because of damping. Finally, we show that at the $r = 1$ resonance there is a second instability, which has (a) Re $\omega \neq 0$ and (b) a range of unstable frequencies with no overlap to that of the nonoscillatory instability and with both upper and lower bounds (these are given explicity in the case of a strong field).

The last instability, which was studied by DuBois and Goldman in the limit of weak fields, has not been further investigated here; it is due to the parametric coupling of the applied field with an ion acoustic and an electron plasma wave, and the bounds for ω_0 arise from the stringent requirements of frequency matching for such coupling. The nonoscillatory instability, on the other hand, must have an entirely different physical origin because (a) there is no lower bound for ω_0 and (b) Re $\omega = 0$. Our results indicate that it is caused by the field-induced streaming of electrons with respect to ions, and is, therefore, a two-stream instability.[18]

The results of Sec. IV for the threshold and the upper bound of k and $\mathbf{k}\cdot\mathbf{v}_E$ (as well as the condition for maximum growth rate when $\omega_0/\omega_{pe} \ll 1$) are similar to results of the theory of the two-stream instability, whose physical origin, as is well known, is a process of charge bunching. In the present case, of course, the streaming velocity, $\mathbf{v}_E \cos \omega_0 t$, is time modulated, and this gives rise to results with no equivalence to the usual case of uniform streaming. First, there are now two frequencies, $\mathbf{k}\cdot\mathbf{v}_E$ and ω_0, so that to the usual condition $\mathbf{k}\cdot\mathbf{v}_E \lesssim \omega_{pe}$, we should add $\omega_0 \lesssim \omega_{pe}$ (the meaning of either condition being that the frequencies of the bunching mechanism should not exceed the highest natural frequency in the plasma). Second, a resonance should develop when ω_0 (or its harmonics) are close to ω_{pe} (in the same way that the maximum growth rate for uniform streaming occurs when $\mathbf{k}\cdot\mathbf{v}_E \approx \omega_{pe}$). The results on the upper bound of ω_0 (Sec. IV) and on the resonance effects (Sec. V) agree with this physical discussion.

The field intensities required to excite the nonoscillatory instability are well within present capabilities; the minimum threshold Eq. (44), for $\beta = 1$, is 0.6^{-1} times smaller than the threshold of Ref. 1 (actually, a more detailed analysis of the DuBois–Goldman instability shows that its threshold is substantially lower than the value originally given in Ref. 1; see Refs. 3 and 6). The instability may be of relevance in the heating of plasmas produced by laser irradiation of solid particles, since initially the plasma frequency is larger than the laser frequency, ω_0 (penetration is made possible by nonlinear effects or by the diffuse nature of the vacuum–plasma interface)[19]; for $N_e = 10^{21}$ cm^{-3} and $\kappa T_e = 10^2$ eV, Eq. (44) gives 1.5×10^{13} W/cm^2 as minimum power for excitation. [We point out here that for an electromagnetic wave field our uniformity assumption is equivalent to the usual dipole approximation, and also that, even if the plasma has a sharp boundary and nonlinear effects on the penetration are neglected, penetration is possible in a narrow range of unstable frequencies, since the upper bound of ω_0 exceeds ω_{pe}; for the range $\omega_0 < \omega_{pe}$, the skin depth must be assumed large compared with k^{-1} and $\boldsymbol{\varepsilon}_e$.]

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APPENDIX

For $\sigma = r + \delta$, where r is any integer and $|\delta| \ll 1$, we have [according to Eq. (26)],

$$\sum_{p\neq\pm r} \frac{J_p^2\sigma^2}{\sigma^2 - p^2} \equiv \frac{\pi\sigma J_\sigma J_{-\sigma}}{\sin \pi\sigma} - \frac{2J_r^2\sigma^2}{\sigma^2 - r^2}$$

$$= \frac{\pi\sigma(-1)^r}{\sin \pi\sigma}\left[J_r J_{-r} + \delta \frac{\partial}{\partial\sigma} J_\sigma J_{-\sigma}\bigg|_{\sigma=r} + O(\delta^2)\right]$$

$$- \frac{2J_r^2\sigma^2}{\delta(2\sigma - \delta)}. \qquad (64)$$

Using the relations[15]

$$\frac{\partial J_\sigma}{\partial\sigma}\bigg|_{\sigma=r} = \frac{\pi}{2} Y_r + \frac{r!}{2}\left(\frac{x}{2}\right)^{-r} \sum_{m=0}^{r-1} \frac{J_m(x/2)^m}{m!\,(r-m)},$$

$$Y_r = \pi^{-1}\left\{\frac{\partial J_\sigma}{\partial\sigma}\bigg|_{\sigma=r} - (-1)^r \frac{\partial J_{-\sigma}}{\partial\sigma}\bigg|_{\sigma=r}\right\},$$

in (64) we get

$$\sum_{p\neq\pm r} \frac{J_p^2\sigma^2}{\sigma^2 - p^2} = -\frac{J_r^2}{2} + r!\,rJ_r\left(\frac{x}{2}\right)^{-r} \cdot \sum_{m=0}^{r-1} \frac{J_m(x/2)^m}{m!\,(r-m)} + O(\sigma - r),$$

from which Eq. (38) follows.

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[9] For Im $\omega \leq 0$, the analytical continuation of $\chi_\alpha{}^n(\omega)$ is obtained by following the usual Landau contour. $\chi_\alpha{}^0$ is the electric susceptibility of the α species for $E = 0$.

[10] Note that in any bounded region in the complex ω plane, we have $\chi_\alpha{}^n = O(n^{-2})$ as $|n| \to \infty$, for any physically admissible $F_\alpha{}^0$. Thus Δ is absolutely and uniformly convergent, since both $\sum_n|\chi_e{}^n| + \sum_n|\chi_i{}^n|$ and $\sum_n\sum_p|\chi_e{}^n J_{n-p}| + \sum_n\sum_p|\chi_i{}^n J_{p-n}|$ are convergent; this is also true of $\Delta_0{}^e$ if the set $\{A_e{}^n, A_i{}^n\}$ is bounded. See E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis* (Cambridge University Press, London, 1947), 4th ed., Chap. 2.

[11] Experimental results are available in this range; see E. M. Barkhudarov,. N. A. Kervalishvili, V. P. Kortkhondzhiya, N. L. Tsintsadze, and D. D. Tskhakaya, Zh. Eksp. Teor. Fiz. Pis. Red. **9**, 278 (1969) [JETP Lett. **9**, 163 (1969)].

[12] Tables of functions directly related to g, R, and I are available; see B. Fried and S. Conte, *The Plasma Dispersion Function* (Academic, New York, 1961).

[13] Silin's off-resonance growth rate may be obtained by taking $v_e = v_i = 0$.

[14] These numerical bounds appear in other problems; see T. H. Stix, *The Theory of Plasma Waves* (McGraw-Hill, New York, 1962), Sec. 9–14.

[15] Bateman Manuscript Project, *Higher Transcendental Functions*, edited by A. Erdelyi (McGraw-Hill, New York, 1953), Vol. II, Chap. 7.

[16] In the study of one instability in Ref. 7 it was assumed that either $\chi_i{}^{\pm1}/D_e{}^1$ or $\chi_i{}^{\pm1}/D_e{}^{-1}$ was not small [note that otherwise Eq. (35) of that paper would read $1 \approx 0$]. Jackson's study was based on an oversimplified dispersion relation; this will be discussed at the end of this section.

[17] Since we assume $\mu \ll \bar{\nu}$, we cannot have $\bar{\nu}_c$ much smaller than 10^{-4}.

[18] See Ref. 14, Chaps. 6, 7, and 9.

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